0

Technical Report No. 32-655

(Part II)

Mathematical Models of Missile Launching

Alfred C. Dahlgren

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

August 21, 1964

Technical Report No. 32-655
(Part II)

Mathematical Models of Missile Launching

Alfred C. Dahlgren

W Hamilton

T. Hamilton, Chief
Systems Analysis Section

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

August 21, 1964

Prepared Under Contract No. NAS 7-100
National Aeronautics & Space Administration

CONTENTS

ABSTRACT

10820

By means of a mathematical model of the situation, including assumptions about the nature of the delays encountered, an estimate is presented of the expected number of days required to launch three missiles, each from its own pad, allowing simultaneous (i.e., same-day) countdowns but no simultaneous launches.

author

I. INTRODUCTION

In space mission feasibility studies it is often necessary to have an estimate of the number of days required to launch a given number of missiles from another given (and possibly different) number of pads.

This report (Part II of three parts) is one of a series whose purpose is to investigate this question with the aid of probability theory, for different launching configurations (see Refs. 1 and 2). The particular launching configuration examined in this report is that one of three missiles being fired, each from its own pad, allowing simultaneous count-downs on all three missiles, but no simultaneous (i.e., same-day) launches. The following assumptions constitute the launching model.

II. THE MATHEMATICAL LAUNCH MODEL

1. There are three identical pads and three identical missiles, each on its own pad.
2. For each missile, a complete countdown and launching in one day is possible, though not necessary. But once the counting has started on a given missile, the counting continues until that missile fires, no matter how many days this requires. There is a probability p (a specified constant) that a missile will successfully count down and fire in one day. There is thus also a constant probability $q = 1 - p$ that the missile will fail to complete a countdown on that day, and will incur a one-day delay. In other words, if the countdown on a given missile stops on any day of counting, the counting on that missile must start over from the beginning the following morning.
3. Because p is a constant, failures to fire a given missile do not influence the value of p in subsequent

attempts to fire the same missile or others. Each missile is thus probabilistically independent of the other two.

4. Prior to the start of the N-day period, the three missiles are erected, each on its own pad and ready to start counting. On the first day of the N-day period, the three missiles all start counting down simultaneously, and continue to count together for as many days as are required until one of the missiles fires. Starting the next day, the two remaining missiles then count down together until one of them fires. Starting the day after that, the remaining missile then counts for as many days as are required to fire it on the Nth day. Of course, with all three missiles counting together, the simultaneous (i.e., same-day) launching of two or three missiles definitely could occur, but we consciously choose to consider only the nonsimultaneous-launch sequences in the sample space for the problem (i.e., the set of all possibly resulting launch sequences, simultaneous and nonsimultaneous). This is done by retaining only the nonsimultaneous-launch probability expressions, after suitably arranging them to form a new problem having a new sample space containing only nonsimultaneous launch sequences. This arranging is called the normalization of the (original) sample space. See Section IV (Derivations) for the detailed discussion of this point.

   The third and last missile is always assumed to fire on the Nth day, whatever the positions of the first and second missile firings within the N-day period.

5. Since each missile is on its own pad, it is not necessary in this launch model to take into account any turnaround time between missile firings, as has been done in other reports in this series.

III. RESULTS

1. The probability $P(N)$ that all three missiles will be launched in exactly N days, allowing simultaneous countdowns but no simultaneous launches, is

$$P(N) = (1 + q + q^2) p q^{N-3} [1 - q^{N-1} - q^{N-2} + q^{2N-3}]$$

2. The mean day μ (counted from the beginning of the N-day period), on which one can expect to launch the third and last missile, is given by

$$\mu = \frac{3 + 4q + 3q^2 + q^3}{(1 - q^3)(1 + q)}$$

3. The variance σ^2 of the $P(N)$ distribution (see Section IV) is

$$\sigma^2 = \frac{q(1 + 5q + 11q^2 + 15q^3 + 11q^4 + 5q^5 + q^6)}{(1 + 2q + q^2)(1 - 2q^3 + q^6)}$$

4. The moment-generating function $M(\theta)$ is obtained as

$$M(\theta) = (1 - q^3)\left[\frac{e^{3\theta}}{(1 - qe^{\theta})} - \frac{(q^2 + q)e^{3\theta}}{(1 - q^2 e^{\theta})} + \frac{q^3 e^{3\theta}}{(1 - q^3 e^{\theta})}\right]$$

IV. DERIVATIONS

In this launch model we are interested only in the ways in which the three missiles fire one after the other even though having counted down together. However, all possible launch sequences for the problem (simultaneous and nonsimultaneous) are contained in the sample space that accompanies the problem, by definition of a sample space. We must look at the possible simultaneous-launch sequences in this sample space, as well as the nonsimultaneous, and calculate the probabilities for both types. This will permit us to retain only the nonsimultaneous-launch probabilities, while arranging them so that they will constitute a new problem with a new sample space containing only nonsimultaneous-launch sequences. This procedure is called normalizing the (original) sample space.

