NASA TECHNICAL REPORT

NASA TR R-212

NASA TR R-212

c.1

THEORETICAL ANALYSIS OF SEVERAL LINEAR STEADY-FLOW PLASMA ACCELERATORS WITH CROSSED ELECTRIC AND MAGNETIC FIELDS

by Clarence W. Matthews

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • DECEMBER 1964

THEORETICAL ANALYSIS OF SEVERAL LINEAR STEADY-FLOW PLASMA ACCELERATORS WITH CROSSED ELECTRIC AND MAGNETIC FIELDS

By Clarence W. Matthews

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Office of Technical Services, Department of Commerce,
Washington, D.C. 20230 -- Price $1.50

THEORETICAL ANALYSIS OF SEVERAL LINEAR STEADY-FLOW PLASMA ACCELERATORS WITH CROSSED ELECTRIC AND MAGNETIC FIELDS

By Clarence W. Matthews
Langley Research Center

SUMMARY

The ideal one-dimensional equations which express the motion of an inviscid, compressible, conducting, single-fluid plasma through crossed electric and magnetic fields are written in a nondimensional form. Solutions of these equations are then presented for conditions wherein the cross-sectional area is held constant or is allowed to vary in such a way that the velocity, pressure, or density is held constant. These conditions are considered for a number of variations of the current density with velocity. Several special cases are also presented in which the velocity, pressure, temperature, or Mach number is held constant in a constant-area channel.

For illustrative purposes computed results are presented for most of the solutions. These results are used to show the nature of the various modes of channel operation and may be used to compare modes and to make a preliminary selection of the mode of operation required to achieve a given purpose. These computations show whether a channel will accelerate or decelerate the flow, whether the flow is regular or forbidden, whether the velocity is limited or unlimited as the channel length becomes infinite, or whether other unexpected peculiarities exist.

INTRODUCTION

The need for high-velocity gas flow has in the past several years promoted studies of the use of crossed electric and magnetic fields to accelerate a conductive gas to the high velocities required. Examples of such studies are found in references 1 to 9. Several solutions of the equations for this flow are available, such as the one for a constant area and constant temperature given in references 1, 2, or 8. Another example is the solution for constant pressure given in reference 9. However, no systematic study of the various possible modes of operation of such a channel appears to have been made. The results of such a study should be useful for the understanding, evaluation, and design of plasma accelerators of this type.

The purpose of this paper, therefore, is to present a study of a number of solutions of the equations of the one-dimensional flow of an inviscid, compressible, conducting, single-fluid plasma through a channel having mutually perpendicular electric-current and constant-magnetic-field vectors, both normal to the channel flow direction. For purposes of illustration and comparison, a few calculations are presented for most of the modes of operation investigated.

SYMBOLS

All values are given in rationalized mksck system.

$$a = \frac{(\gamma - 1)j_o}{B\sigma u_o} = \alpha j_o$$

A cross-sectional area

$b = aM_o^2$

B magnetic flux density normal to flow, assumed to be constant

C constant of integration

c velocity of sound, $(\gamma p/\rho)^{1/2}$

c_p specific heat at constant pressure

E ratio of effective electric field within plasma to u_oB

f( ) function

$\vec{i},\vec{j},\vec{k}$ orthogonal unit vectors

j current density

M local Mach number of flow, u/c

p local pressure

$P = p/p_o$

t temperature

$T = t/t_o$

u velocity in x-direction

$U = u/u_o$

$(U)_{y_{max}}$ value of U for a stationary value of y x flow axis in one-dimensional coordinate system (see fig. 1)

y nondimensional length, $Bj_o(x - x_o)/\rho_o u_o^2$

$\alpha = \frac{\gamma - 1}{B\sigma u_o}$

γ ratio of specific heats, considered as 1.4 in all computations

δ increment

ρ local density

$\rho' = \rho/\rho_o$

σ electrical conductivity of gas, assumed constant

Subscript:

o an initial or reference condition

THEORY

A schematic diagram of the system considered is shown in figure 1. It is seen that the electric current and the magnetic field are normal to each other and to the flow direction along the channel axis. Although these flows may be expressed with the Navier-Stokes equations, analytical solutions of those equations cannot be obtained; thus, if such solutions are desired, it is necessary to make the following simplifications:

The effects of both viscosity and thermal conductivity are considered sufficiently small so that they may be neglected. The density of the gas must be sufficiently large so that the mean free path is much smaller than any of the dimensions of the channel. The number of particles entering and leaving the channel is assumed to be constant; that is, the percentages of ionization and dissociation do not change as the temperature and pressure change. The actual electric field using both normal and axial components is assumed to be applied in such a manner (ref. 2) that the current density is normal to the axis and is assumed to be applied through separate,

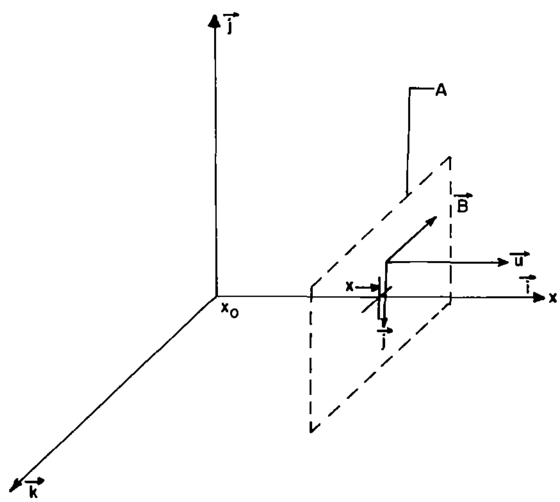


Figure 1.- Schematic diagram of channel.

individually powered electrodes distributed over the conducting wall. The normal component of the applied electric field is the component of the electric field used to determine the current density from Ohm's law. The externally applied magnetic field is considered to be constant, normal to both the electric field and the flow direction, and much greater than the induced magnetic field due to the current through the gas. Because the magnetic Reynolds number is assumed to be much less than one, the induced component of the magnetic field is neglected. The electrical conductivity is also assumed to be constant throughout the channel. Finally, the effect of ion slip is neglected, although it is shown in reference 2 that under certain conditions such effects can be fairly large.

The nature of these assumptions, especially those dependent on constant temperatures, such as constant ionization, dissociation and electrical conductivity, reduces the applicability of the results presented herein. However, if greater accuracy of prediction of results for a specific channel is desired, the channel could be broken into a number of segments over which the variation of all the parameters involved is sufficiently small so that the results of this analysis are applicable to the segment under consideration. The conditions at the end of the segment may then be used to compute the new initial conditions for the next segment. By continuation of this process, a solution could be built up for a specific channel which would show the effects of changing the magnetic field, the conductivity, the percentage of ionization and dissociation, and the ratio of specific heats. Another limitation is seen in the fact that if the area changes are too great, serious departures from the one-dimensional treatment given herein will occur. For example, in the case where the flow is supersonic, the resultant wave structure, especially if the waves are shocks, can cause such serious departures of the properties of the actual flow from the theoretical flow that the results of an analysis would be made inapplicable. Thus, care must be taken in applying the results of the variable-area studies to any channel design to use values of the parameters B, M_o, and $\rho_o u_o^2$ for which the rate of change of area with respect to the actual channel length is sufficiently small to reduce the two- or three-dimensional effects to values that are small compared with the one-dimensional effects.

With proper consideration of these assumptions, the equations for the quasi one-dimensional flow of an inviscid gas through crossed electric and magnetic fields are

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0 \tag{1a}$$

which expresses the continuity of the flow,

$$\rho u \; du + dp = jB \; dx \tag{1b}$$

which expresses the change of momentum,

$$\rho u \; d\left(c_p t\right) - u \; dp = \frac{j^2 dx}{\sigma} \tag{1c}$$

which represents the heat added to the stream, and

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dt}{t} \tag{1d}$$

which is the equation of state. The additional relations derived from the definition of M and relations (1a) and (1d)

$$\frac{2\ dM}{M} = \frac{du}{u} - \frac{dp}{p} - \frac{dA}{A} = \frac{2\ du}{u} - \frac{dt}{t} \tag{1e}$$

are convenient for determining the Mach number M. It is convenient to eliminate du and dt from equations (1b) and (1c) with the use of the values of du and dt from equations (1a) and (1d). If these eliminations are made and c^2 (the square of the velocity of sound) is substituted for $\gamma p/\rho$, it follows that

$$\frac{-\rho u^2 d\rho}{\rho} + dp = jB\ dx + \rho u^2\ \frac{dA}{A} \tag{2a}$$

and

$$\frac{-\rho c^2 u\ d\rho}{\rho} + u\ dp = \frac{(\gamma - 1)j^2}{\sigma}\ dx \tag{2b}$$

These equations may be solved for dp and dρ to obtain differential equations in dp and dρ. Corresponding equations for du, dt, and dM are obtained by use of equations (1a), (1d), and (1e).

If proper manipulations are made on these differential equations and the quantities $U = \frac{u}{u_o}$, $P = \frac{p}{p_o}$, $\rho' = \frac{\rho}{\rho_o}$, $T = \frac{t}{t_o}$, and $\alpha = \frac{\gamma - 1}{B\sigma u_o}$ are introduced, the following equations result:

$$\frac{dU}{U\ dx} = \frac{1}{M^2 - 1}\left[\frac{M^2 BjA}{\rho_o {u_o}^2 U^2 A_o}(U - \alpha j) + \frac{dA}{A\ dx}\right] \tag{3a}$$

$$\frac{dP}{P\ dx} = -\frac{\gamma M^2}{M^2 - 1}\left[\frac{BjA}{\rho_o {u_o}^2 U^2 A_o}(U - \alpha M^2 j) + \frac{dA}{A\ dx}\right] \tag{3b}$$

$$\frac{dT}{T\ dx} = -\frac{(\gamma - 1)M^2}{M^2 - 1}\left[\frac{BjA}{\rho_o {u_o}^2 U^2 A_o}\left(U - \frac{\gamma M^2 - 1}{\gamma - 1}\alpha j\right) + \frac{dA}{A\ dx}\right] \tag{3c}$$

$$\frac{d\rho'}{\rho'\ dx} = -\frac{M^2}{M^2 - 1}\left[\frac{BjA}{\rho_o u_o{}^2 U^2 A_o}(U - \alpha j) + \frac{dA}{A\ dx}\right] \tag{3d}$$

$$\frac{dM}{M\ dx} = \frac{\gamma + 1}{2(M^2 - 1)}\left\{\frac{M^2 BjA}{\rho_o u_o{}^2 U^2 A_o}\left[U - \frac{(\gamma M^2 + 1)\alpha j}{\gamma + 1}\right] + \left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1}\right]\frac{dA}{A\ dx}\right\} \tag{3e}$$

The differential relations (1a), (1d), and (1e) may be integrated to give

$$\rho' U \frac{A}{A_o} = 1 \tag{4a}$$

$$P = \rho' T \tag{4b}$$

$$\frac{M^2}{M_o{}^2} = \frac{U^2}{T} = \frac{U^2 \rho'}{P} = \frac{UA_o}{PA} \tag{4c}$$

With the use of these relations only equations (3a) and (3b) need to be integrated once the area distribution A is known. Inspection of equations (3a) to 3(e) shows that there are two more dependent variables (that is, U, P, ρ', T, M, A, and j) than there are equations; hence, it is necessary to specify any two of these variables before solutions can be obtained. For this analysis either the variation or the constancy of two of them will be specified.

Several important features should be noted concerning the limits of integration and methods of computing any of the solutions considered in this analysis. Since the flow must always start from the initial conditions and proceed to the local condition under consideration, the limits of integration must also extend from the initial value of 1 to the local value; or if the equation contains a constant of integration, that equation must be evaluated at the initial flow values. Also since the flow cannot reverse upon itself, all computations must start from small increments of the initial values and then follow the direction required to render x positive. This procedure is necessary since in many of the equations it will be convenient to treat U or P as the independent variable and to solve for x. Thus, if U is considered to be $1 + \delta$ where δ is small compared with 1, the sign of δ must be chosen to render x positive, and then the computations may continue with U decreasing or increasing as indicated by the sign of the term δ.

A general note should also be made concerning the singularities which appear in many of the integrals required for the evaluation of x. In all such integrals the value of the integral approaches infinity as the variable of integration approaches the singularity. Since x cannot proceed beyond an infinite value, the integration need not be carried beyond the first singularity. Care must be taken therefore that the integrand is not infinite for any value of the variable of integration falling between the considered local value and the initial value.

VARIABLE-AREA CHANNELS

Constant Velocity

The velocity in a channel may be held constant by allowing the area to vary such that dU is always equal to zero. Such an area variation is attained by setting the term within the brackets of equation (3a) equal to zero,

$$\frac{dA}{A} = -\frac{M^2ABj(1 - \alpha j)}{\rho_o u_o^2 A_o} dx \tag{5}$$

wherein the constant value $U = 1$ is substituted for U. If equation (5) and the relation $M^2 = \frac{A_o M_o^2 U}{PA}$ from equation (4c) are substituted into equation (3b), it follows that

$$\frac{dP}{P} = \frac{\gamma M_o^2}{P} \frac{Bj}{\rho_o u_o^2} dx \tag{6}$$

Since the flow in any channel must necessarily start at the reference conditions, the limits of integration must cover the region from $P = 1$ to P and from $x = x_o$ to x. With the velocity U constant, j cannot be a function of U; however, the pressure integral can be readily evaluated if j is chosen as a function of P such as $j = j_o f(P)$ where $f(1) = 1$. The solution for the condition that $U = 1$ becomes

$$\frac{1}{\gamma M_o^2} \int_1^P \frac{dP}{f(P)} = \frac{Bj_o}{\rho_o u_o^2}(x - x_o) = y \tag{7}$$

where y is a nondimensional quantity proportional to the channel length. The parameter y will be used throughout this analysis to compare channel flows under various conditions.

If equation (5) is expressed in terms of dP/P rather than in terms of dx, the result is

$$\frac{dA}{A} = -\frac{1 - \alpha j}{\gamma} \frac{dP}{P} \tag{8}$$

Again, if j is a known function of P, equation (8) may be integrated and the limits evaluated to give

$$\log \frac{A}{A_o} = -\frac{1}{\gamma} \int_1^P \left[1 - \alpha j_o f(P)\right] \frac{dP}{P} \tag{9}$$

Thus A can be expressed as a function of P and therefore of x.

Constant Pressure

General solution.- For the general solution the pressure will be assumed to be constant $(P = 1)$ throughout the channel. This condition appears to be important since this is the area distribution that the flow will take if the channel walls are free fluid surfaces. Thus, these walls are naturally occurring walls and no artificial restraint need be used to cause the flow to conform to a given area distribution. In all the other cases, the flow must be constrained to a given tailored wall distribution. The pressure ratio P may be rendered constant by setting $\frac{dP}{P} = 0$ in equation (3b). Then with the substitution of $M^2 = \frac{M_o^2 U A_o}{A}$

$$dA = -\frac{BjA}{\rho_o u_o^2 A_o} \frac{\left(A - \alpha M_o^2 A_o j\right)}{U} dx \tag{10}$$

With the substitution of equation (10) into equation (3a) it follows that

$$dU = \frac{BjA}{\rho_o u_o^2 A_o} dx \tag{11}$$

The elimination of $\frac{BjA}{\rho_o u_o^2 A_o} dx$ from equations (10) and (11) results in

$$\frac{dA}{A - \alpha M_o^2 A_o j} = -\frac{dU}{U} \tag{12}$$

This equation is a first-order differential equation of the form

$$\frac{dA}{dU} + \frac{A}{U} = \frac{\alpha M_o^2 A_o j}{U} \tag{13}$$

Its integral is given in most standard differential equation texts (ref. 10, for example) and is

$$\frac{A}{A_o} = \frac{\alpha M_o^2}{U} \int_1^U j(U) dU + \frac{1}{U} \tag{14}$$

The velocity is now given by equation (11) as

$$dU = \frac{B}{\rho_o u_o^2}\left[\frac{\alpha j M_o^2}{U}\int_1^U j(U)\,dU + \frac{1}{U}\right]dx \tag{15}$$

In the three subsequent sections j will be taken as the following functions of U: $j = j_o$, $j = j_oU$, and $j = B\sigma u_o(E - U)$ where E is constant. The current distribution $j = B\sigma u_o(E - U)$ is the simplified form of Ohm's law expressed in terms of the nondimensional parameters of this paper. If E, which is proportional to the electric field within the plasma, is held constant along the channel, it may then be considered that the electric field within the plasma is held constant. Also the problems are treated wherein the current density varies inversely as the cross-sectional area of the channel.

