NASA TECHNICAL NOTE

NASA TN D-2216

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NASA TN D-2216

ON FREQUENCY MEASUREMENTS AND RESOLUTION

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • DECEMBER 1964

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SUMMARY

This article relates the resolution to which the frequency of a sine wave can be measured (using zero-crossing techniques) to the signal-to-noise ratio of the wave and the time available to make the wave measurement. It demonstrates the need for making measurements over a constant time interval, instead of a constant number of cycles, if the resolution of measurement is to be independent of frequency.

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INTRODUCTION

The resolution to which the frequency of a sine wave perturbed by noise can be measured, by zero-crossing counting techniques, is related to the signal-to-noise ratio (S/N), the length of time over which the measurement is made, and the frequency to be measured. Familiarization with these relations facilitates the optimum use of the measurement technique.

The resolution of the measurement may be defined in a number of ways. The frequency to be measured is known to within two bounds, f_{max} and f_{min}. If this range is divided into 100/m equal increments, and the measurement is able to distinguish only between one of these increments and its neighbors, then the measurement of f is said to be *resolved* to m percent.

RELATION BETWEEN RESOLUTION AND NOISE

Noise fluctuations added to a sine wave of voltage affect the measurement of the frequency of that voltage. If the frequency is measured by zero-crossing techniques, the effects of the noise on the measurement can be determined under two assumptions. The first is that only $2n + 1$ zero crossings are detected per n cycles. The second assumption is that the perturbing additive noise has a Gaussian distribution. Let

$e = a \sin 2\pi ft =$ the input signal when it is free of noise,

$m \triangleq$ resolution, in percent of full scale, to which the measurement must be made,

$f \triangleq$ the frequency to be resolved to m percent of its total possible range B,

$B \triangleq f_{max} - f_{min}$ (See Figure 1),

$c \triangleq \frac{mB}{100} =$ the number of cps between resolvable elements,

$n \triangleq$ the number of cycles over which a measurement is made.

Since there are c cps between resolvable elements, the measurement is to determine if the frequency is f_1, its neighbor f_2 which equals $f_1 + c$, or its other neighbor f_3 which equals $f_1 - c$.

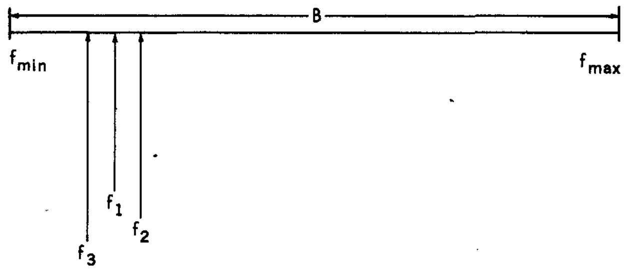

Figure 1—The frequency, as constrained within a specified range B.

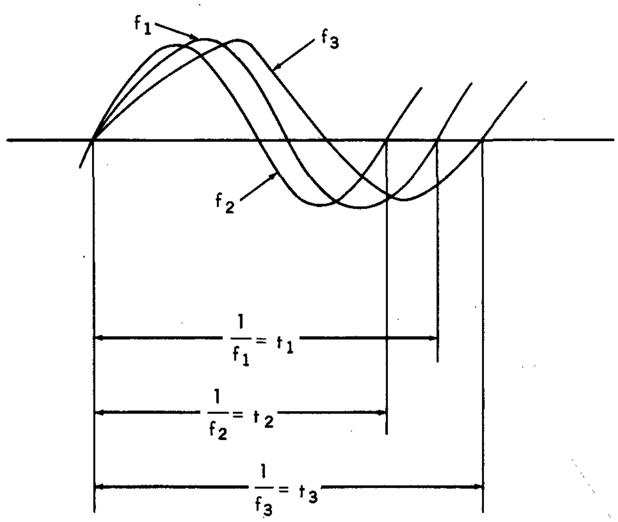

Figure 2—Determination of frequency by measurement of the elapsed time per cycle.

Frequency is determined by measuring the elapsed time per cycle (Figure 2). The time measurement t_1 of one frequency f_1 and the time measurement t_2 of its neighboring resolvable frequency f_2 differ by $1/f_1 - 1/f_2$. In general the magnitude of the difference in a 1 cycle time measurement between two adjacent resolvable elements of frequency is

$$\Delta t\Big|_{n=1} = \frac{1}{f} - \frac{1}{f+c} \text{ or } -\frac{1}{f} + \frac{1}{f-c} \tag{1}$$

In order to resolve the measurement of frequency to c cps, i.e., discriminate between f and its neighboring frequency $f+c$ or $f-c$, the measurement of elapsed time for 1 cycle must be in error by no more than $|\Delta t|_{n=1}$. If the measurement is to be made over n cycles, the required resolution can be maintained with n times the time measurement error for 1 cycle.

