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by Larry D. Pinson

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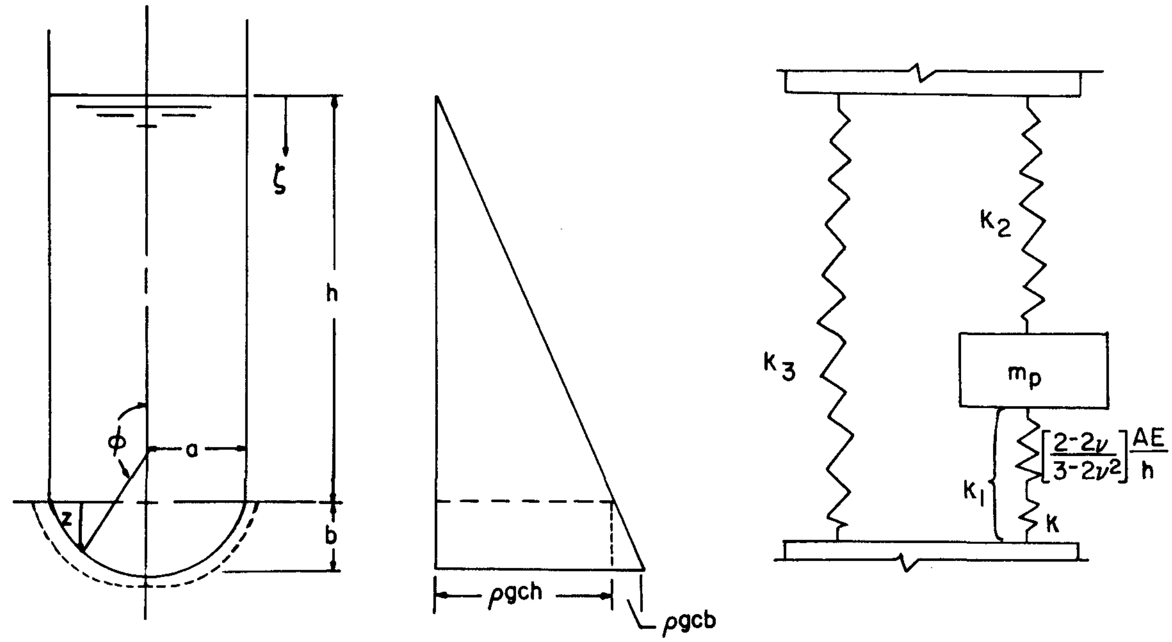
SUMMARY

An analysis using linearized membrane theory has been made to obtain spring constants for ellipsoidal bulkheads to be used in longitudinal vibration analyses of liquid-propellant launch vehicles. A closed-form solution is presented for the volume increment and first moment of the volume increment for two types of loading, namely, constant pressure and hydrostatic pressure. Plots are presented from which the volume increment, first moment of the volume increment, and spring constants for ellipsoidal bulkheads can be obtained. Use of the equations and plots is illustrated by an example.

INTRODUCTION

Stability problems involving interaction between the propulsion system and the structure have recently been encountered in launch vehicles. These problems have led to increased interest in the accurate determination of longitudinal modes and frequencies of the vehicles. However, knowledge of the elastic characteristics of structural components is required before the modes and frequencies can be calculated.

Among the structural components whose elastic properties must be determined are the propellant tanks. These tanks, for liquid-propellant launch vehicles, are typically cylindrical with ellipsoidal ends. Wood, in reference 1, presents a spring-mass representation of such a tank and shows how the spring constant can be determined by considering the bulging of the cylindrical part of the tank and neglecting the flexibility of the bulkhead. Wood then shows how the spring constant for the flexible bulkhead, once obtained, may be incorporated into this spring-mass model. The model of reference 1 is reproduced in figure 1. The spring constant associated with the bulkhead is determined from a consideration of the structural properties of shells of revolution. Treating the shell as a membrane, this consideration involves the determination of the volume change and first moment of the volume change due to pressure loading. Sylvester, in reference 2, derives a general nonlinear expression for the volume change in a shell of revolution and applies the linearized form of this expression to an ellipsoidal bulkhead subjected to constant internal pressure and



(a) Geometry and nomenclature.

(b) Pressure distribution.

(c) Equivalent spring-mass model.

Figure 1.- Ellipsoidal bulkhead under hydrostatic pressure and its equivalent spring-mass model.

hydrostatic internal pressure. Sylvester did not, however, derive expressions for the first moment of the volume increment.

The purpose of the present investigation is to determine the spring constant for an ellipsoidal bulkhead for use in a spring-mass model such as that presented in reference 1. The spring constant for an ellipsoidal bulkhead is determined by considering the change in volume, or volume increment, and the first moment of the volume increment by using an approach somewhat different to that of reference 2. Constant and hydrostatic pressures are considered in determining the deformations of the bulkhead, and the results are presented as plots which may be used to determine the spring constants for ellipsoidal bulkheads with arbitrary depth-to-radius ratios.

SYMBOLS

A	cross-sectional area of cylindrical portion of tank
a	length of semimajor axis
b	length of semiminor axis
C	constant of integration
c	positive constant denoting number of g units of axial acceleration

E Young's modulus

$F(n,\nu)$ function defined by equation (26)

$G(n,\nu)$ function defined by equation (29)

g acceleration of gravity

$H(n,\nu)$ function defined by equation (37)

h height of liquid in cylindrical portion of tank

k spring constant

M first moment of liquid volume

ΔM first moment of volume increment

m_p total mass of propellant

N_θ stress resultant in circumferential direction

N_ϕ stress resultant in meridional direction

n depth-to-radius ratio for ellipsoid of revolution, b/a

p arbitrary pressure

q ratio of liquid height h to tank radius a

$r_o = r_2 \sin \phi$

r_1 radius of curvature in meridional direction

r_2 radius of curvature in circumferential direction

S surface

t bulkhead thickness

V total liquid volume

ΔV volume increment, change in volume of bulkhead due to its deformation under load

v displacement in meridional direction

W total weight of liquid

w displacement normal to shell, positive outward

z	distance measured from base of shell
ϵ_θ	unit strain in circumferential direction
ϵ_ϕ	unit strain in meridional direction
$\bar{\zeta}$	distance from free surface of liquid volume to centroid of volume before deformation
$\Delta\bar{\zeta}$	change in location of centroid of liquid volume due to deformation of bulkhead
θ	coordinate of longitude
ν	Poisson's ratio
ξ	dummy variable
ρ	mass density
ϕ	colatitude
ω	natural circular frequency

Subscripts:

c	constant pressure
H	hydrostatic pressure
z	reference to coordinate z
ζ	reference to distance from free surface of liquid volume before deformation

ANALYSIS

The spring constant for an ellipsoidal bulkhead which is part of a liquid-propellant tank is determined by using linearized membrane theory for shells. The bulkhead is assumed to be attached to a cylindrical shell as shown in figure 1. The depth of the liquid in the cylindrical portion of the tank is h, the radius of the cylinder and the length of the semimajor axis of the ellipsoid is a, and the length of the semiminor axis of the ellipsoid is b. Only the displacements of the ellipsoid are considered herein - the contribution of the cylinder to the longitudinal spring constant is treated elsewhere (ref. 1, for example). The assumptions used herein are the same as those of reference 1; however, a brief discussion of the model is given below.

Spring-Mass Model

The spring-mass model used to represent the propellant-tank combination is shown in figure 1(c). The deflection of the mass m_p is taken to be the change in location of the centroid of the liquid due to the tank deformation. The liquid is assumed to be incompressible and to act essentially as a rigid body insofar as it becomes allowable to assume that all its mass is concentrated at its centroid. The cylindrical shell is assumed to be very thin so that bending stresses are negligible in calculating deflections. The spring constant k represents the stiffness of the bulkhead. When this spring constant is found, it is incorporated into the model as a spring in series with another spring calculated on the basis of an infinitely rigid tank bottom. The result of this combination is the spring k_1. On a qualitative basis, the model corresponds approximately to the propellant-tank interaction.