Constant current density, $j = j_o$.- The channel which has a constant current density, $j = j_o$, is one of the more practical of the channels investigated in this paper because of the ease of approximating constant current density. One relatively simple method of approximation of constant current density would be the location of a large number of electrodes, or arcs, over the channel surface such that the area allotted to each electrode is constant. Then, of course, it is necessary only to adjust the driving potential on each electrode so that the currents are the same. If $j = j_o$ and if the term $\alpha M_o^2 j_o$ is set equal to b, the evaluation of the integrals in equation (14) results in the following area distribution:

$$\frac{A}{A_o} = \frac{b(U - 1) + 1}{U} \tag{16}$$

The substitution of $j = j_o$ and $b = \alpha M_o^2 j_o$ into equation (15) and the subsequent integration result in the following nondimensional velocity distribution:

$$y = \frac{Bj_o}{\rho_o u_o^2}(x - x_o) = \frac{1}{b}\left\{U - 1 - \left(\frac{1 - b}{b}\right)\log\left[b(U - 1) + 1\right]\right\} \tag{17}$$

Since $P = 1$, the relations between the remaining dependent variables from relations (4c), (4b), and (4a) follow:

$$\left.\begin{aligned} \frac{M^2}{M_o^2} &= \frac{UA_o}{A} \\ T &= \frac{AU}{A_o} \\ \rho' &= \frac{A_o}{UA} \end{aligned}\right\} \tag{18}$$

Several velocity, area ratio, and Mach number distributions computed from equations (17), (16), and (18), respectively, are presented in figures 2, 3, and 4. It is interesting to note that the velocity increases regardless of the value of b, which includes initial Mach number, magnetic field, conductivity, and initial velocity. (See fig. 2.) This mode of operation is therefore indicated if velocity acceleration is desired in the channel. It is also observed that if b is made small, that is, less than about 0, then the accelerator becomes insensitive to changes in b. However, if area changes are considered to be undesirable, operation in the region of b = 1 for which the area is constant would seem more desirable, especially if a smooth and uniform flow is desired. The areas are also seen to approach the constant b as y or U increases. (See fig. 3.) In equation (16) if U becomes large, $A/A_o = b$.

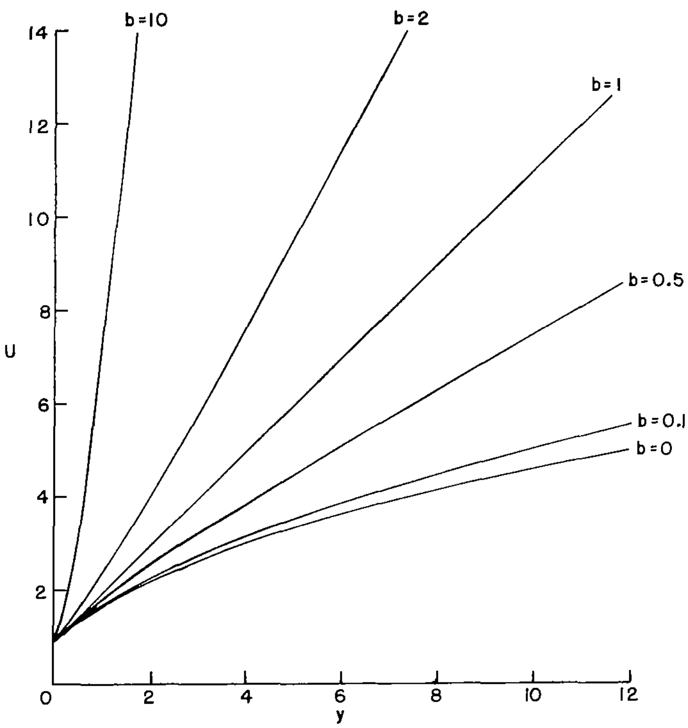


Figure 2.- Velocity ratio distribution in a magnetohydrodynamic channel with P, B, σ, and j held constant.

Examination of figure 4 shows that the Mach number increases. However, the trend for large values of b (see b = 10 curve in fig. 4) indicates a reduction in Mach number which will be recovered later. This operation should also be desirable for a high Mach number output since Mach number ratio gains of 2 to 3 are reasonable. For example, if the input Mach number is 4 or 5, output Mach numbers of an order of 10 to 20 should be reasonably attainable, especially if the conductivity and initial velocity are sufficiently large, so that b is of the order of 0.1 or thereabouts. Also the Mach number curves indicate that Mach number can be controlled by changing the value of b; however, such changes would also change the final velocity, and only a detailed study would show whether such an operation is feasible. The reduction in Mach number for the b = 10 curve occurs because the increase in temperature which is proportional to UA/A_o is sufficiently greater than the increases in U and A to lower the value of the Mach number.

<u>Current density proportional to U, $j = j_oU$.</u>- The mode of operation when the current density is proportional to U, $j = j_oU$, is included because of its mathematical interest rather than because it represents a practical channel. However, as is shown under the special cases, this mode is required in a constant-area, constant Mach number channel. This mode is impractical because it requires velocity measurements and tailoring of the electrode current readings to those measurements and because, as is shown by the analysis, the velocity and current approach infinity within a finite accelerator distance, a condition impossible to meet.

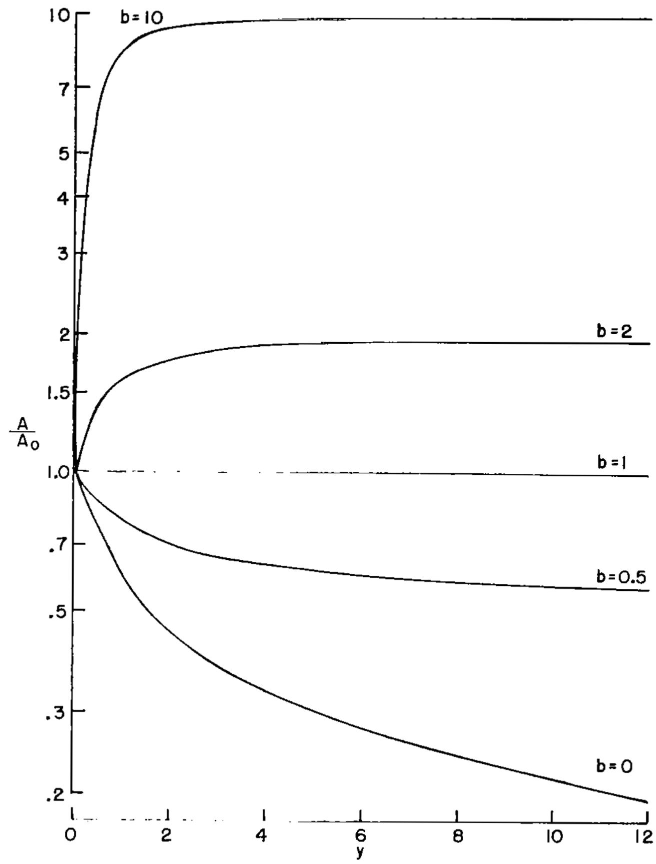


Figure 3.- Area ratio distribution in a magnetohydrodynamic channel with P, B, σ, and j held constant.

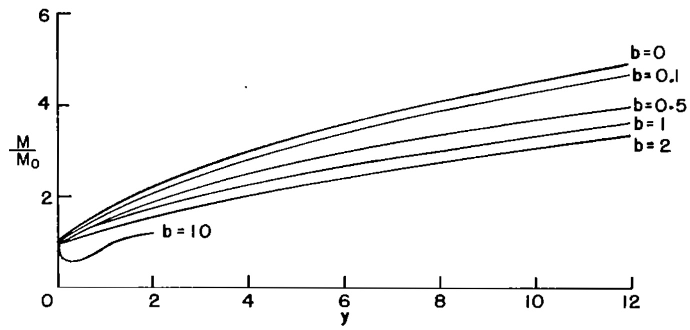


Figure 4.- Mach number ratio distribution in a magnetohydrodynamic channel with P, B, σ, and j held constant.

The flow conditions are determined by the substitution of $j = j_o U$ into equations (14) and (15). Evaluation of appropriate limits and constants of integration results in the following distribution for A/A_o and U:

$$\frac{A}{A_o} = \frac{1}{U}\left[1 + \frac{b}{2}\left(U^2 - 1\right)\right] \tag{19}$$

If $b < 2$,

$$y = \frac{Bj_o}{\rho_o U_o^2}(x - x_o) = \frac{2}{\sqrt{b}\sqrt{2 - b}}\left(\tan^{-1}U\sqrt{\frac{b}{2 - b}} - \tan^{-1}\sqrt{\frac{b}{2 - b}}\right) \tag{20a}$$

if $b \doteq 2$,

$$y = \frac{U - 1}{U} \tag{20b}$$

and if $b > 2$,

$$y = \frac{1}{\sqrt{b(b - 2)}} \log \frac{U\sqrt{b} - \sqrt{b - 2}}{U\sqrt{b} + \sqrt{b - 2}}\,\frac{\sqrt{b} + \sqrt{b - 2}}{\sqrt{b} - \sqrt{b - 2}} \tag{20c}$$

Equation (18) again follows for the other dependent variables.

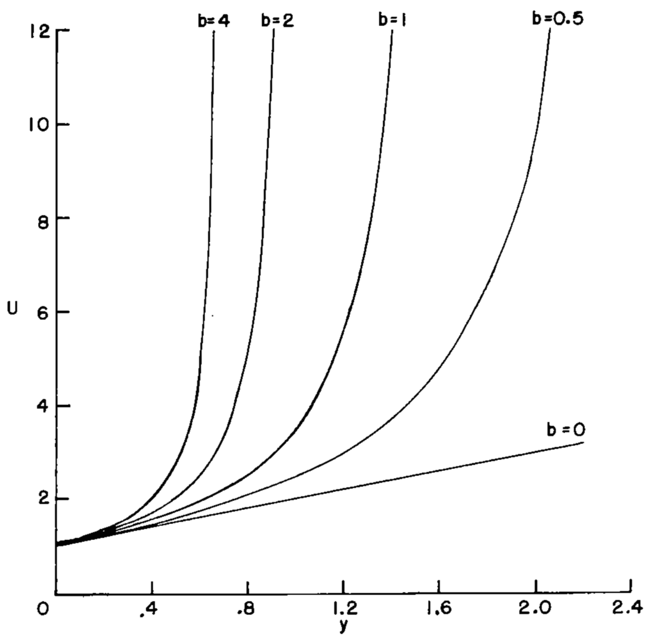


Figure 5.- Velocity ratio distribution in a magnetohydrodynamic channel with P, B, σ, and j/U held constant.

Computed values of U, A/A_o, and M/M_o for various values of b are presented in figures 5, 6, and 7, respectively. The surprising point seen in the velocity curves (fig. 5) is the fact that for values of $b > 0$, the velocities approach infinity at some definite value of y. This phenomenon appears to be reasonable when it is recalled that the current density increases with U; hence, the acceleration forces also increase. Thus, the two combine to produce an infinite velocity. This phenomenon is also easily seen in equations (20a) to (20c) by letting U become very large; then y approaches

$$\frac{2}{\sqrt{b}\sqrt{2 - b}}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{b}{2 - b}}\right),\ 1,\ \text{and}$$

$$\frac{1}{\sqrt{b}\sqrt{b - 2}} \log \frac{\sqrt{b} + \sqrt{b - 2}}{\sqrt{b} - \sqrt{b - 2}} \quad \text{for the}$$

various ranges of b.

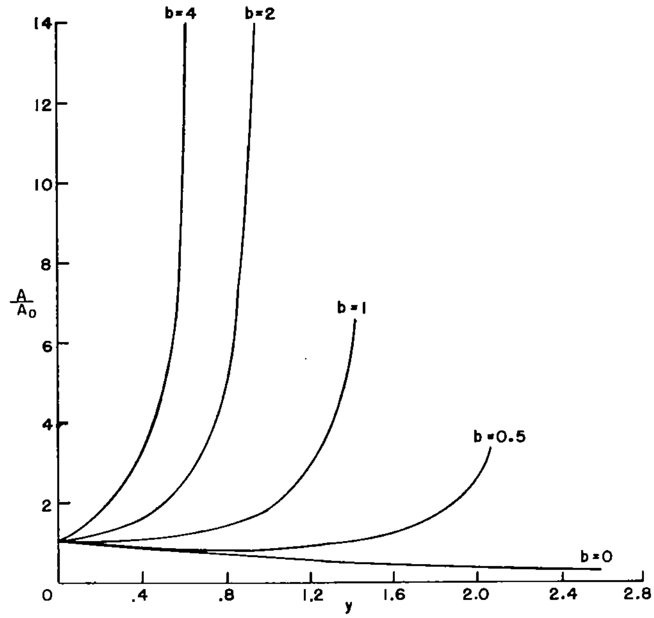


Figure 6.- Area ratio distribution in a magnetohydrodynamic channel with P, B, σ, and j/U held constant.

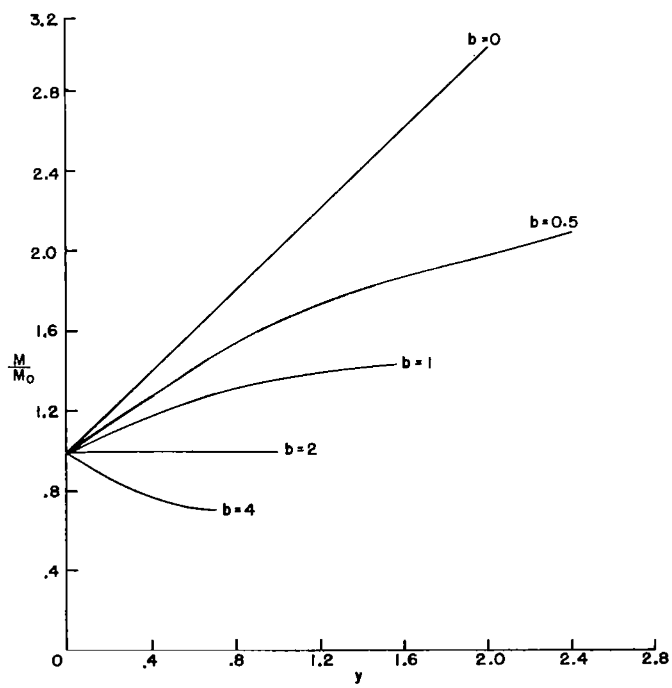


Figure 7.- Mach number ratio distribution in a magnetohydrodynamic channel with P, B, σ, and j/U held constant.

Correspondingly, it is seen that the area ratio (fig. 6) also approaches infinity. Since the area and velocity both become infinite for this current-density distribution at some given length, any channel involving this mode of operation must necessarily be ended before that length is attained.

This mode of operation will increase the Mach number ratio (fig. 7) provided b is kept less than 2. Greater values of b will reduce the Mach number ratio.

Constant electric field, $j = B\sigma u_o(E - U)$.- Operation of a channel with a constant potential is often proposed because in the ideal channel only the electrode voltages need to be held constant. This mode of operation has the advantage at high Mach numbers that as M increases, the heating rate j^2/σ decreases. This property is desirable because at high Mach numbers a high heating rate will rapidly increase the entropy. In the application of these constant electric field equations, the actual electric field within the plasma should be used rather than the electric field given by the potential between the electrodes, because of sheaths.

The equations representing the effects of a constant electric field are obtained with the substitution of $j = B\sigma u_o(E - U)$ into equations (14) and (15). These substitutions result in the following relations:

$$\frac{A}{A_o} = \frac{1}{2U}\left[(\gamma - 1)M_o^2(U - 1)(2E - U - 1) + 2\right] \tag{21}$$

$$y = \frac{Bj_o}{\rho_o u_o^2}(x - x_o) = \frac{E - 1}{(\gamma - 1)M_o^2 Q^2}\left[-2E \log \frac{E - U}{E - 1} + (E + Q)\log \frac{E + Q - U}{E + Q - 1} + (E - Q)\log \frac{E - Q - U}{E - Q - 1}\right] \tag{22}$$

where

$$Q^2 = (E - 1)^2 + \frac{2}{(\gamma - 1)M_o^2} \tag{23}$$

and

$$j_o = B\sigma u_o(E - 1) \tag{24}$$

It is observed that these equations are functions of two parameters E and M_o rather than of the single parameter b as in the previous cases. Thus, there will be a family of constant E curves for each initial M_o or a family of constant M_o curves for each initial E. Plots of a few computed values of U, A/A_o, and Mach number obtained from equations (22), (21), and (18) are shown in figures 8, 9, and 10, respectively. The velocity ratios (fig. 8) are seen to approach E, as may be expected, since the current density is zero if $U = E$ and no further acceleration can occur. Thus, the constant-pressure mode with a constant E is limited in its acceleration. This is not, however, serious since E can readily be made large.

The final velocity ratio can be shown from equation (22) to be independent of the initial Mach number. The initial Mach number does, however, affect the distance in which the velocity ratio approaches its limiting value. These curves indicate that a high initial Mach number should be used if rapid acceleration of the gas is required.

It is interesting to note that the conductivity appears only in the term $j_o = B\sigma u_o(E - 1)$ of the scale factor $Bj_o/\rho_o u_o^2$ which converts y into an actual length.

Readily traceable trends are not so easily seen in the area distribution and the Mach number curves (figs. 9 and 10, respectively) except that both of them tend to approach constant values as y increases. The Mach number is seen to increase for all values of E.

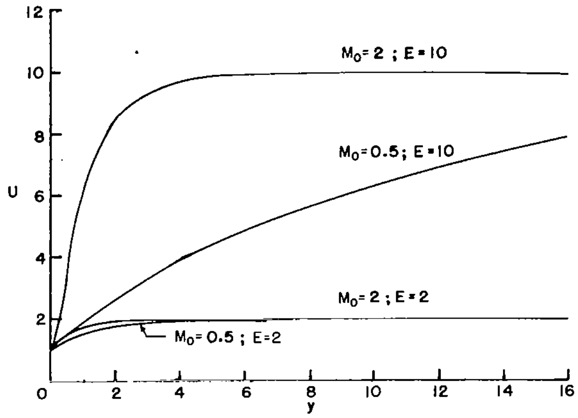


Figure 8.- Velocity ratio distribution in a magnetohydrodynamic channel with P, B, σ, and E held constant.

