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ON THE CAPTURE, STABILITY, AND PASSIVE DAMPING OF ARTIFICIAL SATELLITES

by Ralph Pringle, Jr.

Prepared under Grant No. NsG-133-61 *by*
STANFORD UNIVERSITY
Stanford, Calif.
for

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ABSTRACT

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This dissertation treats analytically the dynamics of passively damped, gravity stabilized artificial satellites. Methods are devised to analyze the following phases of satellite evolution: (1) tumbling motions after launch, (2) capture into a bounded libration motion, (3) stability of equilibrium solutions, (4) damping of the libration motion. These methods are applied to three typical satellite designs of current importance: Vertistat, Beam, and TRAAC.

The results of the specific analyses of passively damped satellites show that the methods are capable of handling non-linear problems of considerable complexity. The methods tend to give much insight into the qualitative behavior of systems; they also provide useful numerical data with only hand computation.

Passive damping techniques are evaluated on the basis of the transient response of several specific designs. Certain conclusions about physical structure are drawn from the response formulae.

Author

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LIST OF SYMBOLS

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Chapter I

None

Chapter II

C	center of mass of all the particles
B_α	subsystem of particles ($\alpha = 1,2,\ldots,N$)
$\underline{R}^c$	vector radial distance from the center of attraction to C
$\underline{R}^\alpha$	vector distance from C to the center of mass of B_α
$\underline{\rho}^\alpha$	vector distance from center of mass of B_α to particle of B_α
$\underline{R}^{\alpha\beta}$	vector distance from C.M. of B_α to C.M. of B_β
$dm_\alpha(\underline{\rho}^\alpha)$	mass of particle at $\underline{\rho}^\alpha$ in B^α
m_α	mass of B_α: $m_\alpha = \int_{B_\alpha} dm_\alpha$
M	total mass of the system $\sum_{\alpha=0}^{N} m_\alpha$
$D(\underline{\rho}^\alpha)$	density of particles of body B_α: $dm_\alpha = Ddv$
$dv(\underline{\rho}^\alpha)$	volume increment of B_α at $\underline{\rho}^\alpha$
k	gravitational constant
$\hat{\hat{1}}$	unit dyadic (idemfactor)
T	kinetic energy
V	potential energy
$P_{\alpha\beta}$	reduced mass matrix (defined in (2.10) ($P^{-1}_{\alpha\beta}$, inverse of $P_{\alpha\beta}$)
$\mathbb{K}_{\alpha\beta}$	dyadic of inertia for the m_α
$\delta_{\alpha\beta}$	Kronecker delta $\delta_{\alpha\alpha} = 1$; $\delta_{\alpha\beta} = 0$ ($\alpha \neq \beta$)
$d\underline{F}^\alpha(\underline{\rho}^\alpha)$	force on particle at ρ^α in B_α

Chapter II (Cont)

$\underline{F}^{\alpha}$	total force on B_{α}
$\underline{L}$, $\underline{H}^{\alpha}$	angular momenta defined in (2.14)
$\underline{M}$	total moment on system
T^{α}	moment on B_{α}
$\underline{F}_g^{\alpha}$	gravity force on B_{α}
$\underline{F}_g$	total gravity force on the system
$\underline{T}_g^{\alpha}$	gravity torque on B_{α}
$\underline{M}_g$	total gravity torque on the system
$T_{applied}$	applied torques
$F_{applied}$	applied forces

Chapter III

$\underline{x}$	state vector with components $(p_1, p_2, \ldots, p_N;\ q_1, q_2, \ldots, q_N)$
$\underline{x}_o$	equilibrium point of $\underline{x}$ $(\underline{f}(\underline{x}_o) = \underline{0})$
$\lVert \underline{x} \rVert$	norm of $\underline{x}$
p_i	components of generalized momentum vector
q_i	components of generalized coordinate vector
Q_i	generalized forces
H	Hamiltonian function
T	kinetic energy
V	potential energy
U	dynamic potential defined in (3.8)

Chapter III (Cont)

L	Lagrangian function ($L = T - V$)
E	total energy ($E = T + V$)
N	number of generalized coordinates (degrees of freedom)
$\underline{F}^k$	force on k^{th} particle of a system of particles expressed in an inertial reference frame
$\underline{R}^k$	position of k^{th} particle expressed in an inertial reference frame
$\alpha_{j\ell}, \beta_j, \gamma$	defined in (3.6)
T_n	part of T which is homogeneous of degree n in $\dot{q}_i$
$\underline{f}(\underline{x})$	vector function of $\underline{x}$
$\underline{\phi}(\underline{x}(t_o);t)$	a motion of $\underline{x}$, starting at $\underline{x} = \underline{x}(t_o)$
η, ϵ	small constants
R	region of space of description of mechanical system
R_ϵ	ϵ neighborhood of R
W, W_1, W_2	scalar functions
S	region of positive definiteness of H
S_c	ultimate region of convergence
S_c'	bound on S_c ($S_c' \subset S_c$)
P	power into the mechanical system ($P \triangleq \sum_{i=1}^{N} Q_i \dot{q}_i$)
h, ω_1, ω_2	parameters in (3.13)
a_{ij}	Hessian matrix of U

Chapter III (Cont)

D_j	j^{th} principle minor of a_{ij}
λ_j	j^{th} eigenvalue of a_{ij}

Chapter IV

τ	normalized time $(\tau = nt)$
n	orbit angular rate
$\frac{d}{d\tau}(\ )=(\ )'$	$\frac{d(\ )}{n\ dt}$, differentiation with respect to τ
γ	satellite pitch angle
θ	sphere pitch angle relative to the main body
Ω	θ' sphere angular rate
$\hat{1},\hat{2},\hat{3}$	principal axes of the satellite
I_1,I_2,I_3	principal moments of inertia about the yaw, roll and pitch axes
I	moment of inertia of the sphere about any axis
k,r,η,b,ω_o	parameters
λ	large perturbation parameter
ω	defined as $\gamma' \triangleq \lambda\omega$
H	Hamiltonian
$\omega_c\gamma_c$	variables at capture
$\bar{\Omega},\bar{\gamma},\bar{\theta},\bar{\omega}$	new "average" variables
D	parameter defined in (4.6) and (4.16)
α,β	parameters defined in (4.16)
N	$N = \lambda\bar{\omega}$, average angular rate of γ

Chapter IV (Cont)

N_o, N_c	N at $\tau = 0$ and at $\tau = \tau_c$
τ_c	time of capture
c	torque applied to vehicle by gas jets is $I_3 n^2 c$
ω_s	angular rate of "synchronization"
τ_d	time constant of synchronization
$\delta N, \delta c$	small changes from synchronization conditions
c_s	torque at synchronization
P	system angular momentum defined in (4.25)

Chapter V

$\hat{1}, \hat{2}, \hat{3}$	unit vectors of rotating reference frame
$\hat{1}b, \hat{2}b, \hat{3}b$	principal axis reference frame
θ_1, θ_2, ψ	Euler angles for Case I
θ, ϕ, ψ	Euler angles for Case II
T,V	kinetic and potential energy
A,C	moments of inertia about $\hat{1}b$ and $\hat{3}b$, respectively
r	C/A
ℓ	p_ψ / C
p_ψ	angular momentum about $\hat{3}b$ axis
n	orbit angular rate
H	Hamiltonian
H_o	Hamiltonian at $t = 0$

Chapter V (Cont)

R_2	part of H, quadratic in the angular velocities
U	dynamic potential function
$\theta_o,\emptyset_o,\theta_{10},\theta_{20}$	equilibrium points of $\theta,\emptyset,\ \theta_1,\theta_2$
$\mathcal{H}_1,\mathcal{H}_2$	Hessian matrices for Cases I and II
$\lvert\mathcal{H}\rvert$	determinant of $\mathcal{H}$
$\lambda_1\lambda_2$	eigenvalues of $\mathcal{H}$
$U_{q_1},U_{q_1q_2}$	first and second partial derivatives of U with respect to q_1,q_2
x,y	projections of the unit sphere on a plane
$\theta_s,\emptyset_s$	defined in Section E

Chapter VI

t	time variable
$q_j,\bar{q}_j$	a generalized coordinate and its complex representation
Q_j	a generalized force component
a_{ij},b_{ij},c_{ij}	coefficients in equations of motion
N	number of degrees of freedom of problem
H	Hamiltonian
E	total energy
P	power into mechanical system due to damping
A_j	amplitude of an oscillation of the form $e^{i\omega_j t}$
λ	complex frequency variable

Chapter VI (Cont)

$M_{ij}(\lambda)$	matrix defined in (6.4)
ω_j	real frequency variable
A_{jk}	defined in (6.5), (6.6)
Λ_k, ψ_k	defined in (6.5)
$\overline{f(q)}$	average of $f(g)$ holding ψ_k, Λ_k constant
f^*	complex conjugate of f
$\alpha_{ij}, \beta_{ijk}, \gamma_{jk}$	defined in (6.10)
H_k	defined in (6.8)
$F_k, G_{jk}, J_{jk\ell}$	defined in (6.11)
W	defined in (6.12)
$g_n(\Lambda, t)$	defined in (6.13)
ω_o^2	natural frequency of Example A, see (6.16)
α, b, c	coefficients in power and force expressions of A, A, see (6.17)
q_{max}	maximum oscillation of q in Example A, see (6.18)
ω_1, ω_2, h	parameters in Example B, see (6.22)
Ω_1, Ω_2	natural frequencies of B, see (6.23)
D_1, D_2	coefficients of damping in (6.29)
S_1, S_2, μ, R_1, R_2	defined in (6.34)

Chapter VI (Cont)

$\Lambda_1^o, \Lambda_2^o, \Lambda_1^1, \Lambda_2^1$	zero and first-order coefficients in the expansion of Λ_1, Λ_2 see (6.36)
τ	"time constant": time for $\Lambda_1(t)/\Lambda_1(o)$ to reach $1/e$

Chapter VII:

T_{RB}^a, T_{RB}^b	kinetic energy of rigid body motion of bodies "a" and "b"
V_{RB}^a, V_{RB}^b	potential energy of rigid body motion of bodies "a" and "b"
ω^a, ω^b	angular velocity of bodies "a" and "b" with respect to inertial space
k	gravitational constant
Π^a, Π^b	inertia dyadic of bodies "a" and "b"
$\gamma_1^a, \gamma_2^a, \gamma_3^a, \gamma_1^b, \gamma_2^b, \gamma_3^b$	Euler angles of bodies "a" and "b". Rotations about $\hat{1}b$, $\hat{2}b$, $\hat{3}b$ axes for small angles
$\hat{1}, \hat{2}, \hat{3}$	reference axes
$\hat{1}a, \hat{2}a, \hat{3}a, \hat{1}b, \hat{2}b, \hat{3}b$	unit vectors along principal axes of bodies "a" and "b"
n	mean orbit angular velocity

Further lists of symbols in the text on pages 95, 99, 100, 120, 121, 134, 135

Chapter VIII

None

CHAPTER I: INTRODUCTION

A. PASSIVE SATELLITE ATTITUDE CONTROL

In the design of spacecraft there is considerable need for attitude controls which are very reliable and simple with a high probability of long life, but which do not necessarily control the attitude to a high accuracy. This requirement can often be satisfied by mechanical dampers that do not involve man-made energy storage, generation, or conversion devices; these are called passive satellite attitude control systems. This dissertation is in large measure directed toward a better understanding of the dynamics of such systems.

There are many kinds of passively-stabilized space vehicles, including, spin-stabilized vehicles, gravity-gradient stabilized satellites, and magnetic-field-oriented satellites. In each of these a physical phenomenon gives position stabilization which, when coupled with a damping device, leads to convergent, stable motions. The various damping devices adopted usually depend upon the dissipation of energy, either due to friction between moving parts or to the interaction of a field (e.g., magnetic) with the satellite.

This dissertation will focus attention upon the motions of the very useful gravity-gradient stabilized satellites and upon the methods of damping their motions. These satellites tend always to point the same axis toward the center of the earth; the same physical effect causes the Moon always to turn the same face toward the Earth.*

The history of investigation of gravitational effects upon a rigid body dates to Newton, who first realized that both the "precession of the equinoxes" and the attitude motion of the Moon (lunar librations) could be explained on the basis of the gravity-gradient torques exerted on a body in the natural gravity field. The problem of rigid body

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* The gravitational effect on bodies can be understood by considering a single rigid body in a Newtonian force field and noticing that separate particles of the body experience different forces (because of their differing distances from the center of attraction); this causes torques on the body and these torques may be used for the stabilization of the body's attitude.

motion under the influence of gravity torques has been well known since Newton (TISSERAND 1, LAGRANGE 2, ROUTH 1, PLUMMER 1) but the investigations were of a very special nature, dealing with a small class of bodies performing motions of a type known by prior observations. The objective of most of the ancient investigations of attitude motion was to verify the universal law of gravitation with high accuracy; those calculations, therefore, tended to be characterized more by precision than by the diversity of phenomena studied qualitatively.

Modern investigations are more concerned with systems consisting of several connected bodies, performing much more general motions which only need be predicted approximately. The history of the various developments of satellite attitude control devices until 1962 is outlined in the dissertation of D. B. DeBra (DEBRA 1) and in the WADD report edited by R. E. Roberson (ROBERSON 1).

The first known analysis of a passive control device using externally joined or connected bodies is that of J. V. Breakwell (BREAKWELL 1), who by early 1954 had analyzed the attitude motions in orbit of a pair of rigid bodies connected together by a torsion bearing having viscous friction. This particular design was called the "hinged satellite" and was patented by Breakwell and R. E. Roberson. The design is certainly similar to those vehicles now called "vertistats." Breakwell analyzed the transient motion and the forced motion excited by a nearly circular orbit; he found that the motions could be damped under certain stability conditions and that, for certain settings of the parameters, "vibration absorption" could be achieved in such a manner that the "main body" would stand still while the second body moved in response to the orbit excitation. This analysis, which has not been published in the literature, stimulated parts of the present work.

Since about 1959 there has been considerable effort directed toward the problem of passive gravity stabilization using connected systems of bodies. One of the earliest investigations in this period was that of R. E. Fischell, et al., at the Applied Physics Laboratory of the Johns Hopkins University (FISCHELL 1, FISCHELL 2, FISCHELL 3). Fischell developed two designs which were both orbited. These use a limber spring for damping (elastic hysteresis) as well as ferromagnetic damping rods. Kamm (KAMM 1) suggested the "vertistat" design which was later investigated by Tinling and Merrick (TINLING 1) and by Bell Telephone Laboratories engineers (FLETCHER 1, PAUL 1). The "vertistat" design, similar to Breakwell's "hinged satellite," looks quite promising. These designs and others are discussed in detail in Chapter VII.

B. PROBLEMS REQUIRING DYNAMIC ANALYSIS

One convenient way of illuminating the dynamical problems in designing a passive attitude control system is to consider the evolution of a particular satellite from launch to death. This evolution takes place in the following phases: (1) launch and separation of the satellite from the final stage booster rocket, usually in tumbling motion, (2) decay of the tumbling motion either by internal dissipation or by special momentum removal devices, (3) "acquisition" or "capture" of the libration region -- the beginning of librations, (4) damping of the librations to a stable equilibrium point, (5) steady-state motion about equilibrium in response to torque and force disturbances (occasional tumbling due to meteoroid impacts), (6) death of the satellite because of component failure or orbit decay. Phases (2)-(5) each imply a need for particular techniques for analysis of the salient features of the motion.