The magnitude of the total allowable time measurement error is (the neighbor $f+c$ is used in this example),

$$\Delta T = n\left(\frac{1}{f} - \frac{1}{f+c}\right). \tag{2}$$

An approximation for the voltage error corresponding to a time error of ΔT is (see Figure 3):

$$\Delta V = \Delta T \frac{de}{dt}\Big|_{t=0} = \Delta T \ (2\pi f a) \tag{3}$$

$$|\Delta V| = 2\pi \ fna\left(\frac{1}{f} - \frac{1}{f+c}\right) \tag{4}$$

The validity of this approximation depends on a relatively large S/N. This error could be caused either by a measurement error due to the measurement instrument itself or by noise which is perturbing the input signal. The analysis here neglects the instrument error.

These measurement errors, because they are due to random noise, are themselves random processes, and therefore must be considered from a statistical point of view. We shall consider ΔT and ΔV to be no longer distinct and constant, but rather random variables which can assume, any value and have associated with them a probability density distribution.

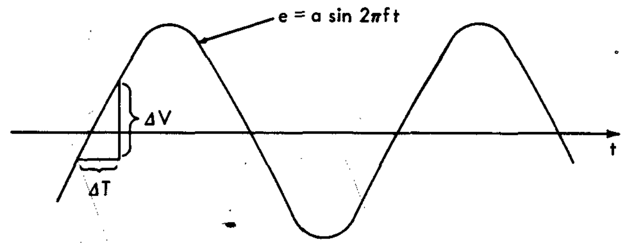

Figure 3—Signal voltage error displacement as a function of the time error displacement.

The error ΔT, and hence ΔV, is an algebraic summation of the errors due to noise from two measurements, the measurements of time at the first zero-crossing and at the zero-crossing at the end of n cycles. Therefore, by assuming a stationary process and a statistical independence between the readings, the standard deviation of one measurement and the algebraic sum of the two measurements are related by

$$\sigma(\text{sum}) = \sqrt{2}\,\sigma(\text{one}) \, , \tag{5}$$

where $\sigma(\text{sum})$ is the standard deviation of ΔV in Equation 4 and $\sigma(\text{one})$ is the rms voltage of the additive Gaussian noise which is causing the error in each measurement.

When a resolution c is prescribed for the measurement, the percentage of measurements that fall within c must also be prescribed. The probability (or percentage of measurements) that the time error ΔT will be found within $\pm K\sigma$ of its mean (which is zero) can be written in terms of the error function

$$\operatorname{erf}\left(\frac{K}{\sqrt{2}}\right) \triangleq \frac{2}{\sqrt{\pi}} \int_0^{K/\sqrt{2}} e^{-v^2}\, dv \, , \tag{6}$$

where $\operatorname{erf}(K/\sqrt{2}) = \text{Probability } (-K\sigma < \Delta T < K\sigma)$ and $v \triangleq$ a dummy variable. The function $1 - \operatorname{erf}(K/\sqrt{2})$ is plotted vs. K on Figure 4.

Combining the notions of Equations 4-6 gives

$$\Delta V = K\sigma\,(\text{sum}) = \sqrt{2}\,K\sigma\,(\text{noise}) \, ,$$

$$\sigma(\text{noise}) = \frac{\Delta V}{K\sqrt{2}} = \frac{2\pi\, fna}{K\sqrt{2}} \left(\frac{1}{f} - \frac{1}{f + c}\right) ,$$

$$\left(\frac{S}{N}\right)_{\substack{\text{rms} \\ \text{voltage}}} = \frac{\frac{a}{\sqrt{2}}}{\sigma(\text{noise})} = \frac{\frac{a}{\sqrt{2}}}{\frac{2\pi\, fna}{K\sqrt{2}} \left(\frac{1}{f} - \frac{1}{f + c}\right)} ,$$

$$\left(\frac{S}{N}\right)_{\substack{\text{rms} \\ \text{voltage}}} = \frac{K}{2\pi\, fn \left(\frac{1}{f} - \frac{1}{f + c}\right)} \, , \tag{8}$$

$$\frac{S}{N}\,(\text{db}) = 20\log\left[\frac{0.159K}{n\left(1-\frac{f}{f+c}\right)}\right].$$