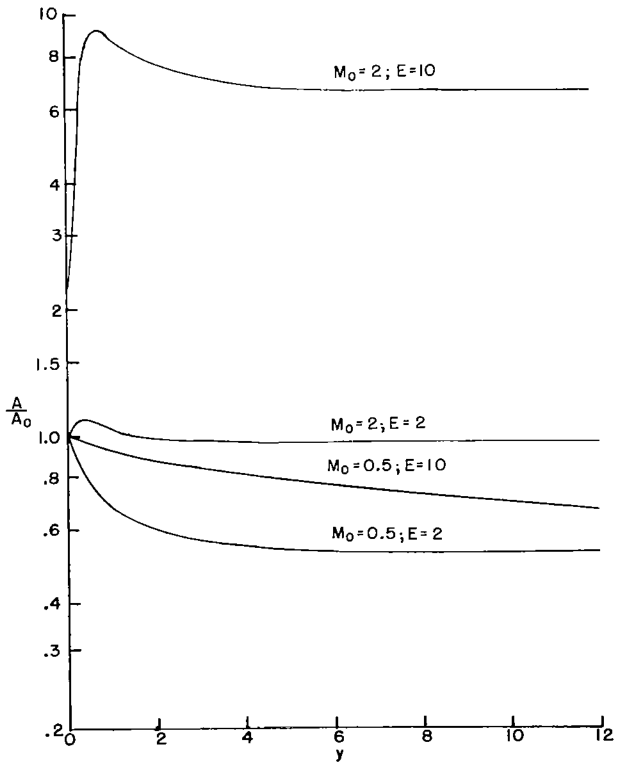


Figure 9.- Area ratio distribution in a magnetohydrodynamic channel with P, B, σ, and E held constant.

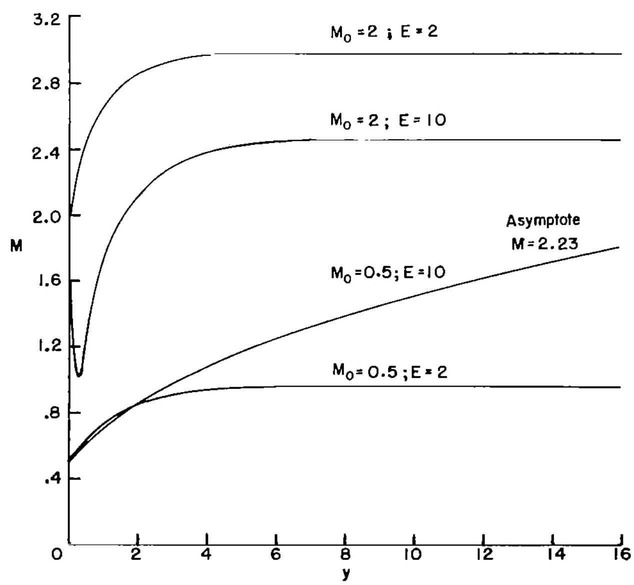


Figure 10.- Mach number distribution in a magnetohydrodynamic channel with P, B, σ, and E held constant.

However, for the condition that $M_o = 2$ and $E = 10$, the Mach number first decreases (a condition indicating a rapid heating of the plasma) before it increases to a value greater than the initial Mach number.

Current density inversely proportional to the area, $jA = j_oA_o$.- The operation of a channel with the current density inversely proportional to the cross-sectional area has an interesting feature. If the channel height is constant, the current through any section of the channel of length δx is equal to the current through any other section of the same length. Such a channel could be called a constant-current channel. This current distribution is approximately attained by equally spacing the electrodes or sets of electrodes along the channel axis and making the currents in each set equal to the current in every other set.

Since j is a function of A rather than of U, equations (14) and (15) are not directly applicable. However, the substitution of $j = \frac{j_oA_o}{A}$ into equation (11) results in

$$dU = \frac{Bj_o}{\rho_o u_o^2}\,dx = dy \tag{25}$$

which may be immediately integrated to obtain, after evaluation of the limits,

$$y = U - 1 \tag{26}$$

or

$$U = 1 + y$$

Also the substitution of $jA = j_oA_o$ into equation (12) results in

$$\frac{A}{A^2 - bA_o^2}\,dA = -\frac{dU}{U} \tag{27}$$

which upon integration and evaluation of limits and after substitution of equation (26) becomes

$$\frac{A}{A_o} = \left[b + (1 - b)(1 + y)^{-2}\right]^{1/2} \tag{28}$$

Typical computed values of the velocity ratios, area ratios, and Mach number ratios are presented in figures 11, 12, and 13, respectively. As seen from equation (26) or figure 11, U is a linear function of y with no other parameters involved, except,

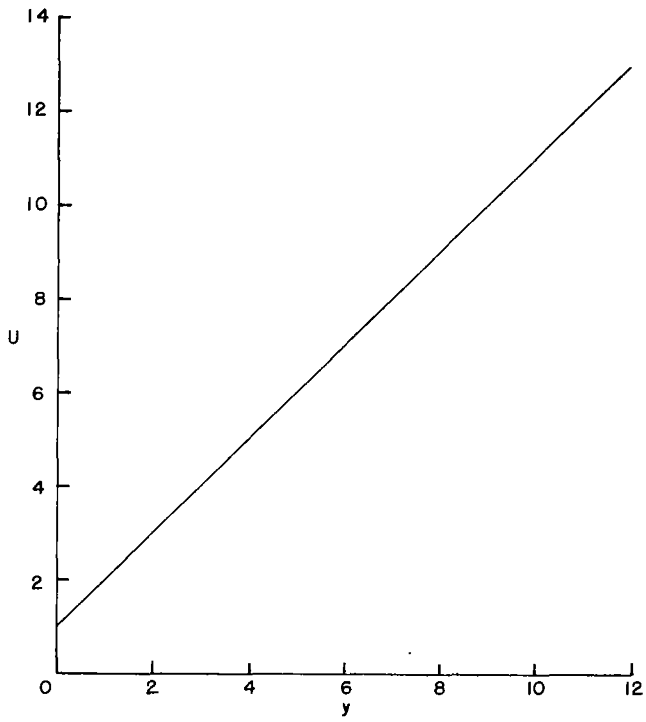


Figure 11.- Velocity ratio distribution of a magnetohydrodynamic channel with P, B, σ, and jA held constant.

of course, the scaling factor $\frac{\rho_o u_o^2}{Bj_o}$ which converts y into actual length. The area ratio (fig. 12) does involve the parameter b, the ratio increasing or decreasing depending on whether b is greater than or less than 1. Also the area ratio appears to approach a constant value as y becomes large. This value is seen to be equal to $b^{1/2}$ by letting y be large in equation (28). This mode of operation increases the Mach number for all the values of b considered, that is, 10 or less. It can be seen however that regardless of the value of b, the Mach number ratio will eventually increase provided y is sufficiently large. This is true because by equation (18) $M = M_o\sqrt{UA_o}/A$; therefore, A eventually becomes constant but U continues increasing and M must eventually vary as $U^{1/2}$. Thus, M/M_o must finally become greater than 1. An accelerator operating under this mode should increase the Mach number substantially if b is rendered small by a large value of the conductivity σ.

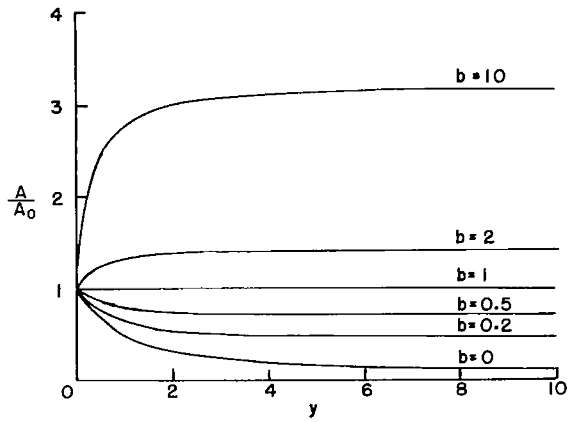


Figure 12.- Area ratio distribution in a magnetohydrodynamic channel with P, B, σ, and jA held constant.

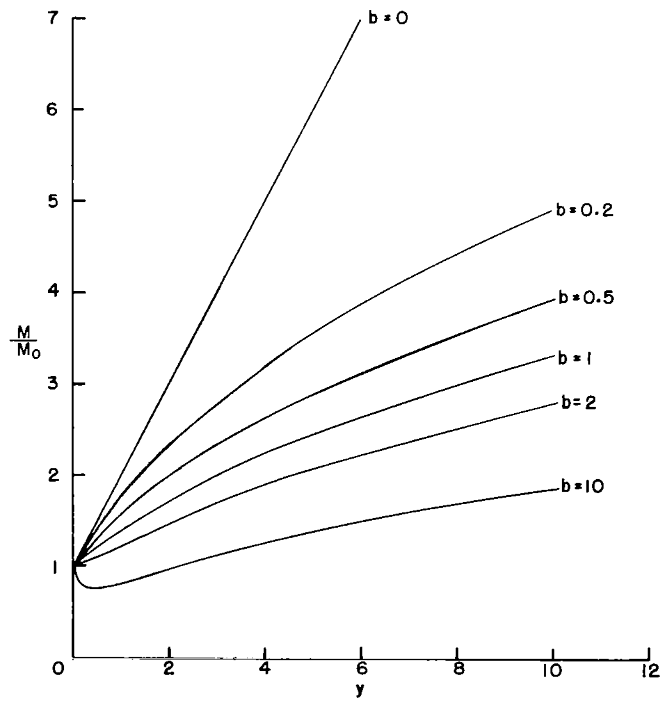


Figure 13.- Mach number ratio distribution in a magnetohydrodynamic channel with P, B, σ, and jA held constant.

Constant Density Solutions

General solution.- The density ratio ρ' rather than the pressure ratio P may be held constant. This mode of solution may be obtained by determining the value of dA/dx from equation (3d) with the condition that $d\rho'$ equals zero. However, the area ratio for constant density is more readily determined from equation (4a). Since the constant density is expressed by letting $\rho = 1$, equation (1a) becomes

$$\frac{dA}{A} = -\frac{dU}{U} \tag{29}$$

which upon integration becomes

$$\frac{A}{A_o} = \frac{1}{U} \tag{30}$$

The use of equation (29) in equations (3a) and (3b) will give the equations in dU and dP which express the condition that ρ is held constant. After these substitutions and appropriate reductions are made, it is found that

$$\frac{U^2}{U - \alpha j}\, dU = \frac{Bj}{\rho_o u_o^2}\, dx \tag{31}$$

and

$$\frac{U}{\alpha \gamma M_o^2 j}\, dP = \frac{Bj}{\rho_o u_o^2}\, dx \tag{32}$$

The elimination of $\frac{Bj}{\rho_o u_o^2}\, dx$ from equations (31) and (32) results in

$$\frac{1}{\gamma M_o^2}\, dP = \frac{\alpha U j}{U - \alpha j}\, dU \tag{33}$$

It is seen by inspection that the variables U and P are separated if j is a function of U and hence P can be expressed as an integral.

Constant current density, $j = j_o$.- In this section, the current density is assumed to be constant along the length of the channel. The substitution of $j = j_o$ into equations (31) and (33) results in

$$\frac{U^2}{U - a}\, dU = \frac{Bj_o}{\rho_o U_o^2}\, dx = dy \tag{34}$$

and

$$\frac{1}{\gamma M_o^2}\, dP = \frac{\alpha j_o U}{U - \alpha j_o}\, dU = \frac{aU}{U - a}\, dU \tag{35}$$

where $a = \alpha j_o = \dfrac{(\gamma - 1) j_o}{B\sigma u_o}$. If equations (34) and (35) are integrated and the constants of integration evaluated from the appropriate reference values, then

$$y = \frac{U^2 - 1}{2} + a(U - 1) + a^2 \log \frac{U - a}{1 - a} \tag{36}$$

and

$$\frac{P - 1}{\gamma M_o^2} = a(U - 1) + a^2 \log \frac{U - a}{1 - a} \tag{37}$$

where $a = \alpha j_o = \dfrac{(\gamma - 1) j_o}{B\sigma u_o}$. It is seen that U is independent of M_o, whereas P depends on M_o as well as on a and U.

The other variables, M, T, and A, are obtained from equations (4c), (4b), and (30), respectively, as

$$M^2 = \frac{M_o^2 U^2}{P} \tag{38a}$$

$$T = P \tag{38b}$$

$$A = \frac{A_o}{U} \tag{38c}$$

These equations show that the Mach number and the temperature ratio of the flow in a magnetohydrodynamic channel in which the density and the current density are held constant are dependent on the initial Mach number, whereas the area ratio is not.

Several curves showing values of U, A/A_o, and $\frac{P - 1}{\gamma M_o^2}$ as functions of a and y are presented in figures 14, 15, and 16, respectively. Examination of the curves (fig. 14) presenting the velocity ratio U shows two important features. First, that acceleration occurs only if a is less than 1 in contrast to the constant-pressure mode of operation in which acceleration occurs for all values of b or aM_o^2. Second, it is seen that if a is greater than 1, the flow velocity is reduced to zero. Such reduction involves, of course, the filling of an infinite sink since the area becomes infinite when the velocity is reduced to zero. Thus the flow is literally stopped in this mode of operation if a is greater than 1. Also, it is observed that the velocity ratio becomes a very sensitive function of y whenever a has values close to 1 as indicated by the large changes of U with respect to y when a changes from 0.99 to 1.01.

The area ratio curves (fig. 15) are obtained from the reciprocal of the velocity ratios (fig. 14) and show that for $a > 1$, the area ratio approaches infinity as U approaches zero and y remains finite. In figure 16, which presents the pressure ratio function $\frac{P - 1}{\gamma M_o^2}$, it is seen that P and hence T remain finite as U approaches zero when a is greater than 1. Hence, if a is greater than 1, this mode of operation not only reduces the velocity to zero but also restricts the pressure and temperature ratios that can be obtained. However, the values of U, P, and T can all be increased if σ and B are sufficiently large so that a is less than 1.

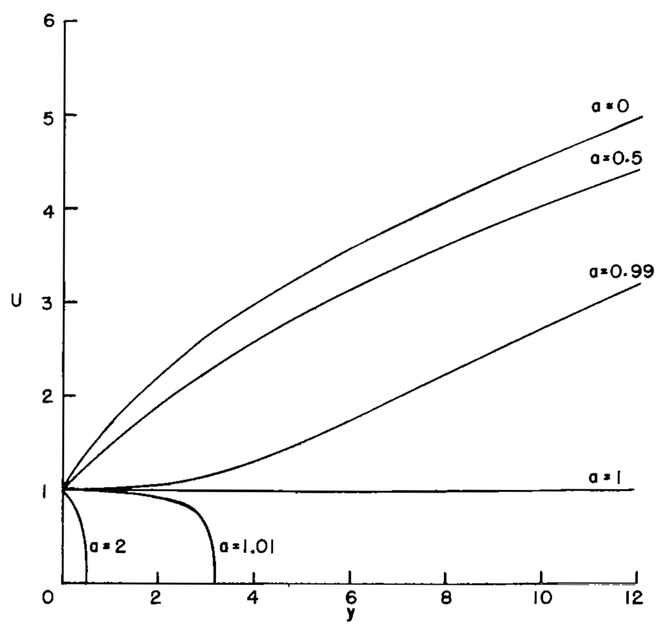


Figure 14.- Velocity ratio distribution in a magnetohydrodynamic channel with ρ', B, σ, and j held constant.

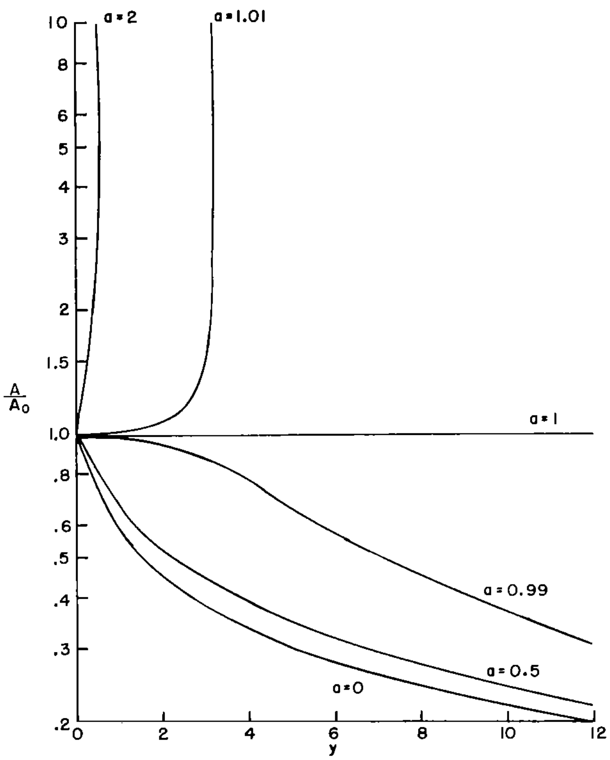


Figure 15.- Area ratio distribution in a magnetohydrodynamic channel with ρ', B, σ, and j held constant.