The initial angular rate after launch may be such that the satellite tumbles, i.e., one or more of the body angular rates has a constant or slowly-decaying component. The vehicle design may incorporate some special scheme for damping the tumbling motions or may employ the passive attitude control system for this purpose. If the latter method or a combination of the two methods is adopted, there arises a problem of analysis, whether or not a computer is used, because the equations of motion are non-linear and the machine times for direct integration can easily become excessively long. A method of perturbation analysis is developed in Chapter IV using the "high" pitch tumbling rate approximation as a generating solution. This leads to some interesting results and begins an essentially new line of analysis in this phase of satellite dynamics.

The libration motion after "capture" is analyzed in Chapter VI using a light-damping approximation which is quite valid for passive attitude control systems of interest. This approximation allows use of an energy method which handles non-linear damping forces. The method is especially convenient and effective for approximating the effects of elastic or magnetic hysteresis forces.

The stability of the equilibrium points of the equations of motion is investigated in Chapter III by employing Lyapunov's direct method. This application leads to a method of determining asymptotic stability for mechanical systems with damping and gyroscopic forces. The regions of "capture" are also rigorously established using a technique based upon Lyapunov's method.

The effectiveness of an attitude control system in steady-state oscillation is determined by a small-oscillation analysis of forced motion due to disturbances. In case the disturbances are "large" the linear methods break down and possible instabilities may occur, giving rise to large excursions away from equilibrium. A large-oscillation analysis of the equations of motion of a rigid body in an eccentric orbit about an oblate body was carried out by DeBra (DEBRA 1) using digital quadrature. At the present time the only effective technique for investigating the large-amplitude oscillations of satellites seems to be digital quadrature.

Connected vehicles, i.e., vehicles with rigid or elastic members joined to a main body, are quite useful for obtaining stabilization and damping of gravity-gradient satellites. Chapter II discusses methods of writing the equations of motion and the energy expressions in a very convenient manner, using general vector expressions encompassing arbitrary systems of bodies with gravity forces acting. The analysis of several important specific connected satellites using passive stabilization is carried out in Chapter VII. The methods outlined above are used to advantage in the analysis of the examples.

C. CONTRIBUTIONS

In this dissertation the author will present an exposition of some significant problems affecting satellite attitude control using passive, gravity stabilization and achieving damping by means of connected systems of bodies. New methods of analysis will be presented to cope with certain of the problems. There are a number of essentially novel examples which both illustrate the analytical methods and lead to conclusions of engineering significance. It is hoped that a proper mixture of general techniques and examples will expose the salient features of the problem of passive attitude control in an effective way.

The following are the major contributions of the dissertation:

1. A discussion of the dynamics of satellites consisting of connected bodies is given. General vector expressions for the equations of motion are presented, and the energy relations for arbitrary satellite systems of bodies under the action of gravity forces are given. (Chapter II)

2. Examples of connected, passive, gravity-stabilized satellites are analyzed to determine their usefulness and properties, and also to illustrate certain analysis procedures. An example of a flexible body under gravity forces is given. (Chapter VII)

3. New methods of analysis are devised and applied to the above examples. These methods include: (a) a technique for approximating the motion of a tumbling satellite with internal dissipation, (Chapter IV), (b) a perturbation approximation for light non-linear damping based upon the undamped motions (Chapter VI), (c) an application of Lyapunov's stability theory to mechanical systems with damping, and a detailed discussion of the available methods for treating satellite stability and capture problems. (Chapter III)

4. A complete solution of the stability problem in the large for the symmetrical satellite under gravity forces is presented to illustrate the techniques of stability theory as they apply to satellite mechanics. (Chapter V)

CHAPTER II: EQUATIONS OF MOTION AND ENERGY THEOREMS

A. INTRODUCTION

This chapter contains general theorems useful in the mathematical description of satellite motion. These theorems place the general equations of particle motion in a form convenient for describing systems of bodies.

There are two distinct methods of deriving the equations of motion of a mechanical system: (a) the vector method based directly upon Newton's laws of motion and (b) the Lagrangian or energy method based on the principle of virtual work or Hamilton's principle. Each of these methods finds application in this work, but usually the Lagrangian method is preferred because of the automatic elimination of holonomic constraints. Often, however, the ease of conceptualization and manipulation of the vector method makes it superior. Vectors are used to great advantage in this chapter for both the Newtonian equations of motion and the energy expressions.

B. THEOREMS ON SYSTEMS OF RIGID BODIES: ENERGY THEOREMS

Because the use of Lagrange's equations implies a need for the kinetic and potential energy expressions, the energy expressions will be derived here for systems of bodies in a gravitational field. Two forms of these expressions will be given, one in which energy is expressed in terms of vectors from the system center of mass to body centers of mass, and one in which energy is expressed in terms of vectors from the center of mass of a "main" body to other body centers of mass. The latter expressions are very useful in subsequent applications.

Consider a system of particles with center of mass at a point C described by a vector $\underline{R}^c$ with respect to a Newtonian reference frame $\mathcal{N}$ as in Fig. 2.1. This system of particles is split arbitrarily into sub-systems B_α $(\alpha = 0,1,\ldots,N)$. Each sub-system, B_α, has a center of mass described by a vector $\underline{R}^\alpha$ originating at C. The particles of B_α are of mass $dm_\alpha(\underline{\rho}^\alpha)$ located by a vector $\underline{\rho}^\alpha$ with respect to the center of mass of B_α.

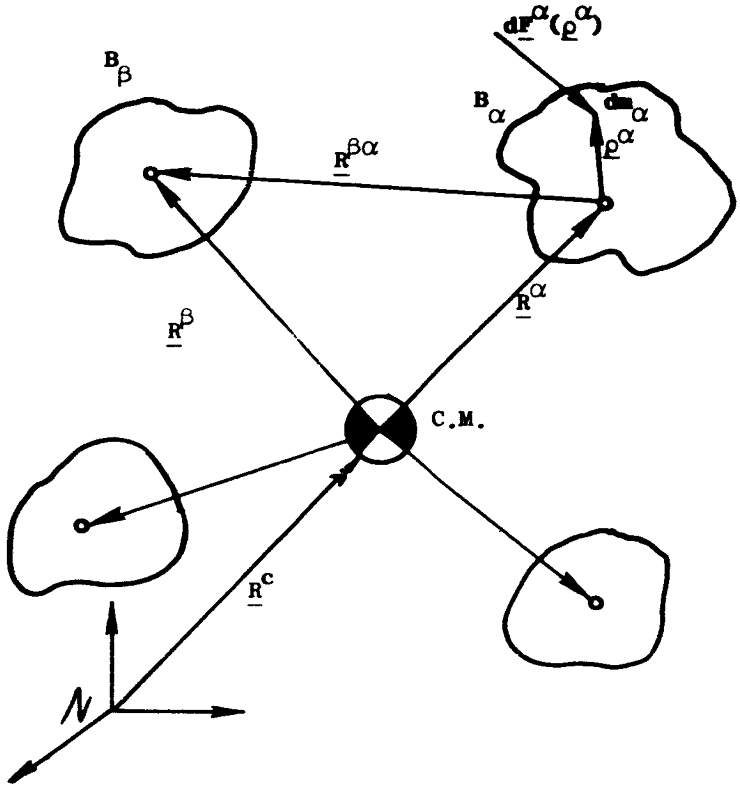

FIG. 2.1. GEOMETRY OF A SYSTEM OF PARTICLES

These facts are summarized by relations defining the centers of mass:

$$C: \quad \sum_{\alpha} \underline{R}^{\alpha} m_{\alpha} = 0$$

$$B_{\alpha}: \quad \int_{B_{\alpha}} dm_{\alpha} \, \underline{\rho}^{\alpha} = 0 \tag{2.1}$$

where

$$M = \sum_{\alpha} \int_{B_{\alpha}} dm_{\alpha} = \sum_{\alpha} m_{\alpha}$$

$$\underline{R}^{\alpha\beta} = \underline{R}^{\alpha} - \underline{R}^{\beta} \tag{2.2}$$

The integrals are to be taken in the Riemann-Stieltjes sense. If $m_{\alpha}(\underline{\rho}^{\alpha})$ is differentiable then dm_{α} can be expressed as $D(\underline{\rho}^{\alpha})dv(\underline{\rho}^{\alpha})$ where $D(\underline{\rho}^{\alpha})$ is the density at $\underline{\rho}^{\alpha}$ and dv is the volume element at $\underline{\rho}^{\alpha}$.

We may now write down the kinetic energy expression

$$T = \frac{1}{2} \sum_{\alpha} \int_{B_{\alpha}} \left\{ \dot{\underline{R}}^{c} + \dot{\underline{R}}^{\alpha} + \dot{\underline{\rho}}^{\alpha} \right\}^{2} dm_{\alpha}(\underline{\rho}^{\alpha})$$

$$T = \frac{M}{2} (\dot{\underline{R}}^{c})^{2} + \frac{1}{2} \sum_{\alpha} m_{\alpha} (\dot{\underline{R}}^{\alpha})^{2} + \frac{1}{2} \sum_{\alpha} \int_{B_{\alpha}} (\dot{\underline{\rho}}^{\alpha})^{2} dm_{\alpha} \tag{2.3}$$

Here $\underline{R}^{c}$ is the vector distance from the origin of $\mathcal{N}$ to the point C. The dots denote differentiation with respect to time in inertial space ($\mathcal{N}$-frame). The second of (2.3) follows from the first using the definitions of the centers of mass given by (2.1) and (2.2).

It can easily be shown that the gravity potential energy of the entire system of particles is

$$V = - \sum_{\alpha} \int_{B_\alpha} \frac{k \, dm_\alpha(\underline{\rho}^\alpha)}{|\underline{R}^c + \underline{R}^\alpha + \underline{\rho}^\alpha|} \tag{2.4}$$

Expanding the denominator of (2.4), keeping terms up to order $(\underline{\rho}^\alpha/|\underline{R}^c|)^2$, we get

$$\begin{aligned} V = &- \frac{kM}{|\underline{R}^c|} + \frac{k}{|\underline{R}^c|^3} \left\{ - \frac{1}{2} \operatorname{tr} \mathbb{A} + \frac{3}{2} \hat{R}^c \cdot \mathbb{A} \cdot \hat{R}^c \right\} \\ &+ \frac{k}{|\underline{R}^c|^3} \sum_{\alpha} \left\{ - \frac{1}{2} \operatorname{tr} \mathbb{I}^\alpha + \frac{3}{2} \hat{R}^c \cdot \mathbb{I}^\alpha \cdot \hat{R}^c \right\} \end{aligned} \tag{2.5}$$

where the dyadics $\mathbb{A}$ and $\mathbb{I}^\alpha$ are defined as

$$\begin{aligned} \mathbb{A} &= \sum_{\alpha} m_\alpha \left[(\underline{R}^\alpha)^2 \, \hat{\hat{1}} - \underline{R}^\alpha \, \underline{R}^\alpha \right] \\ \mathbb{I}^\alpha &= \int_{B_\alpha} dm_\alpha \left[(\underline{\rho}^\alpha)^2 \, \hat{\hat{1}} - \underline{\rho}^\alpha \, \underline{\rho}^\alpha \right] \end{aligned} \tag{2.6}$$

These are the moments of inertia which will be important throughout this discussion. Here the notation $\hat{R}^c$ means the unit vector $\underline{R}^c/|\underline{R}^c|$, tr means the trace of the dyadic following and $\hat{\hat{1}}$ is the unit dyadic or idemfactor. (The theory of vectors and dyadics is given in GIBBS 1.)

The above expressions for T and V refer to the vectors $\underline{R}^\alpha$ and therefore describe the system most naturally in coordinates relative to C. It is often useful to use coordinate vectors with their origin at one particular body. Call this body B_o and the location of its center of mass, $\underline{R}^o$. It can easily be seen using (2.1), (2.2), (2.3), (2.5), etc., that

$$T = \frac{M}{2}(\dot{\underline{R}}^c)^2 + \frac{1}{2}\sum_{\alpha}\int_{B_\alpha}(\dot{\underline{\rho}}^\alpha)^2\, dm_\alpha + \frac{1}{2}\sum_{\alpha,\beta} P_{\alpha\beta}\, \dot{\underline{R}}^{\alpha 0}\cdot \dot{\underline{R}}^{\beta 0} \tag{2.7}$$

$$\begin{aligned} V = -\frac{kM}{|\underline{R}^c|} &+ \frac{k}{|\underline{R}^c|^3}\sum_{\alpha}\left\{-\frac{1}{2}\,\mathrm{tr}\,\Pi^\alpha + \frac{3}{2}\hat{R}^c\cdot \Pi^\alpha \cdot \hat{R}^c\right\} \\ &+ \frac{k}{|\underline{R}^c|^3}\sum_{\alpha,\beta} P_{\alpha\beta}\left\{-\frac{1}{2}\,\mathrm{tr}\,\mathbb{R}_{\alpha\beta} + \frac{3}{2}\hat{R}^c\cdot \mathbb{R}_{\alpha\beta}\cdot \hat{R}^c\right\} \end{aligned} \tag{2.8}$$

where the tensor $\mathbb{R}_{\alpha\beta}$ is defined as

$$\mathbb{R}_{\alpha\beta} = \underline{R}^{\alpha 0}\cdot \underline{R}^{\beta 0}\,\hat{\hat{1}} - \underline{R}^{\alpha 0}\,\underline{R}^{\beta 0} \tag{2.9}$$

and the matrix $P_{\alpha\beta}$, called the "reduced mass matrix," is defined as

$$\begin{aligned} P_{\alpha\beta} &= m_\alpha \delta_{\alpha\beta} - \frac{m_\alpha m_\beta}{M} \\ \delta_{\alpha\beta} &= 1(\alpha = \beta) \\ &= 0(\alpha \neq \beta) \end{aligned} \tag{2.10}$$

It is seen, then, that the expressions (2.7) and (2.8) constitute another way to write the equations of T and V, which is useful in certain cases.

C. THEOREMS ON SYSTEMS OF RIGID BODIES: VECTOR EQUATIONS OF MOTION

The equations of motion of a system of rigid bodies may usefully be written directly from considerations based on Newton's second law of motion. In this section equations are written using the two forms of expression given in the preceding discussion of energy.

It should be pointed out that forces of constraint must be included in the equations of motion and eliminated algebraically. This is not so in the Lagrangian method for holonomic constraints (LANCZOS 1, pp. 80-86).

Using the definitions of (2.1) and (2.2) it is desired to describe the dynamics of a system of orbiting rigid bodies in terms of vector equations of motion derived from Newton's second law of motion for each particle which is, for a mass dm_α and force $d\underline{F}^\alpha$,

$$dm_\alpha(\ddot{\underline{R}}^c + \ddot{\underline{R}}^\alpha + \ddot{\underline{\rho}}^\alpha) = d\underline{F}^\alpha(\underline{\rho}^\alpha) \tag{2.11}$$

If we sum over all particles and bodies, we get, by the definitions of centers of mass,

$$M\ddot{\underline{R}}^c = \underline{F} = \sum_\alpha \int_{B_\alpha} d\underline{F}^\alpha \tag{2.12}$$

This gives the center of mass motion or the orbit motion; there are three scalar equations implicit in (2.12).

Integrating (2.11) over B_α only, gives

$$\begin{aligned} m_\alpha(\ddot{\underline{R}}^c + \ddot{\underline{R}}^\alpha) &= \underline{F}^\alpha = \int_{B_\alpha} d\underline{F}^\alpha \\ m_\alpha \ddot{\underline{R}}^\alpha &= \underline{F}^\alpha - \frac{m_\alpha \underline{F}}{M} \end{aligned} \tag{2.13}$$

There are $N+1$ bodies and therefore (2.13) represents $3N+3$ scalar equations of motion.