The sample space must be normalized, since otherwise the total probability of getting the three missiles launched nonsimultaneously would not equal 1. However, we have assumed that we would keep counting until all missiles fired, which is equivalent to having an upper limit of infinity for N. Thus, we will eventually achieve the desired aim of the problem, the launching one after the other of all three missiles. The total probability of achieving this aim must therefore equal 1, by definition from probability theory. Thus, normalization of the sample space is necessary.

A. Nonsimultaneous-Launch Sequences

Let us look first at the possible nonsimultaneous-launch sequences. We shall number the missiles arbitrarily but permanently as 1, 2, and 3 prior to the start of the N-day period. The order 1, 2, and 3 and the five permutations of this order, are the only distinct possible nonsimultaneous-launch sequences (the number order below indicates the order of firing):

1, 2, 3

1, 3, 2

2, 1, 3

2, 3, 1

3, 1, 2

3, 2, 1

But since the missiles are identical, the six firing orders are all formally equivalent (as may be seen by a simple renumbering of the missiles). Thus, we need to find only the probability for one particular firing order above, and multiply it by six.

Let us compute the probability for the 1, 2, 3 case. The probability p_1 that missile 1 will fire on the kth day (counted from the start of the N-day period) is equal to the probability that it will fail to fire for $(k-1)$ days, times the probability that it will fire in one day, or $p_1 = pq^{k-1}$. Similarly, if missile 2 fires on the mth day of counting (it has counted from the first day of the N-day period), $p_2 = pq^{m-1}$. By assumption, missile 3 always fires on the Nth day, and it has counted for the entire N days. Thus, $p_3 = pq^{N-1}$.

Out of a given N-day period, k and m can assume various different values, the limits on these values being determined by N. In order to express this situation mathematically, we will make m dependent on k, and then permit k to range over its maximum and minimum values within the N-day period. Doing this will automatically take care of the maximum and minimum values for m. The limits on k are as follows. The quantity k must be at least one day, but if it is almost as large as N, at the end of the N-day period we must leave at least one day each to launch missiles 2 and 3. Thus, out of a given N days, k can range only over $1 \leqq k \leqq N-2$. Missile 2 has counted from the first day of the N-day period, but by assumption it must fire one or more days after missile 1. We must again include the requirement that at least the Nth day must be left in order to launch missile 3. Thus, $k+1 \leqq m \leqq N-1$.

The situation that missile 1 will take a particular k_0 days, with missile 2 taking a particular m_0 days (and missile 3 taking the N days in any case), is a compound event, whose probability p_c is the probability that the three subevents will occur together, or $p_c = p_1 p_2 p_3$. Moreover, there can be different specific combinations of k's and m's within the N-day period. The compound firing event associated with a particular k_0 and m_0 set is an event mutually exclusive of those firing events for other particular k_0 and m_0 sets. Thus, we must sum the compound probability $p_1 p_2 p_3$ over all possible k and m sets within a given N-day period, in order to arrive at an expression for the general probability $P_f(N)$ of firing all three missiles in N days with no simultaneous launches. $P_I(N)$ is six times the probability for the 1, 2, 3 case, or

$$P_I(N) = 6 \sum_{k=1}^{N-2} \sum_{m=k+1}^{N-1} p_1 p_2 p_3$$

$$= 6 \sum_{k=1}^{N-2} \sum_{m=k+1}^{N-1} pq^{k-1} pq^{m-1} pq^{N-1}$$

$$= 6p^3 q^{N-1} \sum_{k=1}^{N-2} q^{k-1} \sum_{m=k+1}^{N-1} q^{m-1}$$

or

$$P_I(N) = 6pq^N \left[\frac{1 - q^{N-1} - q^{N-2} + q^{2N-3}}{1+q} \right]$$

The total probability P_{It} of getting a nonsimultaneous-launch sequence at all is given by the sum of $P_I(N)$ over all admissible values of N. A minimum of 1 day each is necessary to launch the three missiles. N thus has a lower limit of 3. From above, N has an upper limit of infinity. Thus