It is the purpose of this paper to establish values of k when the bulkhead is an ellipsoid of revolution.

Spring Constant

For the purpose of finding the spring constant for the bulkhead, the cylindrical portion of the tank is assumed to be infinitely rigid. If the liquid is then assumed incompressible, any movement of the centroid will be the result of deformation of the bulkhead.

The loading on the bulkhead consists of a hydrostatic component of pressure and a constant component equal to the pressure due to the weight of the liquid above the bulkhead-cylinder connection. The loading due to tank pressurization (ullage pressure) is assumed to be constant with time and therefore may be neglected in the dynamic analysis.

The spring constant k used herein is defined as the ratio of the total force acting on the bulkhead due to the weight of the liquid to the displacement of the centroid of the liquid volume. This total force is equal to ρcgV where V is the volume of the mass of liquid, c is a positive constant denoting the number of g units of acceleration, and ρ is the mass density.

With this definition of k, the problem is to find the resultant force acting on the bulkhead and the displacement of the centroid of the mass of the liquid due to the deformation of the bulkhead.

The resultant force is determined as the weight of the mass of liquid in the tank under an acceleration relative to the tank. That is

$$W = \rho cgV \tag{1}$$

where W is the weight of the mass of liquid.

The change in centroidal distance is determined by considering the volume change of the bulkhead and the first moment of this volume change.

The equation which locates the centroid of the liquid is

$$\bar{\zeta} = \frac{M_\zeta}{V} \tag{2}$$

where $\bar{\zeta}$ is the distance from the free surface of the mass of liquid to its centroid before tank deformation and M_ζ is the first moment of the volume occupied by the mass of liquid with the reference axis located at the free surface of the mass of liquid. (See fig. 1.)

After the bulkhead deforms, the volume of the bulkhead will change by some amount ΔV. The free surface of the liquid in the tank will drop by an amount which gives a volume change ΔV in the cylindrical portion of the tank. Let ΔM_Z be the first moment of the volume change in the bulkhead about an axis located at the junction of the bulkhead and cylinder. Then the change in the first moment of the volume about an axis located at the free surface of the mass of liquid before tank deformation is

$$\Delta M_\zeta = \Delta M_Z + h\Delta V - \frac{(\Delta V)^2}{2\pi a^2} \tag{3}$$

where ΔM_ζ is the change in first moment of the volume occupied by the mass of liquid about an axis located at the free surface of the mass of liquid before tank deformation, and a is the radius of the cylinder. Since the volume of the liquid does not change, the change in centroidal distance is

$$\Delta\bar{\zeta} = \frac{\Delta M_Z}{V} + \frac{h\Delta V}{V} - \frac{(\Delta V)^2}{2\pi a^2 V} \tag{4}$$

where $\Delta\bar{\zeta}$ is the change in the centroidal distance of the mass of liquid due to deformation of the bulkhead.

By the previous definition the spring constant for the bulkhead is

$$k = \frac{\rho c g V^2}{\Delta M_Z + h\Delta V - \frac{(\Delta V)^2}{2\pi a^2}} \tag{5}$$

where k is the spring constant. The quantity $(\Delta V)^2/2\pi a^2$ is small when compared with the other terms of the denominator of equation (5) and may be neglected so that

$$k = \frac{\rho c g V^2}{\Delta M_Z + h\Delta V} \tag{6}$$

The neglect of this quantity is consistent with other linearizing assumptions made herein.

The quantity V is found from the geometry of the tank but the quantities ΔM_Z and ΔV must be found by integrating the normal displacements due to the load over the surface of the bulkhead.

Volume Increment

The equation for the volume increment ΔV for a membrane shell of revolution with arbitrarily shaped meridian is derived in this section. A typical shell of revolution and the coordinate system is illustrated in figure 2.

It is assumed in the following derivation that bending stresses are negligible and that changes in the radii of curvature are negligible. A consequence of the assumption that bending stresses are negligible is that there is no control over the normal displacement at the boundary of the shell. This condition makes it impossible to match the displacements of the bulkhead and the cylinder at the boundary. It is considered however that this is a localized effect and will not substantially affect the overall distribution of the displacements.

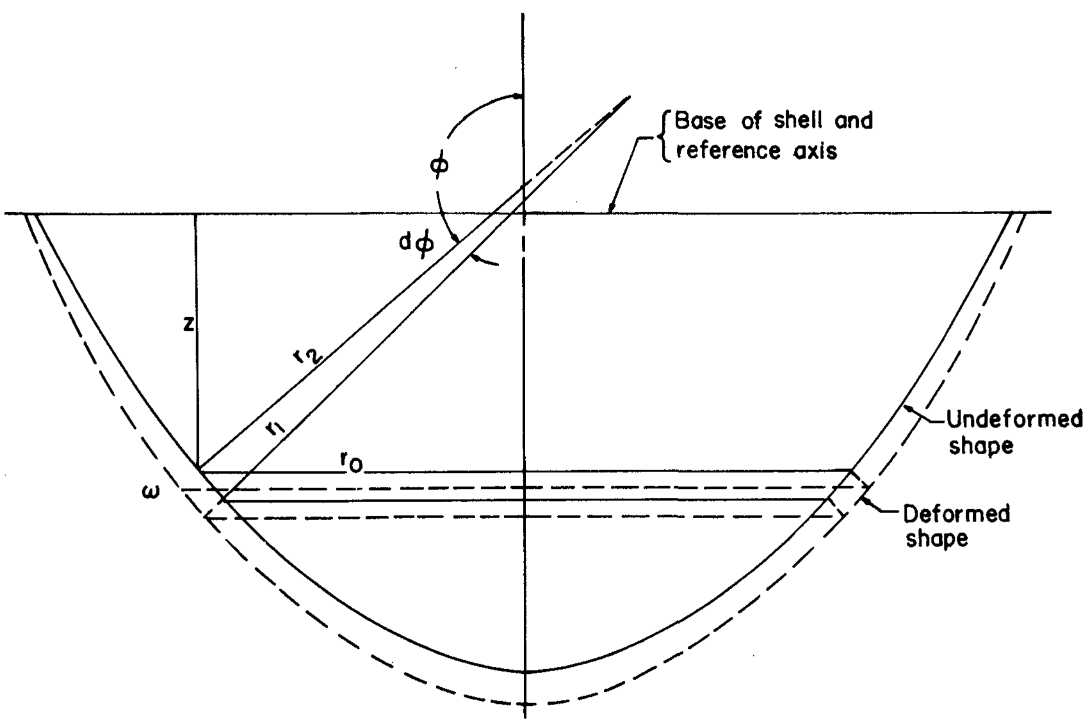


Figure 2.- Typical shell of revolution.