Taking moments around C, we get, after summing over all particles,

$$\sum_\alpha \int_{B_\alpha} dm_\alpha(\underline{R}^\alpha + \underline{\rho}^\alpha) \times (\ddot{\underline{R}}^c + \ddot{\underline{R}}^\alpha + \ddot{\underline{\rho}}^\alpha)$$

$$= \sum_\alpha \int_{B_\alpha} (\underline{R}^\alpha + \underline{\rho}^\alpha) \times d\underline{F}^\alpha$$

Using the definitions of the centers of mass we get

$$\sum_{\alpha} \dot{\underline{H}}^{\alpha} + \dot{\underline{L}} = \underline{M} \tag{2.14}$$

where

$$\underline{H}^{\alpha} = \int_{B_\alpha} \underline{\rho}^{\alpha} \times \dot{\underline{\rho}}^{\alpha} \, dm_{\alpha}$$

$$\underline{L} = \sum_{\alpha} m_{\alpha} \underline{R}^{\alpha} \times \dot{\underline{R}}^{\alpha}$$

$$\underline{M} = \sum_{\alpha} \int_{B_\alpha} \underline{\rho}^{\alpha} \times d\underline{F}^{\alpha} + \sum_{\alpha} \underline{R}^{\alpha} \times d\underline{F}^{\alpha}$$

Vector equation (2.14) represents three scalar equations.

The remaining equations are obtained by taking moments about the center of the mass of each body giving

$$\dot{\underline{H}}^{\alpha} = \underline{T}^{\alpha} = \int_{B_\alpha} \underline{\rho}^{\alpha} \times d\underline{F}^{\alpha} \tag{2.15}$$

Equations (2.12-15) give us $6(N+1) + 3$ equations for $N+1$ rigid bodies (and/or particles).

It remains to calculate the expressions implied in the above equations for vectors $\underline{R}^{\alpha} = \underline{R}^{\alpha 0} + \underline{R}^{0}$. This process, using the matrix $P_{\alpha\beta}$, gives

$$\sum_{\beta} P_{\alpha\beta} \ddot{\underline{R}}^{\beta 0} = \underline{F}^{\alpha} - \frac{\underline{F}\, m_{\alpha}}{M} \tag{2.16}$$

or

$$\ddot{\underline{R}}^{\alpha 0} = \sum_{\alpha} P^{-1}_{\alpha\beta} \left(\underline{F}^{\beta} - \frac{m_{\beta}}{M} \underline{F}^{\beta}\right) \tag{2.17}$$

where

$$\sum_{\alpha,\beta} P_{\alpha\beta} P^{-1}_{\beta\gamma} = \delta_{\alpha\gamma}$$

The vector $\underline{R}^{\alpha 0}$ is the radius vector from the center of mass of B_o to the center of mass of B_α. Equation (2.17) is more convenient for numerical integration and only requires one inversion of an N x N matrix $P_{\alpha\beta}$. Also using $P_{\alpha\beta}$ we may write

$$\underline{L} = \sum_{\alpha,\beta} P_{\alpha\beta}\, \underline{R}^{\alpha 0} \times \dot{\underline{R}}^{\beta 0} \tag{2.18}$$

For the cases where gravity forces and torques are important we may calculate these effects on the bodies. For each particle we have a force

$$d\underline{F}^{\alpha} = - \frac{k m_\alpha [\underline{R}^c + \underline{R}^\alpha + \underline{\rho}^\alpha]}{\left|[\underline{R}^c + \underline{R}^\alpha + \underline{\rho}^\alpha]\right|^3} \tag{2.19}$$

using the binomial expansion of the denominator of (2.19) and keeping only terms up to order $(d/|\underline{R}^c|)^2$ (where d is the largest linear dimension of the system of particles and rigid bodies) we get for the various forces and torques

$$\underline{F}_g = - \frac{kM}{|\underline{R}^c|^3} \underline{R}^c$$

$$\underline{F}_g^{\alpha} - \frac{m_\alpha}{M} \underline{F}_g = - \frac{k}{|\underline{R}^c|^3} \sum_{\beta=0}^{N} P_{\alpha\beta} \underline{R}^{\beta 0} \cdot (\hat{\mathbb{1}} - 3\hat{R}^c \hat{R}^c)$$

$$\underline{T}_g^{\alpha} = \frac{3k}{|\underline{R}^c|^3} \hat{R}^c \times \mathbb{I}^{\alpha} \cdot \hat{R}^c \tag{2.20}$$

$$\underline{M}_g = \sum_{\alpha} \underline{T}_g^{\alpha} + \frac{3k}{|\underline{R}^c|^3} \hat{R}^c \times \mathbb{A} \cdot \hat{R}^c$$

$$= \sum_{\alpha} \underline{T}_g^{\alpha} + \frac{3k}{|\underline{R}^c|^3} \hat{R}^c \times \sum_{\alpha,\beta} P_{\alpha\beta} \mathbb{R}_{\alpha\beta} \cdot \hat{R}^c$$

We see that the subscripts "g" denote the gravity parts of the total forces and moments acting. We get the complete equations of motion by combining (2.20) with (2.12-18).

In summary the equations of motion of the system of bodies of Fig. 2.1 are (sums over α, $\beta,\gamma = 0,1,2,\ldots,N$):

Center of mass motion (3 equations),

$$\boxed{M\,\ddot{\underline{R}}^{c} = \sum_{\alpha} \underline{F}^{\alpha} = \underline{F}_{\text{applied}} - \frac{kM}{|\underline{R}^{c}|^{3}}\,\underline{R}^{c}} \tag{2.21a}$$

Moments about each body center of mass (3(n+1) equations),

$$\boxed{\dot{\underline{H}}^{\alpha} = \underline{T}^{\alpha} = \underline{T}^{\alpha}_{\text{applied}} + \frac{3k}{|\underline{R}^{c}|^{3}}\,\hat{R}^{c} \times \Pi^{\alpha} \cdot \hat{R}^{c}} \tag{2.21b}$$

Moments about composite center of mass (3 equations),

$$\boxed{\begin{aligned}\sum_{\alpha} \dot{\underline{H}}^{\alpha} + \dot{\underline{L}} = \underline{M} = \underline{M}_{\text{applied}} &+ \\ + \frac{3k}{|\underline{R}^{c}|^{3}}\,\hat{R}^{c} \times &\left[\sum_{\alpha} \Pi^{\alpha} + \sum_{\alpha,\beta} P_{\alpha\beta}\,\mathbb{R}_{\alpha\beta}\right] \cdot \hat{R}^{c}\end{aligned}} \tag{2.21c}$$

Relative motion of body center of mass with respect to main body (3 N equations),

$$\boxed{\begin{aligned}\ddot{\underline{R}}^{\alpha 0} = \sum_{\beta} P^{-1}_{\alpha\beta} &\left[\underline{F}^{\beta} - \frac{m_{\beta}}{M}\,\underline{F}\right]_{\text{applied}} \\ + \frac{k}{|\underline{R}^{c}|^{3}} \sum_{\beta,\gamma} &P^{-1}_{\alpha\beta}\,P_{\beta\gamma}\,\underline{R}^{\gamma 0} \cdot (3R^{c}R^{c} - \hat{\hat{1}})\end{aligned}} \tag{2.21d}$$

In total there must be at least $6(N+1)$ equations to describe the motion of $(N+1)$ bodies. We have, including all the previous equations, $6(N+1) + 3$ equations of motion. This means we can view three of these as redundant. This fact allows a freedom of choice of mathematical description; all the $6(N+1) + 3$ equations will be satisfied, however, if $6(N+1)$ equations are satisfied.

A. INTRODUCTION

The concept of stability is an intuitive one to most people who care to reflect upon it. Since it is quite intuitive, it might be expected that the history of stability theory would be old; this is, in fact, quite true. Lagrange observed that stability of a mechanical vibration can be determined by observing the behavior of the potential function near an equilibrium point. He deduced that a sufficient condition for stability of such a system is that the potential be minimum at equilibrium. Dirichlet (LAGRANGE 1) proved this theorem rigorously and Thomson and Tait (THOMSON 1) indicated an extension to systems with dissipation. Poincaré (POINCARÉ 1) used these theorems to discuss problems in celestial mechanics using his bifurcation theory. It remained, however, for Lyapunov (LYAPUNOV 1) to rigorously state and prove the main theorems of this energy approach to stability. His treatise appeared in 1895 and set the stage for more recent investigations by Russian and American mathematicians (MALKIN 1, CHETAYEV 1, KRASOVSKII 1, KALMAN 1, LEFSCHETZ 1). In summary, the main ideas of stability theory are intuitive and old, but their rigorous incorporation into mathematics was largely due to Lyapunov and Poincaré about 1900.

In recent years engineers in the automatic control systems field (KALMAN 1) have found the stability theorems of Lyapunov's direct (or second) method to be useful. Application of the method to actual problems has been impeded by the difficulty of constructing certain functions. This difficulty can be circumvented in mechanical systems by choosing the Hamiltonian function as a Lyapunov testing function. This is usually the same as choosing the total energy except in certain systems involving rotating or cyclic coordinates. It happens that these are the systems of interest in space dynamics problems and therefore a clear understanding of how easily to apply Lyapunov's direct method to the multiplicity of space dynamics problems has not been common.

This chapter attempts to give a clear exposition of the application of Lyapunov's direct method to mechanical problems. This includes a theorem on damped systems which is very useful and which leads to certain results in the automatic control of mechanical systems. The testing of a function for positive definiteness is discussed because it is crucial in application of the basic theorems. In this connection the bifurcation theory of Poincaré is introduced in a simplified form, a form which is nevertheless sufficient for the present purposes.

Examples of the application of the theory are given in connection with the material of Chapters IV, V, and VII.

B. THE EQUATIONS OF MOTION IN MECHANICS

The equations of motion of a mechanical system take a very special form, a form elegantly expressed by Hamilton's canonical equations. Let us describe the system by a state vector $\underline{x}$, valid in a region R. This state vector has components $(p_1,p_2,\ldots,p_N,q_1,q_2,\ldots,q_N)$ and describes the motions completely. If $\underline{x}$ is given at some initial time, then, given forces Q_i acting on the system, the motion of $\underline{x}(t)$ is determined uniquely for all subsequent instants.

The equations of Hamilton are

$$\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i} + Q_i \\ \dot{q}_i &= \frac{\partial H}{\partial p_i} \qquad (i = 1,2,3,\ldots,N) \end{aligned} \tag{3.1}$$

where $H = H(p,q,t)$ is a scalar called the Hamiltonian. If the system is described in terms of a kinetic energy, T, and a potential energy, V, the Hamiltonian is defined as

$$H = \sum_i p_i\dot{q}_i - L \tag{3.2}$$

where, $L = T - V$ is the Lagrangian. L is to be viewed as a function of q and $\dot{q}$ and $p_i = \partial L/\partial \dot{q}_i$. The space of p_i is called "momentum space" and the space of q_i is called "configuration space."

It is important to calculate the total time derivative of the Hamiltonian. This is from (3.1) and (3.2)

$$\dot{H} = \frac{\partial H}{\partial t} + \sum_{i=1}^{N} \left\{ \frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i \right\}$$

$$\boxed{\dot{H} = \frac{\partial H}{\partial t} + \sum_{i=1}^{N} Q_i \dot{q}_i} \tag{3.3}$$

The term $\partial H/\partial t$ is due to time varying parameters in H and the term $\sum_{i=1}^{N} Q_i \dot{q}_i$ is the power into the system by forces not included in the Hamiltonian. This second term can be written

$$\sum_i Q_i \dot{q}_i = \sum_{k,i} \underline{F}^k \cdot \frac{\partial \underline{R}^k}{\partial q_i} \dot{q}_i$$

where $\underline{F}^k$ is the force on the "k-th particle" and $\underline{R}^k(q,t)$ is the position vector of the "k-th particle" in inertial space coordinates. The energy theorem for the inertial space representation is $\dot{E} = \dot{T} + \dot{V} = \sum_k \underline{F}^k \cdot \dot{\underline{R}}^k$, where $\dot{\underline{R}}^k = \partial \underline{R}^k/\partial t + \sum_i (\partial \underline{R}^k/\partial q_i)\dot{q}_i$ is the velocity.

If the system is represented in terms of a coordinate system q_i that depends on $\underline{R}^k$ by a relation, $\underline{R}^k = \underline{R}^k(q,t)$, that is an explicit function of time, then we may write

$$\dot{E} - \dot{H} = \sum_k \underline{F}^k \cdot \frac{\partial \underline{R}^k}{\partial t} \tag{3.4}$$

and, integrating,

$$E - H = \int \sum_k \underline{F}^k \cdot \frac{\partial \underline{R}^k}{\partial t}\, dt \tag{3.5}$$

we see that $E - H$ is a non-zero function of time, in general. This means that when using generalized coordinates, q_i, to describe a system's configuration, we must distinguish between F and H.

C. THE NATURE OF THE HAMILTONIAN IN MECHANICS

It is important to investigate the properties of the Hamiltonian, H, in mechanical systems problems. Because certain momenta are constants of the motion $(\partial L/\partial q_i = 0;\ p_i = \partial L/\partial \dot{q}_i)$, and because we may use rotating coordinate frames, the kinetic energy often takes the form (Appendix A)

$$T = \frac{1}{2} \left\{ \sum_{j,\ell=1}^{N} \alpha_{j\ell} \dot{q}_j \dot{q}_\ell + \sum_{j=1}^{N} 2\beta_j \dot{q}_j + \gamma(q) \right\} \tag{3.6}$$

where the $\alpha_{j\ell}$, β_j, γ are functions of the q_j only and β_j, γ are "gyroscopic" terms. If we form H using (3.2) and (3.6), we get

$$H = \frac{1}{2} \sum_{j,\ell=1}^{N} \alpha_{j\ell} \dot{q}_j \dot{q}_\ell + \left[V(q) - \frac{\gamma}{2}(q) \right] \tag{3.7}$$

If we define the following terms

$$T_2 = \frac{1}{2} \sum_{j,\ell=1}^{N} \alpha_{j\ell} \dot{q}_j \dot{q}_\ell$$

$$T_1 = \sum_{j=1}^{N} \beta_j \dot{q}_j$$

$$T_0 = \frac{1}{2} \gamma(q)$$

(3.7) becomes $(T = T_2 + T_1 + T_0)$

$$H = T_2 + U \tag{3.8}$$

U is defined as $U \overset{\Delta}{=} V - T_0$ and is called the "dynamic potential."

It is important to note that

$$E - H = T_1 + 2T_0. \tag{3.9}$$

This means that there is a difference between E and H that varies in time for "gyroscopic" systems (see result of (3.5)).

D. THE DIRECT METHOD OF LYAPUNOV

We are concerned with stability about a point of equilibrium defined by the vector $\underline{x}_o$ and the equation of motion, $\underline{f}(\underline{x}_o) = 0$.*
This is a possible motion for the system of (3.1), and it is disturbances about this motion that we wish to investigate. It is proper to introduce the definitions of stability here and then discuss them later.

A system is said to be stable if it obeys (3.1) and whenever, given any $\epsilon > 0$, there is an $\eta(\epsilon; t_o) > 0$ such that if for a motion $\underline{\phi}(\underline{x}(t_o); t)$ we have $\|\underline{x}(t_o) - \underline{x}_o\| < \eta$ then $\|\underline{\phi}(\underline{x}(t_o); t) - \underline{x}_o\| < \epsilon$ for any $t > t_o$.**

This definition means that, given any subregion $R_\epsilon \subset R$, we can always find a small enough initial condition so that the solution, $\underline{\phi}$, remains in R_ϵ. If we cannot find such an initial condition the motion is said to be unstable. If the choice of $\eta(\epsilon; t_o)$ is independent of t_o, then we say the system is uniformly stable. The motion is said to be asymptotically stable if it is stable and if in addition $\|\underline{\phi}(\underline{x}(t_o); t) - \underline{x}_o\| \to 0$ as $t \to \infty$.