$$P_{It} = \sum_{N=3}^{\infty} P_I(N)$$

or

$$P_{It} = \frac{6q^3}{1 + 2q + 2q^2 + q^3}$$

and this result checks intuitively, for if q were zero, then the total probability of getting a nonsimultaneous-launch sequence would have to be zero, because all the missiles would fire simultaneously on the first day. It may be noted, as discussed above, that P_{It} does not identically equal one, because we are still in the unnormalized sample space containing both the simultaneous- and nonsimultaneous-launch sequences.

B. Simultaneous-Launch Sequences

We now examine the simultaneous-launch sequences. The possible cases here are (in order of firing, with a dash indicating simultaneous launching):

a. $\left.\begin{array}{l} \text{1, 2-3} \\ \text{2, 1-3} \\ \text{3, 1-2} \end{array}\right\}$ all formally equivalent

b. 1-2-3

c. $\left.\begin{array}{l} \text{1-2, 3} \\ \text{3-2, 1} \\ \text{3-1, 2} \end{array}\right\}$ all formally equivalent

and the formal equivalence of the subcases is readily seen by a renumbering of the missiles. Looking at the above sequences case by case:

Case a. The probability $P_{IIa}(N)$ of getting a case a type of launch sequence would be three times the probability of getting a sequence like the subcase (1, 2–3). If missile 1 counts for k days, again $p_1 = pq^{k-1}$. By assumption, at least the Nth day must be left, in order to launch both missiles 2 and 3 on that day. Thus, k can range only over $1 \leq k \leq N-1$. Missiles 2 and 3 have counted from the beginning, thus $p_2 = p_3 = pq^{N-1}$. We multiply $p_1 p_2 p_3$ by three and sum over all possible values of k, or

$$P_{IIa}(N) = 3 \sum_{k=1}^{N-1} p_1 p_2 p_3$$

$$= 3 \sum_{k=1}^{N-1} pq^{k-1} pq^{N-1} pq^{N-1}$$

or

$$P_{IIa}(N) = 3\, p^2 q^{2N-2} (1 - q^{N-1})$$

Case b. In this case, taking the viewpoint that all three missiles launch simultaneously on the Nth day (where now $N = 1, 2, \cdots, \infty$) takes care of all possible all-three-together launch sequences. Each missile counts for the N days, and thus $p_1 = p_2 = p_3 = pq^{N-1}$, and there is no k or m to sum over. We have

$$P_{IIb}(N) = (pq^{N-1})^3 = p^3 q^{3N-3}$$

Case c. The probability $P_{IIc}(N)$ of getting a case c type of sequence would be three times the probability of getting a typical subcase sequence like (1-2, 3). If missiles 1 and 2 both fire on the kth day, $p_1 = p_2 = pq^{k-1}$, and again the Nth day must be left to shoot missile 3. Thus, $1 \leq k \leq N-1$, and $p_3 = pq^{N-1}$ as usual. Summing over all possible k's, we have

$$P_{IIc}(N) = 3 \sum_{k=1}^{N-1} pq^{k-1} pq^{k-1} pq^{N-1}$$

$$= 3p^3 q^{N-1} \sum_{k=1}^{N-1} q^{2k-2}$$

or

$$P_{IIc}(N) = \frac{3p^2q^{N-1}(1-q^{2N-2})}{1+q}$$

From probability theory, the total probability of getting one of a number of mutually exclusive events is the sum of the total probabilities of each event. Since the 7 subcases within cases a, b, and c are all mutually exclusive and are contained within the expressions $P_{IIa}(N)$, $P_{IIb}(N)$, and $P_{IIc}(N)$, the total probability P_{IIt} of getting any type of simultaneous-launch sequence is therefore

$$P_{IIt} = \sum_{N=2}^{\infty} P_{IIa}(N) + \sum_{N=1}^{\infty} P_{IIb}(N) + \sum_{N=2}^{\infty} P_{IIc}(N)$$

The lower limits on N are as indicated for the different sums because only one day minimum is needed to achieve a case b 1-2-3 sequence, but at least two days are needed to have either a case a or case c sequence. The upper limit is infinity as above; the countdowns continue until all three missiles are off. One finds that

$$P_{IIt} = \frac{1+2q+2q^2-5q^3}{1+2q+2q^2+q^3}$$

It may be noticed that if $q = 0$, then $P_{IIt} = 1$; this result checks intuitively, because $q = 0$ implies $p = 1 - q = 1$, and all three missiles would fire together on the first day. This makes the total probability of getting any type of simultaneous launch necessarily 1.

