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by Harold A. Hamer

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SUMMARY

A method is developed for manually determining a position fix in space. Basically, the method employs four nonsimultaneous angular measurements of celestial bodies, along with precalculated data for a nominal trajectory, to determine vehicle position. The method is simple and accurate, with all onboard calculations required for one position fix being adequately performed with a slide rule in less than 15 minutes.

The various equations required by the navigator, as well as those essential for the precalculations, are outlined in detail. Although the method is developed and applied to Earth-Moon space, it could serve equally as well for interplanetary space. The important details of the method discussed are the linearity characteristics, the effect of star selection on accuracy, and the procedure to convert the nonsimultaneous measurements to a common time.

The method is shown to have a relatively high degree of linearity; therefore, it does not require the actual injection time to be close to a nominal injection time for good accuracy in determining position. On the basis of an error analysis of the method, the root-sum-square position error over most of the Earth-Moon distance is shown to be about 35 kilometers for a standard-deviation error of 10 arc-seconds in the angular measurements. Also, in relation to the accuracy characteristics, almost any set of stars selected for the measurements will give good results, providing that the star used in the range determination lies within ±30° of the Earth-Moon-vehicle plane.

INTRODUCTION

In manned space missions the navigation and guidance will normally be accomplished by automatic or semiautomatic procedures. (For example, see refs. 1, 2, and 3.) The inclusion of manual procedures (for example, ref. 4) for these operations would be a desirable and, in some cases, a necessary feature. With a manual system the astronaut could supposedly perform the essential parts of the midcourse navigation and guidance in the event of a malfunction in the space computer or power supply. Also, the manual system could provide a check on some of the results produced by the more complex automatic systems.

The main requirements for any manual system would be simplicity, good accuracy, and fast calculation procedures.

As a part of the overall navigation and guidance problem, a method is developed in this report which meets the requirements for manual position fixing. Basically, the method employs four nonsimultaneous angular measurements (taken by the same type of optical-measuring instrument) to determine range from a body center and then to determine the vehicle-position coordinates. The method is based on the astronaut's possession of nominal-trajectory information and on certain precalculated data retained either in chart or table form. The astronaut can adequately perform all necessary calculations with a slide rule and a scratch pad.

In the present report the method is applied to an Earth-Moon trajectory, although it could serve equally as well for interplanetary trajectories. Important details discussed are the linearity characteristics of the method and the effect of star selection on accuracy. The procedure to convert the nonsimultaneous measurements to a common time is also given.

An error analysis is performed to show the accuracy characteristics of this method. The accuracy, of course, cannot be expected to approach that of automatic systems which use statistical reduction of a large number of various types of nonsimultaneous measurements.

SYMBOLS

A	angle formed at vehicle by line of sight to Earth center and line of sight to Moon center
$\Delta A, \Delta B, \Delta D, \Delta \theta$	difference between actual values and nominal values
B	angle formed at Earth center by line to vehicle and line to Moon center
b	magnitude of vector perpendicular to plane containing $\bar{r}_{ev}$ and $\bar{h}$
C	angle formed at Moon center by line to vehicle and line to Earth center
c	constant (see eq. (11))
D,E,F	measurement quantities (see eqs. (15))
h	magnitude of vector perpendicular to instantaneous Earth-Moon-vehicle plane
l,m,n	direction cosine of line of sight to star with respect to X-, Y-, and Z-axis, respectively
P,Q,R,S	constants for any selected set of three stars (see eqs. (17))
r	range or distance, $\left(x^2 + y^2 + z^2\right)^{1/2}$

Δr incremental range, $r_a - r_n$

rss root sum square (used as measure of position-fixing accuracy), $\left[(\sigma_{\Delta x})^2 + (\sigma_{\Delta y})^2 + (\sigma_{\Delta z})^2\right]^{1/2}$

R_b body radius

T time from injection

t time

Δt increment in time between common time (time for "common" or representative value measurement) and time at which measurement is taken

X,Y,Z vehicle-centered rectangular right-hand axis system in which X-axis is in the direction of Aries, XY-plane is parallel to Earth equatorial plane, and Z-axis is in the direction of north celestial pole

X_r,Y_r,Z_r rotating rectangular right-hand axis system in which X_r-axis lies along Earth-Moon line, X_rY_r-plane is in Earth-Moon plane, and Z_r-axis is in northerly direction

x,y,z position coordinates in rectangular right-hand axis system

x_r,y_r,z_r position coordinates in rotating rectangular right-hand axis system

$\dot{x},\dot{y},\dot{z}$ velocity coordinates in rectangular right-hand axis system

$\Delta x,\Delta y,\Delta z$ position off nominal trajectory in direction of x-, y-, and z-axis, respectively

α one-half of angular diameter of body as viewed from vehicle

δ angle formed at vehicle by line to star and its projection in the instantaneous Earth-Moon-vehicle plane

θ angle formed at vehicle by line of sight to star and line of sight to body center

Θ angle formed at vehicle by line to Earth and projection of line to star in the instantaneous Earth-Moon-vehicle plane

σ standard-deviation value

Subscripts:

1,2,3 star 1, star 2, and star 3

I,III,IV	specific stars
a	actual (measured) value
A	angle formed at vehicle by line of sight to Earth center and line of sight to Moon center
em	distance between Earth center and Moon center
ev	distance or range from vehicle to Earth center
mv	distance or range from vehicle to Moon center
n	nominal value
r	range
Δr	incremental range, $r_a - r_n$
unc	uncorrected
x,y,z	component in direction of X-, Y-, and Z-axis, respectively
$\Delta x, \Delta y, \Delta z$	position off nominal trajectory in direction of x-, y-, and z-axis, respectively
θ	angle formed at vehicle by line of sight to star and line of sight to body center

Notation:

$[\;]^{-1}$	inverse of square matrix
$\{\;\}$	column matrix
$\|\;\|$	absolute value

Bar over a symbol indicates a vector.

BASIC METHOD

There are many combinations of optical measurements which can be used for position fixing. For example, reference 5 shows 13 different combinations of angular measurements which incorporate sightings on the Earth, Moon, Sun, or stars to yield solutions of vehicle position in Earth-Moon space. The basic requirements for a manual position-fixing procedure are that is must be simple enough to permit rapid calculations (several minutes) and still give accurate results. Of the various combinations of measurements, the only one that would seem applicable to this case is one that uses star-to-body-center angular measurements, together with a range measurement, as illustrated in figure 1.

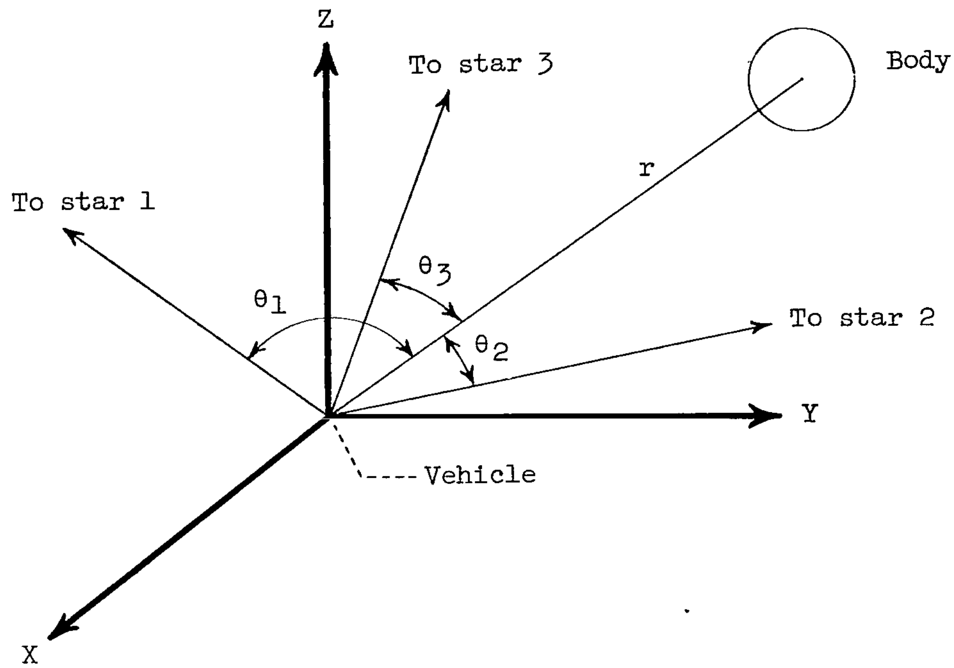


Figure 1.- Required measurements for a position fix.

The set of three simultaneous equations which leads to the solution of the vehicle-position coordinates is:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}^{-1} \begin{Bmatrix} r \cos \theta_1 \\ r \cos \theta_2 \\ r \cos \theta_3 \end{Bmatrix} \tag{1}$$

where l, m, and n are the direction cosines of the respective stars. It is readily seen that the set of equations (1) can be, for the most part, precalculated since the values of l, m, and n are known in advance (by selecting certain stars). The solution of the set of equations (1), however, would involve calculating numbers up to six significant figures for Earth-Moon space. This problem can be circumvented by employing a nominal (or reference) trajectory for which vehicle position and various angles are known for any time along the trajectory. In this approach, the incremental values Δx, Δy, and Δz (distances off the nominal) could then be easily and accurately calculated from the differences between the measured values of the angles and the values pertaining to the nominal trajectory. The equations in this form and some representative accuracies are shown in subsequent sections.

As shown by the set of equations (1), the vehicle position can be determined by four simultaneous measurements, those of r, θ_1, θ_2, and θ_3. In any practical case, simultaneous measurements are not possible, but a simple procedure can be used to convert nonsimultaneous measurements to a common time, as is subsequently shown. The range r in the set of equations (1) could be obtained by measuring the angular diameter of the body, that is, by replacing the quantity r with $R_b/\sin\alpha$ in which R_b is a known value of the body radius. However, this type of measurement is relatively inaccurate at large distances from a body and in Earth-Moon space can lead to position errors on the order of several hundred kilometers, as shown in reference 6. Therefore some other method must be used to obtain a measure of the range. For lunar trajectories, a triangulation procedure based on the Earth-Moon-vehicle triangle seems to be the most simple and accurate. Here again, this procedure is to be used in conjunction with a nominal trajectory for which the range is known for any given time along the trajectory. The details are presented in the following section.