It is necessary to define a "positive definite" function. If a scalar function of the state vector, $W(\underline{x})$, obeys the following relations in a region S: (a) $W(\underline{x} = 0) = 0$, (b) $W(\underline{x}) > 0$ $(\|\underline{x}\| \neq 0)$

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*The equations of motion (3.1) are written $\dot{\underline{x}} = \underline{f}(\underline{x})$.

**$\underline{\phi}(\underline{x}(t_o); t)$ is a "motion," $\underline{x}$, of $\dot{\underline{x}} = \underline{f}(\underline{x})$ starting at $t = t_o$, $\underline{x} = \underline{x}(t_o) = \underline{\phi}(\underline{x}(t_o); t_o)$.

it is called positive definite in S. If it obeys (a) and if $W(\underline{x}) < 0$ $(\|\underline{x}\| \neq 0)$ it is called negative definite in S. If the function $W(\underline{x})$ changes sign in any neighborhood of $\underline{x} = 0$ and if $W(\underline{0}) = 0$ it is called sign variable. If (a) holds but (b) is $W(\underline{x}) \geq 0$ $(\|\underline{x}\| \neq 0)$ the function W is called positive semi-definite.

With the above definitions the main theorems of Lyapunov may be stated.

Theorem I. If for the differential equations and conditions (3.1) and (3.2) it is possible to find a scalar function $W(\underline{x})$, positive definite in $S \subset R$, whose total derivative with respect to time, $\dot{W}$, is negative semi-definite in S then the motion $\underline{\phi}(\underline{x}(t_o);t)$ is stable.

Theorem II. If for the differential equations and conditions (3.1) and (3.2) it is possible to find a scalar function $W_1(\underline{x})$, positive definite in $S \subset R$, whose total derivative with respect to time, $\dot{W}_1$ is negative definite in S then the motion $\underline{\phi}(\underline{x}(t_o);t)$ is uniformly asymptotically stable.

Theorem III. If for the differential equations and conditions (3.1) and (3.2) it is possible to find a scalar function $W_2(\underline{x})$ such that in R its total time derivative, $\dot{W}_2$, is negative definite and the function itself is sign variable or negative in R then the motion, $\underline{\phi}(\underline{x}(t_o);t)$ will be unstable.

The preceding discussion has been rather formal because it is necessary to take care in defining the concepts involved. The proofs of the above are to be found in Malkin (MALKIN 1). Malkin has an excellent geometrical discussion of the theorems. It will be seen that the functions W are energy-like functions. W_1 is called a Lyapunov function.

E. APPLICATIONS OF THE DIRECT METHOD TO MECHANICAL SYSTEMS

In the application of the direct method of Lyapunov to mechanical or Hamiltonian systems we make use of the Hamiltonian function as a possible Lyapunov function. This uses the intuitive idea of the Lyapunov function as an "energy-like" function. It turns out that H has nice properties that make it an ideal Lyapunov function for mechanical systems.

The equilibrium points of a mechanical system can be found by applying Lagrange's equations and setting the generalized velocities to zero (by definition of an equilibrium point). This gives the following conditions on the q_i.

$$\frac{\partial U}{\partial q_i} = 0 \qquad (i = 1,2,\ldots,N) \tag{3.10}$$

where $U = V - T_0$ as explained in Section C.
The subsequent analysis assumes that the above equations are satisfied by the condition $q_{io} = 0$ and that the system is stationary, i.e., $\partial H/\partial t = 0$.

1. Systems with Damping

For systems with damping we can make some strong and surprising statements regarding asymptotic stability. A system is said to have damping if the power into the mechanical parts, P, is negative definite, i.e., energy is always being lost except at equilibrium. This is the key assumption of the stability theory of mechanical systems with damping. Notice that we defined

$$P = \sum_{i=1}^{N} Q_i \dot{q}_i \ .$$

With the above statement as a hypothesis that is always fulfilled in well-designed systems (natural or man-made), we state:

Theorem IV. If for the (autonomous) mechanical systems described by the differential equations (3.1), the power, $P = \dot{H}$, is negative definite in a region S of the (p_i, q_i) space, then the motions are (a) asymptotically stable if $H(p,q)$ is positive definite in S or (b) unstable if $H(p,q)$ is sign variable or negative definite in S.

Part (a) of the theorem is proved using Theorem II and identifying $H = W_1$. Part (b) is likewise proved by letting $H = W_2$ in Theorem III.

It can now be clearly seen that the positive definite property of H is necessary and sufficient for asymptotic stability if P is negative definite.

It is therefore obvious that H is a suitable Lyapunov function and that stability depends upon the testing of H for positive definiteness. To prove H is positive definite we test it directly or notice the formula (3.10) which is

$$H = T_2 + U$$

This allows us to use the fact that T_2 is a positive definite quadratic form in the $\dot{q}_i$ based upon its definition via (3.6). The conditions for positive definiteness of H are then satisfied if and only if U is a positive definite function of the q_i. This reduces the theorem to a test of the "dynamic potential function," U, for positive definiteness in the q_i. The above reasoning can be stated as a corollary to Theorem IV.

Corollary: If the hypothesis of Theorem IV is satisfied, then the motions are (a) asymptotically stable if $U(q)$ is positive definite in the q_i in S or (b) unstable if $U(q)$ is sign variable or negative definite in S.

The proof is immediate from Theorem IV if it is observed that $H = T_2(q,\dot{q}) + U(q)$ where T_2 is a positive definite quadratic form in the $\dot{q}_i$. H is positive definite if and only if $U(q)$ is positive definite by virtue of the fact that if all the $\dot{q}_i$ are zero then T_2 is zero.

2. Systems without Damping

If $P = 0$ we call the system undamped. In this case the expression $\dot{H} = 0$ holds and the system is said to be conservative. If we take H as a possible Lyapunov function we see immediately that Theorem I gives the result that if H is positive definite then the system is stable. The theorem on instability (Theorem III) does not apply because of the conservative nature of the system. One may, however, construct a function W_2 and prove that if $T = T_2$ then the positive definiteness of H is necessary and sufficient for stability but, if the system is gyroscopic, only Theorem I can be applied. This is Chetaev's theorem (MALKIN 1, CHETAYEV 1). The application of Theorem I to the conservative case is called Lagrange's Theorem.

3. Bounds on the Convergence Region

While the theorems give us much useful information about stability, the nature of the definitions of stability make such answers of the "yes or no" variety; furthermore, the stability defined above is only "local" in nature. Another question one must ask is: How large is the region of convergence $S_c \subset S$? This is a difficult question, but we may establish useful lower bounds to S_c, i.e., establish sets $S'_c \subset S_c \subset S$ where we are sure to have convergence to the particular equilibrium point in question.

Consider the equation (3.3) where $P < 0$, and integrate it

$$\dot{H} = P$$

$$H(p(t), q(t)) = H(p(0), q(0)) + \int_0^t P \, dt \tag{3.11}$$

then clearly (let $H(t = 0) = H_o$, $H(p,q) = H$)

$$U \leq H \leq H_o \tag{3.12}$$

If we imagine closed, bounded surfaces of $H = H_o$ or $U = H_o$ and if U and H are positive, these surfaces will enclose the equilibrium point, p_{io}, q_{io}. The H surfaces will enclose the equilibrium point in the full p,q-space and the U surfaces will enclose the equilibrium in the q-space. These surfaces are bounds on the region of convergence

S_c, since if $H > H_o$ for some fixed H_o the inequality (3.12) is violated, leading to a contradiction. Usually the dimensionality and other practical considerations make it expedient to use U and limit ourselves to the configuration space. A consideration of the convex surfaces, U or H gives a bound, S'_c, on the convergence region which is pessimistic, but nevertheless useful, i.e., $S'_c \subset S_c$.

F. USE OF THE THEOREMS ON STABILITY

The hypothesis of Theorem IV is that the power, $\dot{H}$, is negative definite with respect to either (p_i, q_i) or $(\dot{q}_i, q_i)$. This hypothesis is not fulfilled in the majority of applications but it is equally effective for the power to be negative semi-definite if certain conditions are fulfilled. These conditions amount to having the equations of motion "coupled"; that is, every phase variable is dependent upon the phase variables contained in the power expression. A system which illustrates this question is the following "gyroscopic" system.

$$\ddot{q}_1 + \omega_1^2 q_1 + h\dot{q}_2 = 0$$

$$\ddot{q}_2 + \omega_2^2 q_2 - h\dot{q}_1 = -b\dot{q}_2 \tag{3.13}$$

$$H = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2)$$

$$\dot{H} = -b\dot{q}_2^2 \qquad b > 0$$

Here q_1, q_2 are the independent position variables and ω_1^2, ω_2^2, h, and b are parameters. Notice that the Hamiltonian, H, is positive definite for $\omega_1^2 > 0$, $\omega_2^2 > 0$, while $\dot{H}$ is only semi-definite, i.e., it goes to zero for phase points not at the origin. Notice further that if $h = 0$ the system is composed of two decoupled harmonic oscillators with damping in only one of them. Thus, in this case the q_1 coordinate does not damp down but continues to oscillate as a harmonic oscillator while the q_2 variable damps down. If $h \neq 0$ the system is "coupled" and both q_1 and q_2 decay to zero. These anomalous results are directly related to the semi-definiteness of $\dot{H}$. Kalman and Bertram (KALMAN 1) have covered the above case with a theorem (see also LEFSCHETZ 1).

Theorem V. The condition of the negative definiteness of P in Theorem IV may be replaced by (a) $\dot{H}(p,q) \leq 0$ for all p_i, q_i, and (b) $\dot{H}$ does not vanish identically for motion not at the origin of phase space (equilibrium point).

This means that if we set P identically equal to zero (equal to zero for all $t \geq t_o$) that it will imply a particular solution which is the equilibrium point and only this solution will be possible. In (3.13) we see that $P = \theta$ implies that $q_2 = 0$ and in the second differential equation (if $h \neq 0$) that q_1 is identically equal to zero; thus the origin is the only point in phase space where $\dot{H}$ vanishes identically. If $h = 0$ then this is not so and we see that the first differential equation holds regardless of the identical vanishing of $\dot{H}$. This modification to Theorem IV is quite important for applications of the theory of Section D and Section E to mechanical systems.

The above theorem has a corollary of considerable power and usefulness for the qualitative discussion of mechanical stability.

Corollary: If a system obeys the hypotheses of Theorems IV or V, its stability behavior cannot depend upon the magnitude or the analytical form of the power loss function, P.

The proof is immediate, if it is recognized that the test of Theorem V for stability reduces to examining the function H (or U) under the hypothesis of negative P. The function H, however, is determined by the mechanical part of the system and not by the damping mechanism. This last fact was shown in Section B, Part 1 when the expression for H was derived from the kinetic energy of the particles and the potential energy with no reference to the damping law. The damping law only enters in the generalized forces Q_i of (3.3) which only effect the power loss function P.

If a system of bodies, acted upon by forces derivable from a potential function, is to be investigated for stability on the assumption of a loss of energy due to damping which is dependent upon the motion of the bodies, then the stability cannot depend on the amount of damping present. The stability only depends upon whether or not the power function, P, is negative. One of the consequences of the above statements is that if we are faced with an unknown dissipation mechanism which always keeps $P \leq 0$, then we can be sure that the results of a test of $U(q)$ for positive definiteness is satisfactory for determining the stability behavior of the system.

G. TESTING U FOR POSITIVE DEFINITENESS -- BIFURCATION THEORY

As discussed in Section E, the stability of mechanical systems reduces to a "testing" of U for positive definiteness. This test is well known in the theory of quadratic forms and leads to a simple algebraic criterion for positive definiteness. Consider

$$U = \sum_{i,j=1}^{N} a_{ij} q_i q_j + U_3 + U_4 + \ldots \tag{3.14}$$

where $a_{ij} = \partial^2 U / \partial q_i \partial q_j$ and U_n is an n-th degree form, homogeneous in the q_i. For a vanishingly small displacement the positive definite property of U is equivalent to the matrix a_{ij} being positive definite (BELLMAN 1) or

$$D_1 = a_{ii} > 0 \qquad (i = 1,2,\ldots,N)$$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0 \tag{3.15}$$

$$D_j = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{ij} \\ a_{21} & & & \\ \vdots & & & \\ a_{j1} & & \cdots & a_{jj} \end{vmatrix} > 0$$

$$(j = 1,2,\ldots,N)$$

If $D_N = 0$, the matrix a_{ij} is singular; then the higher orders in U_n must be investigated.

The testing of U is interesting to discuss in another way , a way due to Poincaré. Consider a diagonalizing transformation which reduces a_{ij} to a sum of squares. If $\lambda_i (i = 1,2,\ldots,N)$ are eigenvalues of a_{ij} and if q_i' are components of a new coordinate frame, it is possible to reduce U_2 to the form

$$U_2 = \sum_{i=1}^{N} \lambda_i (q_i')^2$$

In this frame the determinant, D_N, of a_{ij} is left invariant (see BELLMAN 1).

$$D_N = \prod_{i=1}^{N} \lambda_i = \left| \frac{\partial^2 U}{\partial q_i \partial q_j} \right| \tag{3.16}$$

This relation (3.16) is extremely important in that it specifies the way in which the system stability (for $D_N \neq 0$) changes. If the λ_i are all positive, $D_N > 0$ and U_2 is positive definite. If any of the λ_i change sign this change of sign is reflected in D_N directly and it changes sign. Thus the examination of (3.16) evaluated at the equilibrium point in question suffices to determine the points in the parameter space and the space of q_{io} where stability changes occur. A point at which $D_N = 0$ is called a "point of bifurcation." This is extremely useful in many investigations of stability behavior via Lyapunov's techniques in mechanical systems and will be illustrated by Chapter V.

CHAPTER IV: ON THE TUMBLING MOTIONS AND CAPTURE OF AN ARTIFICIAL SATELLITE

A. INTRODUCTION

This chapter introduces a new class of problems involving the damping of the tumbling motions of artificial satellites. Simple examples of these problems are treated to indicate a general approach to the calculation of the motions and decay rate using perturbation theory. The use of perturbation methods may be necessary because of the extremely long time intervals of the solutions in comparison with the length of integration steps necessary for a digital integration of the equations of motion. Analog computation methods are of little use for the same reasons, i.e., the long times of solution necessary to obtain the desired solutions allow the amplifiers to drift and this causes large errors.

The system to be studied is a satellite with a tumbling rate in the orbit plane, of one or more times orbit rate, which is to be damped by internal friction. That is, the effect of the gravity torques on the system is to disturb internal moving parts and thus dissipate energy, causing a gradual slowing of the tumbling until the satellite is captured into the libration region.

If one considers an orbiting system of particles with internal damping, and if one neglects the gravity and other external torques, he may use the law of conservation of angular momentum to show that after a period of transient decay of the relative motion of particles, the general motion will be a tumbling motion with no relative motion between particles. The presence of gravity torques, however, invalidates the momentum conservation and produces ever-present relative motion between particles, and thus ever-present friction. This friction damps the tumbling motion; systems with internal friction cannot remain tumbling if only acted upon by gravity torques and forces. If the tumbling rate is high compared to orbit rate, the kinetic energy is large compared to changes in the potential energy of gravity, and the tumbling motion is only slightly affected over an orbit period. This explains the slow attenuation of the tumbling for high initial rates.