From figure 2 the volume increment due to a loading which produces a normal displacement w is

$$\Delta V = 2\pi \int_{\phi_1}^{\phi_2} r_o r_1 w \, d\phi \tag{7}$$

where

$$r_o = r_2 \sin \phi \tag{8}$$

r_1 is the principal radius of curvature in the meridional direction ϕ; r_2 is the principal radius of curvature in the circumferential direction θ; and ϕ_1 and ϕ_2 are the lower and upper limits of the shell, respectively. Reference 3 gives the displacements for axisymmetric loading as:

$$w = r_2 \epsilon_\theta - v \cot \phi \tag{9}$$

$$v = \sin \phi \left[\int \frac{f(\phi)}{\sin \phi} \, d\phi + C \right] \tag{10}$$

where C is a constant of integration which is determined from the requirement that the meridional displacement v have some known value at a particular value of ϕ and

$$f(\phi) = r_1 \epsilon_\phi - r_2 \epsilon_\theta \tag{11}$$

In these equations ϵ_ϕ and ϵ_θ denote unit strains in the meridional and circumferential directions, respectively. Substitution of the equations for r_o, w, and v into equation (7) yields the following expression for the volume increment

$$\Delta V = 2\pi \int_{\phi_1}^{\phi_2} r_1 r_2^2 \epsilon_\theta \sin \phi \, d\phi - 2\pi \int_{\phi_1}^{\phi_2} r_1 r_2 \left[\int \frac{f(\phi)}{\sin \phi} \, d\phi + C \right] \sin \phi \cos \phi \, d\phi \tag{12}$$

If it is required that the meridional displacement v be zero at the lower limit of the shell, the constant C may be eliminated by putting limits of integration on the bracketed integral. Introduce a dummy variable of integration into the integrand of the bracketed integral and place the limits ϕ_1 and ϕ on the integral. The result is

$$\Delta V = 2\pi \int_{\phi_1}^{\phi_2} r_1 r_2^2 \epsilon_\theta \sin \phi \, d\phi - 2\pi \int_{\phi_1}^{\phi_2} r_1 r_2 \left[\int_{\phi_1}^{\phi} \frac{f(\xi)}{\sin \xi} \, d\xi\right] \sin \phi \cos \phi \, d\phi \tag{13}$$

where ξ is a dummy variable of integration.

First Moment of the Volume Increment

In the development of the equation for the first moment of the volume increment, the same basic assumptions which were made for the development of the equation for the volume increment are made.

From figure 2, the first moment of the volume increment about the base of the shell due to a loading which produces a deflection w is

$$\Delta M = 2\pi \int_{\phi_1}^{\phi_2} r_o r_1 z w \, d\phi \tag{14}$$

where z is measured from the base of the shell. Following the same procedure as used in the previous section for determining volume increment, the expression for the first moment of the volume increment about the base of the shell for axisymmetric loading is found to be

$$\Delta M = 2\pi \int_{\phi_1}^{\phi_2} r_1 r_2^2 z \epsilon_\theta \sin \phi \, d\phi - 2\pi \int_{\phi_1}^{\phi_2} r_1 r_2 z \left[\int_{\phi_1}^{\phi} \frac{f(\xi)}{\sin \xi} \, d\xi\right] \sin \phi \cos \phi \, d\phi \tag{15}$$

where the requirement has again been made that the meridional displacement v is zero at the base of the shell.

Application to the Ellipsoidal Bulkhead

In order to apply the general expressions for the volume increment and the first moment of the volume increment (eqs. (13) and (15)), several quantities which are interrelated through the loading and shell geometry must be established. The quantities in equations (13) and (15) which are related to the loading are the unit strains ϵ_ϕ and ϵ_θ. These strains are related to the stress resultants through Hooke's law and the stress resultants are related to the loading through the equilibrium of forces on the shell and the geometry of the shell. Specifically, Hooke's law states that

$$\epsilon_\phi = \frac{1}{Et}\left(N_\phi - \nu N_\theta\right) \tag{16}$$

and

$$\epsilon_\theta = \frac{1}{Et}\left(N_\theta - \nu N_\phi\right) \tag{17}$$

where E is Young's modulus; t is the thickness of the shell; ν is Poisson's ratio; and N_ϕ and N_θ are the stress resultants in the meridional and circumferential directions, respectively. To find the stress resultants the radii of curvature of the shell must be known. Also needed to apply the expression for the first moment of the volume increment is the quantity z which is the distance from the base of the shell to any point on the bulkhead. For the ellipsoid of revolution these quantities are: (See ref. 3.)

$$r_1 = \frac{n^2 a}{\left(\sin^2\phi + n^2\cos^2\phi\right)^{3/2}} \tag{18}$$

$$r_2 = \frac{a}{\left(\sin^2\phi + n^2\cos^2\phi\right)^{1/2}} \tag{19}$$

$$z = \frac{-n^2 a \cos \phi}{\left(\sin^2\phi + n^2\cos^2\phi\right)^{1/2}} \tag{20}$$

where n is the depth-to-radius ratio of the ellipsoidal bulkhead. The pertinent range of ϕ is $\frac{\pi}{2} \leq \phi \leq \pi$.

Stresses in an ellipsoidal bulkhead.- The stress resultants for the ellipsoidal bulkhead are found from the equilibrium equations and the external forces. These forces are related to the pressure distribution. The loadings, or pressures, being considered herein are divided into two components as illustrated in figure 1 - a constant normal pressure equal to ρcgh and a hydrostatic pressure equal to ρcgz. For these loadings, the membrane theory for shells of revolution (ref. 3, for instance) gives the following stress resultants for the two components of the load on the ellipsoidal bulkhead:

For constant pressures -

$$N_\phi = \frac{\rho cgha}{2\left(\sin^2\phi + n^2\cos^2\phi\right)^{1/2}} \tag{21}$$

and

$$N_\theta = \frac{\rho cgha}{\left(\sin^2\phi + n^2\cos^2\phi\right)^{1/2}} - \frac{\rho cgha}{2n^2}\left(\sin^2\phi + n^2\cos^2\phi\right)^{1/2} \tag{22}$$

For hydrostatic pressure -

$$N_\phi = \frac{\rho cga^2n}{3}\left[\frac{n^3\cos^3\phi + \left(\sin^2\phi + n^2\cos^2\phi\right)^{3/2}}{\sin^2\phi\left(\sin^2\phi + n^2\cos^2\phi\right)}\right] \tag{23}$$

and

$$N_\theta = \frac{-\rho cga^2n^2\cos\phi}{\left(\sin^2\phi + n^2\cos^2\phi\right)} - \frac{\rho cga^2}{3n}\left[\frac{n^3\cos^3\phi + \left(\sin^2\phi + n^2\cos^2\phi\right)^{3/2}}{\sin^2\phi}\right] \tag{24}$$

Having obtained the stress resultants, the strains may be found by substituting the stress resultants for the particular component of loading being considered into equations (16) and (17).

<u>Volume increment for an ellipsoidal bulkhead</u>.- The volume increment for an ellipsoidal bulkhead is obtained by substituting the strains for the appropriate loading condition into equation (13), the general expression for the volume increment. If the thickness of the bulkhead is variable the strains must contain t as a function of ϕ. In the present analysis the bulkhead thickness is assumed to be constant. It is also assumed that the bulkhead is one-half of an ellipsoid of revolution, thus giving the range of integration $\frac{\pi}{2} \leqq \phi \leqq \pi$. The integrations are straightforward, and give the following results for the volume increment.

For constant internal pressure, after making the substitution $\xi = \cos\phi$, it is found that

$$\Delta V_c = \frac{\pi\rho cgha^4}{2Et}\left[F(n,\nu)\right] \tag{25}$$

where

$$F(n,\nu) = \left(\frac{15 - 12\nu}{8} - \frac{3}{4n^2}\right) + \left[\frac{(15 - 12\nu)n^2}{8} - 2 + \nu + \frac{1}{n^2}\right]\frac{\ln\left(\frac{1 + \sqrt{1 - n^2}}{1 - \sqrt{1 - n^2}}\right)}{2\sqrt{1 - n^2}} \tag{26}$$

The function $F(n,\nu)$ is plotted against n in figure 3 for various values of Poisson's ratio. Equation (25) agrees with the result of reference 2. The volume increment for a hemispherical dome can be found by taking the limit of ΔV_c given in equation (25) as n approaches unity. The result for a hemisphere is found to be

$$\Delta V_c = \frac{\pi\rho cgha^4}{Et}(1 - \nu) \tag{27}$$

The simplicity of the geometry of the hemisphere makes the calculation of the volume increment for this case quite simple and the calculation provides an independent check, though not an absolute proof, of the validity of the expression for the volume increment for the ellipsoidal bulkhead. An entirely separate calculation was made for the hemisphere and the result agreed with equation (27).