Equation 8 is plotted in its logarithmic form on Figure 5. Rewriting Equation 8 yields

$$c = f\left(\frac{1}{1-\frac{K}{2\pi n\frac{S}{N}}} - 1\right). \tag{9}$$

Note that with this technique, whereby a prespecified constant number of cycles is measured, the resolution to which the measurement can be made depends on the frequency of the input signal. This dependence can be changed by making the measurement over a constant interval of time T regardless of the frequency, where $T = n/f$. Substituting fT for n in Equation 8 yields:

$$\frac{S}{N} = \frac{K}{2\pi T}\,\frac{f+c}{cf}, \tag{10}$$

$$\frac{S}{N}\,(\text{db}) = 20\log\left(\frac{0.159K}{T}\,\frac{f+c}{cf}\right),$$

or

$$c = \frac{1}{\frac{S}{N}\frac{2\pi T}{K} - \frac{1}{f}}. \tag{11}$$

Equation 10 is plotted in its logarithmic form on Figure 6. Equation 11 is plotted on Figure 7. Note that with this method of measurement the resolution c is nearly independent of frequency when f is large compared with c, or the S/N required to make a measurement of specified resolution c is independent of frequency when f is large compared with c.

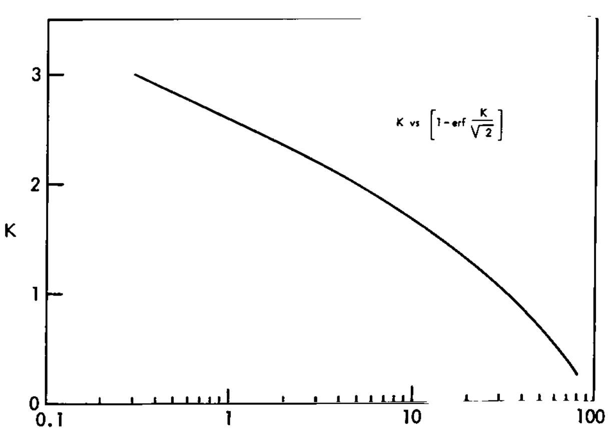

Figure 4—The error function (K vs $1-\text{erf}\,K/\sqrt{2}$).

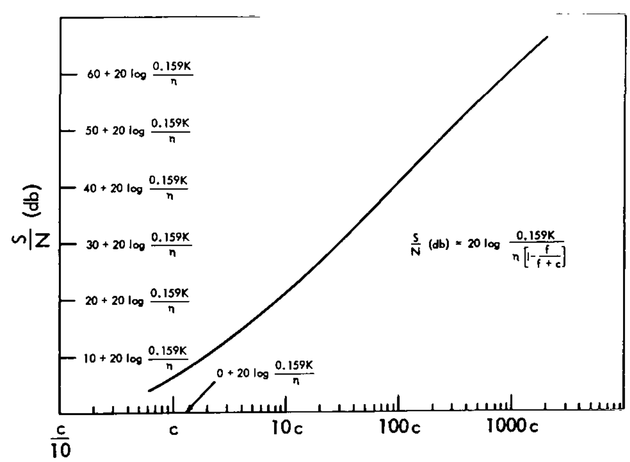

Figure 5—The relation between S/N and frequency for a constant c, n, and K.

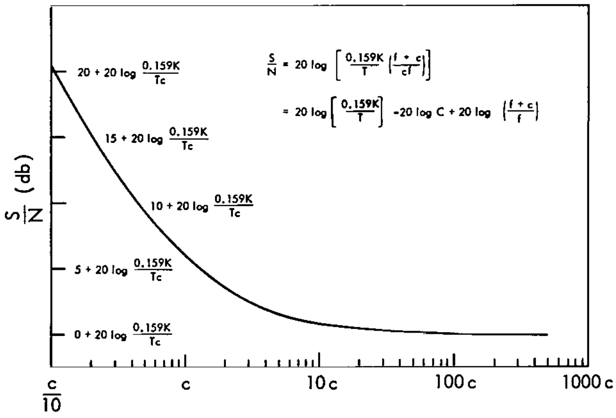

Figure 6—The relation between S/N and frequency for a constant c, T, and K.