This problem is of interest to engineers because satellites may have initial conditions corresponding to tumbling motion when they are released from the booster rocket or when they are struck by a meteorite. The problem also has interest as an application of the theory of perturbations, i.e., the method of averaging. (Appendix A and BOGOLIUOV 1, MINORSKY 2.)

It has been shown by A. E. Sabroff of Space Technology Laboratories, in computer simulation studies, that the roll and yaw motions during tumbling decay rapidly, leaving the satellite tumbling only in the pitch (orbit) plane. For this reason the motion of principal concern is studied in this analysis, which treats only the case of pitch tumbling. However, the method used may be extended to apply to more general motions.

A solution is given for a simple problem of the class just described, namely, a tumbling satellite in the pitch plane (orbit plane) with an internal inertia sphere which is viscously coupled to the main body. The motions are given and, in particular, the decay of the average tumbling rate is calculated. It is shown that an optimum damping coefficient exists and that this optimum coefficient increases monotonically with the initial tumbling rate. It is also shown that the damping time increases sharply with the initial tumbling rate.

A modification of the above system is also studied; in this modification a linear spring is added to restrain the sphere in the pitch direction. The addition of the spring causes a more violent motion of the sphere as the tumbling rate reaches a certain "resonance" region; this increased oscillation in turn causes additional dissipation and faster decay. There exist optimum damping and spring constants for the problem. Addition of "resonances" may result in much-decreased decay times.

By addition to the spring-mass-damping system of a certain small, constant pitch torque, e.g., by gas jets, we may cause the satellite angular velocity of tumbling to "synchronize" at a certain angular velocity which is largely determined by the "resonance" frequency of the spring and sphere inertia. In this way one may regulate the tumbling rate without complex sensing equipment. The calculation done in connection with the "synchronization" effect serves also to evaluate the final tumbling rate in the case of gas leakage.

The results discussed above are obtained by using the asymptotic methods of approximation due to N. Krylov, N. Bogoliubov and Y. A. Mitropolsky (BOGOLIUBOV 1, MINORSKY 1, MINORSKY 2). The method of averaging (Appendix B) is used to eliminate the rapid but predictable oscillatory motions excited by the gravity torque. The end result of employing the averaging method is a set of non-linear differential equations for the averaged motion; the integration of these equations can be accomplished with much less effort than that necessary to integrate the complete equations of motion. If we use a digital computer the integrations can be accomplished in many fewer steps than a direct integration of the equations of motion.

B. EQUATIONS OF MOTION

Let us consider a rigid body moving in the orbit plane in a circular orbit about a spherical attracting body. Its attitude is described by an angle γ relative to the radius vector, Fig. 4.1. Within the satellite body is a homogeneous sphere mounted in a viscous fluid.* The angular velocity of the sphere relative to the satellite (and in the orbit plane) is $\dot{\theta}$. The torque due to the viscosity of the fluid is $-b\dot{\theta}$. The kinetic and potential energies of the system are then (see Chapter II)

$$\begin{aligned} T &= \frac{1}{2}\left[I_3(\dot{\gamma} + n)^2 + I(\dot{\gamma} + \dot{\theta} + n)^2\right] \\ V &= kn^2 I_3 \sin^2\gamma \end{aligned} \tag{4.1}$$

From (4.1) Lagrange's equations may be used to obtain the differential equations of motion:

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* This can also be viewed as an inertia wheel with its axis mounted along the pitch axis. "Viscous" in the sense intended here means any force or moment which is linearly dependent on velocity.

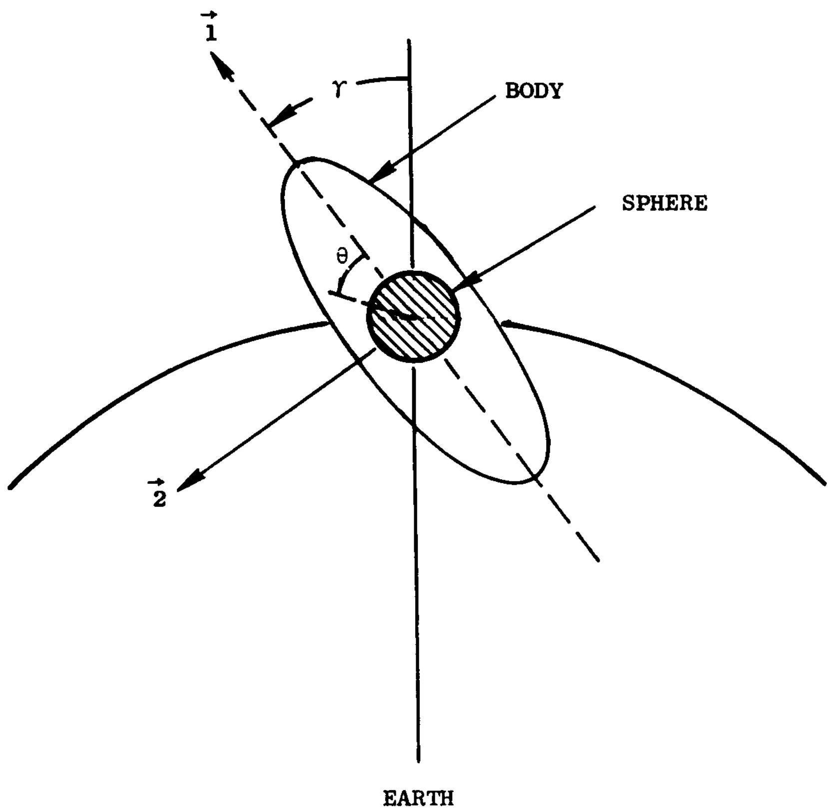

FIG. 4.1. COORDINATES OF TUMBLING VEHICLE

$$I_3\ddot{\gamma} + I(\ddot{\gamma} + \ddot{\theta}) + kn^2 I_3 \sin 2\gamma = 0$$

$$I(\ddot{\theta} + \ddot{\gamma}) + b\dot{\theta} = 0$$

With some normalization and algebraic manipulation the above equations may be written:

$$\gamma'' = r\eta\theta' - k \sin 2\gamma$$

$$\theta'' = -\eta\theta' + k \sin 2\gamma$$

The following notation has been used

$\tau = nt$ = normalized time

n = orbit angular rate (constant)

$\frac{d}{d\tau}(\ ) = (\ )' = \frac{1}{n} \frac{d}{dt}(\ )$

γ = satellite body pitch angle (angle from radius vector to $\hat{1}$ principal axis)

θ = sphere position relative to the satellite body (about pitch)

$\Omega \triangleq \theta'$ = sphere angular velocity relative to the satellite

$\hat{1}, \hat{2}, \hat{3}$ = principal axes of the satellite

I_1, I_2, I_3 = moments of inertia about the yaw, roll, and pitch principal axes respectively

I = moment of inertia of the sphere about any axis

$$k = \frac{3}{2}\left(\frac{I_2 - I_1}{I_3}\right)$$

$$\eta = \frac{b}{n}\left(\frac{I_3 + I}{II_3}\right)$$

$$r = \frac{I}{I + I_3}$$

Finally we introduce the new variables λ and ω by defining

$$\lambda\omega \triangleq \gamma'$$

the normalized tumbling rate. Furthermore, λ is defined to equal the final (capture) angular velocity, so that $\omega > 1$ is a variable. $1/\lambda$ will be used as a small parameter in perturbation series.

The differential equations of motion can now be written in first order form as:

$$\begin{aligned} \omega' &= \frac{1}{\lambda}\left[r\eta\Omega - k \sin 2\gamma\right] \\ \gamma' &= \lambda\omega \qquad\qquad (4.2) \\ \Omega' &= -\eta\Omega + k \sin 2\gamma \end{aligned}$$

C. CAPTURE

From the equations of motion (4.2), or directly from (4.1), we can write the Hamiltonian, H, an energy-like quantity, which is useful in solving the capture problem and in giving limits of validity to the perturbation series expansions. This function and its derivative are, using (4.1) and (3.3), (3.8)

$$H = \left[\frac{1}{2} \ \lambda^2\omega^2 + 2r\ \lambda\omega\Omega + r\Omega^2\right]$$

$$+ \ (1 - r)\ k\ \sin^2\gamma \tag{4.3}$$

$$H' = - \ \frac{b}{n}\left(\frac{\Omega^2}{I_3 + I}\right) < 0; \qquad (b > 0)$$

The integration of the second of (4.3) leads easily to the inequality (Chapter III):

$$H \leq H_o \tag{4.4}$$

where H_o is H of (4.3) evaluated at the initial instant.

Physically H' is negative and H decreases during the tumbling motion. When H reaches a value $(1 - r)k$ then the subsequent motion will be bounded. This means that in less than one tumble capture will occur.

If $r \ll 1$ then the equation of energy at capture is

$$\frac{1}{2}\ \lambda^2\omega_c^2 = (1 - r)k\ \cos^2\gamma_c$$

where γ_c, ω_c are γ, ω evaluated at capture.. This, by analogy with a simple pendulum, is the largest closed energy contour surrounding the equilibrium point $\gamma = 0$. The capture angular velocity is, therefore,

$$\lambda\omega_c = \lambda = \pm\sqrt{2k(1 - r)}\ \cos\gamma_c$$

where, as previously discussed, $\lambda = (\gamma')_c$. If the motion reaches $\lambda = \pm\sqrt{2k(1-r)}$, then in one revolution it will surely be captured because of the energy loss. The preceding arguments mean that the value of λ should be taken as $\lambda = \pm\sqrt{2k(1-r)}$ for r small.

D. PERTURBATION SOLUTION FOR THE TUMBLING MOTION

Approximate solutions will be obtained for the angular velocity $\gamma' = \lambda\omega$ large compared to one. Actually the results will be approximately correct anywhere out of the capture region. We use $1/\lambda$ as a small parameter in the expansions, assuming that $\omega > 1$. If one uses the averaging method of Appendix B, he obtains series solutions in powers of $1/\lambda$ and can thus approximate the actual motion. To begin, we must get the equations in "standard form" by a transformation. We use for this purpose the forced solution with constant ω. The reasoning is that for large λ, $\omega' \approx 0$ and we have just a sinusoidally-forced set of equations. The transformation becomes,

$$\Omega = \bar{\Omega} + D(\eta \sin 2\bar{\gamma} - 2\lambda\bar{\omega} \cos 2\bar{\gamma}) + \bar{\bar{\Omega}}$$

$$\omega = \bar{\omega} + \frac{k}{2\lambda^2\bar{\omega}} \cos 2\bar{\gamma} - \frac{\eta r D}{2\lambda^2\bar{\omega}} (\eta \cos 2\bar{\gamma} + 2\lambda\bar{\omega} \sin 2\bar{\gamma}) + \bar{\bar{\omega}} \qquad (4.6)$$

$$\gamma = \bar{\gamma} + \frac{k}{4\lambda^2\bar{\omega}^2} \sin 2\bar{\gamma} - \frac{\eta r D}{4\lambda^2\bar{\omega}^2} (\eta \sin 2\bar{\gamma} - 2\lambda\bar{\omega} \cos 2\bar{\gamma}) + \bar{\bar{\gamma}}$$

where

$$D \triangleq \frac{k}{\eta^2 + (2\lambda\bar{\omega})^2} .$$

Here the barred variables represent the "secular" terms caused by damping and the $\sin 2\bar{\gamma}$, $\cos 2\bar{\gamma}$ terms are the forced response due to gravity, for large λ. The double-barred variables represent additional perturbations to be defined in (4.9). They will be set equal to zero for the present.

The above equations are to be viewed as a transformation from Ω, ω, γ, variables to the barred variables $\bar{\Omega}$, $\bar{\omega}$, $\bar{\gamma}$. The oscillating terms in $2\bar{\gamma}$ are the result of using the $\sin 2\gamma$ terms in (4.2) to force the equations in a periodic oscillation (as if $\lambda\omega$ were a constant). It must be emphasized that no approximation is implicit in (4.6) but merely a transformation based on an approximate way of viewing (4.2).

Now we are going to use (4.6) to get the differential equations (4.2) in terms of the barred variables. By direct substitution of (4.6) into (4.2) (with the $\bar{\bar{\Omega}}$, $\bar{\bar{\omega}}$, $\bar{\bar{\gamma}}$ set equal to zero) the following differential equations in the barred variables are obtained carrying terms up to fourth order in $1/\lambda$.

$$\bar{\omega}' = \frac{\eta r\bar{\Omega}}{\lambda} - \frac{1}{4\lambda^3\bar{\omega}^2}\left\{k^2 \sin 4\bar{\gamma} + \eta krD(2\lambda\bar{\omega})(1 + \cos 4\bar{\gamma})\right\} + 0\,(1/\lambda^4)$$

$$\bar{\gamma}' = \lambda\bar{\omega} - \frac{k^2 \cos 4\bar{\gamma}}{8\lambda^3\bar{\omega}^3} + 0\,(1/\lambda^4) \tag{4.7}$$

$$\bar{\Omega}' = -\eta\bar{\Omega} + \frac{1}{4\lambda^2\bar{\omega}^2}\left\{k^2 \sin 4\bar{\gamma} - k\eta^2 r\, D \sin 4\bar{\gamma} + k\eta rD(2\lambda\bar{\omega})(1 + \cos 4\bar{\gamma})\right\} + 0\,(1/\lambda^4)$$

In the above expressions it has been assumed that $\bar{\Omega}$ is of order $1/\lambda^3$. This is in keeping with the "steady-state" assumption of the next section.

Equations (4.7) are the equations (4.2) in terms of the barred variables and these differential equations possess the "standard form" of Bogoliuboff and Mitropolsky (BOGOLIUBOFF 1). That is to say, the right-hand sides are small and thus the barred variables vary slowly and the averaged differential equation may then be solved to find the secular variations in the barred variables. The averaged equations in the barred variables are:

$$\bar{\omega}' = \frac{\eta r}{\lambda}\,\bar{\Omega} - \frac{\eta rDk}{2\lambda^2\bar{\omega}}$$

$$\bar{\gamma}' = \lambda\bar{\omega} + 0\,(1/\lambda^4) \tag{4.8}$$

$$\bar{\Omega}' = -\eta\bar{\Omega} + \frac{k\eta rD}{2\lambda\bar{\omega}}$$

Based on the transformed variables as defined by (4.8) we may write the approximate solutions including the effects of the oscillating terms in (4.7). These are:

$$\bar{\bar{\Omega}} = - \frac{1}{16\lambda^3\bar{\omega}^3} \left[k^2 \cos 4\bar{\gamma} - k\eta r D(\eta \cos 4\bar{\gamma} + 2\lambda\bar{\omega} \sin 4\bar{\gamma}) \right] \qquad (4.9)$$

$$\bar{\bar{\omega}} = \frac{k^2}{16\lambda^4\bar{\omega}^3} \cos 4\bar{\gamma}$$

$$\bar{\bar{\gamma}} = 0\ (1/\lambda^5)$$

These expressions (4.9) are of order $1/\lambda^3$ or smaller. The barred variables are solutions to the averaged equations (4.8) and together with (4.6) and (4.9) comprise the approximate solution to (4.2) up to order $(1/\lambda^2)$ (BOGOLIUBOV 1). The differential equations of average motion (4.6) are all that remain to be solved. These are solved analytically in the next section. In a more complicated case these would have to be integrated on a digital computer. This could be done with many fewer steps of integration than a pointwise integration of the original equations of motion, because the velocity and position variables are smoothed by the averaging process.