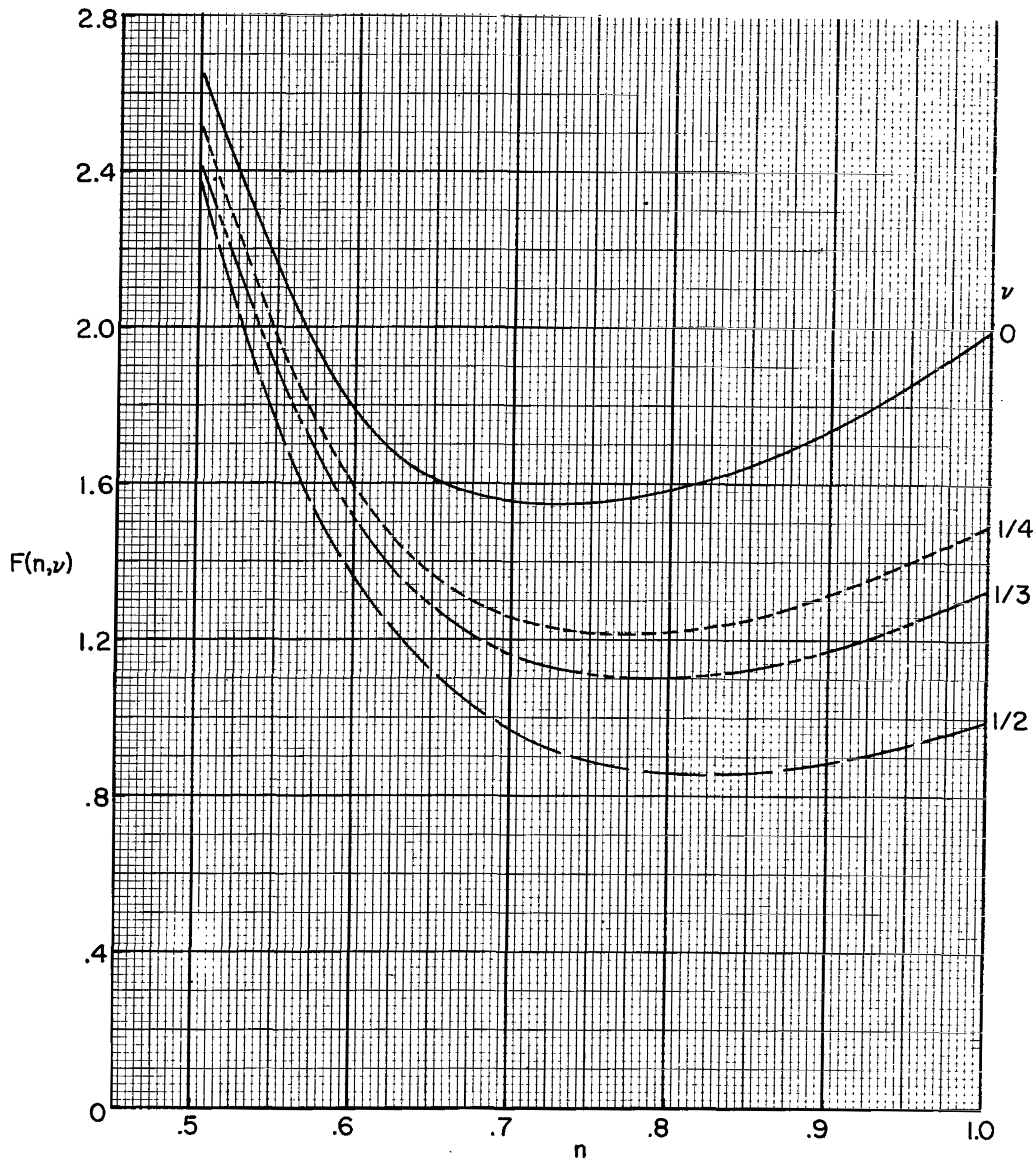


Figure 3.- Variation of $F(n,\nu)$ with depth-to-radius ratio for various values of Poisson's ratio.

For hydrostatic pressure, after making the substitution $\xi = \sin^2\phi$, it is found that

$$\Delta V_H = \frac{\pi\rho cga^5}{2Et}\left[G(n,\nu)\right] \tag{28}$$

where

$$G(n,\nu) = \frac{-13 + 37n^2 + 10n^3 + \frac{22n^4}{1 + n}}{45n} - \frac{2\nu n}{9}\left(4 + \frac{n^2}{1 + n}\right)$$

$$+ \left(2 - n^2 + 2\nu n^2\right)\frac{\sqrt{1 - n^2}}{6n} \ln\left(\frac{1 + \sqrt{1 - n^2}}{1 - \sqrt{1 - n^2}}\right) \tag{29}$$

Equation (29) is plotted in figure 4 against n for several values of Poisson's ratio. The volume increment for the hemisphere subjected to hydrostatic pressure is found by letting n approach unity in equation (28). The result is

$$\Delta V_H = \frac{\pi\rho cga^5}{2Et}(1 - \nu) \tag{30}$$

Separate calculation of the volume increment for the hemisphere subjected to hydrostatic pressure yields a result which is in agreement with equation (30).

Equation (29) differs from the result presented in reference 2 in that the quantity $2\nu n^2$ in the last term of the equation is νn^2 in reference 2. Communication with the author of reference 2 confirmed the validity of the expression as given in equation (29).

First moment of volume increment for an ellipsoidal bulkhead.- Before proceeding with the development of the relations for the first moment of the volume increment the following theorem is considered which is presented in reference 4 on the first moment of the volume increment in any shell: Let p be an arbitrary pressure distribution and w_H be the normal displacements due to a hydrostatic pressure resulting from a liquid whose unit weight is ρcg. The first moment of the volume increment due to the arbitrary pressure distribution is the surface integral

$$\Delta M = \int_S p \frac{w_H}{\rho cg} dS \tag{31}$$

This theorem can be used to determine the first moment due to constant pressure. If p is selected to be the constant pressure ρcgh, equation (31) becomes

$$\Delta M_c = h \int_S w_H \, dS \tag{32}$$

Note that, by definition, the volume increment due to hydrostatic pressure can be written as