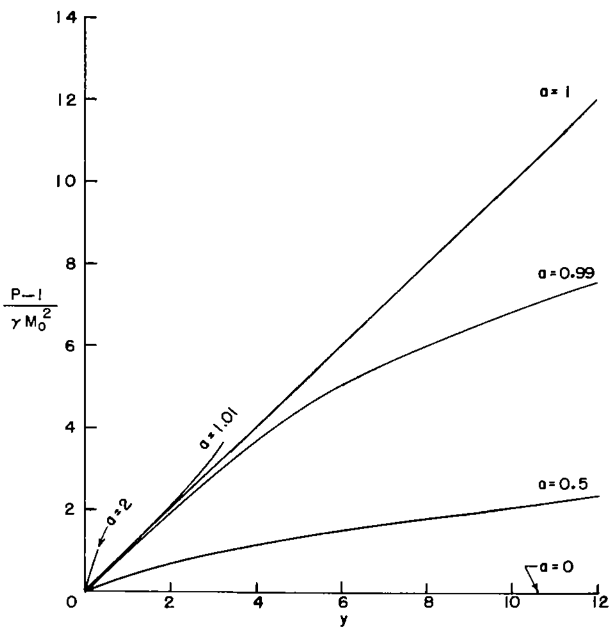


Figure 16.- Distribution of pressure function in a magnetohydrodynamic channel with ρ' B, σ, and j held constant.

Current density proportional to velocity, $j = j_oU$.- In this section the current density will be considered to vary directly with U (that is, $j = j_oU$). Moreover, because in a constant-density channel, U varies as A_o/A, the relation $jA = j_oA_o$ is also satisfied.

The functions U and P are obtained with the substitution of $j = j_oU$ into equations (31) and (33). This substitution results in

$$\frac{dU}{1 - a} = dy \tag{39}$$

and

$$\frac{dP}{\gamma M_o^2} = \frac{a}{1 - a} U\, dU \tag{40}$$

The integration of equation (39) and the evaluation of the constant result in

$$y = \frac{U - 1}{1 - a}$$

or

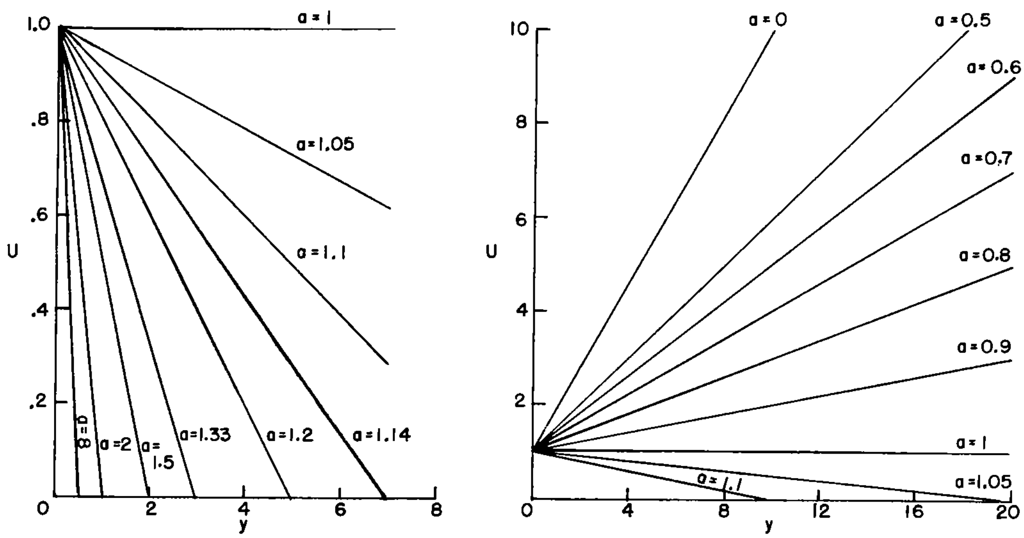
$$U = 1 + (1 - a)y \tag{41}$$

The integration and evaluation of equation (40) with the substitution of equation (41) result in

$$\frac{P - 1}{\gamma M_o^2} = \frac{ay[2 + (1 - a)y]}{2} \tag{42}$$

Equations (41) and (42) along with equations (38a) to (38c) fully express the flow properties of this particular mode of operation of a magnetohydrodynamic channel, that is, with ρ', B, and σ constant.

Several curves are shown in figure 17 which presents U as a function of y for various values of a. Again, U becomes zero within a finite distance for values of a greater than 1 but accelerates without limit for values of a less than 1. Correspondingly, the values of the area ratio A/A_o which are not shown either increase to infinity or asymptotically approach zero as y becomes infinite, depending upon whether a is greater than or less than zero.

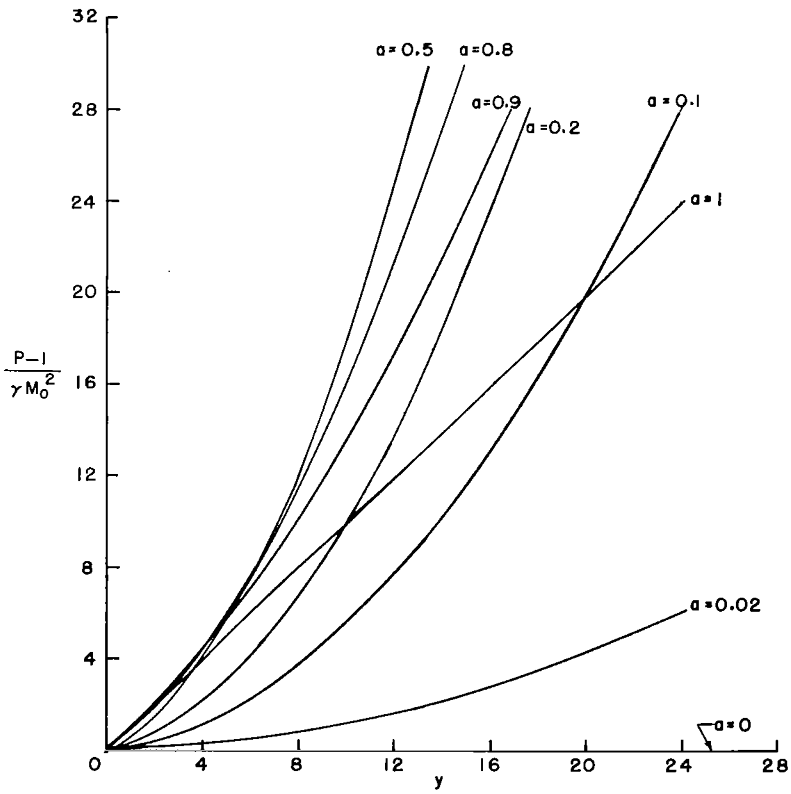


(a) $a \geqq 1$.

(b) $a \leqq 1.1$.

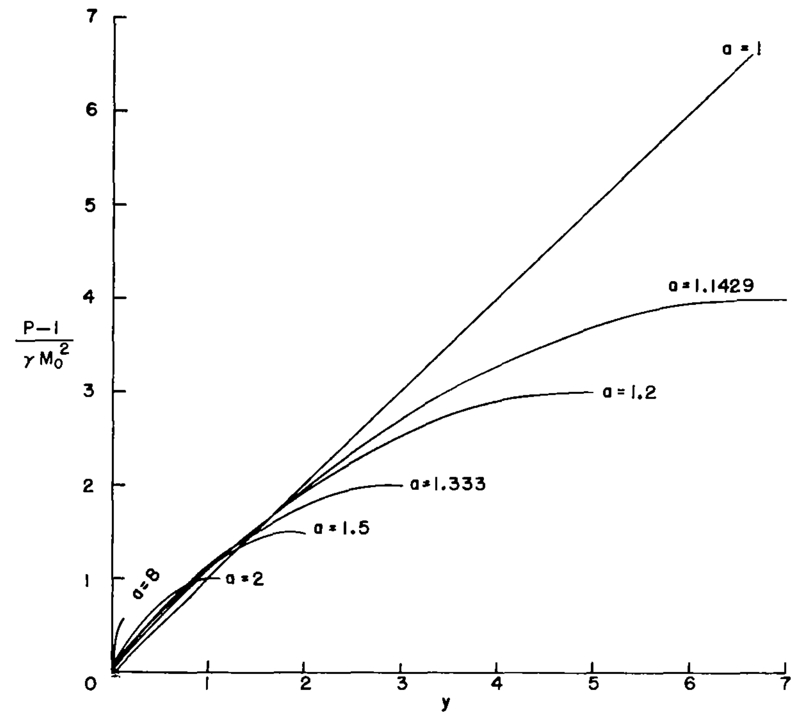
Figure 17.- Velocity ratio distribution in a magnetohydrodynamic accelerator with ρ', B, σ, and j/U held constant.

Values of the pressure function for a less than 1 are shown as functions of y in figure 18(a) and for a greater than 1 in figure 18(b). If a is less than 1, it is seen that the pressures increase indefinitely. If a is greater than 1, then, as in the previous case, the pressure function $\frac{P - 1}{\gamma M_o^2}$ and, correspondingly, P and T rise to a limited but finite value. Therefore, this mode of operation has several characteristics similar to the constant current density mode in that the value of a strongly determines the nature of the operation of the channel.



(a) $a \leqq 1$.

Figure 18.- Pressure ratio function in a magnetohydrodynamic channel with ρ', B, σ, and j/U held constant.



(b) $a \geqq 1$.

Figure 18.- Concluded.

Current density inversely proportional to U, $jU = j_o$.- In this section, the current density is considered to vary inversely with respect to U. Such a variation may be expressed as $j = \frac{j_o}{U}$. The values of U and P for this current distribution are determined with the substitution of $j = \frac{j_o}{U}$ into equations (31) and (33). After making the indicated substitution, equations (31) and (33) become, respectively,

$$\frac{U^4}{U^2 - a} \, dU = dy \tag{43}$$

and

$$\frac{1}{\gamma M_o^2} \, dP = \frac{aU}{U^2 - a} \, dU \tag{44}$$

After equations (43) and (44) are integrated and the constants evaluated, they become, respectively,

$$y = \frac{U^3 - 1}{3} + a(U - 1) + \frac{a^{3/2}}{2} \log \frac{\left(U - \sqrt{a}\right)\left(1 + \sqrt{a}\right)}{\left(U + \sqrt{a}\right)\left(1 - \sqrt{a}\right)} \tag{45}$$

and

$$\frac{P - 1}{\gamma M_o^2} = \frac{a}{2} \log \frac{U^2 - a}{1 - a} \tag{46}$$

Equations (38a), (38b), and (38c) may again be used to determine M, T, and A/A_o.

Values of U and $\frac{P - 1}{\gamma M_o^2}$ as functions of y are presented in figures 19 and 20. Examination of figure 19 shows that for this mode of operation if a is less than 1, then $y = \frac{U^3 - 1}{3}$ and the curve for $a = 0$ is an excellent approximation of the velocity distribution. However, when a becomes larger than 1, an almost discontinuous change occurs in which y changes from the $\frac{U^3 - 1}{3}$ distribution to a distribution in which U quickly becomes zero, for example, see the $a = 1.0001$ curve; or if $a = 1.225$, then $U = 0$ at $y = 0.14$. The areas become infinite just as in the previous cases. Perhaps, the interesting point is the fashion in which the velocity curves approach 1 as a approaches 1. This phenomenon is indicated at the base of the $a = 0.9998$ curve. This curve indicates that as a becomes still closer to 1, the U curve will fall on the value $U = 1$ for larger and larger values of y until finally it breaks away and becomes approximately parallel to the $a = 0$ curve.

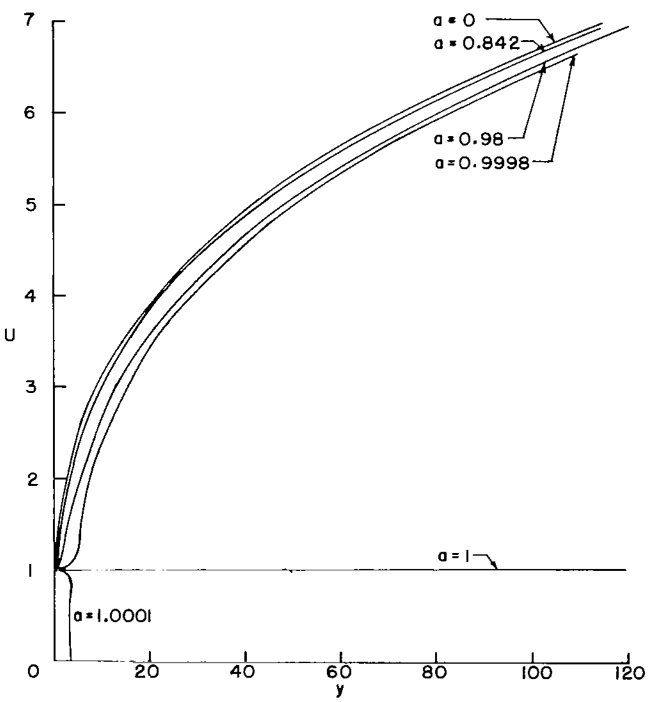


Figure 19.- Velocity ratio distribution in a magnetohydrodynamic channel with ρ', B, σ, and jU held constant.

The pressure function curves for $a < 1$ are shown in figure 20. These curves show that the pressures increase for all values of a less than 1 and greater than 0. At $a = 1$, the pressure function is expressed by $\frac{P - 1}{\gamma M_o^2} = y$. This may be shown with the substitution of $j = \frac{j_o}{U}$ and $U = 1$ into equation (32). If a is greater than 1, the pressure curves are terminated at a finite value of y.

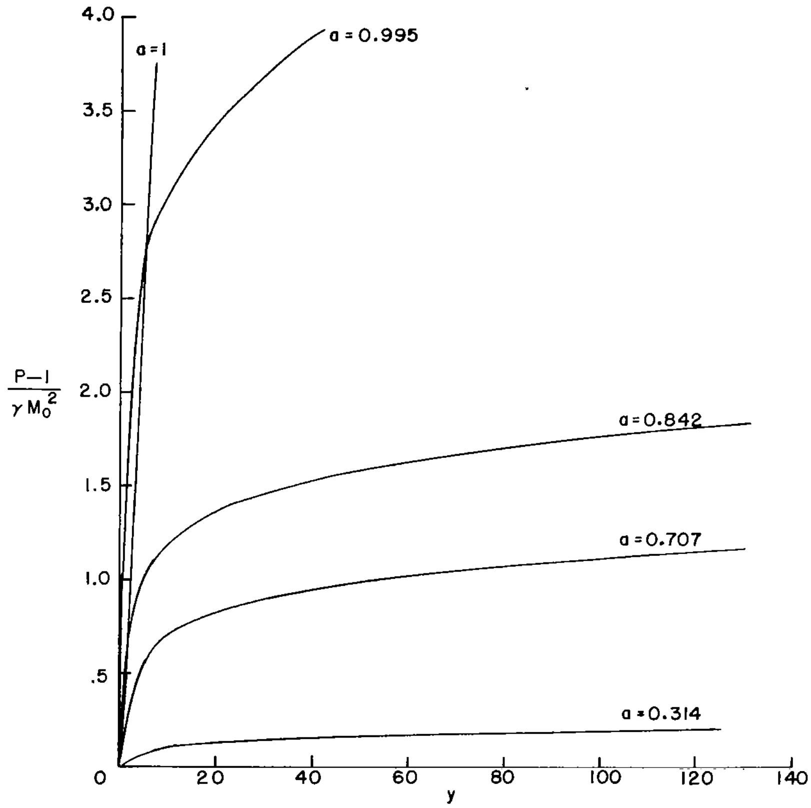


Figure 20.- Pressure distribution function in a magnetohydrodynamic channel with ρ', B, σ, and jU held constant.