DERIVATION OF EQUATIONS

Determination of Range

In the present method for determining vehicle position along a lunar trajectory, a necessary first step is the determination of range to the center of the Earth.

In figure 2 the angle Θ_1, which can vary between 0° and 180°, lies in the instantaneous Earth-Moon-vehicle plane. If figure 2 represents the vehicle position on a nominal trajectory at a given time, then the incremental range

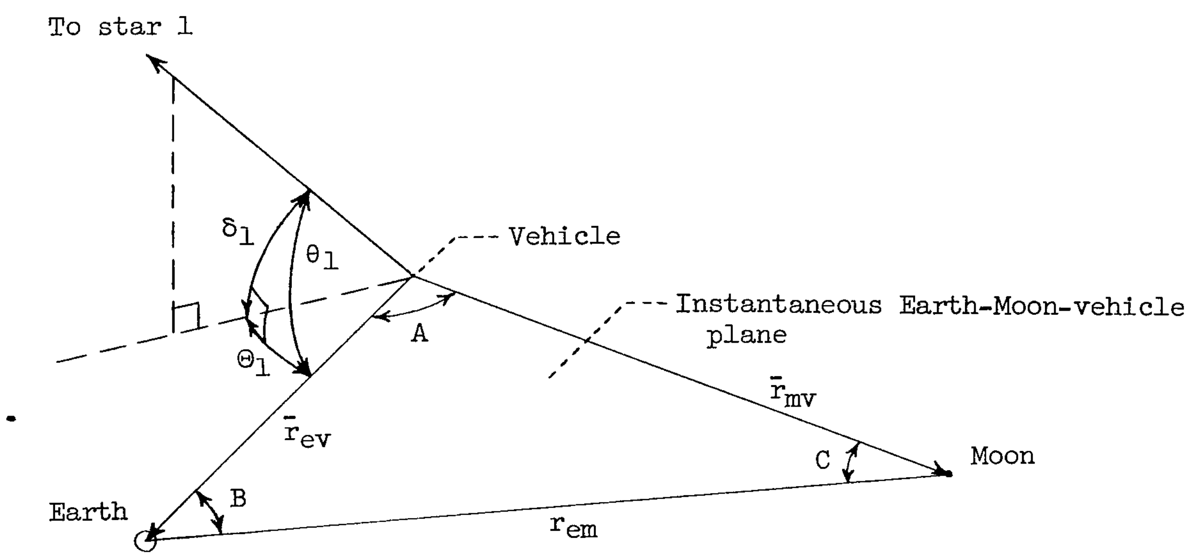


Figure 2.- Variables involved in determining range.

Δr_{ev} (defined as the actual range minus the nominal range) at that particular time is obtained as follows:

$$r_{ev} = \frac{r_{em}}{\sin A} \sin C = \frac{r_{em}}{\sin A} \sin(A + B)$$

or

$$r_{ev} = r_{em} \cos B + r_{em} \sin B \cot A \tag{2}$$

As shown in equation (2), the variables which define r_{ev} at a given time are the angles A and B; therefore:

$$dr_{ev} = \frac{\partial r_{ev}}{\partial A} dA + \frac{\partial r_{ev}}{\partial B} dB \tag{3}$$

From equation (2).

$$\frac{\partial r_{ev}}{\partial A} = -r_{em} \sin B \csc^2 A$$

but

$$\sin B = \frac{r_{mv}}{r_{em}} \sin A \tag{4}$$

so that

$$\frac{\partial r_{ev}}{\partial A} = - \frac{r_{mv}}{\sin A} \tag{5}$$

where

$$\cos A = \frac{r_{ev}^2 + r_{mv}^2 - r_{em}^2}{2 r_{ev} r_{mv}} \tag{6}$$

Furthermore, from equation (2),

$$\frac{\partial r_{ev}}{\partial B} = r_{em}(\cos B \cot A - \sin B) \tag{7}$$

where the angle B is given by equation (4) and the angle A is given by equation (5). The values r_{ev}, r_{mv}, and r_{em} are known values for the nominal trajectory.

The variables A and B in equation (3) refer to angular measurements which must be made onboard the vehicle. The angle B cannot be measured from the vehicle; therefore, an angle which is related to B and which can be measured from the vehicle must be substituted. As shown by figure 2, the angle θ_1 measured between a star and the Earth center can be used in the following manner.

For the case in which the star is in the instantaneous Earth-Moon-vehicle plane, $d\Theta_1/dB = \pm 1$, where the sign is determined by the relative directions of the Earth and the star. Equation (3), therefore, can be written as

$$dr_{ev} = \frac{\partial r_{ev}}{\partial A} dA + c \frac{\partial r_{ev}}{\partial B} d\Theta_1 \tag{8}$$

where $d\Theta_1$ represents the change in the star-to-Earth angle and c is either ± 1. The equation for determining c is given in appendix A. If the star is not in the instantaneous plane, as shown in figure 2, $d\Theta_1$ can be determined from the right spherical trigonometric relationship,

$$\cos \theta_1 = \cos \Theta_1 \cos \delta_1$$

or

$$\Theta_1 = \text{arc} \cos\left(\frac{\cos \theta_1}{\cos \delta_1}\right)$$

Thus,

$$d\Theta_1 = \frac{\sin \theta_1}{\left(\cos^2\delta_1 - \cos^2\theta_1\right)^{1/2}} d\theta_1$$

or

$$d\Theta_1 = \left(\frac{1 - \cos^2\theta_1}{\cos^2\delta_1 - \cos^2\theta_1}\right)^{1/2} d\theta_1 \tag{9}$$

Therefore, in terms of two angles which can be measured from the vehicle (Earth center to Moon center and star-to-Earth center), equation (8) can be written

$$dr_{ev} = \frac{\partial r_{ev}}{\partial A} dA + c \frac{\partial r_{ev}}{\partial B}\left(\frac{1 - \cos^2\theta_1}{\cos^2\delta_1 - \cos^2\theta_1}\right)^{1/2} d\theta_1 \tag{10}$$

or, for the region in which a change in r_{ev} is linear with changes in A and B,

$$\Delta r_{ev} = \frac{\partial r_{ev}}{\partial A}\Delta A + c\,\frac{\partial r_{ev}}{\partial B}\left(\frac{1 - \cos^2\theta_1}{\cos^2\delta_1 - \cos^2\theta_1}\right)^{1/2}\Delta\theta_1 \tag{11}$$

where

$$\Delta A = A_a - A_n$$

and

$$\Delta\theta_1 = \theta_{1,a} - \theta_{1,n} \qquad \left(0^\circ < \theta_1 < 180^\circ\right)$$

The incremental range Δr_{ev}, then, is determined by the two measurements A_a and $\theta_{1,a}$ and by the precalculated values of A_n, $\theta_{1,n}$, $\frac{\partial r_{ev}}{\partial A}$, $\frac{\partial r_{ev}}{\partial B}$, and $c\left(\frac{1 - \cos^2\theta_1}{\cos^2\delta_1 - \cos^2\theta_1}\right)^{1/2}$. Equations for calculating values of θ, δ, and c which pertain to a nominal trajectory are given in appendix A.

Linearity characteristics.- Investigation of the variations of Δr_{ev} with ΔA and ΔB for a typical lunar trajectory showed that at a given time the partials $\partial r_{ev}/\partial A$ and $\partial r_{ev}/\partial B$ can be considered to be essentially linear over a wide region, as indicated in figure 3. The nominal trajectory used (70.6 hours from injection to perilune) was selected from reference 7 and is shown in figure 4. Table I presents pertinent position and velocity information of the trajectory, as well as values of the angles A and B. In figure 3 the curve for ΔB was determined from the relations (see fig. 2)

$$\sin C = \frac{r_{ev}}{r_{em}}\sin A \tag{12}$$

and

$$B = 180^\circ - (A + C) \tag{13}$$

where A is held constant at the nominal value and is given by equation (6). The curve for ΔA was determined from the relation

$$\sin A = \frac{r_{em}}{r_{mv}}\sin B$$

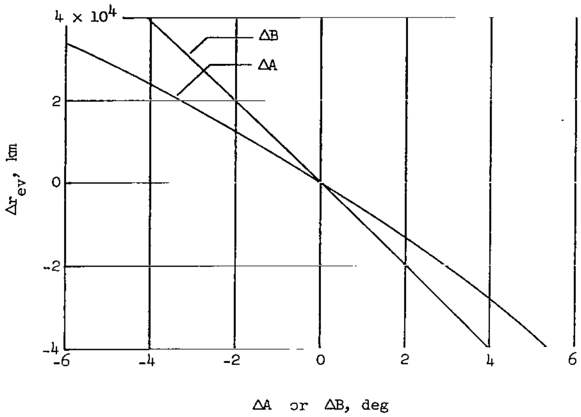


Figure 3.- Linearity characteristics of $\frac{\partial r_{ev}}{\partial A}$ and $\frac{\partial r_{ev}}{\partial B}$ for a nominal distance of 161,981 kilometers from center of Earth to vehicle.

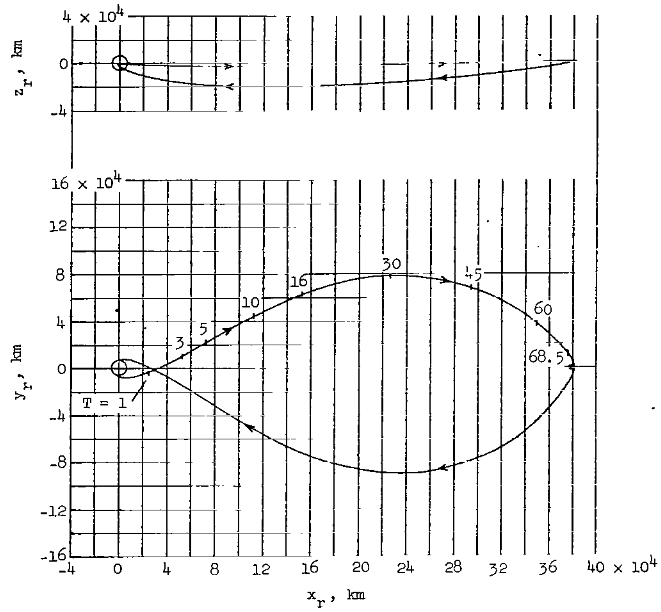


Figure 4.- Nominal trajectory in rotating coordinate system; T denotes hours from injection.

where

$$r_{mv} = \left(r_{ev}^{2} + r_{em}^{2} - 2r_{ev}r_{em}\cos B\right)^{1/2}$$

and B is held constant at the nominal value and is determined by equations (12) and (13). The curves shown in figure 3 are for a nominal time from injection of 16.125 hours ($r_{ev} \approx 162{,}000$ kilometers). At this time a 2-hour difference at injection from the nominal injection time would represent a difference in range from the Earth of about 13,000 kilometers. The data in figure 3 show that this difference can be considered to be within the linear region. Several nominal or reference trajectories, therefore, could suffice for range determination for a given daily launch window.