$$\Delta V_H = \int_S w_H \, dS \tag{33}$$

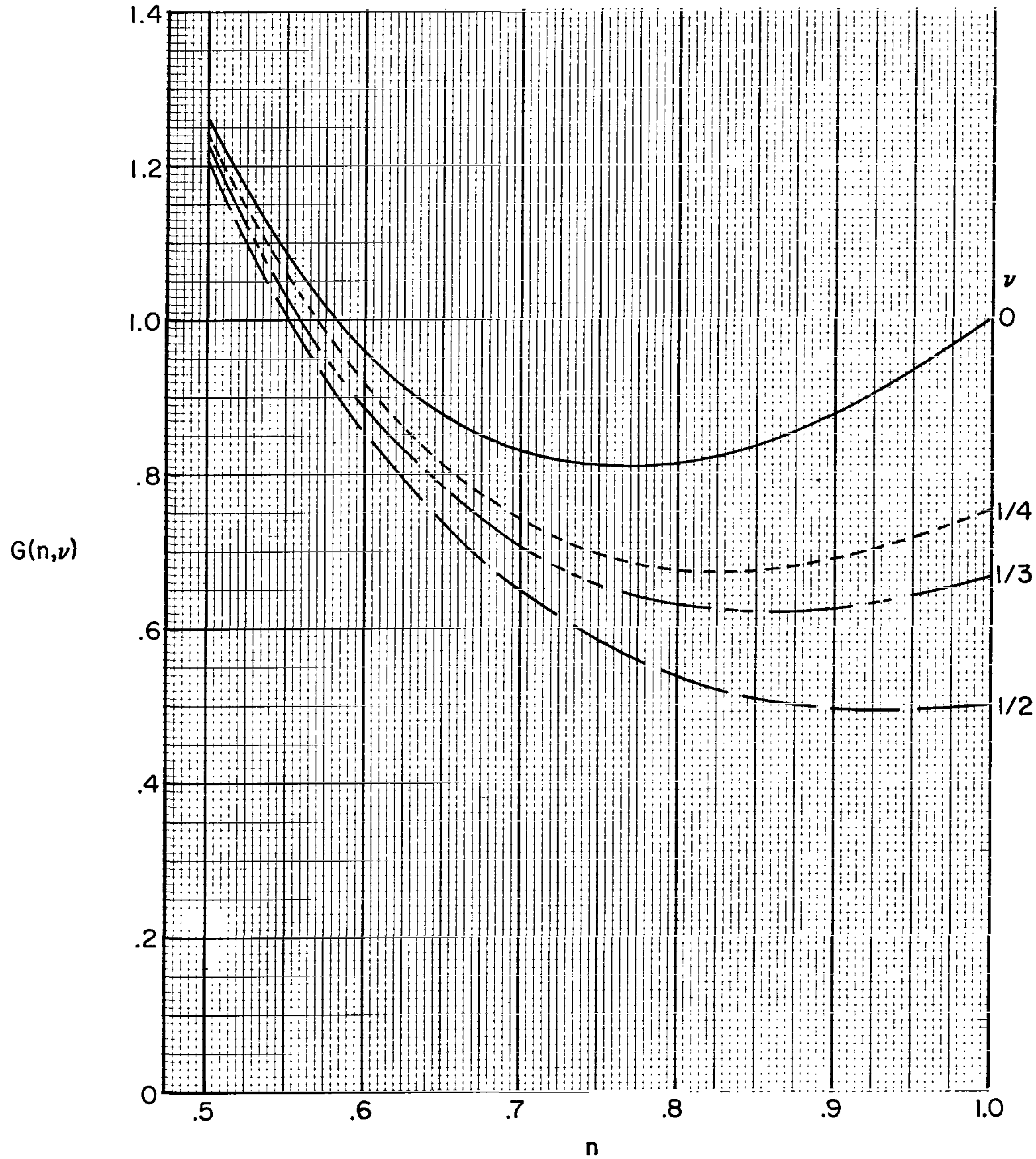


Figure 4.- Variation of $G(n,\nu)$ with depth-to-radius ratio for various values of Poisson's ratio.

Therefore, the first moment of the volume increment due to constant pressure is

$$\Delta M_c = h \Delta V_H \tag{34}$$

Substitution of equation (28) into equation (34) gives the first moment of the volume increment created by the constant component of pressure

$$\Delta M_c = \frac{\pi \rho c g h a^5}{2Et}\left[G(n,\nu)\right] \tag{35}$$

where $G(n,\nu)$ is given by equation (29).

Unfortunately no such simple relationship exists for the first moment of the volume increment caused by the hydrostatic component of the pressure and one must resort to the method used in obtaining the volume increment.

To find the first moment of the volume increment which is caused by the hydrostatic component of the pressure, the equations for the stress resultants, equations (23) and (24) are substituted into equations (16) and (17) to obtain the strains, and the resulting expressions are substituted into equation (15). This expression is then integrated over the range $\frac{\pi}{2} \leqq \phi \leqq \pi$ and it is found that

$$\Delta M_H = \frac{\pi \rho c g a^6}{2Et}\left[H(n,\nu)\right] \tag{36}$$

where

$$H(n,\nu) = \frac{n^2}{27}\left[27 - 16n - \nu(24n - 3)\right] + \left[2 - 6n^2 + 3n^4 - \nu n^2\left(2 - n^2\right)\right]\frac{\ln\left(\frac{1 + \sqrt{1 - n^2}}{1 - \sqrt{1 - n^2}}\right)}{9\sqrt{1 - n^2}} + \frac{n^4(3 - \nu)}{18}\left[\frac{2\sqrt{1 - n^2} - n^2 \ln\left(\frac{1 + \sqrt{1 - n^2}}{1 - \sqrt{1 - n^2}}\right)}{2\left(1 - n^2\right)\sqrt{1 - n^2}}\right] + \frac{8n^2(1 + \nu)}{9}\ln\left(\frac{1 + n}{n}\right) \tag{37}$$

The function $H(n,\nu)$ is plotted in figure 5 against n for various values of Poisson's ratio.

When equation (36) is specialized to the hemisphere by letting n approach unity, the following expression results for the first moment of the volume increment in a hemisphere subjected to hydrostatic pressure:

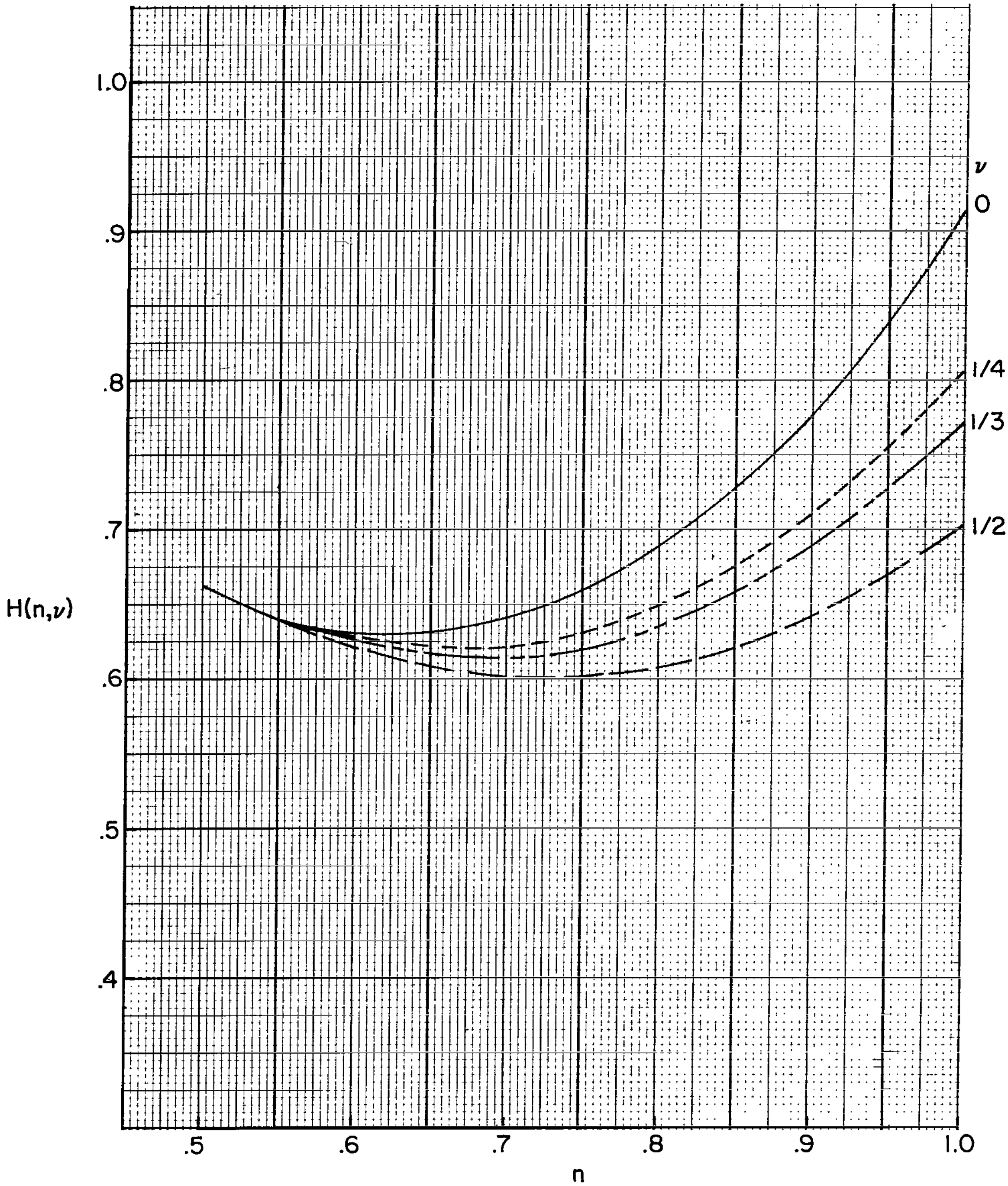


Figure 5.- Variation of $H(n,\nu)$ with depth-to-radius ratio for various values of Poisson's ratio.

$$\Delta M_H = \frac{\pi\rho cga^6}{2Et}\left[\frac{4(2 - 7\nu)}{27} + \frac{8(1 + \nu)}{9} \ln(2)\right] \tag{38}$$

Separate calculations for the hemisphere, as in the checks on the volume increment, yield an expression which is in agreement with equation (38).

Spring Constants for Ellipsoidal Bulkheads

Development of equations.- With expressions for the volume increment and first moment of the volume increment, the equation for the spring constant can be developed according to equation (6).

The volume of liquid within the tank is

$$V = \frac{\pi a^3}{3}(3q + 2n) \tag{39}$$

where q is the ratio of the height of liquid within the tank h to the radius of the tank a, and n is the depth-to-radius ratio of the ellipsoidal bulkhead.

Adding the volume changes for the two components of loading and the first moments of the volume changes and substituting these sums, along with equation (39) for the volume into equation (6) yields the following expression for the spring constant for the ellipsoidal bulkhead

$$k = Et \frac{2\pi(3q + 2n)^2}{9\left[H(n,\nu) + 2qG(n,\nu) + q^2F(n,\nu)\right]} \tag{40}$$

where the functions $F(n,\nu)$, $G(n,\nu)$, and $H(n,\nu)$ are defined by equations (26), (29), and (37), respectively.

Equation (40) gives the spring constant for an ellipsoidal bulkhead subjected to constant and hydrostatic internal pressure. This equation shows that the spring constant is a function of the geometry of the ellipsoid - the ratio n; the height of the liquid in the tank - the ratio q; Young's modulus; the bulkhead thickness; and Poisson's ratio. The spring constant does not depend on the radius of the tank directly - the radius affects the spring constant only through the ratios n and q.

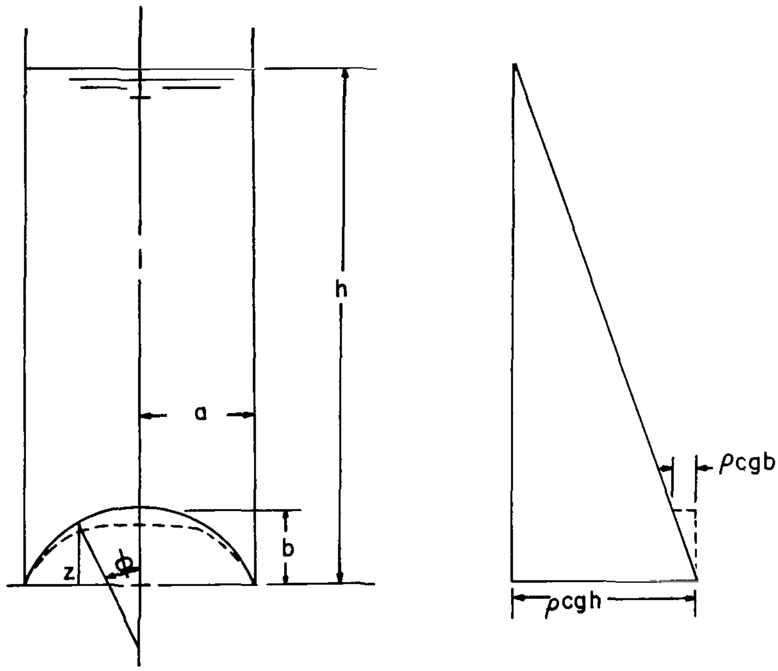
In certain launch vehicles the oxidizer and fuel are separated by a bulkhead which is common to both tanks, and geometry similar to that shown in figure 6 results. The possibility of buckling due to the resulting compressive stresses in the bulkhead is offset by pressurizing the aft tank. For this configuration the volume change and first moment of the volume change may be calculated by using the functions already developed by finding the volume change or first moment of the volume change due to a constant pressure ρcgh and subtracting the volume change or first moment of the volume change due to a hydrostatic pressure (shown by the dotted line on the pressure distribution of fig. 6).

The equation for the volume of the liquid in the tank is

$$V = \frac{\pi a^3}{3}(3q - 2n) \quad (41)$$

where the quantities V, a, q, and n have the same meaning as before.

Repeating the procedure used in the development of equation (40) and noting that the terms associated with the hydrostatic component of pressure are subtracted, rather than summed, the following equation for the spring constant results:



(a) Geometry for an inverted bulkhead.

(b) Pressure distribution.

Figure 6.- Common ellipsoidal bulkhead under hydrostatic pressure.

$$k = Et \frac{2\pi(3q - 2n)^2}{9\left[H(n,\nu) - 2qG(n,\nu) + q^2F(n,\nu)\right]} \quad (42)$$

The range of q in this case must be restricted to the range $q \geqq n$ because it is assumed that the liquid completely covers the bulkhead.

If the height of liquid is very large compared with the depth of the ellipsoidal bulkhead, the contribution of the hydrostatic component of pressure to the spring constant will become small compared with the constant component of pressure and the expression for the spring constant will approach an aysmptotic value. This asymptotic value is found by taking the limit of the expression for the spring constant as q increases without bound. Taking this limit yields

$$k = Et \frac{2\pi}{F(n,\nu)} \quad (43)$$

Properties of equations.- If equation (40) is divided on both sides by the quantity Et, differentiated with respect to q, and the derivative set equal to zero, an equation results which gives the value of q for which the quantity k/Et is a maximum. This equation is

$$q = -\frac{3H(n,\nu) - 2nG(n,\nu)}{3G(n,\nu) - 2nF(n,\nu)} \tag{44}$$

If this value of q is plotted against n for a value of Poisson's ratio of 1/3, two branches result as shown in figure 7. These branches become asymptotic to the value $n = 0.827$. A similar procedure applied to equation (42) yields the same expression except the sign of the right side of the equation is changed from negative to positive. The branch having negative values is of no interest in the discussion of equation (40) because the analysis does not hold when the liquid surface is below the ellipsoid-cylinder connection in which case a separate problem must be solved. However, for values of n greater than 0.827, positive values of q exist where the quantity k/Et is maximum. Figure 8 shows this effect. The curve for $n = 1$, that is, the hemisphere has a maximum at $q = 1.48$. For values of n less than 0.827, no relative maximum exists. Thus, for values of n less than 0.827, and for a Poisson's ratio of 1/3, the spring constant will not exceed the asymptotic value given by equation (43). In discussing equation (42), opposite conclusions may be drawn. Since the sign of the right side of equation (44) is changed the negative branch of the plot of figure 7 becomes the positive branch and the positive branch becomes negative. Thus, for values of n greater than 0.827, no finite value of q exists for which the quantity k/Et is a maximum and it will never exceed the asymptotic value given by equation (43). For values of n less than 0.827, positive values of q which are greater than n exist such that the quantity k/Et is maximum. This effect is shown in figure 9.