A stronger verification of the entire derivation above is the result that

$$P_{It} + P_{IIt} = 1$$

indicating that we have correctly accounted for all possible launch sequences (simultaneous, nonsimultaneous, and otherwise) in the sample space for the problem.

C. Normalization

Now the necessity of normalizing enters. As explained above, P_{It} is the total probability of getting at all a nonsimultaneous-launch sequence, but it is not equal to 1, as it must be. A new P_{It} would equal 1 if we transposed the P_{IIt} over to the right side of the equation,

$$P_{It} = 1 - P_{IIt}$$

and then divided through by $1 - P_{IIt}$,

$$\frac{P_{It}}{1-P_{IIt}} = 1$$

using

$$\frac{1}{1-P_{IIt}}$$

as a constant normalizing factor in front of every $P_I(N)$ in the series that forms P_{It}, or

$$\frac{P_{It}}{1-P_{IIt}} = 1 = \frac{1}{1-P_{IIt}} \sum_N P_I(N)$$

But from above, $1 - P_{IIt} = P_{It}$. Thus,

$$\frac{P_{It}}{1-P_{IIt}} = \frac{P_{It}}{P_{It}} = 1 = \frac{1}{P_{It}} \sum_N P_I(N)$$

The final normalized probability $P(N)$ of getting a nonsimultaneous-launch sequence of the three missiles in N days is therefore

$$P(N) = \frac{1}{P_{It}} P_I(N)$$

or

$$P(N) = (1+q+q^2)pq^{N-3}[1-q^{N-1}-q^{N-2}+q^{2N-3}]$$

D. Calculation of σ and μ

In reality, $P(N)$ plays the role of the probability function for a discrete random variable n, where n is the number of days required to launch all three missiles. In other words, $P(N)$ is the probability that $n = N$. From here, the mean day μ (counted from the first day of the N-day period), on which one can expect to launch the third and last missile, and the variance σ^2 of the $P(N)$ distribution may be calculated by the usual series

$$\mu = E[N] = \sum_{N=3}^{\infty} N\,P(N)$$

and

$$\sigma^2 = \nu_2 - \mu^2 = \left\{ \sum_{N=3}^{\infty} N^2 P(N) \right\} - \mu^2$$

The quantities μ and σ^2 may also be calculated from the moment-generating function $M(\theta)$, where

$$M(\theta) = E[e^{\theta N}] = \sum_{N=3}^{\infty} e^{\theta N} P(N)$$
$$= \frac{1}{P_{It}} \sum_{N=3}^{\infty} e^{\theta N} P_I(N)$$

with the result that

$$M(\theta) = (1-q^3)\left[\frac{e^{3\theta}}{1-qe^{\theta}} - \frac{(q^2+q)e^{3\theta}}{1-q^2e^{\theta}} + \frac{q^3e^{3\theta}}{1-q^3e^{\theta}}\right]$$

From here,

$$\mu = \frac{dM(\theta)}{d\theta}\bigg|_{\theta=0} = \frac{3+4q+3q^2+q^3}{(1-q^3)(1+q)}$$

As a check on the correctness of the $M(\theta)$ expression, this above result for μ is the same as gotten by the series calculation. The μ result also checks intuitively, for if $q=0$, the expected value of the number of days to fire all three missiles would be the minimum necessary for no simultaneous launches, or three days. Also from $M(\theta)$,

$$\sigma^2 = \nu_2 - \mu^2 = \frac{d^2M(\theta)}{d\theta}\bigg|_{\theta=0} - \mu^2$$

or

$$\sigma^2 = \frac{q(1+5q+11q^2+15q^3+11q^4+5q^5+q^6)}{(1+2q+q^2)(1-2q^3+q^6)}$$

REFERENCES

1. Solloway, C. B., A *Simplified Statistical Model for Missile Launching,* Technical Report No. 32-431, Jet Propulsion Laboratory, Pasadena, California, May 1, 1963.

2. Lass, H., and C. B. Solloway, A *Statistical Problem Related to the Launching of a Missile,* Technical Report No. 32-124, Jet Propulsion Laboratory, Pasadena, California, July 20, 1961.