The linearity characteristics of $\partial r_{ev}/\partial A$ and $\partial r_{ev}/\partial B$ illustrated in figure 3 are representative of those over most of the Earth-Moon distance. Actually the linear approximation for $\partial r_{ev}/\partial A$ improves as the Moon is approached, as shown by the derivative of equation (5) which is

$$\frac{\partial^2 r_{ev}}{\partial A^2} = r_{mv}\frac{\cos A}{\sin^2 A}$$

TABLE I.- NOMINAL TRAJECTORY PARAMETERS

[Vehicle position and velocity coordinates are vehicle centered]

Variable	Time from injection, hours						
	1	2	3	4	5.5	10.5	16.125
x_{ev}, km	17,000.141	19,873.013	20,772.521	20,874.339	20,284.239	15,797.195	9,255.8512
y_{ev}, km	13,826.706	28,971.665	41,465.015	52,416.910	66,965.325	106,257.19	141,266.32
z_{ev}, km	8,591.5583	17,141.773	24,114.484	30,189.287	38,219.273	59,747.541	78,788.545
r_{ev}, km	23,537.152	39,091.360	52,271.918	63,989.558	79,727.772	122,922.38	162,016.91
x_{mv}, km	-189,769.15	-183,757.26	-179,697.19	-176,413.57	-172,192.03	-160,316.94	-147,900.10
y_{mv}, km	-254,284.19	-241,070.11	-230,478.64	-221,399.43	-209,604.99	-179,008.56	-152,872.43
z_{mv}, km	-129,595.24	-122,195.85	-116,359.00	-111,404.98	-105,027.68	-88,757.164	-75,157.655
r_{mv}, km	342,735.63	326,823.12	314,564.62	304,221.18	290,886.84	256,170.68	225,594.98
r_{em}, km	365,694.67	365,790.69	365,888.32	365,987.50	366,139.23	366,669.32	367,307.81
$\dot{x}_{ev}$, km/sec	1.3791860	0.42971609	0.11338018	-0.041338963	-0.16327614	-0.30064959	-0.33717387
$\dot{y}_{ev}$, km/sec	4.7862262	3.7612012	3.2252961	2.8804782	2.5318916	1.9188737	1.5703809
$\dot{z}_{ev}$, km/sec	2.7279185	2.1078791	1.7935755	1.5940144	1.3941551	1.0468713	0.85165680
$\dot{x}_{mv}$, km/sec	[a]2.24	[a]1.67	[a]0.973	[a]0.841	[a]0.746	[a]0.626	[a]0.617
$\dot{y}_{mv}$, km/sec	[a]4.40	[a]3.67	[a]2.70	[a]2.38	[a]2.08	[a]1.50	[a]1.15
$\dot{z}_{mv}$, km/sec	[a]2.38	[a]1.82	[a]1.46	[a]1.28	[a]1.12	[a]0.788	[a]0.604
A, deg	166°3055"	175°650"	168°725"	163°1262"	158°958"	148°1881"	142°774"
B, deg	12°1125"	4°1099"	10°449"	13°2801"	17°392"	21°1404"	22°379"

Variable	23.125	31.125	45.125	60	65	68.5	69.5
x_{ev}, km	618.92375	-9,233.1921	-25,632.839	-40,898.372	-44,885.478	-46,341.619	-45,992.488
y_{ev}, km	177,258.93	211,785.18	261,074.69	303,197.37	315,953.95	324,980.53	327,676.34
z_{ev}, km	98,256.097	116,836.74	143,211.87	165,528.05	172,234.22	176,961.26	178,384.29
r_{ev}, km	202,670.61	242,051.73	298,868.83	347,851.77	362,637.86	372,927.85	375,909.62
x_{mv}, km	-132,226.49	-113,475.61	-78,287.308	-37,619.423	-22,760.254	-11,053.984	-6,951.3123
y_{mv}, km	-126,541.42	-101,136.23	-62,762.242	-25,101.552	-12,194.307	-2,572.1041	-367.61865
z_{mv}, km	-61,739.027	-49,059.589	-30,346.349	-12,316.612	-6,164.1073	-1,558.2646	-129.66503
r_{mv}, km	193,153.52	159,725.06	104,827.97	46,872.249	26,546.686	11,455.759	6,962.2336
r_{em}, km	368,158.94	369,198.98	371,160.79	373,389.42	374,161.41	374,706.77	374,863.24
$\dot{x}_{ev}$, km/sec	-0.34476699	-0.33782194	-0.31034989	[a]-0.252	[a]-0.181	[a]0.00410	[a]0.243
$\dot{y}_{ev}$, km/sec	1.3058873	1.1039615	0.87009046	[a]0.735	[a]0.704	[a]0.740	[a]0.744
$\dot{z}_{ev}$, km/sec	0.70449040	0.59265123	0.46342536	[a]0.382	[a]0.369	[a]0.389	[a]0.398
$\dot{x}_{mv}$, km/sec	[a]0.632	[a]0.666	[a]0.725	0.79926153	0.86291157	1.0488207	1.2677780
$\dot{y}_{mv}$, km/sec	[a]0.964	[a]0.830	[a]0.720	0.70492821	0.73557119	0.80363455	0.82276785
$\dot{z}_{mv}$, km/sec	[a]0.469	[a]0.405	[a]0.347	0.33600494	0.35090673	0.38753199	0.40476791
A, deg	136°3204"	132°1683"	126°2528"	119°2995"	113°3074"	98°234"	80°2992"
B, deg	21°260"	18°2197"	13°316"	6°907"	3°2592"	1°2646"	1°184"

[a]Approximate value is given inasmuch as this value was not calculated in the trajectory program used in the present analysis.

As noted from table I, the angle A approaches 90° as the Moon is approached. From the derivative of equation (7),

$$\frac{\partial^2 r_{ev}}{\partial B^2} = -r_{em}(\cot A \sin B + \cos B)$$

it can be determined (by use of the values of A and B from table I) that the linear approximation for $\partial r_{ev}/\partial B$ will also improve as the Moon is approached.

Effect of proximity to Earth-Moon line.- The values of $\partial r_{ev}/\partial A$ and $\partial r_{ev}/\partial B$ which are precalculated and supplied to the astronaut in the form of charts or tables are shown in figure 5 for vehicle positions along the ascent portion of the nominal trajectory. The partials $\partial r_{mv}/\partial A$ and $\partial r_{mv}/\partial C$ required to determine range from the Moon are also shown. These values will approach those shown for $\partial r_{ev}/\partial A$ and $\partial r_{ev}/\partial B$ at distances near the Earth. It is to be noted that for a position fix near the Moon higher accuracy is obtained by measuring range from the Earth rather than from the Moon, inasmuch as the geometry gives lower values for $\partial r_{ev}/\partial A$ and $\partial r_{ev}/\partial B$. From equation (11) it is seen that the error in determining the distance off the nominal Δr is proportional to the magnitudes of the partials.

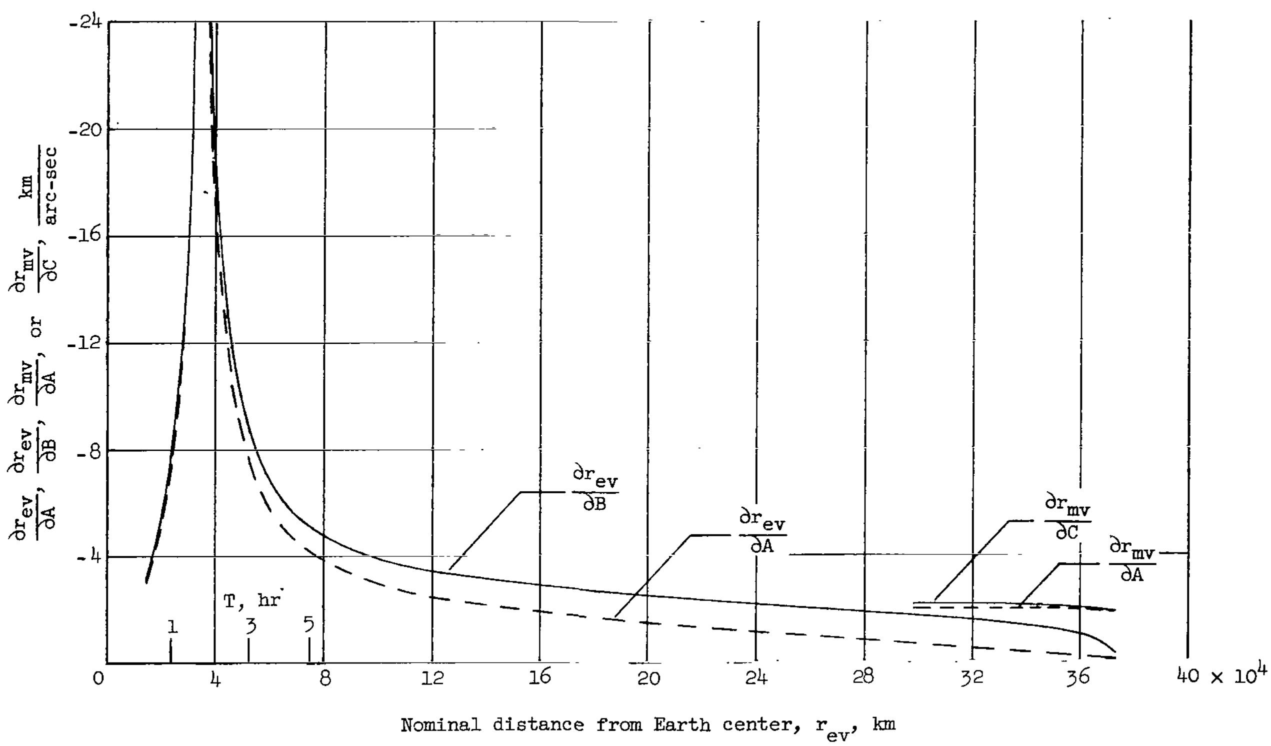


Figure 5.- Variation over Earth-Moon distance in partials required for range determination.