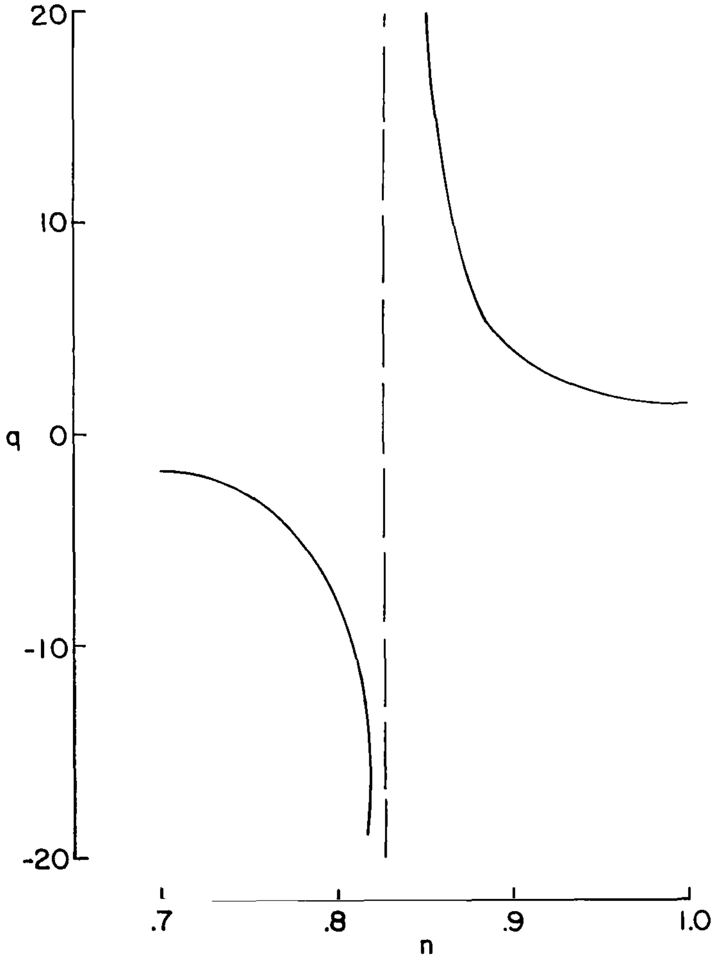


Figure 7.- Variation of critical value of height-to-radius ratio with depth-to-radius ratio for a Poisson's ratio of 1/3.

The derivation did not include the deformations of the cylindrical portion of the tank.

However, this effect can be included in the total representation for the tank as described in reference 1 and illustrated in the following section.

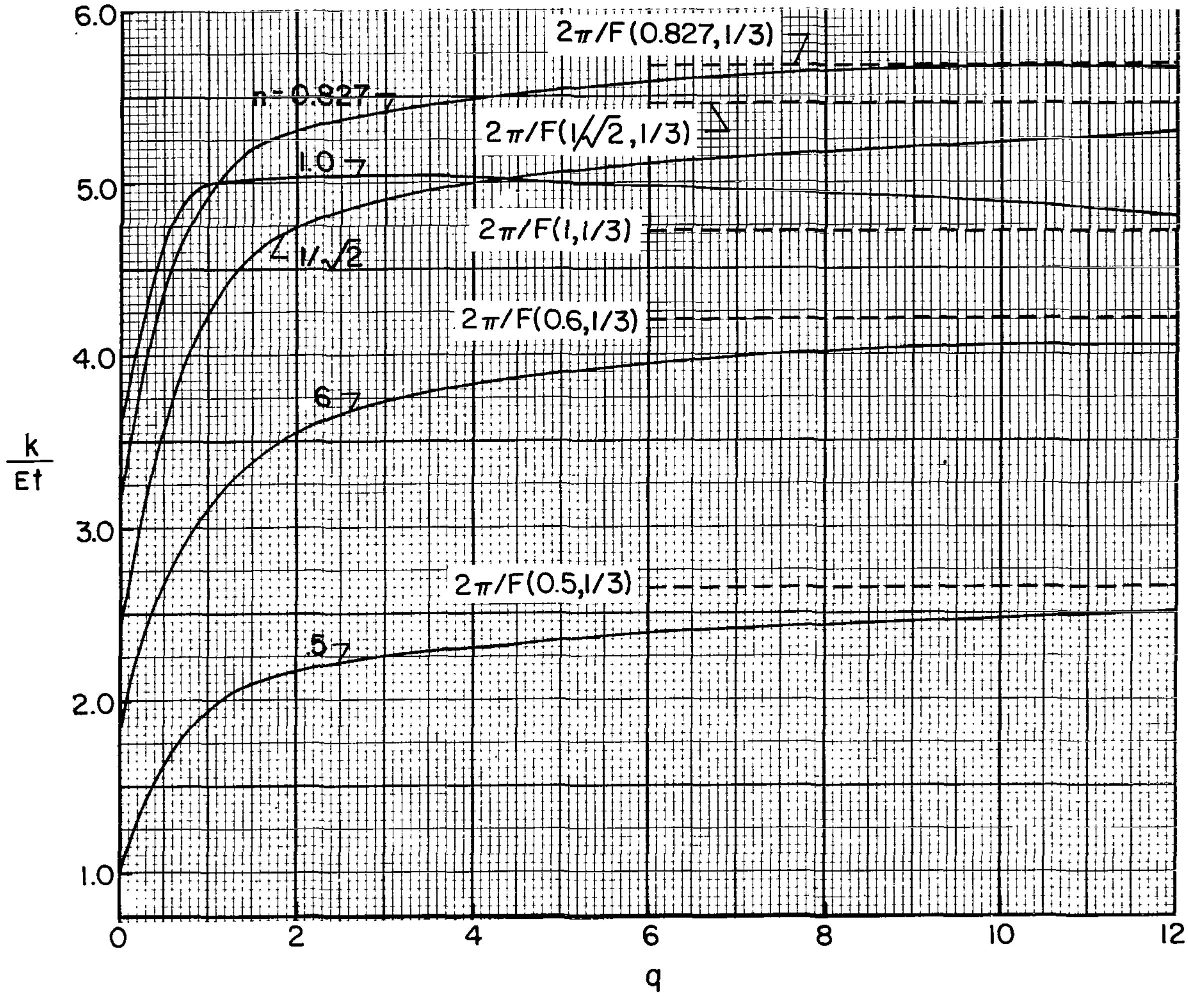


Figure 8.- Variation of spring constant for ellipsoidal bulkhead with height-to-radius ratio for several values of n and Poisson's ratio of 1/3.

Illustrative Example

In order to clarify the procedure used in obtaining the spring constant and the process of combining the bulkhead spring constant with that of the cylindrical portion of the tank, an example is presented using typical parameters.