Constant electric field, $j = B\sigma u_o(E - U)$.- The current density which occurs if the electric field within the plasma is constant has been shown to be $j = B\sigma u_o(E - U)$. The use of this function in equations (31) and (33) results in

$$\frac{U^2 dU}{\left[\gamma U - (\gamma - 1)E\right](U - E)} = -\frac{B^2\sigma}{\rho_o u_o} dx = -\frac{Bj_o}{\rho_o u_o^2(E - 1)} dx = -\frac{dy}{E - 1} \tag{47}$$

and

$$\frac{dP}{\gamma M_o^2} = \frac{(\gamma - 1)(E - U)U}{\gamma U - (\gamma - 1)E} dU \tag{48}$$

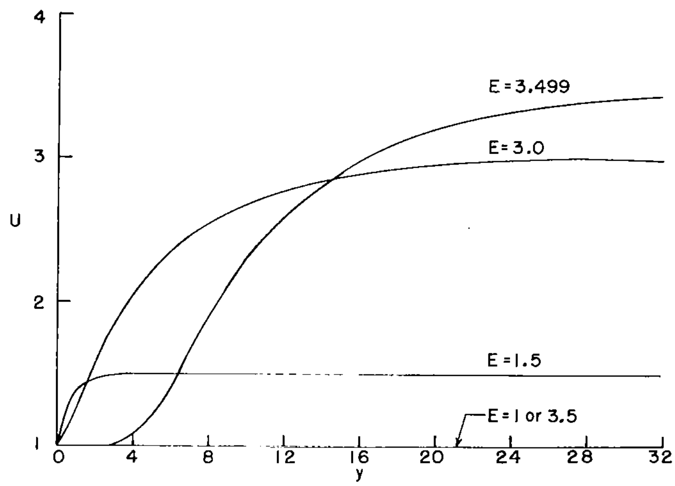
After integration and evaluation of the constants, equations (47) and (48) become, respectively,

$$y = (E - 1)\left[\frac{(\gamma - 1)^2}{\gamma^2} E \log \frac{\gamma U - (\gamma - 1)E}{\gamma - (\gamma - 1)E} - E \log \frac{U - E}{1 - E} - \frac{U - 1}{\gamma}\right] \tag{49}$$

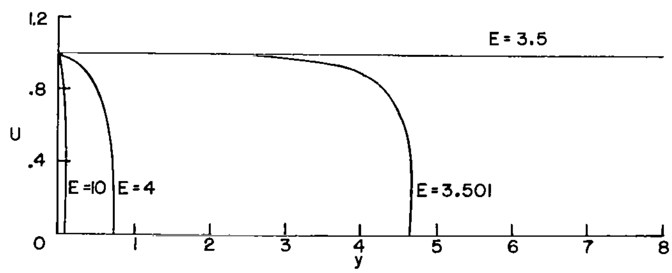
and

$$\frac{P - 1}{\gamma M_o^2} = \frac{\gamma - 1}{2\gamma}\left[1 - U^2 + \frac{2E(U - 1)}{\gamma} + \frac{2(\gamma - 1)E^2}{\gamma^2} \log \frac{\gamma U - (\gamma - 1)E}{\gamma - (\gamma - 1)E}\right] \tag{50}$$

It is interesting to note from equation (49) that U is constant for two values of E; that is, $E = 1$ and $E = \frac{\gamma}{\gamma - 1}$ ($E = 3.5$ for $\gamma = 1.4$). It can also be shown that U approaches E asymptotically as y becomes infinite, provided $1 < E < \frac{\gamma}{\gamma - 1}$. These observations are verified upon examination of figure 21, which presents values of U plotted as a function of y for different values of the parameter E. If E is less than $\frac{\gamma}{\gamma - 1}$ but greater than 1 $\left(\frac{\gamma}{\gamma - 1} = 3.5 \text{ if } \gamma = 1.4\right)$, the U-distribution approaches E as y becomes large. However, it is also observed that as E closely approaches 3.5, the velocity U follows the $U = 1$ curve for a considerable length before it leaves the curve and approaches the $E = 3.5$ value. If $E = 3.5$, the velocity becomes constant at $U = 1$ as may be seen by examination of equation (47). In this equation, if $E = \frac{\gamma}{\gamma - 1}$, dU must be equal to zero. Thus, since dU equals zero, U must equal 1. For larger values of E, dU must be negative; thus, as y increases, U must decrease as is indicated by the curves. Again, infinite cross-sectional area is required to meet continuity conditions. The interesting point here is that if E is made large, the flow is stopped rather than accelerated to the value of E or even $\frac{(\gamma - 1)E}{\gamma}$ as might be expected from equation (49).



(a) $E < 3.5$.



(b) $E > 3.5$.

Figure 21.- Velocity ratio distribution in a magnetohydrodynamic channel with ρ', B, and σ held constant and $j = B\sigma U_o(E - U)$.

The corresponding pressure function $\frac{P - 1}{\gamma M_o^2}$ is presented in figure 22. These curves show that the pressure function approaches a limiting value for values of E between 1 and 3.5. If E is greater than 3.5, the pressure function terminates, as might be expected, at the same value of y at which U becomes equal to zero. At E = 3.5, the value of $\frac{P - 1}{\gamma M_o^2}$ is determined by substitution of $U = 1$, $j = j_o = B\sigma u_o(E - 1)$, and $E = \frac{\gamma}{\gamma - 1}$ or 3.5 into equation (32). The result after integration and evaluation of constants is

$$\frac{P - 1}{\gamma M_o^2} = y \tag{51}$$

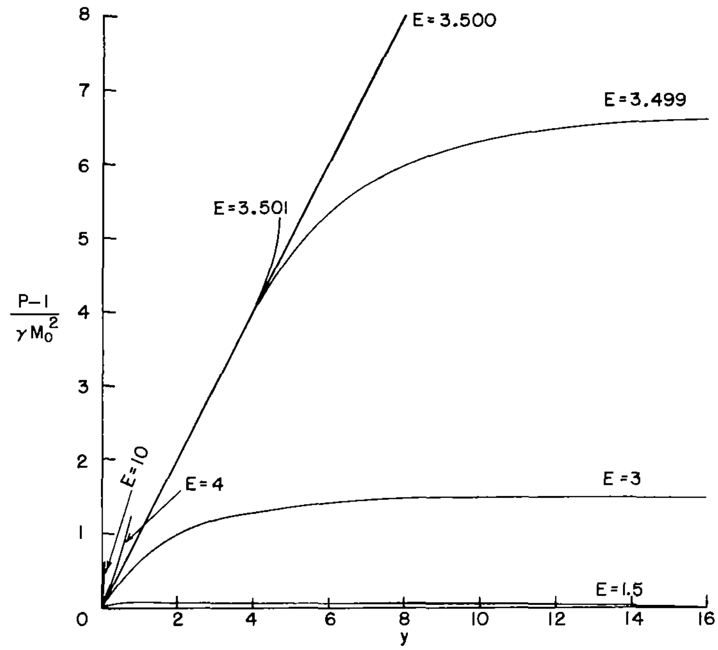


Figure 22. - Pressure function distribution in a magnetohydrodynamic channel with ρ', B, and σ held constant and $j = B\sigma U_o(E - U)$.

Constant Temperature

The operation of a channel at constant temperature is often proposed (see refs. 1 and 2) because, among other reasons, the gas properties such as conductivity, ionization, dissociation, and the specific-heat ratio γ remain constant. This mode of operation is usually obtained by properly varying the current density in a constant-area channel. It is, however, possible to vary the area in such a fashion to insure constant temperature.

This constant-temperature area variation is obtained by setting

$$\frac{1}{A}\frac{dA}{dx} = -\frac{BjA}{\rho_o u_o^2 U^2 A_o}\left(U - \frac{\gamma M^2 - 1}{\gamma - 1}\alpha j\right) \tag{52}$$

which, upon substitution into equation (3c) renders dT equal to zero; and hence, T becomes a constant. Since T is defined as a temperature ratio, its value must equal 1. Also since $T = 1$, $M = M_o U$, from equation (4c). With the substitution of $M = M_o U$, equation (52) becomes

$$\frac{1}{A}\frac{dA}{dx} = \frac{BjA}{\rho_o u_o^2 U^2 A_o}\left[\frac{\gamma M_o^2 U^2 \alpha j - (\gamma - 1)U - \alpha j}{\gamma - 1}\right] \tag{53}$$

If equation (53) is used in equation (3a), that equation becomes

$$\frac{dU}{U\,dx} = \frac{BjA}{\rho_o u_o^2 U^2 A_o}\left[\frac{(\gamma - 1)U + \alpha j}{\gamma - 1}\right] \tag{54}$$

The elimination of $\dfrac{BjA}{\rho_o u_o^2 U^2 A_o}$ from equations (53) and (54) results in the following relation between U and A:

$$\frac{dA}{A} = \frac{\left(\gamma M_o^2 U^2 - 1\right)\alpha j - (\gamma - 1)U}{(\gamma - 1)U + \alpha j}\frac{dU}{U} \tag{55}$$

If j is a function of U, the variables in this equation are separated so that the solution of equation (55) is reduced to a quadrature. Once A/A_o is obtained, its value may be substituted into equation (54) which may then be integrated to obtain both A/A_o and U as functions of x. Unfortunately, the simple current-velocity relations used previously do not give functions of A/A_o which, when substituted into equation (54), will render that equation integrable with elementary functions.

Constant Mach Number

General solution.- The general solution for $M = M_o$ is obtained in a similar fashion. If dA/A is given by

$$\frac{1}{A}\frac{dA}{dx} = -\frac{M_o^2 BjA}{\rho_o u_o^2 U^2 A_o}\left[\frac{(\gamma + 1)U - \left(\gamma M_o^2 + 1\right)\alpha j}{2 + (\gamma - 1)M_o^2}\right] \tag{56}$$

the substitution of this equation into equation (3e) will make $dM = 0$, and hence M is a constant. The substitution of equation (56) and $M = M_o$ into equation (3a) gives the following equation for dU:

$$\frac{dU}{U\,dx} = \frac{M_o^2 BjA}{\rho_o u_o^2 U^2 A_o}\left[\frac{(\gamma - 1)U + \alpha j}{2 + (\gamma - 1)M_o^2}\right] \tag{57}$$

The elimination of $\dfrac{M_o^2 BjA}{\rho_o u_o^2 U^2 A_o\left[2 + (\gamma - 1)M_o^2\right]}$ from equations (56) and (57) results in

$$\frac{dA}{A} = -\frac{(\gamma + 1)U - \left(\gamma M_o^2 + 1\right)\alpha j}{(\gamma - 1)U + \alpha j}\,\frac{dU}{U} \tag{58}$$

Again, if j is a function of U, equation (58) is separated and the solution is reduced to a quadrature. Again U is determined by integrating equation (57) after substituting the value of A/A_o as determined by the integration of equation (58) as follows:

$$\frac{U\,dU}{j\left[(\gamma - 1)U + \alpha j\right]\left[e^{-\int \frac{(\gamma+1)U-\left(\gamma M_o^2+1\right)\alpha j}{(\gamma-1)U+\alpha j}\frac{dU}{U}+C}\right]} = \frac{M_o^2 B\,dx}{\rho_o u_o^2\left[2 + (\gamma - 1)M_o^2\right]} \tag{59}$$

<u>Current density proportional to U, $j = j_o U$.</u>- Of the five functions of U that have been used to express j, only one, current density proportional to U, that is,

$$j = j_o U \tag{60}$$

will render equation (59) readily integrable. The substitution of this function into equation (58) gives

$$\frac{dA}{A} = -\frac{\gamma + 1 - \left(\gamma M_o^2 + 1\right)a}{\gamma - 1 + a}\,\frac{dU}{U} \tag{61}$$

A similar substitution into equation (57) gives

$$\frac{dU}{U} = M_o^2\left[\frac{\gamma - 1 + a}{2 + (\gamma - 1)M_o^2}\,\frac{A}{A_o}\,dy\right] \tag{62}$$

If equation (61) is integrated and the constant evaluated, it becomes

$$\frac{A}{A_o} = U^{-\frac{\gamma+1-\left(\gamma M_o^2+1\right)a}{\gamma-1+a}} \tag{63}$$

If equation (63) is substituted into equation (62), that equation becomes

$$\left[U^{\frac{\gamma+1-\left(\gamma M_o^2+1\right)a}{\gamma-1+a}-1}\right]dU = \frac{M_o^2(\gamma - 1 + a)}{2 + (\gamma - 1)M_o^2}\,dy \tag{64}$$

After equation (64) is integrated and its constant evaluated, it becomes

$$\frac{2 + (\gamma - 1)M_o^2}{M_o^2\left[\gamma + 1 - \left(\gamma M_o^2 + 1\right)a\right]}\left(U^{\frac{\gamma+1-\left(\gamma M_o^2+1\right)a}{\gamma-1+a}} - 1\right) = y \tag{65}$$

or

$$U = \left\{1 + \frac{M_o^2\left[\gamma + 1 - \left(\gamma M_o^2 + 1\right)a\right]}{2 + (\gamma - 1)M_o^2}\,y\right\}^{\frac{\gamma-1+a}{\gamma+1-\left(\gamma M_o^2+1\right)a}} \tag{66}$$

Substitution of equation (66) into equation (63) results in

$$\frac{A}{A_o} = \frac{2 + (\gamma - 1)M_o^2}{2 + (\gamma - 1)M_o^2 + M_o^2\left[\gamma + 1 - \left(\gamma M_o^2 + 1\right)a\right]y} \tag{67}$$

The variables P, T, and ρ' may be determined from equations (4a), (4b), and (4c) by letting $M = M_o$. They are

$$P = U\frac{A_o}{A} \tag{68}$$

$$T = U^2 \tag{69}$$

$$\rho' = \frac{A_o}{AU} \tag{70}$$

Examination of equations (66) and (67) shows that U and A/A_o are dependent on both a and M_o^2. It is also seen that equation (66) becomes indeterminate at $a = \frac{\gamma + 1}{\gamma M_o^2 + 1}$. This indetermination is resolved with the use of the given value of a and equations (62) and (63). The solution of these equations then gives

$$U = e^{\frac{\gamma M_o^2}{\gamma M_o^2+1}y} \tag{71}$$

and A becomes constant.

CONSTANT-AREA CHANNELS

General Solution

The equations which express the properties of the flow in a constant-area magnetohydrodynamic channel are obtained by making $\frac{dA}{A\ dx}$ equal to zero in equations (3a) to (3e); this procedure renders A constant and makes A/A_o equal to 1. With these substitutions, the flow equations for a constant-area channel become

$$\frac{dU}{U\ dx} = -\frac{d\rho'}{\rho'\ dx} = \frac{M^2}{M^2 - 1}\frac{Bj}{\rho_o u_o^2 U^2}(U - \alpha j) \tag{72a}$$

$$\frac{dP}{P\ dx} = -\frac{\gamma M^2}{M^2 - 1}\frac{Bj}{\rho_o u_o^2 U^2}(U - \alpha M^2 j) \tag{72b}$$

$$\frac{dT}{T\ dx} = -\frac{M^2}{M^2 - 1}\frac{Bj}{\rho_o u_o^2 U^2}\left[(\gamma - 1)U - \left(\gamma M^2 - 1\right)\alpha j\right] \tag{72c}$$

$$\frac{dM}{M\ dx} = \frac{M^2}{2\left(M^2 - 1\right)}\frac{Bj}{\rho_o u_o^2 U^2}\left[(\gamma + 1)U - \left(\gamma M^2 + 1\right)\alpha j\right] \tag{72d}$$

and equations (4a), (4b), and (4c) become

$$\rho' = \frac{1}{U} \tag{73a}$$

$$T = UP \tag{73b}$$

$$\frac{M}{M_o} = \left(\frac{U}{P}\right)^{1/2} \tag{73c}$$

If j is a function of either U or P, the integrals of equations (72a) and (72b) may be written as quadratures. These quadratures are obtained by first eliminating M^2 from both equations with the substitution of $M^2 = \frac{M_o^2 U}{P}$. This substitution results in the following relations for dU and dP:

$$\frac{UM_o^2 - P}{U - \alpha j} dU = \frac{M_o^2 Bj}{\rho_o u_o^2} dx \tag{74}$$

$$\frac{UM_o^2 - P}{P - \alpha M_o^2 j} dP = -\frac{\gamma M_o^2 Bj}{\rho_o u_o^2} dx \tag{75}$$

If the term $\frac{M_o^2 Bj}{\rho_o u_o^2} dx$ is eliminated from equations (74) and (75), then

$$\frac{dP}{dU} = -\gamma \frac{\left(P - \alpha M_o^2 j\right)}{U - \alpha j} \tag{76}$$

This equation may be written in two forms:

$$\frac{dP}{dU} + \frac{\gamma P}{U - \alpha j} = \frac{\gamma \alpha M_o^2 j}{U - \alpha j} \tag{77}$$

or

$$\frac{dU}{dP} + \frac{U}{\gamma\left(P - \alpha M_o^2 j\right)} = \frac{\alpha j}{\gamma\left(P - \alpha M_o^2 j\right)} \tag{78}$$

Equations (77) and (78) become first-order linear differential equations if j is assumed to be a function of U in equation (77) or of P in equation (78). The solution for either equation may be found in most differential equation texts (for example, page 116 of ref. 10). The solutions of equations (77) and (78) are, respectively,

$$P = e^{-\int \frac{\gamma\, dU}{U - af(U)}} \left[\int \frac{\gamma a M_o^2 f(U)}{U - af(U)} e^{\int \frac{\gamma\, dU}{U - af(U)}} dU + C \right] \tag{79}$$

where $j = j_o f(U)$ and

$$U = e^{-\int \frac{dP}{\gamma\left[P - aM_o^2 f(P)\right]}} \left[\int \frac{af(P)}{\gamma\left[P - aM_o^2 f(P)\right]} e^{\int \frac{dP}{\gamma\left[P - aM_o^2 f(P)\right]}} dP + C \right] \tag{80}$$

where $j = j_o f(P)$.

The constants must be evaluated from the initial or reference conditions. The expressions of P as a function of U (eq. (79)) or U as a function of P (eq. (80)) may now be substituted into equation (74) or (75), respectively. The resultant equations give U as a function of x if j is a known function of U and correspondingly give P as a function of x if j is a known function of P. Once either of these variables is determined with respect to x and with the use of equations (73a), (73b), and (73c), the density ratio, Mach number, and temperature ratio are readily determined.