In figure 5 the large values shown for $\partial r_{ev}/\partial A$ and $\partial r_{ev}/\partial B$ near the Earth occur because of the close proximity of the vehicle to the Earth-Moon line (i.e., the angle A approaches 180°). The relative position of the vehicle with respect to the instantaneous Earth-Moon line is shown in figure 4. Thus, range or position-fixing determination by use of equation (11) would be inaccurate for about the first 5 hours (or 75,000 kilometers) from injection. These results would be typical for any Earth-Moon trajectory for any method which is based on Earth-Moon measurements. The accuracy for the first 5 hours could be improved considerably by determining range from the angular-diameter measurement of the Earth, as shown in figure 6. The dashed curve represents the error in range for an angular-diameter-measurement σ error of 10 arc-seconds.

For times along the trajectory except for the first 5 hours, the error in determining range by equation (11) is relatively small, as shown by the solid curve in figure 6. This curve represents the error in the incremental range Δr_{ev} along the ascent portion of the nominal trajectory where the angular measurements A and θ_1 are considered to have random uncorrelated errors, each with a standard deviation σ of 10 arc-seconds such that

$$\sigma_{\Delta r_{ev}} = \left[\left(\frac{\partial r_{ev}}{\partial A}\sigma_A\right)^2 + \left(\frac{\partial r_{ev}}{\partial \theta_1}\sigma_{\theta_1}\right)^2\right]^{1/2}$$

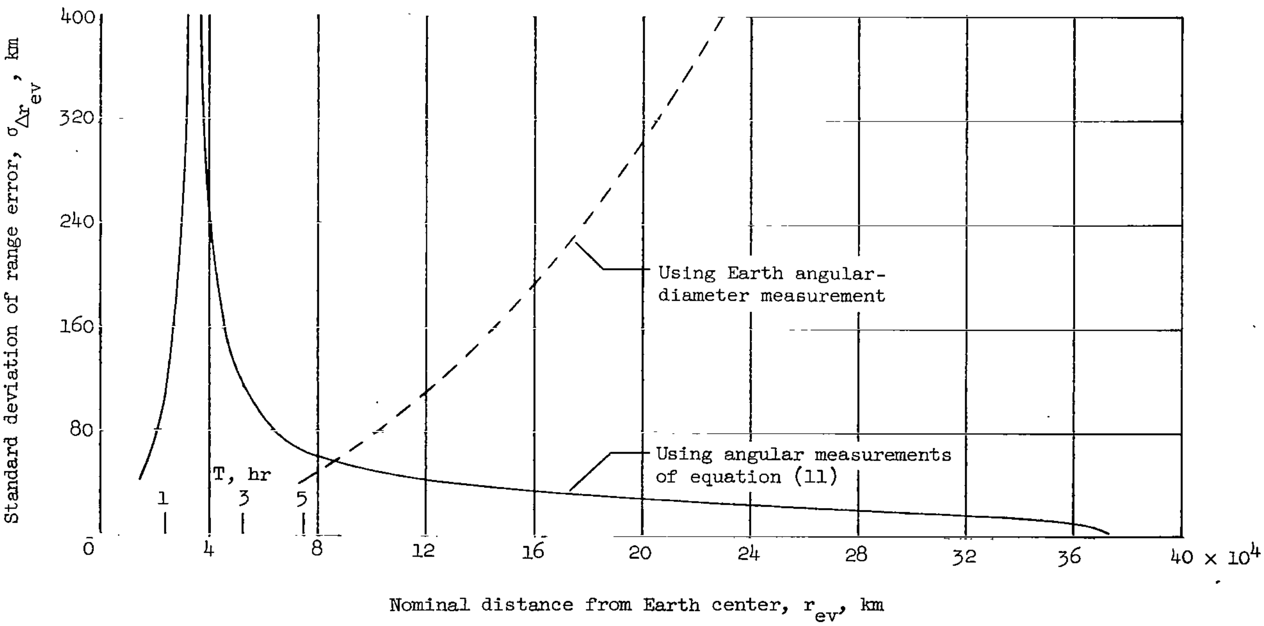


Figure 6.- Variation of error in range determination over Earth-Moon distance; σ of angular-measurement error is 10 arc-seconds.

In figure 6, the star which is used to measure θ_1 is assumed to lie in (or near) the instantaneous Earth-Moon-vehicle plane so that the value of the term $\left(\frac{1 - \cos^2\theta_1}{\cos^2\delta_1 - \cos^2\theta_1}\right)^{1/2}$ in equation (11) is 1 and $\frac{\partial r_{ev}}{\partial B} = \pm \frac{\partial r_{ev}}{\partial \theta_1}$.

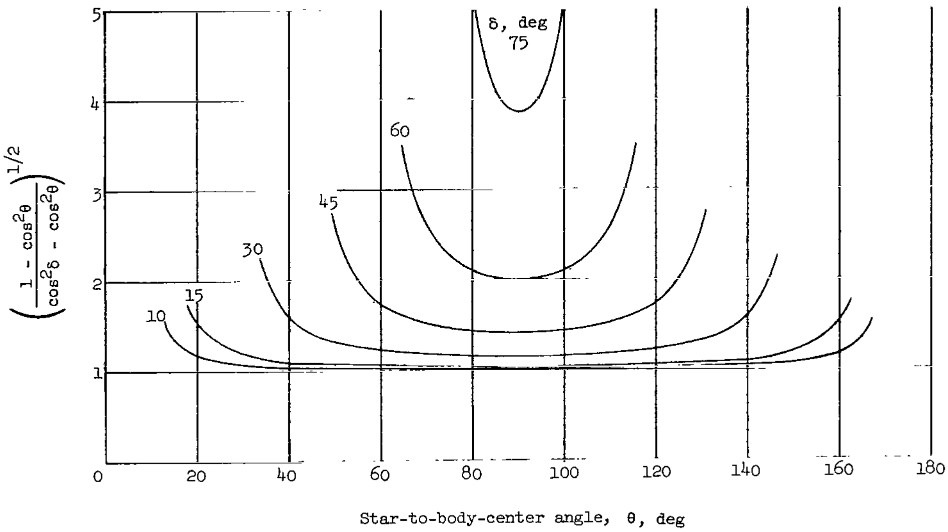
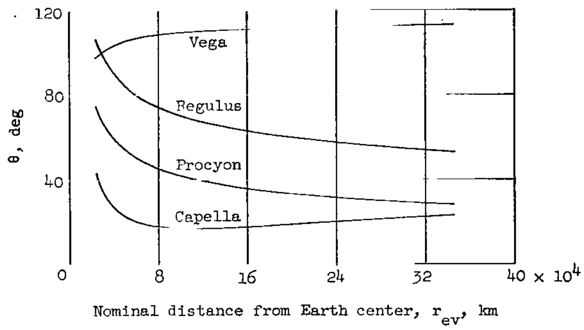


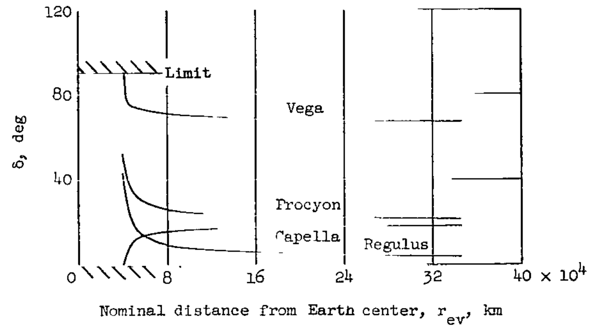
Figure 7.- Effect of star position on accuracy of range determination;

$$\Delta r_{ev} = \frac{\partial r_{ev}}{\partial A}\Delta A + c\,\frac{\partial r_{ev}}{\partial B}\left(\frac{1 - \cos^2\theta}{\cos^2\delta - \cos^2\theta}\right)^{1/2}\Delta\theta .$$

<u>Effect of star selected for measurement</u>.- The value of the quantity $\left(\frac{1 - \cos^2\theta_1}{\cos^2\delta_1 - \cos^2\theta_1}\right)^{1/2}$ can vary from 1 to ∞, depending on the nominal trajectory and the star selected for the measurement of θ_1. The effect of star position on the error in Δr is shown in figure 7. The error in Δr is a minimum when the quantity $\left(\frac{1 - \cos^2\theta_1}{\cos^2\delta_1 - \cos^2\theta_1}\right)^{1/2}$ is a minimum. This quantity has its minimum with respect to θ at $\theta = 90^\circ$ and rises rapidly as the extreme values, $\theta = \delta$ and $\theta = 180^\circ - \delta$, are approached. The extreme values of θ occur when the star position is in the plane containing the vehicle-Earth line and the vertical to the Earth-Moon-vehicle plane. The desired stars, therefore, are those that lie near the Earth-Moon-vehicle plane (δ within about $\pm 30^\circ$) with projections in that plane away from the vehicle-Earth line. In figure 8 values of the



(a) Angle between star and Earth center.



(b) Angle between star and its projection in instantaneous Earth-Moon-vehicle plane.

Figure 8.- Variation of θ and δ over the Earth-Moon distance for nominal trajectory.

angles θ and δ for several stars are shown along the ascent portion of the nominal trajectory.

As indicated in figure 8(b), large changes can occur in δ as the vehicle crosses "under" the Earth-Moon line at $r_{ev} \approx 40{,}000$ kilometers. In this region the Earth-Moon-vehicle plane rotates about the Earth-Moon line through an angle of approximately 180°. For the star Regulus, which forms a star-to-body angle θ of about 90°, the value of δ would go through a range of angles close to 180°.