An investigation of the errors in these perturbation calculations for $\bar{\omega} > 1$, $\lambda = \sqrt{2k(1-r)}$ shows good convergence if $r < 0.1$ regardless of the value of η. The dominant oscillatory term in the expression for ω is of amplitude $1/4(1-r)$ which is certainly smaller than $\bar{\omega} > 1$ for $r < 0.1$. The next term in the expression for ω is $\eta r/2\sqrt{2}$ for $\eta < 1$ or $(k^{1/2}r)/\eta\sqrt{2}$ for $\eta >> 1$. These terms are quite small compared to $\bar{\omega} > 1$. The amplitude of $\bar{\bar{\omega}}$ is 1/64 at capture. These estimates of the oscillatory terms show good convergence for the stated conditions. The conditions $r < 0.1$ and $\lambda \gtrapprox \sqrt{2k(1-r)}$ are quite realistic.

E. APPROXIMATE SOLUTION OF THE AVERAGED EQUATIONS

In the third equation of (4.8) the second term on the right-hand side will vary quite slowly compared to the time constant $1/\eta$; let us assume this and verify it later. This allows us to use the steady state value of $\bar{\Omega}$ as the solution. The result is, upon elimination of $\bar{\Omega}$ in the first of (4.8):

$$\boxed{\begin{aligned} \bar{\omega}' &= \frac{\eta - \eta(1-r)r\,k^2}{2\lambda^2\bar{\omega}(\eta^2 + 4\lambda^2\bar{\omega}^2)} && (4.10) \\ \bar{\Omega} &= \frac{rk^2}{2\lambda\bar{\omega}(\eta^2 + 4\lambda^2\bar{\omega}^2)} && (4.11) \end{aligned}}$$

Using the defined relation $N = \lambda\bar{\omega}$, we solve (4.10) by direct integration to obtain the relationship between the average tumbling rate, N, and the time, τ.

$$\left[2N^4 + \eta^2 N^2\right]_{N_c}^{N_o,\,\tau=0} = -\,\eta r(1-r)k^2\tau \qquad (4.12)$$

This can be used with (4.11), (4.9) to give the complete approximate solution.

For some fixed value of the final angular velocity, N_c, we may solve for the final time, τ_c:

$$\tau_c = \frac{2}{\eta r(1-r)k^2}\left[N_o^4 - N_c^4 + \eta^2(N_o^2 - N_c^2)\right] \qquad (4.13)$$

Under the conditions of (4.13) there exists a minimum time to capture as a function of the damping parameter, η. This is given by the following $(\partial\tau_c/\partial\eta = 0)$ (see Fig. 4.2):

$$\begin{aligned} \eta_{opt}^2 &= 2(N_o^2 + N_c^2) \\ (\tau_c)_{min} &= \frac{2(N_o^2 - N_c^2)}{r(1-r)k^2}\,\eta_{opt} \end{aligned} \qquad (4.14)$$

F. SECULAR SOLUTION WITH SPRING-CONSTRAINED DAMPING SPHERE: THE EXPLOITATION OF RESONANCE

Consider the problem treated in Section D with the addition of a linear spring acting between the sphere and the satellite body. This modifies the equations of motion, giving rise to the possibility of resonance between the tumbling rate and the relative motion of the sphere with respect to the satellite body.

The solution given here makes certain assumptions based upon the previous, more restricted case. The major assumption is that the relative velocity, Ω, is in steady state, forced oscillation, and therefore the secular solution is equivalent to the solution of the differential equation of average motion.

The differential equations of motion with an additional torque of magnitude $I(1-r)\omega_o^2\theta$ applied between the sphere and the body are:

$$\gamma' = \lambda\omega$$

$$\omega' = \frac{1}{\lambda}\left[\eta r\Omega + r\omega_o^2\theta - k\sin 2\gamma\right] \tag{4.15}$$

$$\theta' = \Omega$$

$$\Omega' = -\eta\Omega - \omega_o^2\theta + k\sin 2\gamma$$

where θ stands for the angular displacement of the sphere from the zero torque position of the spring. Assuming that λ is large we get a set of transformation equations analogous to (4.6)

$$\Omega = \bar{\Omega} + 2\lambda\bar{\omega}\left[-\alpha\sin 2\bar{\gamma} + \beta\cos 2\bar{\gamma}\right]$$

$$\omega = \bar{\omega} + \frac{1}{2\lambda^2\bar{\omega}}\Big[k\cos 2\bar{\gamma} + r\omega_o^2(\alpha\sin 2\bar{\gamma} - \beta\cos 2\bar{\gamma})$$

$$+ 2\lambda\bar{\omega}r\,\eta(\alpha\cos 2\bar{\gamma} + \beta\sin 2\bar{\gamma})\Big]$$

$$\theta = \alpha \cos 2\bar{\gamma} + \beta \sin 2\bar{\gamma} + \bar{\theta} \tag{4.16}$$

$$\gamma = \bar{\gamma} + \frac{1}{4\lambda^2\bar{\omega}^2}\left[k \sin 2\bar{\gamma} + r\omega_o^2 (-\alpha \cos 2\bar{\gamma} - \beta \sin 2\bar{\gamma})\right.$$

$$\left. + 2\lambda\bar{\omega}r\,\eta(\alpha \sin 2\bar{\gamma} - \beta \cos 2\bar{\gamma})\right]$$

where,

$$\alpha = -\frac{D\eta}{2\lambda\bar{\omega}}$$

$$\beta = \frac{D(\omega_o^2 - 4\lambda^2\bar{\omega}^2)}{4\lambda^2\bar{\omega}^2}$$

$$D = \frac{k}{\eta^2 + \left[\dfrac{\omega_o^2 - 4\lambda^2\bar{\omega}^2}{2\lambda\bar{\omega}}\right]^2}$$

The value of D given above reduces to that previously given (as do all the formulae of this section) if we let $\omega_o \doteq 0$.

Combining (4.15) and (4.16) we get, upon averaging, the secular differential equations, with $\lambda\bar{\omega} \triangleq N$.

$$\boxed{\begin{aligned} \bar{\theta} &= \frac{rkD\eta}{2\lambda\bar{\omega}\,\omega_o^2} = \frac{rkD\eta}{2N\omega_o^2} \\ N' &= -\frac{(1-r)r\eta Dk}{2N} \end{aligned}} \tag{4.17}$$

The solution of the second of (4.17) gives the time to capture, τ_c, as

$$\tau_c = \frac{1}{k^2(1-r)r\eta}\left[\eta^2(N_o^2 - N_c^2) + 2(N_o^4 - N_c^4)\right.$$

$$\left. + \frac{\omega_o^4}{2}\log_e(N_o/N_c) - 2\,\omega_o^2(N_o^2 - N_c^2)\right] \tag{4.18}$$

where it is assumed that $N_o/N_c > 0$. Notice that in (4.18) τ_c is independent of the sign of N_o. The minimum value of τ_c occurs when the derivatives of τ_c with respect to η and ω_o^2 vanish. This occurs for the following optimum values of the damping and spring resonance frequencies:

$$(\omega_o^2)_{opt} = \frac{2(N_o^2 - N_c^2)}{\log_e(N_o/N_c)}$$

$$\eta^2_{opt} = 2\left[N_o^2 + N_c^2 - \frac{(\omega_o^2)_{opt}}{2}\right] \tag{4.19}$$

The minimum value of the capture time is given from (4.18) and (4.19) as

$$(\tau_c)_{min} = \frac{2(N_o^2 - N_c^2)}{k^2(1-r)r}\,\eta_{opt} \tag{4.20}$$

It is useful to compare the minimum value of τ_c with the value for arbitrary values of damping and spring constant; this gives the ratio

$$\frac{\tau_c}{(\tau_c)_{min}} = \frac{1}{2}\left[\frac{\eta}{\eta_{opt}} + \frac{\eta_{opt}}{\eta} + \frac{(\omega_o^2)_{opt}}{\eta\eta_{opt}}\left(\frac{\omega_o^2}{(\omega_o^2)_{opt}} - 1\right)^2\right] \tag{4.21}$$

Note that (4.19), (4.20), (4.21) reduce to (4.14) if $\omega_o^2 = 0$. Table 1 gives the computed values of the optimum capture time for a particular value of terminal velocity and for both resonant and non-resonant cases.

TABLE 1

$K = k^2(1-r)r$

$N_c = \sqrt{3}$	NON-RESONANT CASE $\omega_o = 0$			RESONANT CASE $\omega_o = (\omega_o)_{opt}$				
N_o	η_{opt}	$K(\tau_c)_{opt}$	K(# Orbits To N_c)	η_{opt}	$(\omega_o^2)_{opt}$	$K(\tau_c)_{opt}$	K(# Orbits To N_c)	# Orbits (k=0.01)
1.8	3.54	1.675	0.270	0.482	12.25	0.231	0.037	3.6
1.9	3.62	4.42	0.705	0.790	12.6	0.965	0.154	15
2.0	3.73	7.47	1.19	1.025	12.96	2.05	0.327	33
3.0	4.90	58.6	9.34	1.414	22.0	16.9	2.69	270
4.0	6.15	160.0	25.5	3.64	31.0	94.7	15.1	1510
5.0	7.47	328.0	52.3	3.80	41.5	167	26.6	2660
10.0	14.4	2,800	447	9.75	110.5	1890	302	30,200

Various physical observations may be made about these results. By reference to Table 1 one can see that, for a given vehicle configuration (K) and for optimum parameters, the number of orbits to reach N_c is considerably larger for the "non-resonant" case; while in both cases the damping time increases sharply from $N_o = N_c$ until for $N_o \geq 5$ it increases approximately as the cube of N_o (Fig. 4.2).

This behavior may be justified on physical grounds. The "resonance" phenomenon causes larger relative motion between the sphere and the satellite body, thus dissipating more energy per orbit and damping γ' more rapidly (Fig. 4.3). The rapid increase of damping time with increasing N_o is, of course, the reason for the perturbation technique; this is a direct result of the "internal" nature of the damping.

If there were no gravity, one could immediately use the angular momentum principle to show that the particles of the system are frozen to some rotating reference axes in steady-state; there would be no relative motion between particles and thus no internal dissipation of energy. If there is gravity torque then one has relative particle motion and thus damping. Notice, however, that this relative motion is very small for large rotation velocity because the kinetic energy of rotation is much larger than the fluctuations in potential energy due to gravity.

Figure 4.4 illustrates how the effect of gravity on the angular velocity γ' increases as we near capture; the effect is to increase the "ripples" of oscillation about the average angular velocity N. This method of perturbations saves computer integration over all these rapidly-varying functions; integration of functions varying as the average velocity, N, can be accomplished with many fewer computer steps.

G. SYNCHRONIZATION OF THE TUMBLING RATE

If a small, constant torque (for example, via gas-jets or gas leakage) of magnitude $n^2 I_3 c$ is added, an interesting phenomenon occurs. The average speed, N, of the main satellite body rises from zero to a certain value determined by a parameter ω_s and then

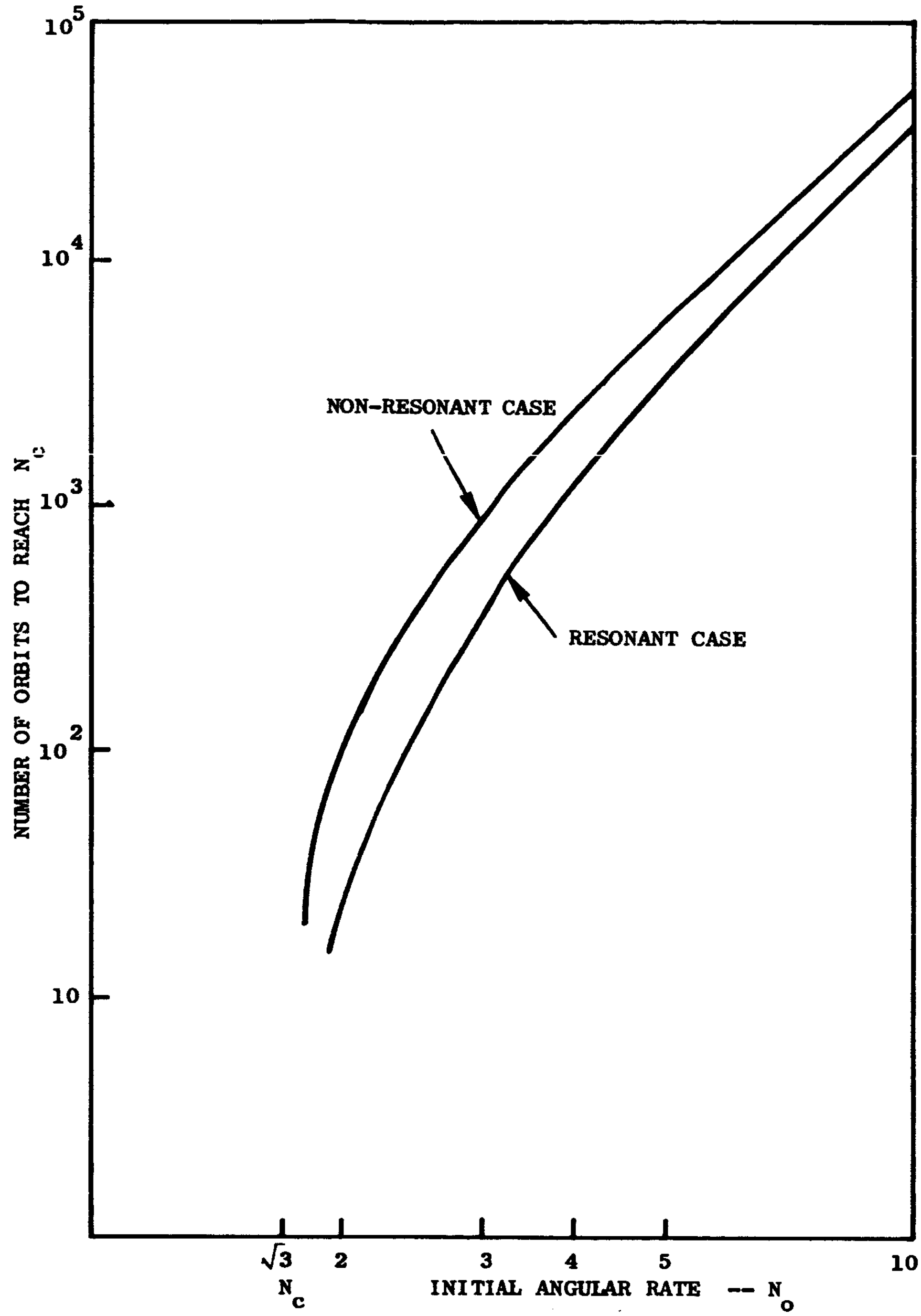

FIG. 4.2. DAMPING TIMES FOR VARIOUS INITIAL RATES: OPTIMUM SPRING AND DAMPING CONSTANTS $K = 0.01$.