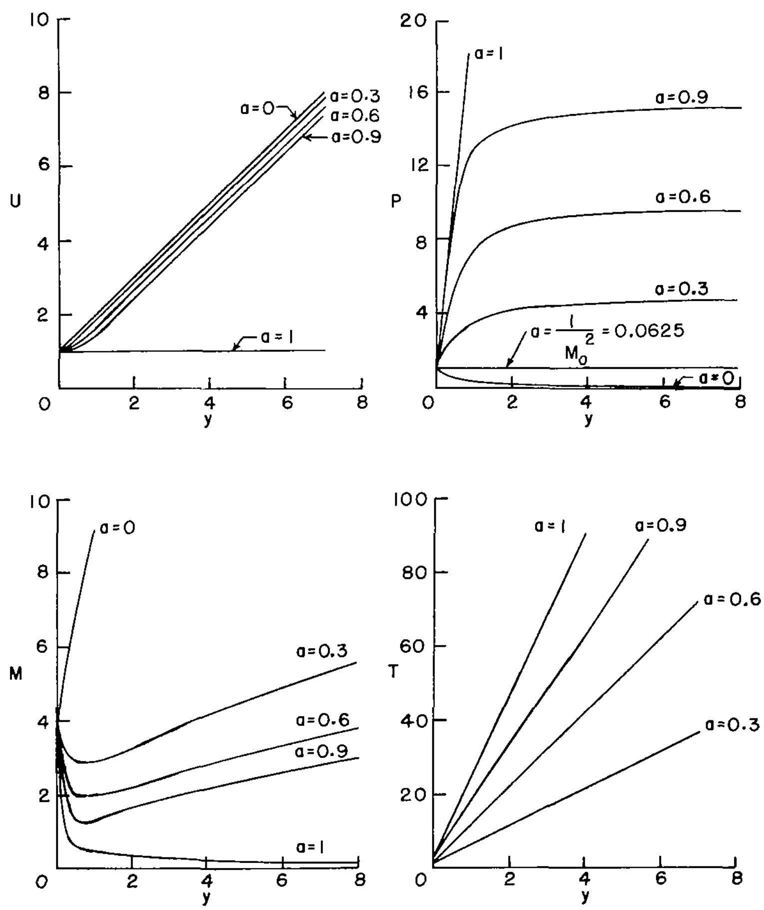
Constant Current Density

The operation of a channel in which the current density and the cross-sectional area are held constant is quite practical for experimental use, since the driving source required is a constant-current system for each electrode used on the channel. If the electrodes are evenly spaced areawise, the current density should be approximately constant.

The solution of the equations which represent the properties of the flow in a constant-area, constant-current magnetohydrodynamic channel is obtained by substituting $j = j_o$ into either equation (79) or equation (80), integrating, and evaluating the constant to determine P or U which may then be substituted into equation (74) or (75), respectively. Integration of the resultant equation then gives U or P as a function of x.

The substitution of $j = j_o$ or $f(U) = f(P) = 1$ into equations (79) and (80) results in

$$P = aM_o^2 - \left(aM_o^2 - 1\right)\left(\frac{U - a}{1 - a}\right)^{-\gamma} \tag{81}$$



(a) $M_o = 4$; $0 < a < 1$.

Figure 23.- Distributions of U, P, M, and T in a magnetohydrodynamic channel with B, σ, j, and A held constant.

and

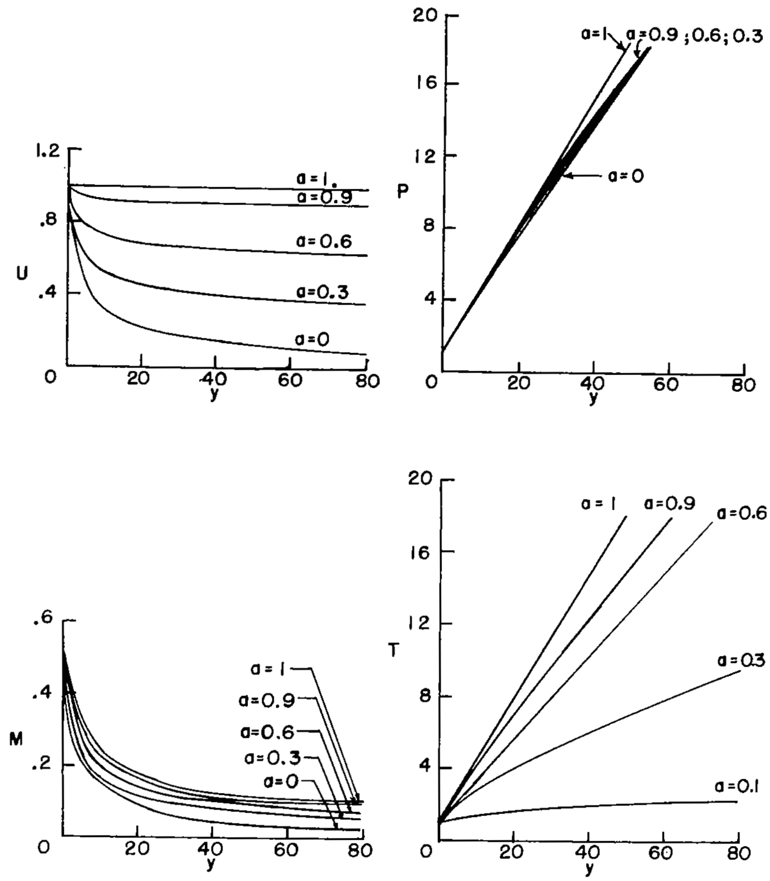
$$U = a - (a - 1)\left(\frac{P - aM_o^2}{1 - aM_o^2}\right)^{-1/\gamma} \tag{82}$$

Equations (81) and (82) may now be substituted into equations (74) and (75), respectively. Integration of the resultant equations and appropriate evaluation of the constants give the following relationships for U and P as functions of y:

$$y = U - 1 - \frac{aM_o^2 - 1}{\gamma M_o^2}\left[\left(\frac{U - a}{1 - a}\right)^{-\gamma} - 1\right] \tag{83}$$

and

$$y = \frac{P - 1}{\gamma M_o^2} - (a - 1)\left[\left(\frac{P - aM_o^2}{1 - aM_o^2}\right)^{-1/\gamma} - 1\right] \tag{84}$$



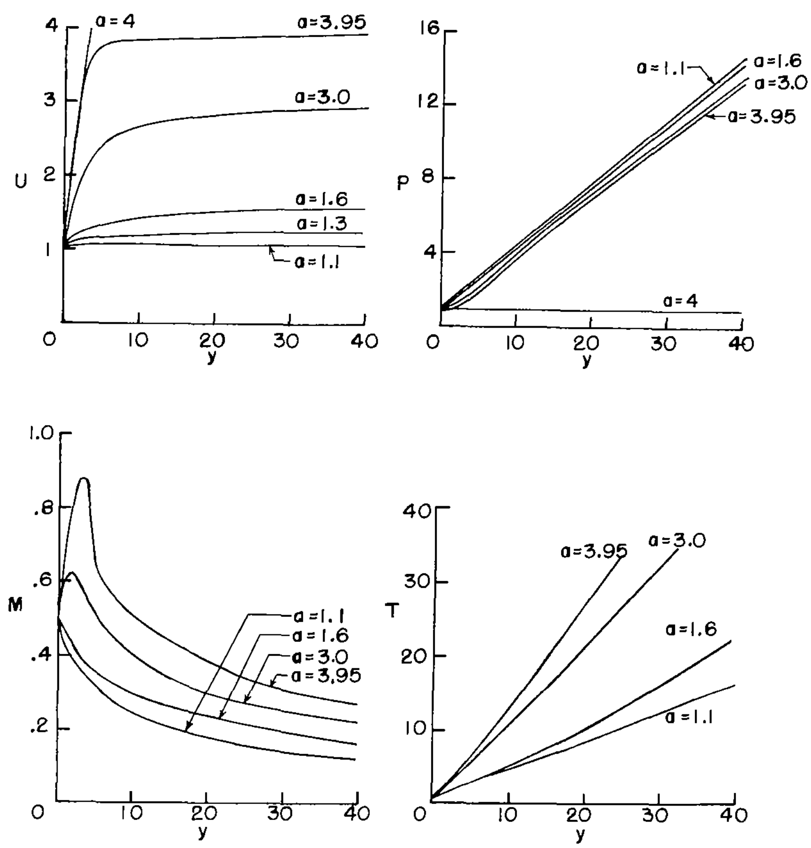
(b) $M_o = 0.5;\ 0 \leqq a \leqq 1.$

Figure 23.- Continued.

Once U and P are determined as functions of y, then ρ', T, and M may be determined from equations (73a), (73b), and (73c). Typical values of U, P, M, and T are presented in figure 23. The curves are presented for a supersonic ($M_o = 4$) reference condition and for a subsonic ($M_o = 0.5$) reference condition.

Several interesting features are seen from the figure. The velocity ratio U increases almost linearly for the initial value $M_o = 4$ provided a is less than about 0.9. Also U as a function of y is seen not to be sensitive to changes in a; thus, if the conductivity σ is sufficiently large so that $a < 0.9$, the velocity will not be sensitive to changes in σ. However, even though U is not sensitive to σ, it is seen that P, M, and T do change a great amount with respect to a and hence are very sensitive to σ.

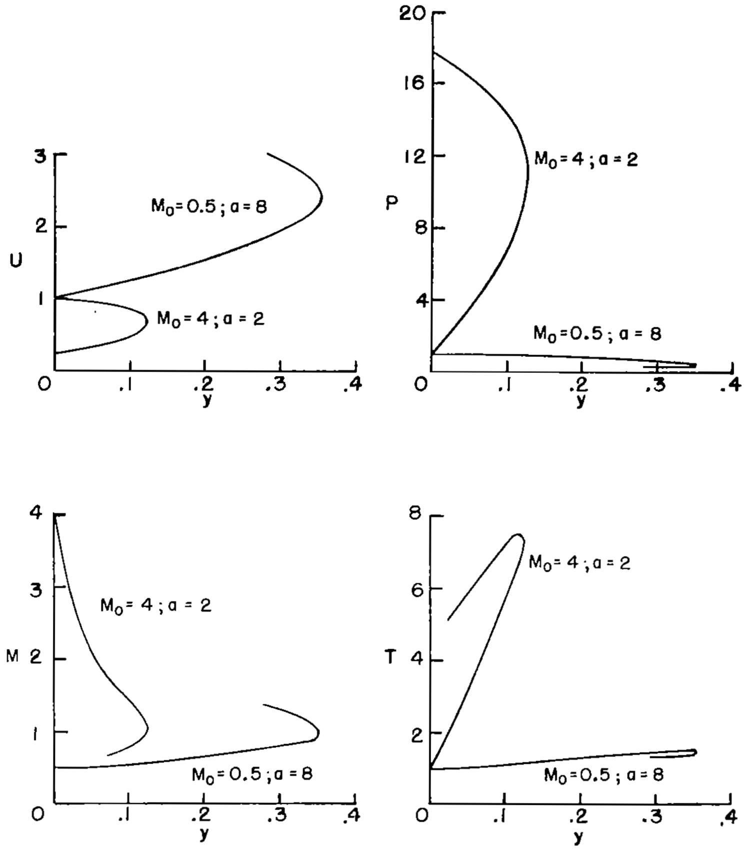
However, in the subsonic case for $a < 1$, it is seen that P and M are insensitive to a whereas U and T become sensitive. It may also be observed that a range of a exists (fig. 23(c)) for which subsonic velocities may be increased, specifically the region $1 < a < 1/M_o^2$. In this region, the velocity ratio asymptotically approaches a and does not increase continuously as for the supersonic initial condition (fig. 23(a)).



(c) $M_o = 0.5$; $1 < a \leqq 4$.

Figure 23.- Continued.

The flow conditions beyond the point of turning represented in figure 23(a) are impossible to attain because the flow cannot reverse its direction as indicated by the figure. Examination of equation (83) shows that this condition occurs for supersonic flow if a is greater than 1 and for subsonic flow if a is greater than $1/M_o^2$. Since these flows are impossible, the flow will automatically adjust itself. In the subsonic initial condition such adjustment will occur because of upstream pressure changes and readjustment of the entire upstream flow to conform to the condition that $M = 1$ at the end of the channel. If the initial flow is supersonic, a normal shock must occur at some point within or ahead of the channel such that the flow will proceed subsonically to the end of the channel and emerge at a Mach number of 1 or less.



(d) $M_o = 4;\ a > 1;\ M_o = 0.5;\ a > \frac{1}{M_o^2}$.

Figure 23.- Concluded.

Current Density Proportional to Velocity

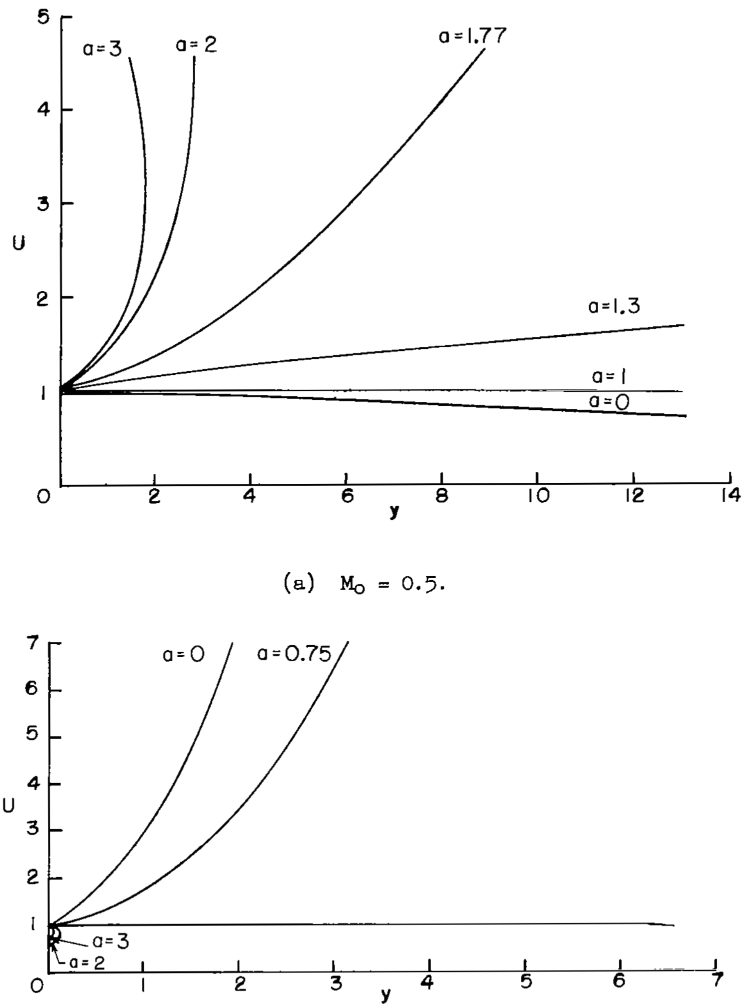
An interesting mode of operation occurs if the current density is assumed to vary directly with the velocity ratio U. Since for this mode $j = j_o U$, the equations are obtained with the substitution of $f(U) = U$ into equations (74) and (79). After the substitution is made,

$$P = \frac{\gamma a M_o^2}{\gamma + 1 - a} U + \frac{\gamma + 1 - a\left(\gamma M_o^2 + 1\right)}{\gamma + 1 - a} U^{-\frac{\gamma}{1 - a}} \tag{85}$$

The use of equation (85) and of $j = j_o U$ in equation (74) results after integration in

$$\frac{Bj_o}{\rho_o u_o^2}\left(x - x_o\right) = y = \frac{\gamma + 1}{\gamma + 1 - a} \log U + \frac{\gamma + 1 - a\left(\gamma M_o^2 + 1\right)}{(\gamma + 1 - a)^2 M_o^2}\left[U^{-\left(\frac{\gamma+1-a}{1-a}\right)} - 1\right] \tag{86}$$

Equations (85) and (86) along with equations (73a), (73b), and (73c) may be used to determine completely the flow in a magnetohydrodynamic channel in which B, σ, and A are held constant and the current density is proportional to the velocity.



(a) $M_o = 0.5$.

(b) $M_o = 2.0$.

Figure 24.- Several velocity ratio distributions in a magnetohydrodynamic channel with B, σ, j/U, and A held constant.

Several typical velocity ratio distributions are given in figure 24 for a subsonic initial Mach number ($M_o = 0.5$) and for a supersonic initial Mach number ($M_o = 2.0$). For the subsonic Mach number, it is seen that if a is less than 1, the flow decelerates; whereas if $1 < a < \frac{\gamma + 1}{\gamma M_o^2 + 1}$ (1.77 where $\gamma = 1.4$), the flow continuously accelerates. However, although it is not obvious from the figure, if a is greater than $\frac{\gamma + 1}{\gamma M_o^2 + 1}$, the flow reverses upon itself. This flow reversal indicates that these initial conditions could not occur if the channel length is too great. Correspondingly, for the supersonic velocity distributions (fig. 24(b)), it is seen that if a is less than 1, the acceleration is positive and the velocity continuously increases with y. In this case, if a is greater than 1, the flow decelerates and reverses direction. (See curves for $a = 2$ and $a = 3$, fig. 24(b).) As previously discussed, the flow will adjust itself to eliminate the reversal condition.