Determination of Vehicle Position

As previously stated, equations (1), in the form given, are not convenient for manual calculation of vehicle position. The positions off the nominal trajectory, however, can be easily calculated from the following set of equations:

$$\begin{Bmatrix} x_a - x_n \\ y_a - y_n \\ z_a - z_n \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}^{-1} \begin{Bmatrix} r_a \cos\theta_{1,a} - r_n \cos\theta_{1,n} \\ r_a \cos\theta_{2,a} - r_n \cos\theta_{2,n} \\ r_a \cos\theta_{3,a} - r_n \cos\theta_{3,n} \end{Bmatrix} \tag{14}$$

where the subscript a represents an actual (measured) value and the subscript n represents a nominal value. After substituting

$$\begin{Bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{Bmatrix} = \begin{Bmatrix} x_a - x_n \\ y_a - y_n \\ z_a - z_n \end{Bmatrix}$$

and

$$\begin{Bmatrix} D \\ E \\ F \end{Bmatrix} = \begin{Bmatrix} r_a \cos \theta_{1,a} - r_n \cos \theta_{1,n} \\ r_a \cos \theta_{2,a} - r_n \cos \theta_{2,n} \\ r_a \cos \theta_{3,a} - r_n \cos \theta_{3,n} \end{Bmatrix} \tag{15}$$

a solution of the set of the three simultaneous equations (14) yields:

$$\left.\begin{aligned} \Delta z &= \left(\frac{Rl_2 - Pl_3}{QR - PS}\right)D + \left(\frac{Pl_1}{QR - PS}\right)F - \left(\frac{Rl_1}{QR - PS}\right)E \\ \Delta y &= \left(\frac{l_2}{P}\right)D - \left(\frac{l_1}{P}\right)E - \left(\frac{Q}{P}\right)\Delta z \\ \Delta x &= \left(\frac{1}{l_1}\right)D - \left(\frac{m_1}{l_1}\right)\Delta y - \left(\frac{n_1}{l_1}\right)\Delta z \end{aligned}\right\} \tag{16}$$

where

$$\left.\begin{aligned} P &= \left(m_1 l_2 - m_2 l_1\right) \\ Q &= \left(n_1 l_2 - n_2 l_1\right) \\ R &= \left(m_1 l_3 - m_3 l_1\right) \\ S &= \left(n_1 l_3 - n_3 l_1\right) \end{aligned}\right\} \tag{17}$$

The parenthetical expressions in equations (16) and (17) are constants for any three given stars and can be precalculated. Various solutions of the set of equations (14) can be obtained, but all would be similar to those given by equations (16).

Therefore, equations (16), or some similar to these, are the equations that an astronaut could use to calculate vehicle position. The star-to-body angle measurement utilized in equation (11) is used, of course, as one of the star measurements in the set of equations (14). As shown by equations (16), a position fix would require, first, a calculation of the quantities D, E, and F from the four measurements r_a, $\theta_{1,a}$, $\theta_{2,a}$, and $\theta_{3,a}$, second, nine slide rule multiplications, and third, several additions and subtractions on a scratch pad. Inspection of the expressions for D, E, and F (eq. (15)), however, indicates calculations involving a large number of significant figures which cannot be accurately performed on a slide rule. A procedure similar to that previously shown for determining Δr can be employed by making use of precalculated values of D, E, and F for a nominal trajectory, as described in the following section.

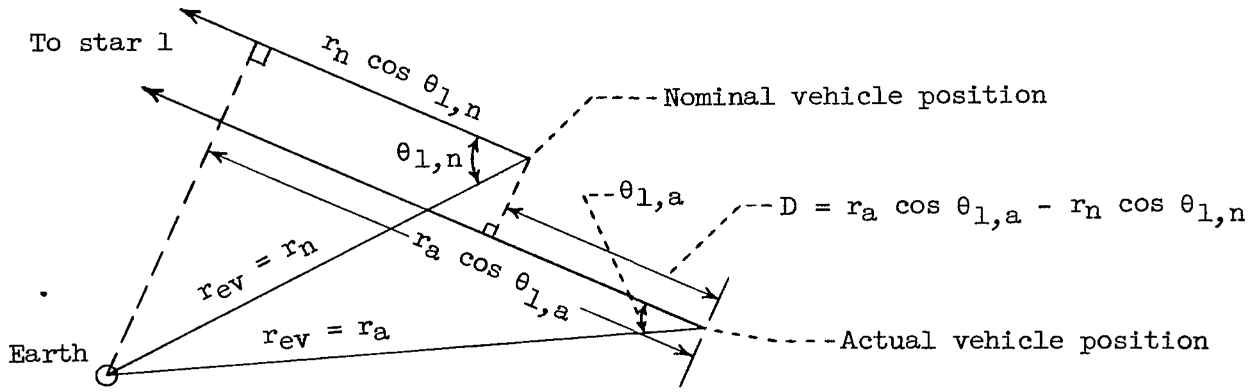


Figure 9.- Example of variables involved in determining vehicle position.

Determination of the measurement quantities D, E, and F.- The quantity D is illustrated in figure 9. By inspection of figure 9, the derivative of D is seen to be:

$$dD = \frac{\partial D}{\partial r_{ev}} dr_{ev} + \frac{\partial D}{\partial \theta_1} d\theta_1$$

or, for the region in which a change in D is linear with changes in r_{ev} and θ_1,

$$\Delta D = \frac{\partial D}{\partial r_{ev}} \Delta r_{ev} + \frac{\partial D}{\partial \theta_1} \Delta\theta_1$$

Since $D = 0$ when $r_a \cos \theta_{1,a} = r_n \cos \theta_{1,n}$, then $\Delta D = D$, so that

$$D = \frac{\partial D}{\partial r_{ev}} \Delta r_{ev} + \frac{\partial D}{\partial \theta_1} \Delta\theta_1 \tag{18}$$

From the expression

$$D = r_a \cos \theta_{1,a} - r_n \cos \theta_{1,n}$$

it is seen that

$$\frac{\partial D}{\partial r_{ev}} = \cos \theta_1 \tag{19}$$

and

$$\frac{\partial D}{\partial \theta_1} = -r_{ev} \sin \theta_1 \tag{20}$$

where r_{ev} and θ_1 are known values for the nominal trajectory. Equations similar to equations (18), (19), and (20) are obtained for E and F by replacing θ_1 with θ_2 and θ_3, respectively.

Values of the partials $\partial D/\partial r_{ev}$ and $\partial D/\partial \theta$ are shown in figure 10 for the example nominal trajectory. The data correspond to the star Regulus, which lies near the instantaneous Earth-Moon-vehicle plane throughout most of the trajectory. (See fig. 8(b).) The astronaut, then, would be supplied with information, such as given in figure 10, concerning three preselected stars. This type of information, along with the values for Δr_{ev}, $\Delta\theta_1$, $\Delta\theta_2$, and $\Delta\theta_3$, would allow him to calculate readily the quantities D, E, and F by equations similar to equation (18). The value of Δr_{ev} is obtainable directly from equation (11).

Linear region for the quantities D, E, and F.- For equation (18) to be exactly correct, the partials $\partial D/\partial r_{ev}$ and $\partial D/\partial \theta_1$ for a given time must be constants. Inspection of equation (19) shows that $\partial D/\partial r_{ev}$ is a constant for all values of r_{ev}. From equation (20),

$$\frac{\partial^2 D}{\partial \theta_1^2} = -r_{ev} \cos \theta_1$$

Hence, at a given time, $\partial D/\partial \theta_1$ will be constant for a value of $\theta_1 = 90^\circ$. Actually, $\partial D/\partial \theta_1$ will be approximately constant over a large range of values of θ_1 (for example, $60^\circ < \theta_1 < 120^\circ$). Star selection should not be necessarily limited to this range of star-to-body angles, however, unless very large deviations (greater than $\pm 2^\circ$) from the nominal θ are expected. This deviation of 2° would correspond roughly to a 2-hour difference between the actual injection time and a nominal injection time.

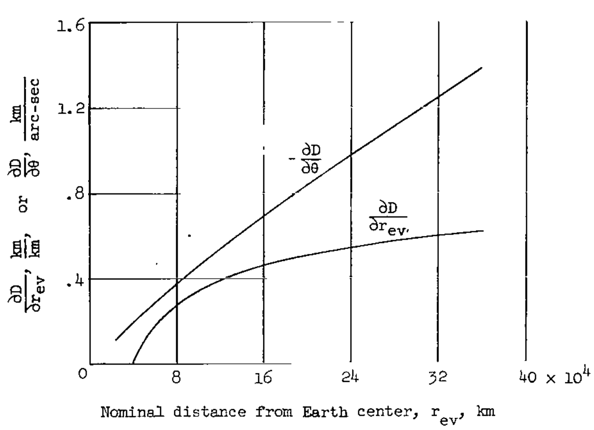


Figure 10.- Variation over Earth-Moon distance of $\frac{\partial D}{\partial \theta}$ and of $\frac{\partial D}{\partial r_{ev}}$ for star Regulus.

CORRECTION FOR NONSIMULTANEOUS MEASUREMENTS

In the equations for determination of range (eq. (11)) and position (eqs. (16)), all four measurements are assumed to have been made simultaneously. Since simultaneous measurements are usually not possible with the same instrument, errors would be introduced in the results unless the measurements were corrected or converted to some common time. This time (for which the position fix is required) could be either before or after the actual time of the measurements. If the time corresponding to one of the measurements is made the common time, only three measurements would have to be converted.

The measurements can be converted to a common time (small increments of time, only) by use of precalculated data on the rate of change of angle with time along the nominal trajectory. Examples of such data are shown in figure 11 for several stars, as well as for the angular measurement between the Earth and the Moon. (Table II identifies the various stars used in this report.) These data would be practically the same for any other trajectory reasonably near the nominal, so that they would apply for making corrections when off the nominal trajectory. It is of interest to note that the curve shown for dA/dt would be about the same for any trajectory to the Moon providing the flight

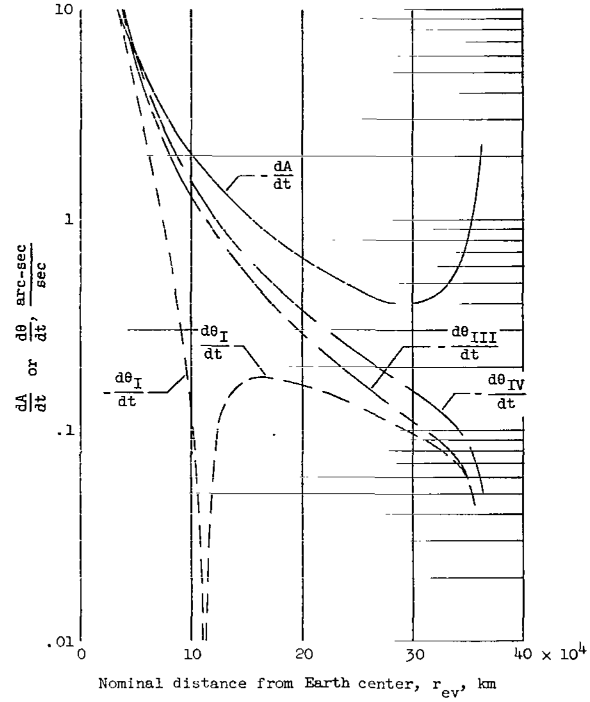


Figure 11.- Rate of change of measurement angles with time along ascent portion of nominal trajectory, where A is angle between Earth and Moon centers and θ_I, θ_{III}, θ_{IV} are angles between different stars and Earth center. (See table II for identification of stars.)