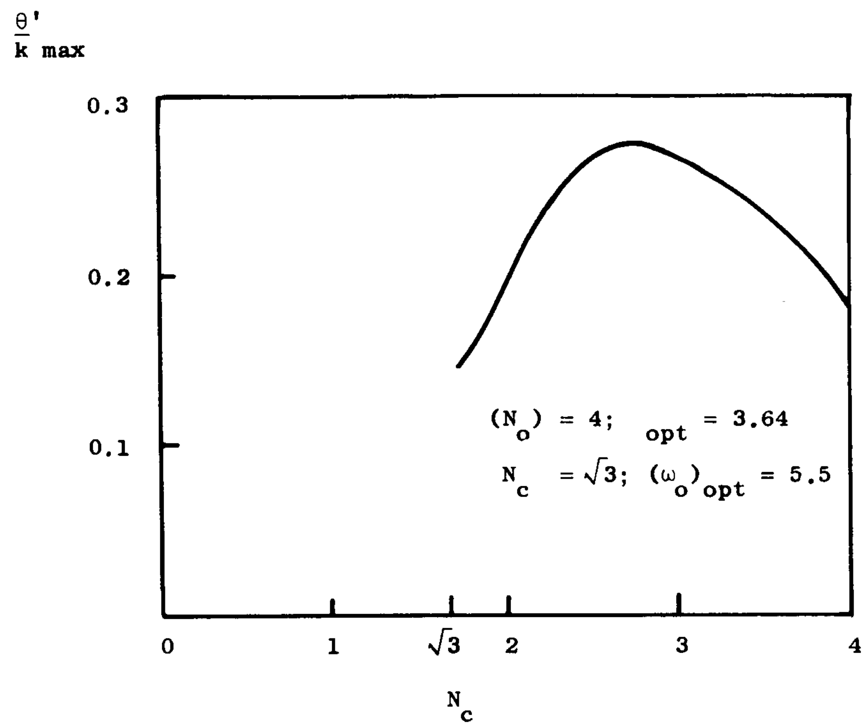

FIG. 4.3. MOTION OF θ'_{max} FOR OPTIMUM RESONANT DAMPING FROM $N_o = 4$

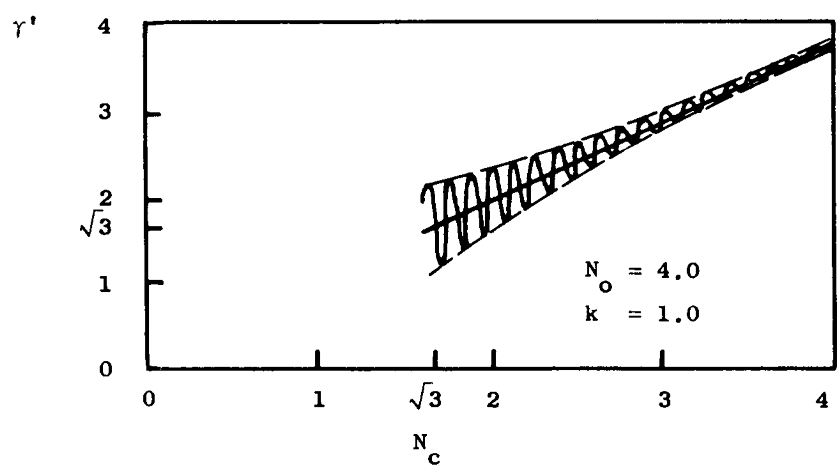

FIG. 4.4. MOTION OF THE TUMBLING RATE FROM $N_o = 4$ (SECULAR PLUS PERIODIC) -- RESONANT CASE

the satellite remains turning at a constant average speed. If the initial speed is too large, this does not occur; the body simply spins up indefinitely.* The equations of average motion are, with the additional torque:

$$N' = -K\eta \left\{ \frac{2N}{4\eta^2 N^2 + (\omega_o^2 - 4N^2)^2} \right\} + (1-r)c$$

$$\bar{\Omega} = -\frac{c}{\eta} + \frac{rkD}{2N} \tag{4.22}$$

where we have defined $K = (1-r)rk^2$.
There are, of course, equilibrium points at which $N' = 0$. These are the points of synchronization that are sought. They are solutions to the equations (4.22) with $N' = 0$. This can be seen graphically by noticing Fig. 4.5, which shows the solution to $N' = 0$. From this solution we may deduce that $N = \omega_s$ is a stable point and $N = \omega_o - \omega_s$ is unstable. The arrows show the transient motions. $N = \omega_o - \omega_s$ is the maximum initial speed for stable behavior.

We may calculate ω_s if $c(1-r)$ is equal to $1/2\ K/\eta\omega_o$. This is

$$\omega_s^2 + \frac{\eta}{2}\omega_s + \frac{\omega_o^2}{2} = 0$$

or,

$$\omega_s = -\frac{\eta}{4} \pm \sqrt{(\eta/2)^2 + \omega_o^2/2} \tag{4.23}$$

$$\Delta\omega_s = \frac{\omega_o}{2} - \omega_s = \frac{\eta\omega_s}{\omega_o + 2\omega_s}$$

where $\Delta\omega_s$ is defined above.

The slope of the curve in Fig. 4.5 at the synchronization point leads to the "linearized" behavior about this point.

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* In this analysis it is assumed that the internal damping is the only damping present. In case other sources are present they may be easily accounted for.

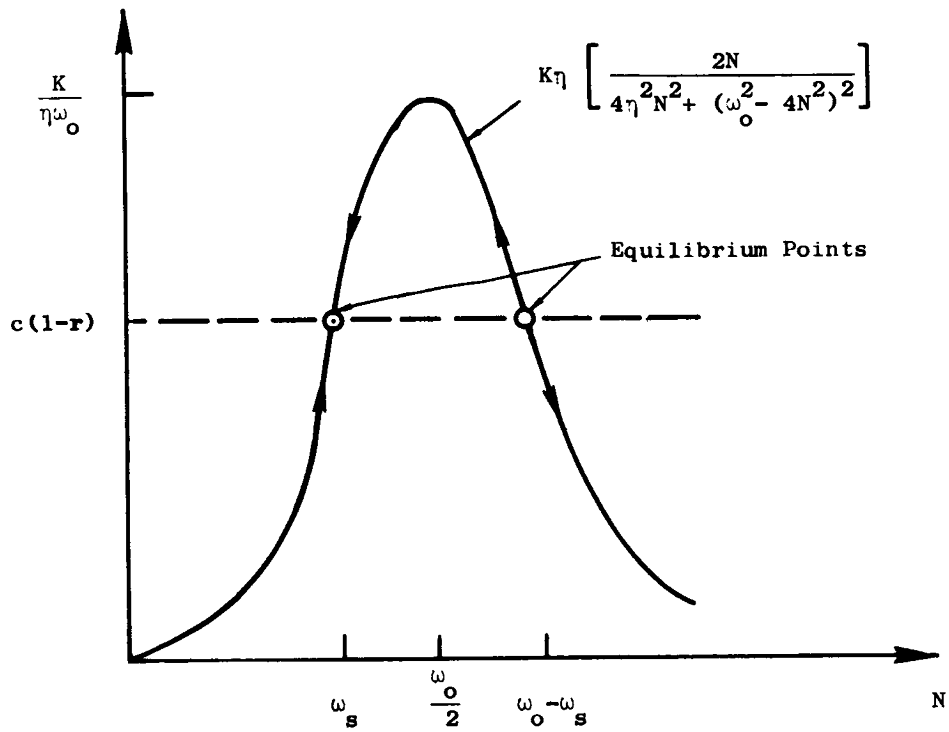

FIG. 4.5. LOCATION OF EQUILIBRIUM POINTS WITH CONSTANT THRUST

$$\delta N' = \frac{1}{\tau_d} \delta N + (1-r)\delta c \qquad (4.24)$$

where,

$$N = \omega_s + \delta N$$

$$c = c_s + \delta c$$

$$\tau_d = \frac{4\,\omega_s^2\,\eta}{K(8\,\omega_s/\eta - 1)} \quad ; \qquad K = k^2(1-r)r$$

$$c_s = K/2(1-r)\eta\,\omega_o$$

The time constant for the transient decay to $1/e$ of its initial value for small $\delta N(\frac{\delta N}{\omega_s} \ll 1)$ is τ_d.

H. CONCLUSIONS

Several problems related to tumbling satellites have been discussed. These problems pose some interesting questions which involve the length of time it takes to damp a satellite with only internal damping between the moving parts. It is instructive to notice, in the light of (4.8) the value of the average rate of change of angular momentum. Letting P denote the total angular momentum we have

$$\overline{P} = N + r\overline{\Omega}$$

$$\overline{P}' = -(1-r)r\,\frac{\eta k D}{2N} = r\,0\,(1/N^3) \qquad (4.25)$$

where $\overline{P}$ denotes the average angular momentum (divided by $I + I_3$). The important observation about (4.25) is that the average torque (or rate of change of total angular momentum) is small of order $1/N^3$. This fact indicates that the average torque about the pitch axis due to the damping torques between particles of the system is of order $1/\lambda^3$.

Because for tumbling rates significantly above the orbit rate the angular momentum decreases as the cube of the reciprocal angular rate, the decrease in tumbling rate is extremely slow. One must notice that this would not necessarily hold if external damping torques, for example gas jets or earth magnetic field damping, were significant. The slow nature of the secular decay of tumbling rate makes direct digital solution of the equations of motion very costly in time and in round-off errors. The perturbation scheme outlined makes the job of integration much less difficult and in certain cases obviates the need for numerical integration entirely.

CHAPTER V: BOUNDS ON THE LIBRATIONS OF A SYMMETRICAL SATELLITE*

A. INTRODUCTION

The problem of rigid body motions in a gravity field dates at least to Newton who explained the first-order precession of the earth on the basis of his new gravitational theory. The precession and nutation of the earth later became important to astronomers in connection with the length of the sidereal day. Hipparchus first observed the effect of these rigid librations of the earth before the birth of Christ.

It was, however, D'Alembert who first gave a complete formulation of the problem for the case of high spin (precession and nutation). Laplace, Poissón, and Tisserand all subsequently worked on the problem and it comes to us largely in the form in which they left it.

These investigators had in mind the verification of Newtonian gravity theory by pure deduction based on Newton's second law and the known orbits of the Sun, Moon, and planets; our interest is in the use of this theory to predict the dynamic behavior of artificial satellites. We are, therefore, interested in the complete range of motions rather than just the high spin case (of interest in the case of the earth).

Since the first artificial earth satellites were contemplated, there has been an increasing interest in the attitude dynamics of rigid bodies in orbit. Work on the stability of a symmetrical rigid body in a circular orbit has been done by Thomson (THOMSON 2), Beletskii (BELETSKII 1), and Auelmann (AUELMANN 1). Thomson considered the stability of the small-amplitude motion of the symmetry axis of a satellite relative to the normal to the orbit plane. Beletskii treated the high-spin case of a satellite with gravity and aerodynamic perturbations. Auelmann attacked the problem of the present chapter, but established results only for the "zero spin" case.

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* This chapter is to be published in AIAA Journal, May 1964.

In this chapter the author will endeavor to show a complete picture of the equilibrium solutions, their stability and regions of stability for a symmetric, rigid body in a circular orbit. This type of analysis is of course preliminary to a complete study of system behavior including librations and periodic solutions.

The analysis draws on the theory of bifurcations of Poincaré (POINCARÉ 1) and on Hill's use of the Jacobian integral in the lunar theory (HILL 1). The problem is interesting because it has (1) a physical motivation and use, (2) a simplicity inherent in two-degree-of-freedom cases, and (3) a wealth of interesting phenomena.

B. EQUATIONS OF MOTION

Consider the angular motion of a symmetrical, rigid body relative to a rotating coordinate system ($\hat{1},\hat{2},\hat{3}$ unit vectors) which is centered at the body mass center. The $\hat{1}$ axis points along the radius vector from the earth, the $\hat{3}$ axis points normal to the orbit plane, and the $\hat{2}$ axis points along the orbit velocity vector and is such that $\hat{2} = \hat{3} \times \hat{1}$ (Fig. 5.1).

Two sets of Euler angles (generalized coordinates) will be used to define orientations of the satellite relative to the $\hat{1},\hat{2},\hat{3}$ reference frame, because each set of Euler angles is singular at a particular point of interest. In each case $\hat{3}b$ is the axis of symmetry of the satellite. For the first set consider (Fig. 5.2) an axis system $\hat{1}b$, $\hat{2}b$, $\hat{3}b$ initially aligned with $\hat{1},\hat{2},\hat{3}$. Then a counterclockwise rotation, θ_2, about the $\hat{2}$ axis and a successive counterclockwise rotation, θ_1, about the $-\hat{1}b$ axis locates the $\hat{1}b$, $\hat{2}b$, $\hat{3}b$ frame relative to $\hat{1},\hat{2},\hat{3}$. The $\hat{1}b$, $\hat{2}b$, $\hat{3}b$ set does not spin with the satellite.

The second set of Euler angles are the spherical angular coordinates of $\hat{3}b$ in Fig. 5.1. The angle φ is a rotation locating the position of the spin axis ($\hat{3}b$) relative to the plane of the orbit and θ is an angle locating the plane of φ-rotation relative to the $\hat{1},\hat{2}$ plane.

In both of the coordinate sets used above the angular rate about the $\hat{3}b$ axis relative to the rotated coordinate axes is defined as $\dot{\psi}$; ψ is a generalized coordinate not appearing in the energy expressions.

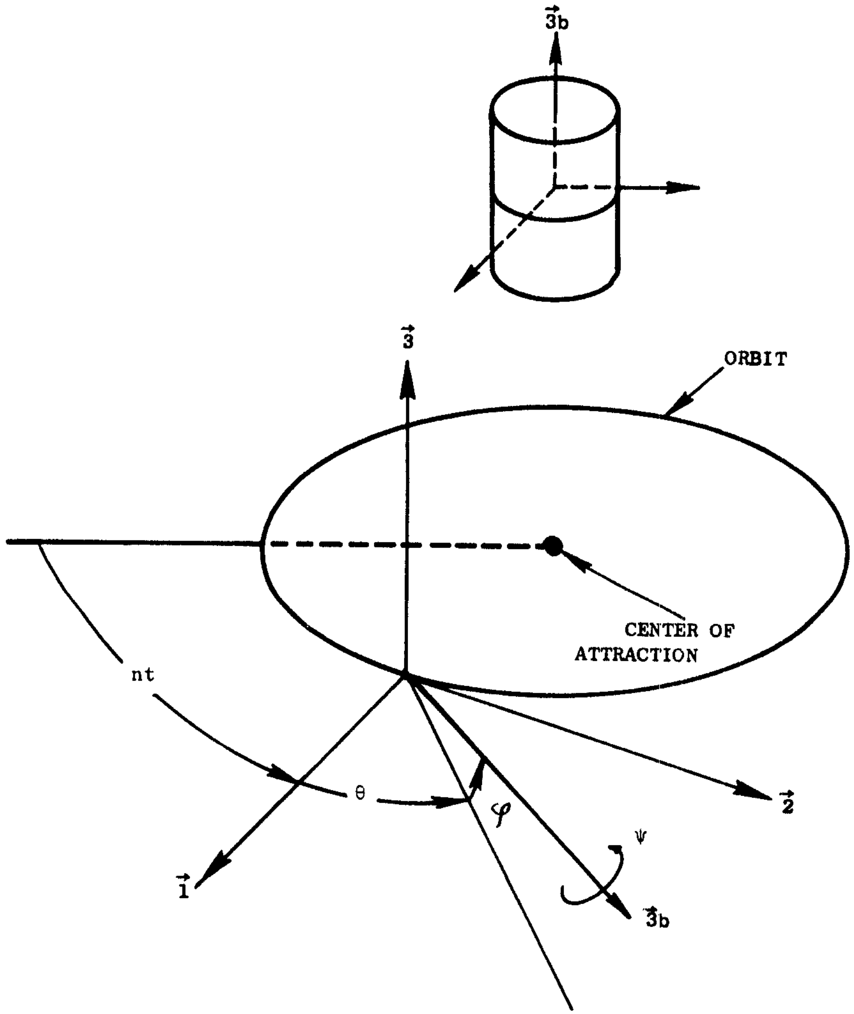


FIG. 5.1. COORDINATE SYSTEM OF A SYMMETRICAL RIGID BODY IN A CIRCULAR ORBIT ABOUT THE EARTH

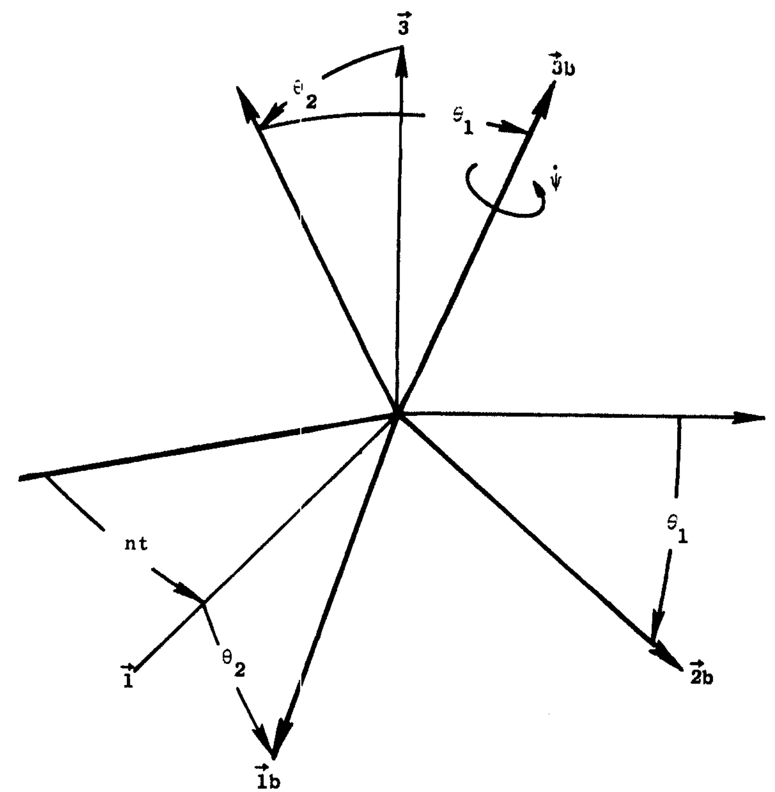


FIG. 5.2. ROTATIONS IN θ_1, θ_2 COORDINATES

The body is specified by a moment of inertia C about the $\hat{3}b$ axis (symmetry axis) and a moment of inertia A about any axis normal to $\hat{3}b$ and passing through the center of mass.