It is of interest to divide the parametric field of a and M_o into regions showing the values of the parameters that determine acceleration or deceleration and direct or reversed flow. This field is mapped in figure 25. The nature of each of the indicated regions is determined by examination of the derivative dy/dU evaluated at $U = 1$ and of the conditions corresponding to $dy/dU = p$. The first expression

$$\left(\frac{dy}{dU}\right)_{U=1} = \frac{1 - M_o^2}{(a - 1)M_o^2}$$

is used to determine whether U increases or decreases at $U = 1$. Examination of the complete expression for dy/dU shows that U is monotonic with respect to y provided flow reversal does not occur; thus, if U increases, y will always increase or if y decreases, U will always decrease for all values of y. The term $dy/dU = 0$ gives the values of U and y for which y is a maximum and hence the point at which y reverses direction and causes a flow reversal. This value $(U)_{y_{max}}$ of U is given by

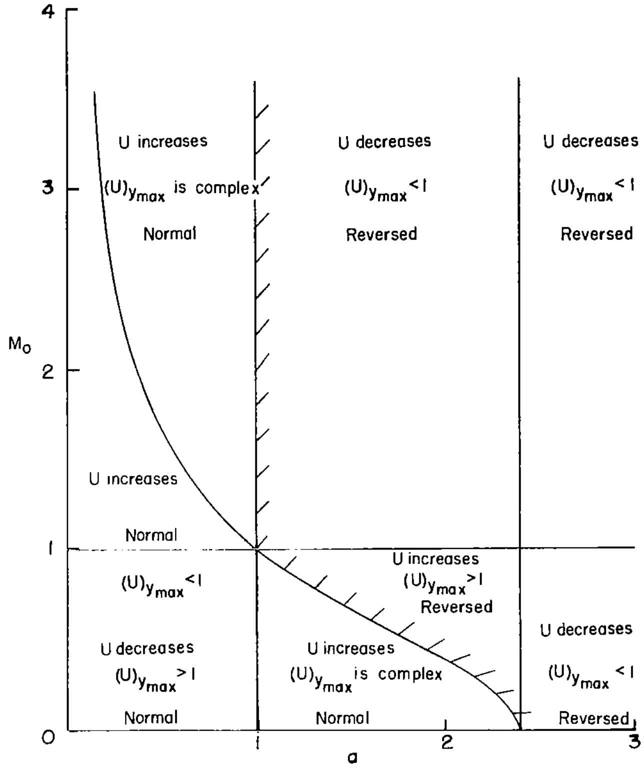


Figure 25.- Regions showing normal flow and reversed flow in a magnetohydrodynamic channel with B, σ, j/U, and A held constant.

$$(U)_{y_{max}} = \left[\frac{\gamma + 1 - a\left(\gamma M_o^2 + 1\right)}{(1 - a)(\gamma + 1)M_o^2}\right]^{\frac{1-a}{\gamma+1-a}}$$

If in a given region U increases and $(U)_{y_{max}}$ is greater than 1 or if U decreases and $(U)_{y_{max}}$ is less than 1 a reversal must occur since U can achieve the value $(U)_{y_{max}}$. If U increases and $(U)_{y_{max}}$ decreases or if U decreases and $(U)_{y_{max}}$ increases, U cannot achieve $(U)_{y_{max}}$ and no flow reversal can occur and the flow will proceed through the channel. There are two regions in which the term within the brackets of the equation for $(U)_{y_{max}}$ is negative, and the exponent is less than 1 so that $(U)_{y_{max}}$ is complex and hence is unattainable. The flow in these regions is therefore normal. The cross-hatched line indicates the division between the regions in which normal flow and reversed flow occur.

Constant Electric Field

A constant-area channel in which the electric field is held constant is also quite practical for experimental consideration even though the voltmeter reading must be corrected for anode and cathode drops before the field in the gas can be determined. However, if the electric field in the gas can be held constant, the current density may be expressed either as $j = B\sigma u_o(E - U)$ or $j = \frac{j_o(E - U)}{E - 1}$. The values of U and P as functions of y may be obtained with the substitution of $f(U) = \frac{E - U}{E - 1}$ into equation (79) to determine P as a function of U and substitution of the resultant function into equation (74) to determine U as a function of x or y. When these substitutions are made, the results are

$$P = \frac{\gamma(\gamma - 1)M_o^2(U - 1)(2E - U - 1) + 2\left[\gamma - (\gamma - 1)E\right]}{2\left[\gamma U - (\gamma - 1)E\right]} \tag{87}$$

$$y = \frac{(1 - E)}{2E}(\gamma + 1)\left\{\frac{\left[(\gamma - 1)^2E^2 - \gamma F\right](U - 1)}{\left[\gamma - (\gamma - 1)E\right]\left[\gamma U - (\gamma - 1)E\right]} + \frac{(\gamma - 1)^2E^2 - \gamma F}{\gamma E}\log\frac{\gamma U - (\gamma - 1)E}{\gamma - (\gamma - 1)E} + \frac{(2 - \gamma)E^2 + F}{E}\log\frac{E - U}{E - 1}\right\} \tag{88}$$

where

$$F = \frac{\gamma(\gamma - 1)(2E - 1)M_o^2 - 2\left[\gamma - (\gamma - 1)E\right]}{(\gamma + 1)M_o^2}$$

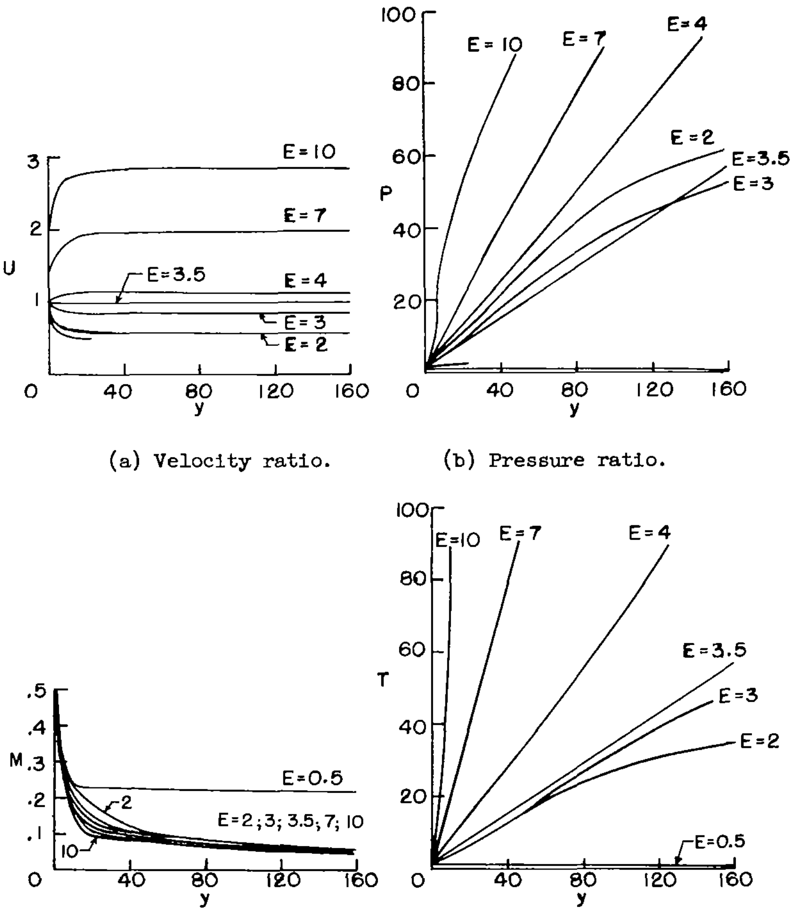
and

$$a = (\gamma - 1)(E - 1)$$

and ρ', M, and T may be determined from equations (73a), (73b), and (73c).

Plots of P, U, M, and T as determined from equations (87), (88), and (73b) and (73c) are presented in figures 26 and 27. A set of values of U, P, M, and T are presented in figure 26 for a subsonic ($M_o = 0.5$) initial condition. Examination of figure 26 shows several interesting points. One point is that if E is less than 3.5 or $\frac{\gamma}{\gamma - 1}$ the velocity is decreased but the pressure ratios and temperature ratios are increased. Contrastingly, if E is greater than $\frac{\gamma}{\gamma - 1}$, the velocity will approach $\frac{(\gamma - 1)E}{\gamma}$ rather than E as might be expected. This point is demonstrated upon examination of equation (88) in which it is seen that two terms occur that make y infinite as

U approaches $\frac{(\gamma - 1)E}{\gamma}$. Hence the acceleration attainable from a subsonic stream is limited not to E but to $\frac{(\gamma - 1)E}{\gamma}$. This limitation of acceleration causes an even more rapid increase in both pressure and temperature. The Mach number is seen to decrease for all values of E and to be rather insensitive to the value of E.

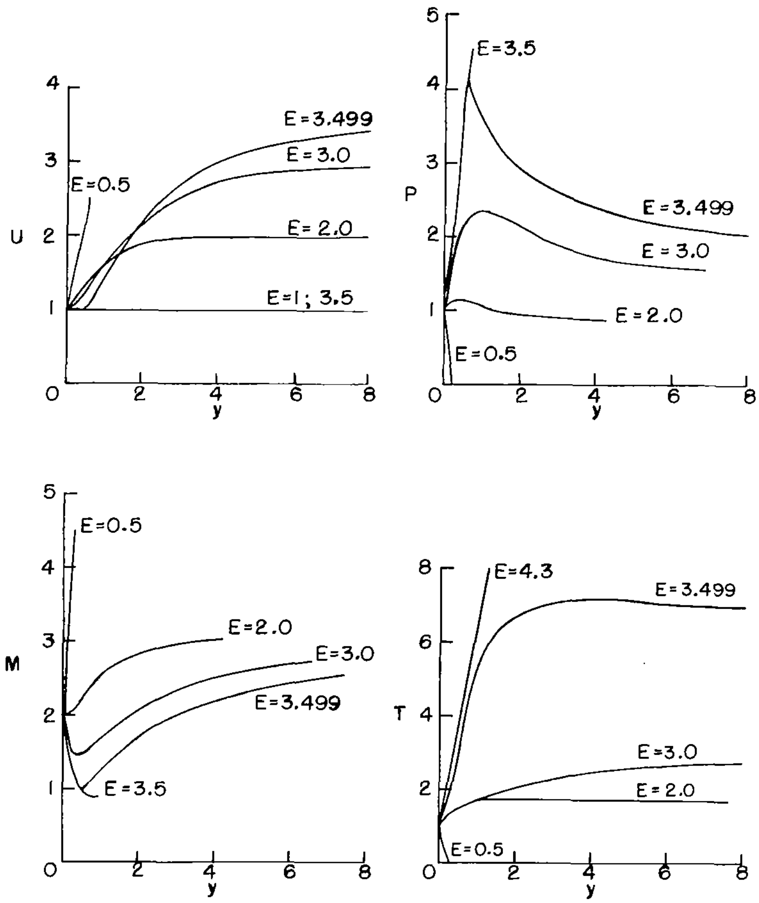


(a) Velocity ratio. (b) Pressure ratio.

(c) Mach number. (d) Temperature ratio.

Figure 26.- Distributions of U, P, M, and T in a magnetohydrodynamic channel with B, σ, and A held constant. $j = B\sigma U_o(E - U)$; $M_o = 0.5$.

Values of U, P, M, and T computed for a supersonic initial Mach number ($M_o = 2.0$) are presented in figures 27(a) and 27(b). If the initial flow is supersonic, the flow properties are greatly different depending on whether E is less than or greater than $\frac{\gamma}{\gamma - 1}$ or 3.5 when $\gamma = 1.4$. The presentation of the results is therefore divided into two sections: (1) E with values less than 3.5 (fig. 27(a)), and (2) E with values greater than 3.5 (fig. 27(b)).

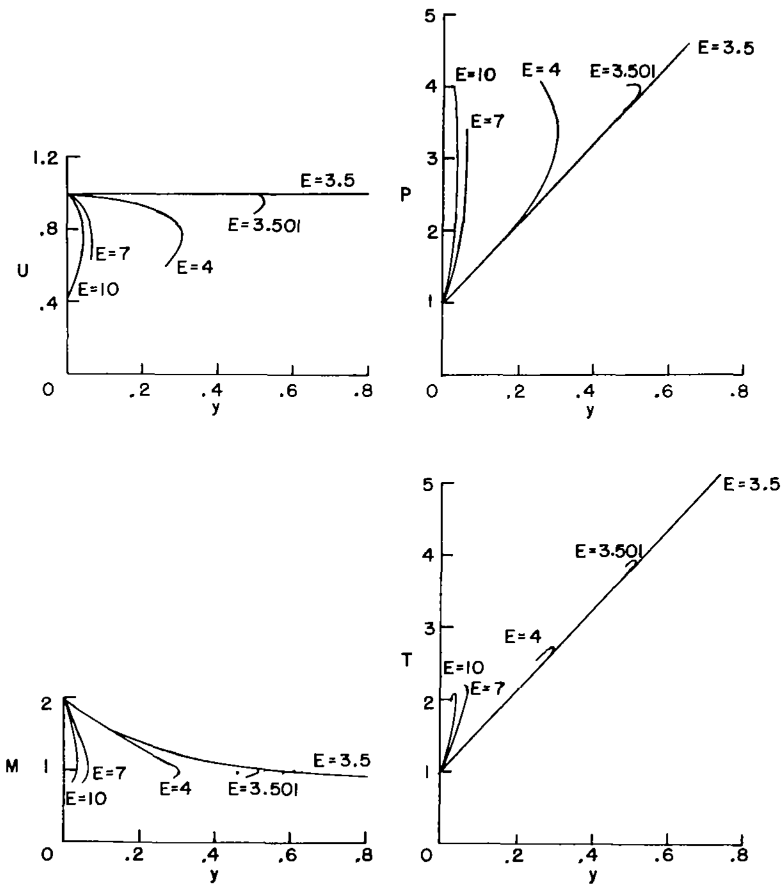


(a) $j = B\sigma U_0(E - U)$, $E < 3.5$, and $M_0 = 2.0$.

Figure 27.- Distributions of U, P, M, and T in a magnetohydrodynamic channel with B, σ, and A held constant.

Examination of figure 27(a) shows that if E is less than $\frac{\gamma}{\gamma - 1}$ or 3.5, the velocity increases asymptotically to $U = E$. Pressures, temperatures, and Mach numbers also increase if E is greater than 1. However, if E is less than 1 (see curves for $E = 0.5$), the flow exhibits several irregularities. It is seen that the velocity increases indefinitely. Since the channel acts as a generator if E is less than 1, such a velocity increase can only be made with the reduction of pressure and temperature to negative values. Such values of both pressure and temperature are impossible; hence, there must exist some value of pressure below which the continuous equations used for this analysis no longer apply. It is also interesting to note that the velocity curve for

$E = 3.499$ indicates a similar mode of transition between the change from an asymptotic approach of U to E for $E < \frac{\gamma}{\gamma - 1}$ to the approach of U to 1 at $E = \frac{\gamma}{\gamma - 1}$ as was observed in figure 21.



(b) $j = B\sigma U_o(E - U)$, $E > 3.5$, and $M_o = 2.0$.

Figure 27.- Concluded.

Examination of the velocity of the flow for E greater than 3.5 (fig. 27(b)) shows that these flows are forbidden inasmuch as the length coordinate y reverses when U attains a certain value. The Mach number curves show that this reversal occurs, as in previous cases, when $M = 1$. It is expected that a shock will occur and reduce the velocity to a subsonic value, from which it may proceed, as shown in figure 26, through the channel.

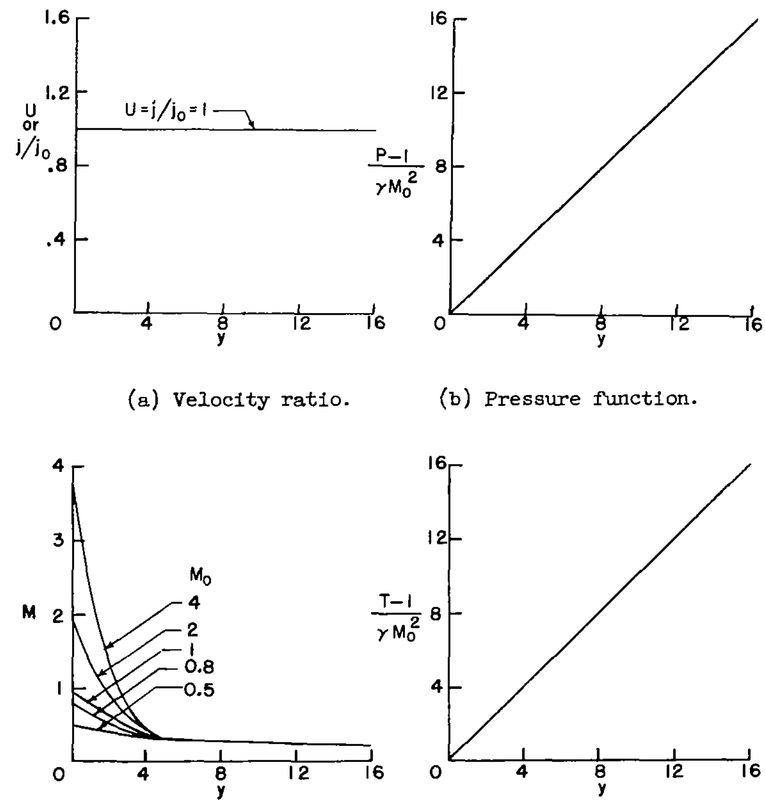
Constant Velocity

The constant-velocity condition is treated by letting $\alpha j = U$; then dU of equation (72a) is equal to zero and U becomes constant and equal to 1. Hence, $\alpha j = \alpha j_o = 1$ and since α is constant and equal to $\frac{\gamma - 1}{B\sigma u_o}$, j must equal j_o and thus j is also constant. The pressure equation (eq. (72b)) may now be written, with these substitutions and $M^2 = M_o^2/P$ as

$$\frac{dP}{P\,dx} = \frac{\gamma M_o^2}{M^2 - 1} \frac{Bj_o}{\rho_o u_o^2} \frac{M^2 - 1}{P}$$

or

$$dP = \gamma M_o^2 dy \tag{89}$$



(a) Velocity ratio. (b) Pressure function.

(c) Mach number. (d) Temperature function.

Figure 28.- Distributions of $\frac{P - 1}{\gamma M_o^2}$, M, and $\frac{T - 1}{\gamma M_o^2}$ in a magnetohydrodynamic channel with B, σ, A, U, and j held constant and $B\sigma U_o = (\gamma - 1)j_o$.