TABLE II.- STAR POSITIONS USED IN ANALYSIS

Star		l	m	n
Number	Name			
I	Capella	0.13862	0.68133	0.71873
II	Sirius	-.17870	.94126	-.28652
III	Procyon	-.40895	.90781	.09295
IV	Regulus	-.85903	.46641	.21104
V	Rigil Kentaurus	-.37937	-.30996	-.87178
VI	Vega	.11866	-.77099	.62570

time is in the neighborhood of 70 hours. Equations for calculating dA/dt and $d\theta/dt$ along a nominal trajectory are given in appendix B.

In converting a measurement to a common time, the corresponding value of $d\theta/dt$ or dA/dt at the given range multiplied by the time increment would be sufficient (for converting over short time intervals) to calculate the incremental angle to be added algebraically to the measured angle. The range would need to be known only approximately. For trajectories launched near nominal injection time, the nominal range (at the time of measurements) would be adequate, whereas for trajectories not launched near nominal injection time the approximate range could be determined according to time from actual injection time. Because the correction to the angle A would ordinarily be the largest (fig. 11) and also because the precalculations for dA/dt are more extensive and require more information on the nominal trajectory (see appendix B), it would be best to use the time at which A is measured as the common time. In this case,

$$\left.\begin{aligned}
A_a &= (A_a)_{unc} \\
\theta_{1,a} &= (\theta_{1,a})_{unc} + \frac{d\theta_1}{dt}\Delta t_1 \\
\theta_{2,a} &= (\theta_{2,a})_{unc} + \frac{d\theta_2}{dt}\Delta t_2 \\
\theta_{3,a} &= (\theta_{3,a})_{unc} + \frac{d\theta_3}{dt}\Delta t_3
\end{aligned}\right\} \tag{21}$$

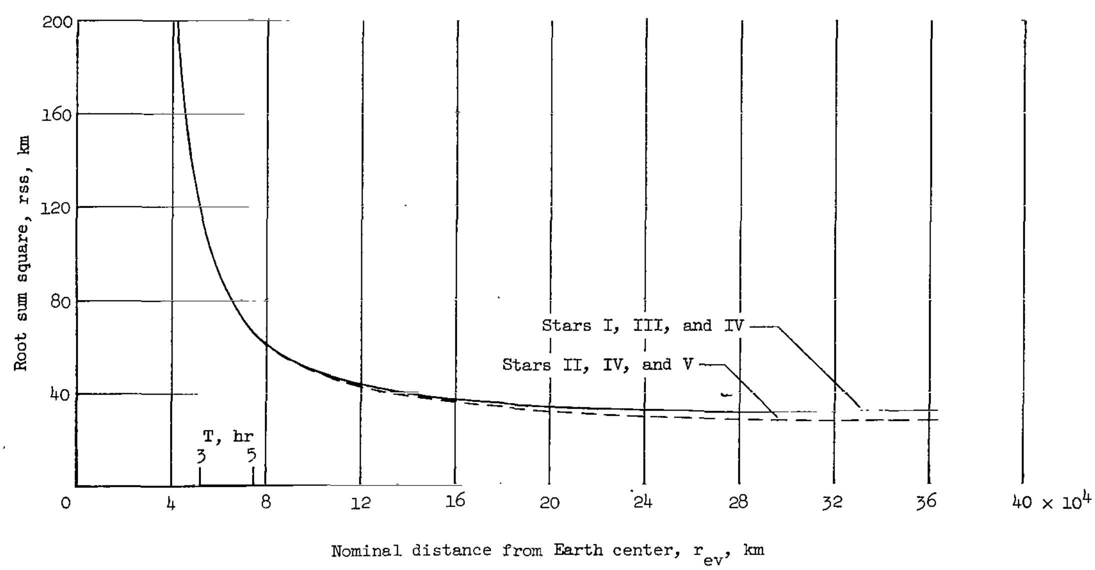


Figure 12.- Accuracy of manual method over Earth-Moon distance; σ of measurement error is 10 arc-seconds. (See table II for identification of stars.)

The calculated results shown in figures 12 and 13 are typical of the accuracy obtainable with the present method, providing the range is determined by the angular measurements given in equation (11). It can be seen that after the first 5 hours the root-sum-square position error is in the neighborhood of 35 kilometers. The large position errors shown for the first 5 hours are due to large errors in range determination (fig. 6) when the vehicle is in close proximity to the Earth-Moon line. If, however, the range in this region is determined by an angular-diameter measurement of the Earth, the root-sum-square position error could be reduced to 40 kilometers or less.

In the analysis the errors in each of the four measurements were assumed to be uncorrelated. The results are for the case in which the star used in determining Δr lies in or near the instantaneous Earth-Moon-vehicle plane; that is, the range error shown in figure 6 would apply in this case. In the error analysis, linear perturbation theory was used to calculate values for the root sum square of the standard deviations of the component position errors. (See appendix C for the pertinent equations.) This root-sum-square value can be considered as a measure of the position error.

The results in figure 12 are for standard deviations of measurement error of 10 arc-seconds in each of the measurements. It should be noted that doubling the measurement error will approximately double the position-determination error. The variation in position error with range is shown in figure 12 for

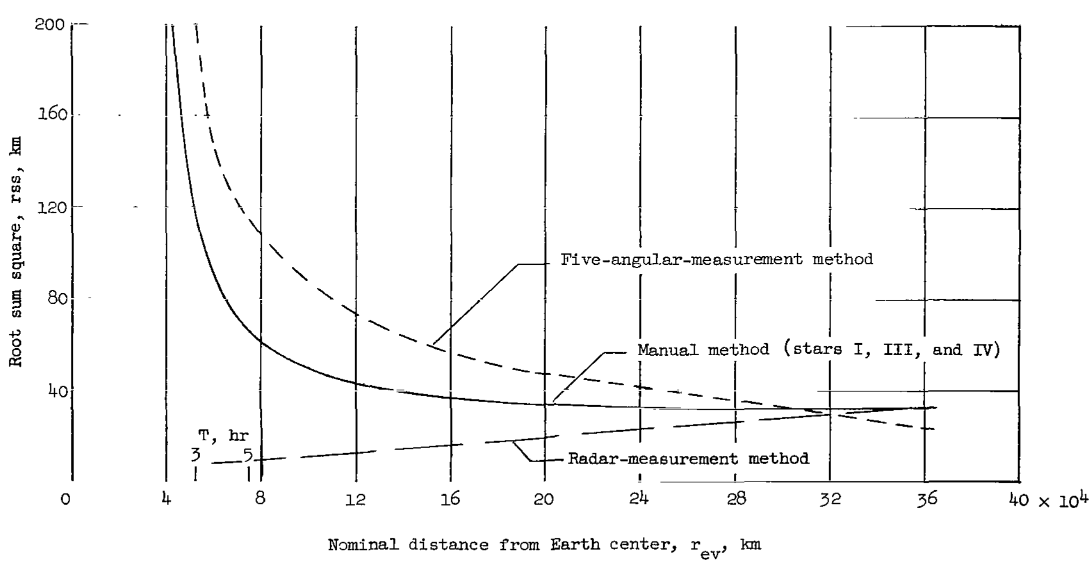


Figure 13.- Comparison of manual method with two other methods of high accuracy; σ of angular-measurement error is 10 arc-seconds and of radar-distance-measurement error is 16 meters.

cases in which two different sets of stars are used for the measurements. The accuracy for either set is about the same, and this fact would be true for any combination of stars selected, with three exceptions. These three exceptions are that (1) the star used for the determination of Δr (eq. (11)) must lie within about $\pm 30^{\circ}$ of the instantaneous Earth-Moon-vehicle plane (fig. 7); (2) the stars selected should be such that the rate of change of θ with time along any part of the trajectory is comparatively large (ref. 8); for example, such as is indicated for the stars Regulus and Procyon in fig. 8(a); (3) the stars should not all be at approximately the same location.

In figure 13 the manual method is shown to be about as accurate over most of the Earth-Moon distance as two other position-determination methods of good accuracy. The results shown for these two other methods were obtained from reference 6. The radar-measurement method consists of the same onboard star-to-body angular measurements as the manual method; however, the range is determined by a highly accurate Earth-based radar measurement. The errors shown for this method are almost entirely due to the errors in the three angular measurements. In the five-angular-measurement method, five onboard angular measurements are used to determime vehicle position. The five angular measurements are the included angles between stars and body centers. More specifically, the measurements are the included angles between each of three stars and Earth (or Moon) center and between two of these stars and Moon (or Earth) center. Of all the optical methods which rely completely on onboard sightings and which do not use statistical procedures or information on a precalculated nominal trajectory, this five-measurement method is the most accurate. (See ref. 6.) However, because of the complexity of the equations, the onboard calculations for this method would require an automatic computer.

SUMMARY OF PROCEDURES FOR MANUAL METHOD

No attempt is made in this report to systematize the present method. The required procedures are simple enough and any system used would be according to individual preference. The various required calculations (both precalculations and onboard calculations) are summarized in the following outline in the proper order, along with the pertinent equation numbers.