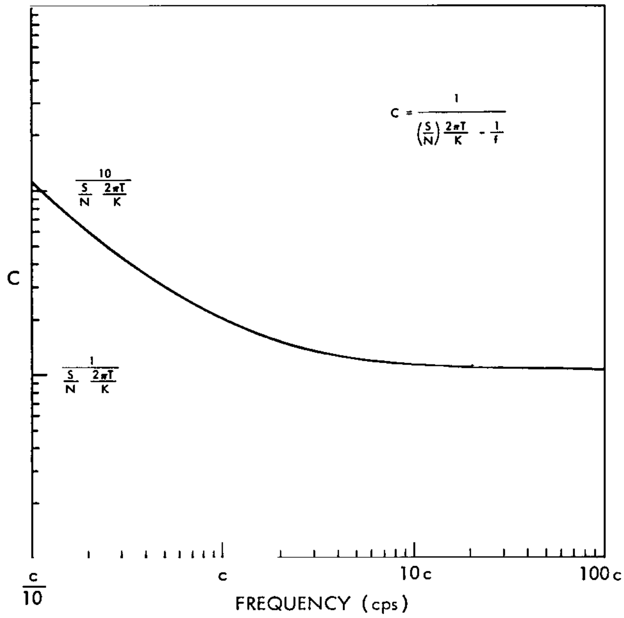

Figure 7—The relation between resolution and frequency for a constant S/N, T, and K.

Figure 6 (Equation 10) can serve the following purpose. Given (1) the resolution required of a measurement (c or m), (2) the probability that a measurement must fall within the required resolution (thus, from Figure 4 deduce K), (3) the relative magnitude of the frequency f , and (4) the time available T to make the measurement, then the input S/N required can be determined. Figure 7 (Equation 11) can serve to determine the resolution c which can be obtained given the order of magnitude f , the probability that the measurement will be within the specified resolution (thus, from Figure 4 deduce K), the amount of time available to make the measurement (T) , and the S/N .

To show the relation between the required S/N and the probability that the measurements will be within the specified resolution c , Figure 8 was constructed. In determining this curve the variable K (the number of standard deviations chosen) was transformed to the probability parameter by the use of Figure 4. It is thus apparent that in order to keep the resolution to which a measurement can be made independent of frequency* for a given S/N , the zero-crossing measurements must be made over a constant interval of time. The heuristic reason for this is that the total energy utilized in making the measurement is made independent of frequency.

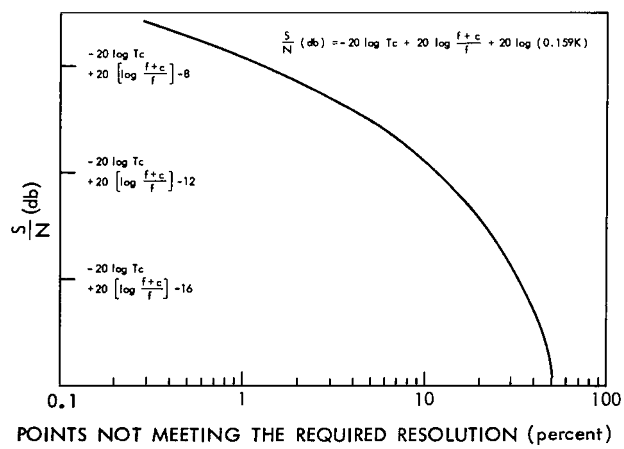

Figure 8—The relation between S/N and the probability that a measurement is resolved to better than c for a constant c, T, and f.

In the case where the measurement is made by determining the elapsed time for n cycles,

* For realistic values of c/f; i.e., f > 10c.

independent of the frequency, the energy utilized in making the measurement is an inverse function of frequency. Hence for a constant S/N the resolution is dependent upon the frequency. Laboratory tests have demonstrated the correctness of Equation 10.*

CONCLUSIONS

When measurements of noisy sine waves are made by zero-crossing techniques, a constant resolution of measurement over all frequencies where $f > 10\,c$ can be made only by measuring for a constant length of time for all these frequencies.

It is possible, by the use of Equations 10 and 11 to compute (1) the required S/N for a specified resolution c, measured time T, and error function argument K, and (2) the possible c for a specified S/N, f, T, and K.

(Manuscript received June 29, 1964)

*Demmerle, A. M., and Heffner, P., "The Resolution of Frequency Measurements in PFM Telemetry," NASA Technical Note D-2217, 1964.

NASA-Langley, 1964 G-590

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