Then, integrating and evaluating the constants gives

$$P = 1 + \gamma M_o^2 y \tag{90}$$

From equations (73b) and (73c)

$$M = \frac{M_o}{\sqrt{P}} = \frac{M_o}{\sqrt{1 + \gamma M_o^2 y}} \tag{91}$$

and

$$T = UP = P = 1 + \gamma M_o^2 y \tag{92}$$

Plots of $\frac{P - 1}{\gamma M_o^2}$, $\frac{T - 1}{\gamma M_o^2}$, and M are presented in figure 28 to give a rapid estimate of the variations in these quantities. The notable feature is that the Mach numbers reduce to subsonic values and approach each other too closely to show on the figure after y is greater than about 4.

Constant Pressure

The mode of operation in which both the area and pressure are held constant has several features of interest applicable to the design of practical channels. Since the pressure is constant, it should be possible to design a constant-area channel having a minimum of wall contact with the plasma.

This constant-area, constant-pressure mode is created by letting $\alpha j M^2 = U$ in equation (72b). The substitution of this value of U renders dP equal to zero and hence P is a constant and therefore equal to 1. Also from equation (73c), $M^2 = \frac{M_o^2 U}{P}$ or $P = 1 = \frac{U M_o^2}{M^2}$; hence $\alpha j = \frac{U}{M^2} = \frac{1}{M_o^2}$ = Constant. Since αj equals a constant, j must equal j_o, and hence $\alpha j = \alpha j_o = a = \frac{1}{M_o^2}$ or $a M_o^2 = 1$. This condition is observed to give constant pressure in figure 23(a). The substitution of $\alpha j = \frac{U}{M^2}$, $M^2 = M_o^2 U$, and $j = j_o$ into equation (72a) results in the following equation for dU:

$$dU = \frac{B j_o}{\rho_o u_o^2} dx = dy \tag{93}$$

Integration and evaluation of the constant gives

$$U = 1 + y \tag{94}$$

From equations (73c),

$$M = M_o(1 + y)^{1/2} \tag{95}$$

and from equation (73b),

$$T = U = 1 + y \tag{96}$$

Plots of the values of U, M, and T for various values of M_o computed from these equations are presented in figure 29. Mach number, velocity, and temperature all increase for this case. It is also interesting to note that the current density is constant for a constant-pressure channel as well as for a constant-velocity channel.

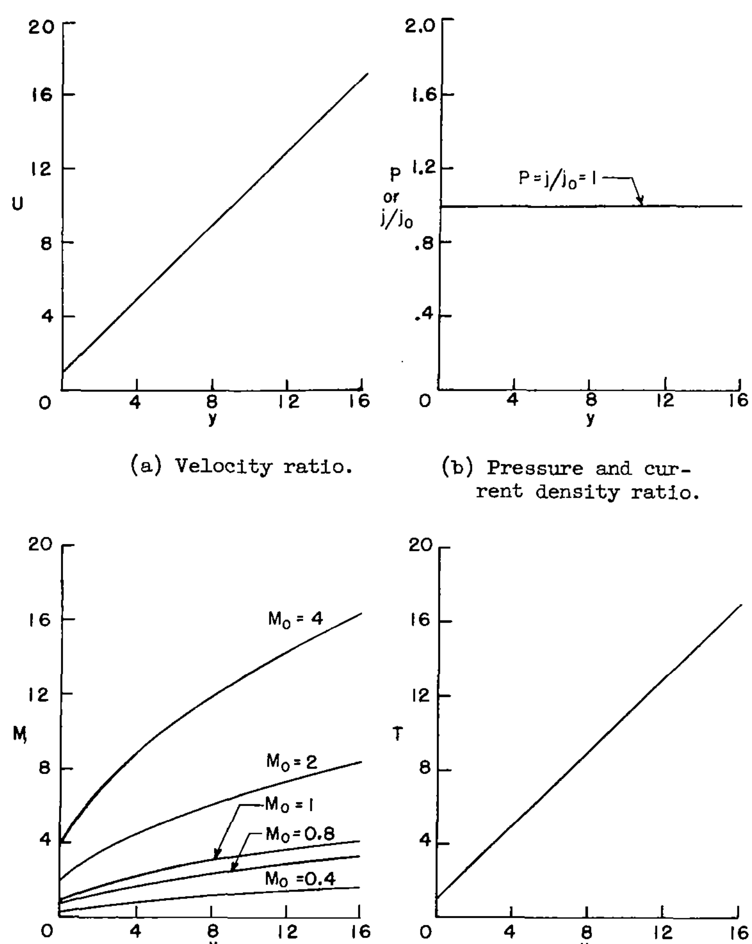


Figure 29.- Distribution of U, M, j/j_o, and T in a magnetohydrodynamic channel with B, σ, P, A, and j held constant and $B\sigma U_o = (\gamma - 1)M_o{}^2 j_o$.

An important point must be observed, however, in case the initial Mach number is subsonic. Although equation (94) shows that for $a = \frac{1}{M_o^2}$ the velocity increases indefinitely, it can be shown with the use of equation (83) that if a is the smallest fraction less than $\frac{1}{M_o^2}$, the flow will exhibit a forbidden singularity and therefore completely change its character if the channel is of sufficient length. Since it is highly improbable that a can be held exactly equal to $\frac{1}{M_o^2}$ over the entire length of the channel and during the entire operation period, it may well be expected that the increase in U will be limited to $\frac{1}{M_o^2}$.

Constant Temperature

This channel is often considered as an example of tailoring the current distribution to give a desired condition. (See ref. 1.) The constant-temperature condition is interesting because the conductivity remains practically constant in such a channel.

The constant-temperature condition is attained by letting $\alpha j = \frac{(\gamma - 1)U}{\gamma M^2 - 1}$. Use of this equation in equation (72c) renders dT equal to zero, and hence T is constant and therefore equal to 1. Since $T = 1$, equation (4c) gives $M = M_oU$ and thus $\alpha j = \frac{(\gamma - 1)U}{\gamma M_o^2U^2 - 1}$. Now $j = j_o$ when $U = 1$; hence, $\alpha j_o = \frac{\gamma - 1}{\gamma M_o^2 - 1}$, from which $j = \frac{\left(\gamma M_o^2 - 1\right)Uj_o}{\gamma M_o^2U^2 - 1}$, and $a = \frac{\gamma - 1}{\gamma M_o^2 - 1}$. These relations give the necessary tailoring of j to achieve a constant temperature. The current density j is seen to vary as U if $\gamma M_o^2U^2$ is much smaller than 1 and to vary as $1/U$ if $\gamma M_o^2U^2$ is much larger than 1. Typical values of this current density variation are shown in figure 30(d).

If these relations for M, αj, and j are substituted into equation (72a), the result after several manipulations is

$$\frac{\left(\gamma M_o^2U^2 - 1\right)^2}{U^3} dU = \gamma M_o^2\left(\gamma M_o^2 - 1\right)dy \tag{97}$$

Integration and evaluation of the constant give

$$y = \frac{1}{\gamma M_o^2\left(\gamma M_o^2 - 1\right)}\left[\frac{\left(\gamma^2M_o^4U^2 + 1\right)\left(U^2 - 1\right)}{2U^2} - 2\gamma M_o^2\log U\right] \tag{98}$$

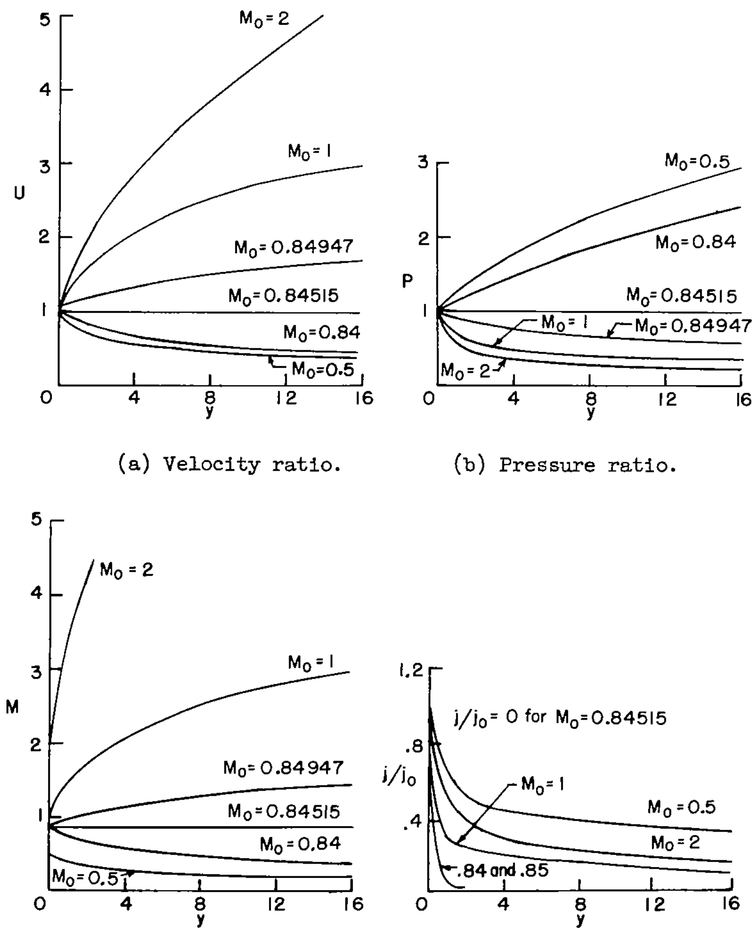
and from equation (73b)

$$P = \frac{1}{U} \tag{99}$$

Also

$$M = M_o U \tag{100}$$

Plots of the values of U, P, and M as calculated from equations (98), (99), and (100) are presented in figure 30. The flow properties are rather sensitive to Mach number when $M_o = \frac{1}{\sqrt{\gamma}} = 0.84515$. Also velocity and Mach number decrease if the initial Mach number is less than $\frac{1}{\sqrt{\gamma}}$ and increase if M_o is greater than $\frac{1}{\sqrt{\gamma}}$.



(a) Velocity ratio.

(b) Pressure ratio.

(c) Mach number.

(d) Current density ratio.

Figure 30.- Distributions of U, P, M, and j/j_o in a magnetohydrodynamic channel with B, σ, A, and T held constant.

Constant Mach Number

The current-density distribution may also be varied such that the Mach number of a constant-area channel will remain constant. This condition is attained by letting $\alpha j = \frac{(\gamma + 1)U}{\gamma M_o^2 + 1}$. The use of this value for αj in equation (72d) makes $dM = 0$; therefore, M becomes a constant. Also at $U = 1$, $j = j_o$, and hence $\alpha j_o = a = \frac{\gamma + 1}{\gamma M_o^2 + 1}$, and hence $j = j_o U$. The substitution of these relations for αj_o and j along with $M = M_o$ into equation (72a) results in the following relation between dU and dy:

$$\frac{dU}{U} = \frac{\gamma M_o^2}{\gamma M_o^2 + 1} dy \qquad (101)$$

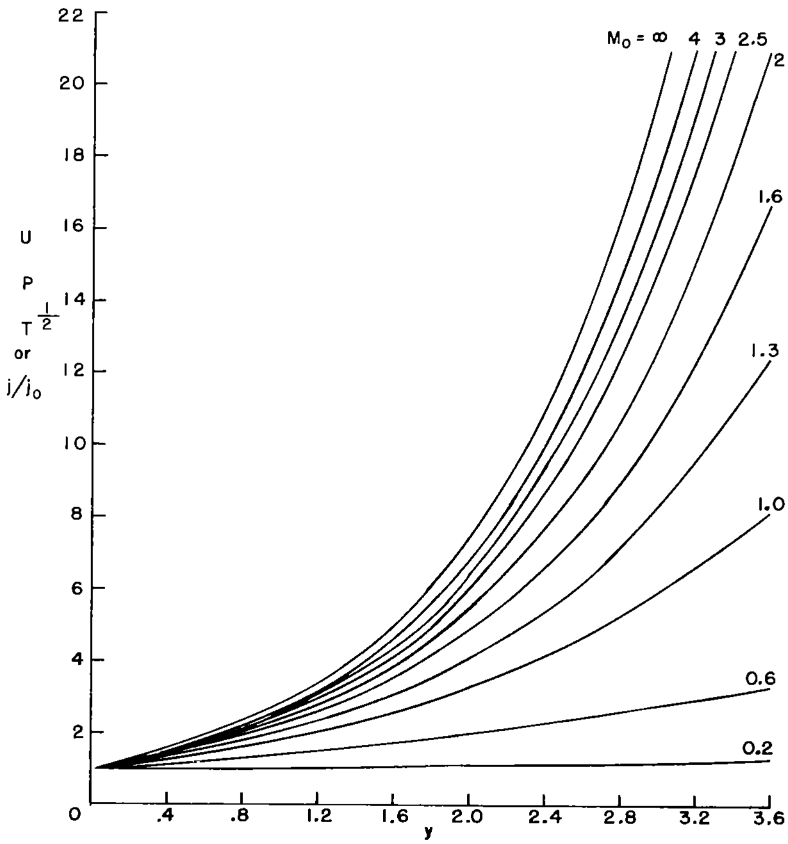


Figure 31.- Distribution of U, P, T, and j/j_o in a magnetohydrodynamic channel with B, σ, M, and A held constant.

Integration and evaluation of the constant gives

$$U = e^{\frac{\gamma M_o^2}{\gamma M_o^2+1}y} \tag{102}$$

The values of P and T may be determined from equations (73b) and (73c) and are

$$P = U = T^{1/2} = e^{\frac{\gamma M_o^2}{\gamma M_o^2+1}y} \tag{103}$$

Plots of the values of P, U, $T^{1/2}$, and j/j_o are presented in figure 31.

SUMMARY OF RESULTS

In an analysis of the equations representing the flow of an inviscid plasma through steady crossed electric and magnetic fields, the following phenomena were found to occur:

(1) The velocities were limited for certain modes of operation. These limitations were usually the result of the balancing of the body forces and the pressure gradients.

(2) The velocity was found to decrease for a number of operational conditions. In certain of the variable-area modes, this deceleration was sufficient to reduce the velocity to zero.

(3) Flow reversals were observed in constant-area channels when the Mach number became equal to 1. Such a situation is impossible; hence, the flow must readjust itself to new values.

(4) Rapid changes in the velocity may occur with small parametric changes in channels operated with parametric values near to those for constant velocity. The flow in such channels may therefore be unstable.

(5) The constant-pressure modes of operation exhibited consistent accelerations, a minimum of the irregular phenomena being found to occur in the constant-density modes of the constant-area modes.

(6) These observations indicate the care that must be exercised in the choice of an operational mode as well as of initial conditions if a desired result is to be obtained.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., May 27, 1964.

REFERENCES

1. Wood, George P., and Carter, Arlen F.: Considerations in the Design of a Steady DC Plasma Accelerator. Dynamics of Conducting Gases, Ali Bulent Cambel and John B. Fenn, eds., Northwestern Univ. Press (Evanston, Ill.), c.1960, pp. 201-212.

2. Wood, George P., Carter, Arlen F., Lintz, Hubert K., and Pennington, J. Byron: A Theoretical Treatment of the Steady-Flow, Linear, Crossed-Field, Direct-Current Plasma Accelerator for Inviscid, Adiabatic, Isothermal, Constant-Area Flow. NASA TR R-114, 1961.

3. Podolsky, B., and Borman, G.: The Electromagnetic Acceleration of a Continuously Flowing Plasma. Plasma Acceleration, Sidney W. Kash, ed., Stanford Univ. Press (Stanford, Calif.), 1960, pp. 12-29.

4. Eckert, A. C., and Weirick, R. G.: Investigation of a Maximum Mach Number Plasma Acceleration Channel. ASD Tech. Rep. 61-224, U.S. Air Force, May 26, 1961. (Available from ASTIA as AD No. 263444.)

5. Kerrebrock, Jack L., and Marble, Frank E.: Constant-Temperature Magneto-Gasdynamic Channel Flow. Jour. Aero/Space Sci. (Readers' Forum), vol. 27, no. 1, Jan. 1960, p. 78.

6. Hogan, William T.: An Experimental Investigation of a Magnetogasdynamic Accelerator. Magnetogasdynamics Lab. Rep. No. 62-1, Dept. Mech. Eng., M.I.T., Jan. 1962.

7. Barger, R. L., Brooks, J. D., and Beasley, W. D.: An Experimental Study of Continuous Plasma Flows Driven by a Confined Arc in a Transverse Magnetic Field. NASA TN D-716, 1961.

8. Erofeev, V. S., and Sanochkin, Yu. V.: Certain Questions in the Magnetohydrodynamics of Quasi-One-Dimensional Plasma Flow. II. Soviet Physics - Tech. Physics, vol. 8, no. 1, July 1963, pp. 49-53.

9. Russell, G. R., Byron, S., and Bortz, P. I.: Performance and Analysis of a Crossed-Field Accelerator. [Preprint] 63-005, American Inst. Aero. and Astronautics, Mar. 1963.

10. Cohen, Abraham: An Elementary Treatise on Differential Equations. Second ed., D. C. Heath and Co., 1933.

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Technical information generated in connection with a NASA contract or grant and released under NASA auspices.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

TECHNICAL REPRINTS: Information derived from NASA activities and initially published in the form of journal articles.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities but not necessarily reporting the results of individual NASA-programmed scientific efforts. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546