Precalculations required for the nominal trajectory for various times from nominal injection time (assuming that nominal values of the quantities listed in table I are known):

Angle A . . . . Equation (6)

Angles θ_1, θ_2, and θ_3 . . . . Equation (A1)

Derivatives $\frac{d\theta_1}{dt}$, $\frac{d\theta_2}{dt}$, $\frac{d\theta_3}{dt}$, and $\frac{dA}{dt}$ . . . . Equations (B1) and (B3)

Partials $\frac{\partial r_{ev}}{\partial A}$ and $\frac{\partial r_{ev}}{\partial B}$ . . . . Equations (5) and (7)

Quantity $c\left(\frac{1 - \cos^2\theta_1}{\cos^2\delta_1 - \cos^2\theta_1}\right)^{1/2}$ by use of . . . . . . Equations (A2) and (A4)

Partials $\frac{\partial D}{\partial r_{ev}}$, $\frac{\partial E}{\partial r_{ev}}$, $\frac{\partial F}{\partial r_{ev}}$, $\frac{\partial D}{\partial \theta_1}$, $\frac{\partial E}{\partial \theta_2}$, and $\frac{\partial F}{\partial \theta_3}$ . . . . Equations (19) and (20)

According to the selected set of stars, calculate the constants P, Q, R, and S (eqs. (17)), as well as the other constants appearing in equations (16)

Onboard calculations required for a position fix (assuming that the four non-simultaneous measurements have been taken over a relatively short time interval):

Convert each measurement to a common time (small increments of time, only) by equations similar to . . . . Equations (21)

Determine the values ΔA, $\Delta\theta_1$, $\Delta\theta_2$, and $\Delta\theta_3$, which are the differences between the actual and nominal angles

Calculate the value Δr_{ev} . . . . Equation (11)

Determine the quantities D, E, and F from equations similar to . . . . Equation (18)

Calculate position $\Delta x, \Delta y, \Delta z$ relative to nominal trajectory . . . . Equations (16)

Calculate true position x,y,z by algebraically adding Δx, Δy, and Δz to nominal trajectory values

The units for the onboard calculations would be arc-seconds for the angles and kilometers for the range and position values.

A trial calculation of a position fix which would apply to an Earth-Moon trajectory was performed in approximately 15 minutes. For this calculation it was assumed that the measurements had been recorded. With two persons working together, the time could be reduced.

CONCLUDING REMARKS

A simple method for manual determination of position of a space vehicle has been presented. The method as presented has been applied to a lunar trajectory; however, it would apply equally as well to an interplanetary trajectory.

The method makes use of the knowledge of the distance between the Earth and the Moon (or the distance between a planet and the Sun) to obtain an accurate measure of position relative to a precalculated nominal or reference trajectory. The linearity characteristics of the method are such that good accuracy can be obtained even though differences as large as 2 hours exist between actual and nominal injection time. In addition to the one type of instrument to measure angles, the only other requirements for the navigator are time-history information on the nominal trajectory, various charts or tables of precalculated information, and several simple equations which can be solved adequately with a slide rule. It has been determined that after the measurements have been made the method would require no more than a total of 15 minutes to calculate a position fix.

From results of an error analysis, the accuracy of the method after about the first 5 hours from the Earth was shown to be approximately constant over the remaining Earth-Moon distance. In this region the error in position determination was shown to be in the neighborhood of 35 kilometers. For the times within the first 5 hours, close proximity of the vehicle to the Earth-Moon line causes large error in position determination unless range is determined by a measurement of the angular diameter of the Earth. Also, in relation to the accuracy characteristics, practically any set of three stars selected for the measurements would give good results providing that the star used in the range determination lies within about $\pm 30^{\circ}$ of the instantaneous Earth-Moon-vehicle plane.

Under linear assumptions, a manual guidance procedure could be developed which makes use of the position measurements. The same number of calculations as for two position fixes could determine the velocity changes for performing a midcourse correction. The main limitation to such a procedure would be the distance the flight trajectory could be allowed to deviate from the nominal trajectory.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., July 20, 1964.

APPENDIX A

EQUATIONS FOR CALCULATING δ, θ, AND c

The angles δ and θ (fig. 14), as well as the value c, pertaining to a nominal trajectory can be precalculated for any given star with known direction cosines l, m, and n by the following equations.

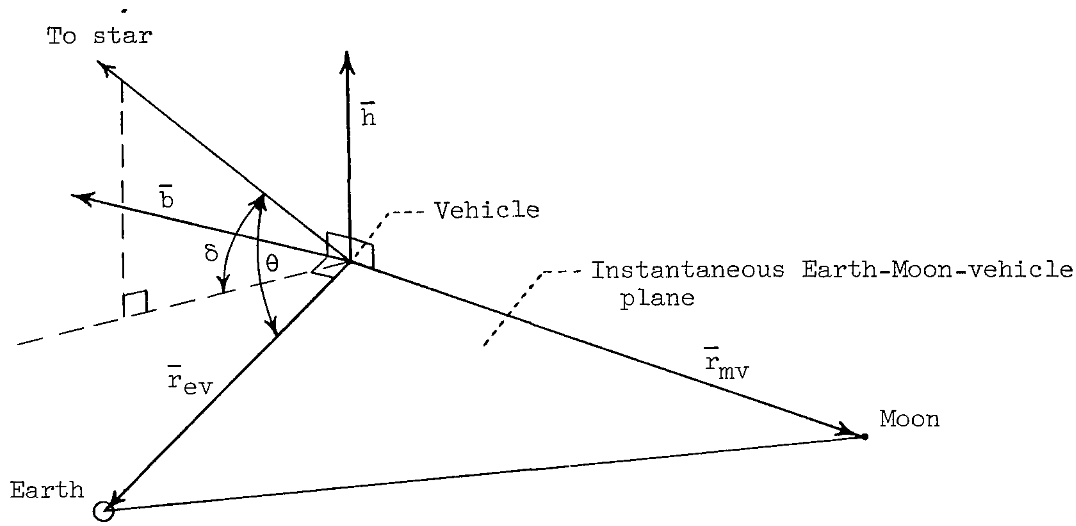


Figure 14.- Illustration showing the vectors $\bar{b}$ and $\bar{h}$.

The angle θ, which is the angle from the vehicle between the lines of sight to a star and a body center (Earth, for example), is obtained from the dot product of the position vectors of the Earth center and star

$$\cos\theta = \frac{l x_{ev} + m y_{ev} + n z_{ev}}{r_{ev}} \qquad \left(0^\circ \leqq \theta \leqq 180^\circ\right) \tag{A1}$$

By defining a vector $\bar{h}$ perpendicular to the instantaneous Earth-Moon-vehicle plane, the angle δ, which is the angle between the line of sight to the star and its projection in this plane, is obtained from the dot product of the vector $\bar{h}$ and the position vector of the star

$$\sin\delta = \left|\frac{l h_x + m h_y + n h_z}{h}\right| \tag{A2}$$

where the absolute value can be used such that $0^\circ \leqq \delta \leqq 90^\circ$ and where the values for h_x, h_y, h_z, and h are obtained from the cross product of the position vectors of the Earth and Moon centers. Thus,

$$\left.\begin{aligned} h_x &= y_{ev}z_{mv} - z_{ev}y_{mv} \\ h_y &= z_{ev}x_{mv} - x_{ev}z_{mv} \\ h_z &= x_{ev}y_{mv} - y_{ev}x_{mv} \end{aligned}\right\} \tag{A3}$$

and

$$h = \left(h_x^2 + h_y^2 + h_z^2\right)^{1/2}$$

The sign of the quantity c in equation (11) is determined by the position of the star with respect to the Earth-vehicle line $\bar{r}_{ev}$. The sign is positive (negative) if, as viewed from the direction of $\bar{h}$, the projection of the line to the star in the instantaneous Earth-Moon-vehicle plane is to the right (left) of the Earth-vehicle line. As shown in figure 14, the relative directions of these two lines at a given time along a nominal trajectory can be determined by first defining a vector $\bar{b}$ which is perpendicular to the plane containing the vectors $\bar{r}_{ev}$ and $\bar{h}$.

The vector $\bar{b}$ is obtained from the cross product of the vectors $\bar{r}_{ev}$ and $\bar{h}$ such that the components of $\bar{b}$ are given by

$$\begin{aligned} b_x &= y_{ev}h_z - z_{ev}h_y \\ b_y &= z_{ev}h_x - x_{ev}h_z \\ b_z &= x_{ev}h_y - y_{ev}h_x \end{aligned}$$

where the values for h_x, h_y, and h_z are given by equations (A3). Next, the dot product of the position vector of the star and the vector $\bar{b}$ will yield the component of the star vector along the vector $\bar{b}$ which is given by the term $\frac{lb_x + mb_y + nb_z}{b}$. If this term is positive, the projection of the star vector in the instantaneous Earth-Moon-vehicle plane is to the right of the Earth-vehicle line $\bar{r}_{ev}$ so that the quantity c is given as

$$c = \frac{lb_x + mb_y + nb_z}{\left|lb_x + mb_y + nb_z\right|} \tag{A4}$$

APPENDIX B

EQUATIONS FOR CALCULATING RATE OF CHANGE OF

MEASUREMENT WITH TIME

The rate of change of the measurement θ (angle between star and Earth center) with time along a nominal trajectory is given by

$$\frac{d\theta}{dt} = \frac{\partial\theta}{\partial x_{ev}} \frac{dx_{ev}}{dt} + \frac{\partial\theta}{\partial y_{ev}} \frac{dy_{ev}}{dt} + \frac{\partial\theta}{\partial z_{ev}} \frac{dz_{ev}}{dt} \tag{B1}$$

where dx_{ev}/dt, dy_{ev}/dt, and dz_{ev}/dt are known values for the nominal trajectory and, as can be determined from equations given in reference 8, the partials for the angular measurement θ are

$$\left.\begin{aligned}
\frac{\partial\theta}{\partial x_{ev}} &= \frac{x_{ev}\left(my_{ev} + nz_{ev}\right) - l\left(y_{ev}^2 + z_{ev}^2\right)}{r_{ev}^2\left[r_{ev}^2 - \left(lx_{ev} + my_{ev} + nz_{ev}\right)^2\right]^{1/2}} \\
\frac{\partial\theta}{\partial y_{ev}} &= \frac{y_{ev}\left(lx_{ev} + nz_{ev}\right) - m\left(x_{ev}^2 + z_{ev}^2\right)}{r_{ev}^2\left[r_{ev}^2 - \left(lx_{ev} + my_{ev} + nz_{ev}\right)^2\right]^{1/2}} \\
\frac{\partial\theta}{\partial z_{ev}} &= \frac{z_{ev}\left(lx_{ev} + my_{ev}\right) - n\left(x_{ev}^2 + y_{ev}^2\right)}{r_{ev}^2\left[r_{ev}^2 - \left(lx_{ev} + my_{ev} + nz_{ev}\right)^2\right]^{1/2}}
\end{aligned}\right\} \tag{B2}$$