Suppose it is desired to find the equivalent spring-mass system for a tank having the following parameters:

Young's modulus, E, psi	10^7
Bulkhead thickness, t, in.	0.05
Depth-to-radius ratio, n	$1/\sqrt{2}$
Tank radius, a, in.	60
Poisson's ratio, ν	1/3
Liquid height, h, in.	45

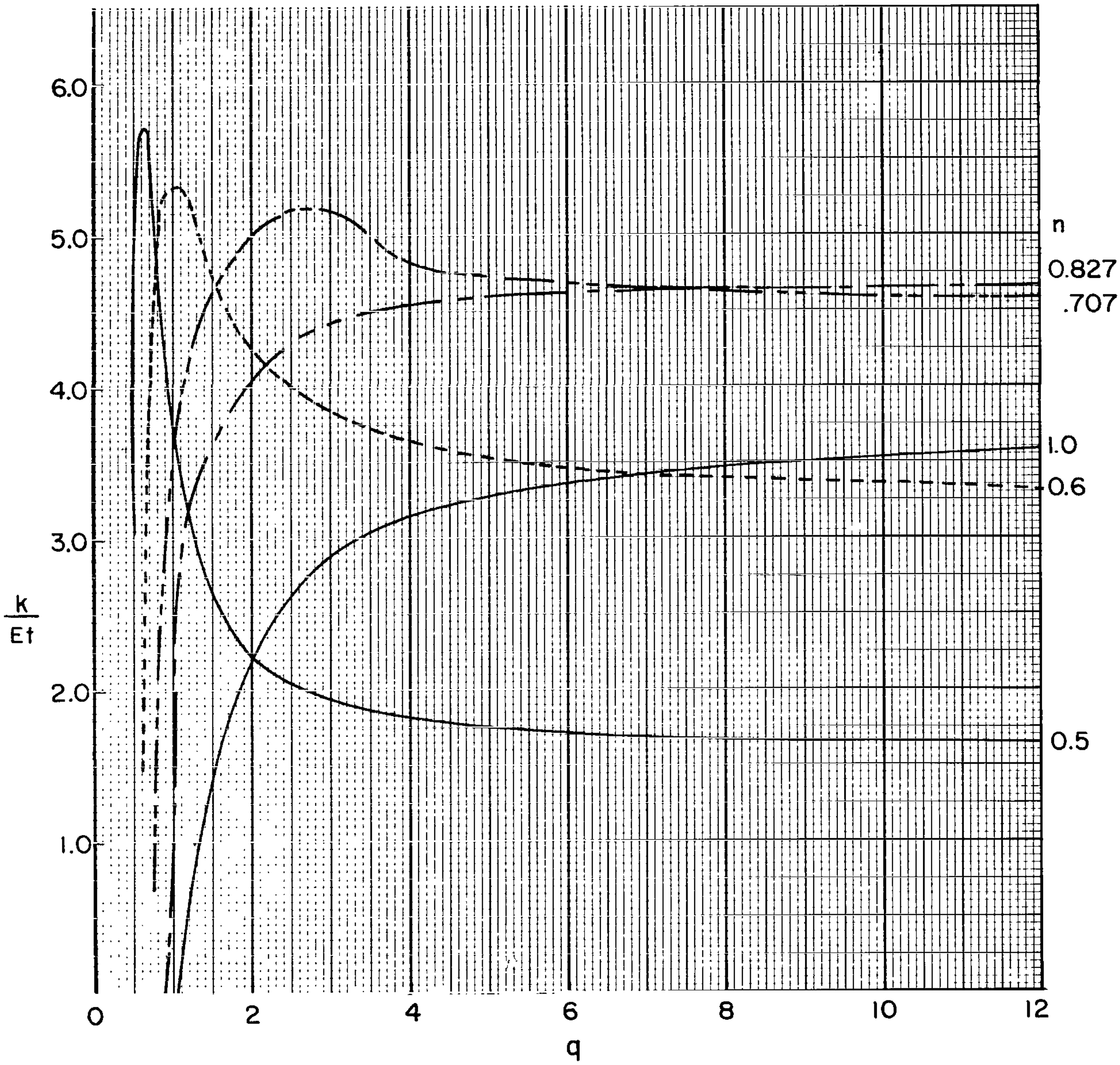


Figure 9.- Variation of spring constant for inverted ellipsoidal bulkhead with height-to-radius ratio for several values of n and Poisson's ratio of 1/3.

The quantity q is $\frac{45}{60} = 0.75$. From figures 3, 4, and 5, it is found that $F(n,\nu) = 1.150$, $G(n,\nu) = 0.695$, and $H(n,\nu) = 0.615$. Substitution of these values into equation (40) yields a spring constant for the bulkhead of 2.034×10^6 lb/in.

The spring constant just determined is only part of the complete stiffness representation for the tank-liquid combination. The complete representation, from reference 1, is illustrated in figure 1. The equations used to obtain the various spring constants are taken directly from reference 1; however, equation A.21 on page 81 of the reference should read $\bar{K}_1 = \frac{\nu \bar{K}_2}{1 - \nu}$, not $\bar{K}_1 = \frac{\bar{K}_2}{1 - \nu}$.

$$k_1 = \frac{\left(\frac{2 - 2\nu}{3 - 2\nu^2}\right)\frac{AE}{h} k}{\left(\frac{2 - 2\nu}{3 - 2\nu^2}\right)\frac{AE}{h} + k} \tag{45}$$

$$k_2 = \frac{\nu k_1}{1 - \nu} \tag{46}$$

$$k_3 = \frac{AE}{h} - \nu k_1 \tag{47}$$

Substituting the given values into equations (45), (46), and (47) and assuming the cylindrical tank wall to have the same thickness as that of the bulkhead, the following results are obtained: $k_1 = 1.011 \times 10^6$ lb/in., $k_2 = 0.506 \times 10^6$ lb/in., and $k_3 = 3.852 \times 10^6$ lb/in.

The effect of incorporating the spring constant for the bulkhead into the spring-mass model of figure 1 may be evaluated by comparing the values obtained from equations (45), (46), and (47) with those obtained by assuming that the bulkhead is infinitely stiff. If the bulkhead is rigid equations (46) and (47) are valid but equation (45) becomes

$$k_1 = \frac{(2 - 2\nu)}{\left(3 - 2\nu^2\right)} \frac{AE}{h} \tag{48}$$

By using equation (48) along with equations (46) and (47), the following results are obtained for the system with a rigid bulkhead: $k_1 = 2.011 \times 10^6$ lb/in., $k_2 = 1.005 \times 10^6$ lb/in., and $k_3 = 3.519 \times 10^6$ lb/in. The effect of including bulkhead flexibility, therefore, is a reduction of 50 percent in both k_1 and k_2 and an increase of 9 percent in k_3.

The effect of bulkhead flexibility on the frequency may be determined by assuming the ends to be fixed. The uncoupled liquid frequency is

$$\omega = \sqrt{\frac{k_1 + k_2}{m_p}} \tag{49}$$

where m_p is the actual liquid mass. Since the frequency varies with the 1/2 power of the sum $k_1 + k_2$, the effect of including the bulkhead spring constant is a 42-percent reduction in the frequency for this example.

CONCLUDING REMARKS

An analysis was made to obtain spring constants for ellipsoidal bulkheads to be used in longitudinal vibration analyses of liquid-propellant launch vehicles. The analysis was based on linear membrane theory of shells. Closed-form expressions for the spring constant for ellipsoidal bulkheads subjected to constant and hydrostatic pressure have been presented. The volume increment and first moment of the volume increment for these pressures were derived. Plots are presented to aid in the evaluation of the spring constants for bulkheads of arbitrary depth-to-radius ratios. The relation of the spring constant of the bulkhead to the entire tank model has been discussed. The procedure has been illustrated by means of an example. The example showed that including the bulkhead flexibility caused a significant change in the frequency of the particular system considered.

The work presented herein could be extended by computing the spring constant for other cases, among which are the following: (1) the case in which the level of the liquid is below the ellipsoid-cylinder connection, or (2) by computing the spring constant for another configuration, say a conical shell. The work could further be extended by directing efforts toward the improvement of the propellant-tank model.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., August 13, 1964.

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