The rate of change of the measurement A (angle between Earth and Moon centers) with time along the nominal trajectory is given by

$$\begin{aligned}
\frac{dA}{dt} = {} & \frac{\partial A}{\partial x_{ev}} \frac{dx_{ev}}{dt} + \frac{\partial A}{\partial y_{ev}} \frac{dy_{ev}}{dt} + \frac{\partial A}{\partial z_{ev}} \frac{dz_{ev}}{dt} + \frac{\partial A}{\partial x_{mv}} \frac{dx_{mv}}{dt} + \frac{\partial A}{\partial y_{mv}} \frac{dy_{mv}}{dt} \\
& + \frac{\partial A}{\partial z_{mv}} \frac{dz_{mv}}{dt}
\end{aligned} \tag{B3}$$

where dx_{ev}/dt, dy_{ev}/dt, dz_{ev}/dt, dx_{mv}/dt, dy_{mv}/dt, and dz_{mv}/dt are known values for the nominal trajectory. The partials of A with respect to the

Earth are determined from equations (B2) by replacing the direction cosines of the stars l, m, and n with the direction cosines of the Moon:

$$\frac{\partial A}{\partial x_{ev}} = \frac{x_{ev}\left(y_{ev}y_{mv} + z_{ev}z_{mv}\right) - x_{mv}\left(y_{ev}^2 + z_{ev}^2\right)}{r_{ev}^2\left[r_{ev}^2 r_{mv}^2 - \left(x_{ev}x_{mv} + y_{ev}y_{mv} + z_{ev}z_{mv}\right)^2\right]^{1/2}}$$

$$\frac{\partial A}{\partial y_{ev}} = \frac{y_{ev}\left(x_{ev}x_{mv} + z_{ev}z_{mv}\right) - y_{mv}\left(x_{ev}^2 + z_{ev}^2\right)}{r_{ev}^2\left[r_{ev}^2 r_{mv}^2 - \left(x_{ev}x_{mv} + y_{ev}y_{mv} + z_{ev}z_{mv}\right)^2\right]^{1/2}}$$

$$\frac{\partial A}{\partial z_{ev}} = \frac{z_{ev}\left(x_{ev}x_{mv} + y_{ev}y_{mv}\right) - z_{mv}\left(x_{ev}^2 + y_{ev}^2\right)}{r_{ev}^2\left[r_{ev}^2 r_{mv}^2 - \left(x_{ev}x_{mv} + y_{ev}y_{mv} + z_{ev}z_{mv}\right)^2\right]^{1/2}}$$

The partials of A with respect to the Moon are determined from equations (B2) by replacing x_{ev}, y_{ev}, z_{ev}, and r_{ev} with x_{mv}, y_{mv}, z_{mv}, and r_{mv}, respectively, and by replacing the direction cosines of the stars l, m, and n with the direction cosines of the Earth:

$$\frac{\partial A}{\partial x_{mv}} = \frac{x_{mv}\left(y_{ev}y_{mv} + z_{ev}z_{mv}\right) - x_{ev}\left(y_{mv}^2 + z_{mv}^2\right)}{r_{mv}^2\left[r_{ev}^2 r_{mv}^2 - \left(x_{ev}x_{mv} + y_{ev}y_{mv} + z_{ev}z_{mv}\right)^2\right]^{1/2}}$$

$$\frac{\partial A}{\partial y_{mv}} = \frac{y_{mv}\left(x_{ev}x_{mv} + z_{ev}z_{mv}\right) - y_{ev}\left(x_{mv}^2 + z_{mv}^2\right)}{r_{mv}^2\left[r_{ev}^2 r_{mv}^2 - \left(x_{ev}x_{mv} + y_{ev}y_{mv} + z_{ev}z_{mv}\right)^2\right]^{1/2}}$$

$$\frac{\partial A}{\partial z_{mv}} = \frac{z_{mv}\left(x_{ev}x_{mv} + y_{ev}y_{mv}\right) - z_{ev}\left(x_{mv}^2 + y_{mv}^2\right)}{r_{mv}^2\left[r_{ev}^2 r_{mv}^2 - \left(x_{ev}x_{mv} + y_{ev}y_{mv} + z_{ev}z_{mv}\right)^2\right]^{1/2}}$$

APPENDIX C

EQUATIONS FOR ERROR ANALYSIS OF METHOD

For a linear perturbation procedure, which is used in this report to analyze position error, the system of equations (1) differentiated with respect to θ_1 yields equations for determining the effect of errors in θ_1 on the position coordinates x, y, and z:

$$\left.\begin{aligned} l_1 \frac{\partial x}{\partial \theta_1} + m_1 \frac{\partial y}{\partial \theta_1} + n_1 \frac{\partial z}{\partial \theta_1} &= -r \sin \theta_1 \\ l_2 \frac{\partial x}{\partial \theta_1} + m_2 \frac{\partial y}{\partial \theta_1} + n_2 \frac{\partial z}{\partial \theta_1} &= 0 \\ l_3 \frac{\partial x}{\partial \theta_1} + m_3 \frac{\partial y}{\partial \theta_1} + n_3 \frac{\partial z}{\partial \theta_1} &= 0 \end{aligned}\right\} \tag{C1}$$

When linear perturbation is used, the partials are, of course, assumed to be constants. Data in reference 6 show that these partials are essentially constant over variations of θ of $\pm 1.5^\circ$. These changes in θ would represent trajectories which are thousands of kilometers off the nominal trajectory.

A solution of the simultaneous equations (C1) yields:

$$\frac{\partial z}{\partial \theta_1} = \frac{r \sin \theta_1 \left(P l_3 - R l_2\right)}{QR - PS}$$

$$\frac{\partial y}{\partial \theta_1} = \frac{-l_2 r \sin \theta_1}{P} - \frac{Q}{P} \frac{\partial z}{\partial \theta_1}$$

$$\frac{\partial x}{\partial \theta_1} = - \frac{m_2}{l_2} \frac{\partial y}{\partial \theta_1} - \frac{n_2}{l_2} \frac{\partial z}{\partial \theta_1}$$

where P, Q, R, and S are defined by equations (17).

In a similar manner, the partials for θ_2, θ_3, and r can be determined as

$$\frac{\partial z}{\partial \theta_2} = \frac{R l_1 r \sin \theta_2}{QR - PS}$$

$$\frac{\partial y}{\partial \theta_2} = -\frac{S}{R}\frac{\partial z}{\partial \theta_2}$$

$$\frac{\partial x}{\partial \theta_2} = -\frac{m_1}{l_1}\frac{\partial y}{\partial \theta_2} - \frac{n_1}{l_1}\frac{\partial z}{\partial \theta_2}$$

$$\frac{\partial z}{\partial \theta_3} = -\frac{P l_1 r \sin \theta_3}{QR - PS}$$

$$\frac{\partial y}{\partial \theta_3} = -\frac{Q}{P}\frac{\partial z}{\partial \theta_3}$$

$$\frac{\partial x}{\partial \theta_3} = -\frac{m_1}{l_1}\frac{\partial y}{\partial \theta_3} - \frac{n_1}{l_1}\frac{\partial z}{\partial \theta_3}$$

$$\frac{\partial z}{\partial r} = \frac{\cos \theta_1 \left(R l_2 - P l_3\right) + l_1\left(P \cos \theta_3 - R \cos \theta_2\right)}{QR - PS}$$

$$\frac{\partial y}{\partial r} = \frac{l_2 \cos \theta_1 - l_1 \cos \theta_2}{P} - \frac{Q}{P}\frac{\partial z}{\partial r}$$

$$\frac{\partial x}{\partial r} = \frac{\cos \theta_1}{l_1} - \frac{m_1}{l_1}\frac{\partial y}{\partial r} - \frac{n_1}{l_1}\frac{\partial z}{\partial r}$$

The standard deviations of the errors in the x-, y-, and z-directions due to errors in the measurements are, then,

$$\left.\begin{aligned}
\sigma_x^2 &= \left(\sigma_{\Delta x}\right)^2 = \left(\frac{\partial x}{\partial \theta_1}\sigma_\theta\right)^2 + \left(\frac{\partial x}{\partial \theta_2}\sigma_\theta\right)^2 + \left(\frac{\partial x}{\partial \theta_3}\sigma_\theta\right)^2 + \left(\frac{\partial x}{\partial r}\sigma_r\right)^2 \\
\sigma_y^2 &= \left(\sigma_{\Delta y}\right)^2 = \left(\frac{\partial y}{\partial \theta_1}\sigma_\theta\right)^2 + \left(\frac{\partial y}{\partial \theta_2}\sigma_\theta\right)^2 + \left(\frac{\partial y}{\partial \theta_3}\sigma_\theta\right)^2 + \left(\frac{\partial y}{\partial r}\sigma_r\right)^2 \\
\sigma_z^2 &= \left(\sigma_{\Delta z}\right)^2 = \left(\frac{\partial z}{\partial \theta_1}\sigma_\theta\right)^2 + \left(\frac{\partial z}{\partial \theta_2}\sigma_\theta\right)^2 + \left(\frac{\partial z}{\partial \theta_3}\sigma_\theta\right)^2 + \left(\frac{\partial z}{\partial r}\sigma_r\right)^2
\end{aligned}\right\} \qquad \text{(C2)}$$

The root-sum-square (rss) of the position error due to errors in the three star-to-body angular measurements and in the range measurement (which includes one of the star-to-body angular measurements and the Earth-Moon angular measurement) is

$$\mathrm{rss} = \left[\left(\sigma_{\Delta x}\right)^2 + \left(\sigma_{\Delta y}\right)^2 + \left(\sigma_{\Delta z}\right)^2\right]^{1/2}$$

Actually, the quantities $\partial x/\partial r$, $\partial y/\partial r$, and $\partial z/\partial r$ need not be calculated since the sum of the last terms in each of equations (C2) is approximately equal to the variance of the range error determined from equation (11), where (for range to the Earth)

$$\left(\sigma_{\Delta r_{ev}}\right)^2 = \left(\frac{\partial r_{ev}}{\partial A} \sigma_A\right)^2 + \left[c \frac{\partial r_{ev}}{\partial B}\left(\frac{1 - \cos^2\theta_1}{\cos^2\delta_1 - \cos^2\theta_1}\right)^{1/2} \sigma_{\theta_1}\right]^2$$

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