We may now write the kinetic and potential energy due to rotation of the rigid body about its mass center (see (2.7) and (2.8)).

$$T = \frac{A}{2}\left\{\dot{\varphi}^2 + (\dot{\theta} + n)^2\cos^2\varphi\right\} + \frac{C}{2}\left\{\dot{\psi} + (\dot{\theta} + n)\sin\varphi\right\}^2$$

$$+ \frac{A}{2}\left\{(\dot{\theta}_1 + n\sin\theta_2)^2 + (\dot{\theta}_2\cos\theta_1 - n\cos\theta_2\sin\theta_1)^2\right\}$$

$$+ \frac{1}{2}C\left\{\dot{\psi} + n\cos\theta_2\cos\theta_1 + \dot{\theta}_2\sin\theta_1\right\}^2 \tag{5.1}$$

$$V = \frac{3}{2}n^2(C - A)\cos^2\varphi\cos^2\theta$$

$$= \frac{3}{2}n^2(C - A)\cos^2\theta_1\sin^2\theta_2$$

It is convenient to define

$$r = C/A$$

$$\ell = p_\psi/C \tag{5.2}$$

where $\dot{p}_\psi = d/dt\ (\partial T/\partial\dot{\psi}) = 0$. p_ψ, the angular momentum about the $\hat{3}b$ axis, is a constant equal to

$$p_\psi = C\left[\dot{\psi} + (\dot{\theta} + n)\sin\varphi\right] = C\left[\dot{\psi} + n\cos\theta_2\cos\theta_1 + \dot{\theta}_2\sin\theta_1\right] \tag{5.3}$$

We may now use Lagrange's equations to find the motion of the satellite under the potential energy, V. These may be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i} \qquad (i = 1,2,3) \tag{5.4}$$

where the q_i are generalized coordinates taken to be either φ, θ, ψ or θ_1, θ_2, ψ. There exists a first integral for the above problem (LANCZOS 1); this is the Hamiltonian, H (which is not the total energy in this problem).

$$H = \frac{\partial T}{\partial \dot{\psi}} \dot{\psi} + \frac{\partial T}{\partial \dot{\varphi}} \dot{\varphi} + \frac{\partial T}{\partial \dot{\theta}} \dot{\theta} - T + V \tag{5.5}$$

Using (5.4) we can easily verify that $dH/dt = 0$ and, therefore, that H is a constant of the motion. In terms of the variables of this problem H becomes,

$$H = R_2 + U \tag{5.6}$$

where,

$$R_2 \triangleq \frac{A}{2} \left(\dot{\varphi}^2 + \dot{\theta}^2 \cos^2\varphi\right)$$

or

$$R_2 \triangleq \frac{A}{2} \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 \cos^2\theta_1\right) \tag{5.7a}$$

$$U \triangleq \frac{3}{2} n^2 A(r - 1) \cos^2\varphi \cos^2\theta - \frac{An^2}{2} \cos^2\varphi - A\ell rn \sin\varphi$$

or

$$U \triangleq \frac{3}{2} n^2 A(r - 1)\cos^2\theta_1 \sin^2\theta_2 - A\ell rn \cos\theta_1 \cos\theta_2 - \frac{An^2}{2}\left(\cos^2\theta_2 \sin^2\theta_1 + \sin^2\theta_2\right) \tag{5.7b}$$

where the R_2 functions are positive definite functions of the velocities and the U are functions of the coordinates φ, θ and θ_1, θ_2. Notice that $\dot{\psi}$ has been eliminated in favor of the constant ℓ by use of (5.3). The function U is called the "dynamic potential" and plays an important role in the stability analysis that follows.

C. EQUILIBRIUM POINTS

An equilibrium point is a point in the φ, θ or θ_1, θ_2 space where the $\hat{3}b$ axis can come to rest relative to reference frame $\hat{1}$, $\hat{2}$, $\hat{3}$. This condition is defined using (5.4) by

$$\frac{\partial U}{\partial \varphi} = U_\varphi = 0 \tag{5.8a}$$

$$\frac{\partial U}{\partial \theta} = U_\theta = 0 \tag{5.8b}$$

Using (5.7), (5.8) gives the equilibrium relations

$$[3(1 - r)\sin\varphi_o \cos^2\theta_o + \sin\varphi_o - \frac{\ell r}{n}]\cos\varphi_o = 0 \tag{5.9a}$$

$$\cos\theta_o \sin\theta_o \cos^2\varphi_o = 0 \tag{5.9b}$$

The above conditions reflect the static balance between gyroscopic and gravity torques. We must investigate the equilibrium points φ_o, θ_o defined by (5.9). We see that two cases obtain:

I. $\varphi_o = \pm\pi/2$, θ_o arbitrary and

II. $\varphi_o \neq \pm\pi/2$

Case II contains two subcases, i.e.,

IIa. $\theta_o = 0, \pi$; $\sin\varphi_o = \ell r/n(4 - 3r)$ and

IIb. $\theta_o = \pm\pi/2$; $\sin\varphi_o = \ell r/n$.

We note that this, due to (5.9b), exhausts the possibilities.

D. STABILITY OF EQUILIBRIUM

We must now discuss the stability of each point and whether U has a maximum, minimum, or saddlepoint. Define the (hessian) matrices

$$\mathcal{H}_1 = \begin{bmatrix} U_{\theta_1\theta_1} & U_{\theta_1\theta_2} \\ U_{\theta_2\theta_1} & U_{\theta_2\theta_2} \end{bmatrix}$$

$$\mathcal{H}_2 = \begin{bmatrix} U_{\varphi\varphi} & U_{\varphi\theta} \\ U_{\theta\varphi} & U_{\theta\theta} \end{bmatrix}$$

where the symbols $U_{q_i q_j} = \partial^2 U/\partial q_i \partial q_j$ are evaluated at the various equilibrium points. We shall use $\mathcal{H}_1$ for the points of Case I and $\mathcal{H}_2$ for points of Case II. Since $\mathcal{H}$ can be thought of as the matrix of a quadratic form, U, in φ, θ or θ_1, θ_2, when φ, θ, θ_1, θ_2 are small displacements from equilibrium, then: (1) if $\mathcal{H}$ is positive definite at φ_o, θ_o, or θ_{10}, θ_{20} there exists a minimum of U, (2) if $\mathcal{H}$ is negative definite at the point of equilibrium there exists a maximum of U, and (3) if $\mathcal{H}$ is sign variable at equilibrium, there exists a saddlepoint of U at the equilibrium point. If $|\mathcal{H}| = 0$, there is a bifurcation at the equilibrium. Note that it suffices to determine the results for $\ell > 0$ only. This is so because in (5.7) U is unchanged if $\ell \rightarrow -\ell$ and $\varphi_o \rightarrow -\varphi_o$.

It can be shown (POINCARÉ 1) that a "point of bifurcation" occurs when the determinant of the hessian of U, $|\mathcal{H}|$, vanishes. At a point of bifurcation the <u>qualitative</u> nature of the stability behavior <u>changes</u>. This can be seen by using the fundamental result that $|\mathcal{H}| = \lambda_1\lambda_2$, where λ_1 and λ_2 are eigenvalues of the quadratic terms in the expansion of U about an equilibrium point. Observe that if $\lambda_1 > 0$, $\lambda_2 > 0$ we have a minimum of U, if $\lambda_1 > 0$, $\lambda_2 < 0$ or if $\lambda_1 < 0$, $\lambda_2 > 0$ we have a saddlepoint of U, and if $\lambda_1 < 0$, $\lambda_2 < 0$ we have a maximum of U; <u>this means that as $|\mathcal{H}| = \lambda_1\lambda_2$ passes through zero, the topology of U changes</u>.

Consider now the stability of the equilibrium points of Cases I and II by using the hessian matrix, $\mathcal{H}$, to test U. It can easily be seen that the elements of $\mathcal{H}_2$ are at θ_o, φ_o:

$$U_{\varphi\varphi} = + 3n^2A(1 - r)\cos 2\varphi_o \cos^2\theta_o + An^2 \cos 2\varphi_o + \ell rAn \sin \varphi_o$$

$$U_{\varphi\theta} = \frac{3}{2} n^2A(r - 1) \sin 2\varphi_o \sin 2\theta_o$$

$$U_{\theta\theta} = \frac{3}{2} n^2A(r - 1) \cos^2\varphi_o \cos 2\theta_o$$

and at $\varphi_o = \pi/2$, the elements of $\mathcal{H}_1$ become

$$U_{\theta_1\theta_1} = An^2 \left(\frac{\ell r}{n} - 1\right)$$

$$U_{\theta_1\theta_2} = 0$$

$$U_{\theta_2\theta_2} = An^2\left(\frac{\ell r}{n} - 4 + 3r\right)$$

For Case I it can be seen that if $\ell/n > 1/r$, $\ell/n > (4/r) - 3$ the function U is minimum. If only one of these is violated, U has a saddlepoint, and if $\ell/n < 1/r$, $\ell/n < (4/r) - 3$ then U has a maximum at $\varphi_o = \div \pi/2(\theta_{10} = 0,\ \theta_{20} = 0)$.

For Case IIa $(\theta_o = 0, \pi$; $\sin \varphi_o = \ell r/n(4 - 3r)$ we see that if $r < 1$ we have a stable (minimum) point. If $1 < r < 4/3$, we have a saddlepoint of U, and if $r > 4/3$ we have a maximum of U. There is a bifurcation at $r = 1$.

For Case IIb $(\theta_o = \pm \pi/2$, $\sin \varphi_o = \ell r/n)$, $r = 1$ is a bifurcation point. If $1 < r < 2$ there is a minimum of U, while if $0 < r < 1$ we have a saddlepoint of U.

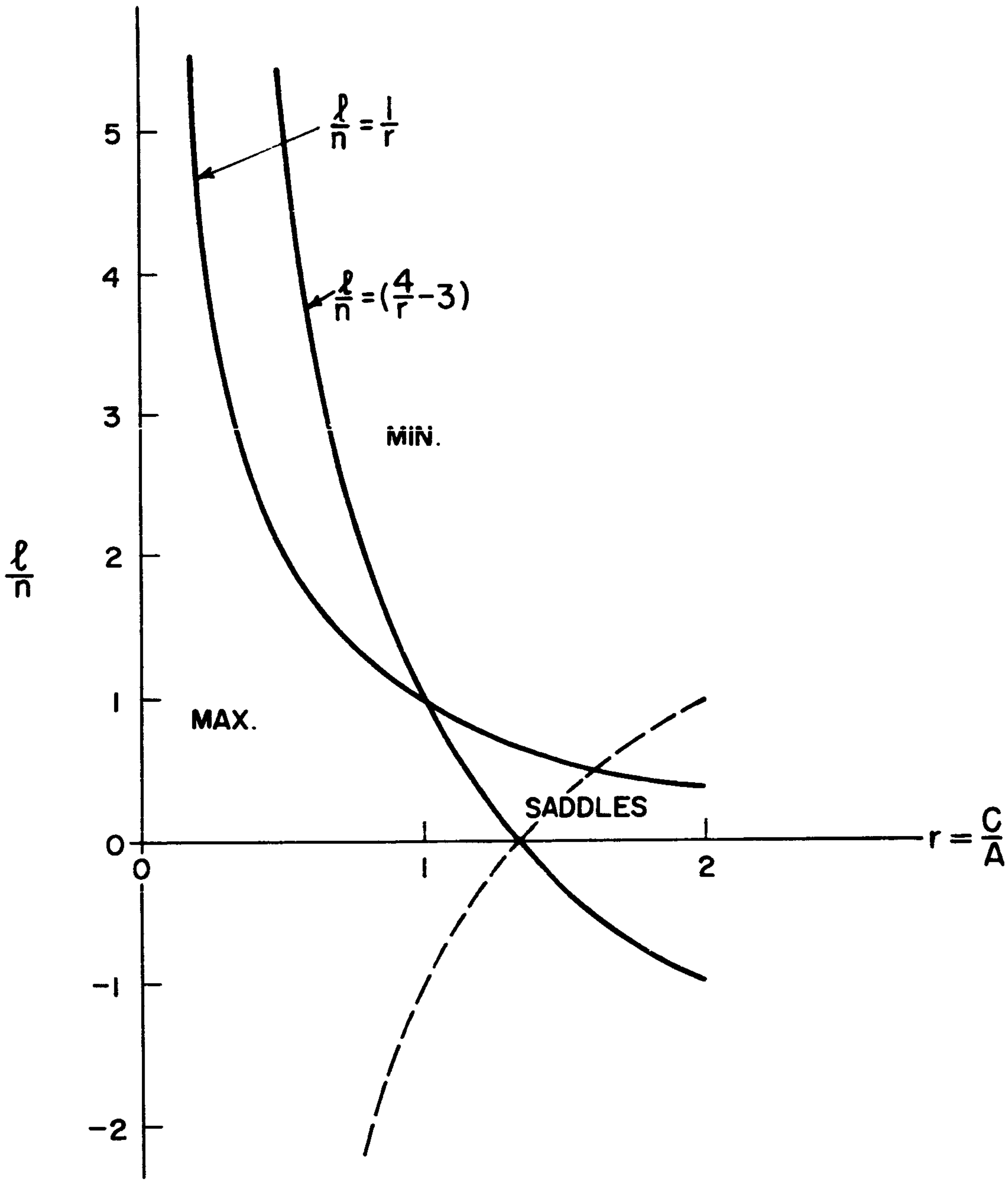


FIG. 5.3. CASE I: $\emptyset = + \pi/2$

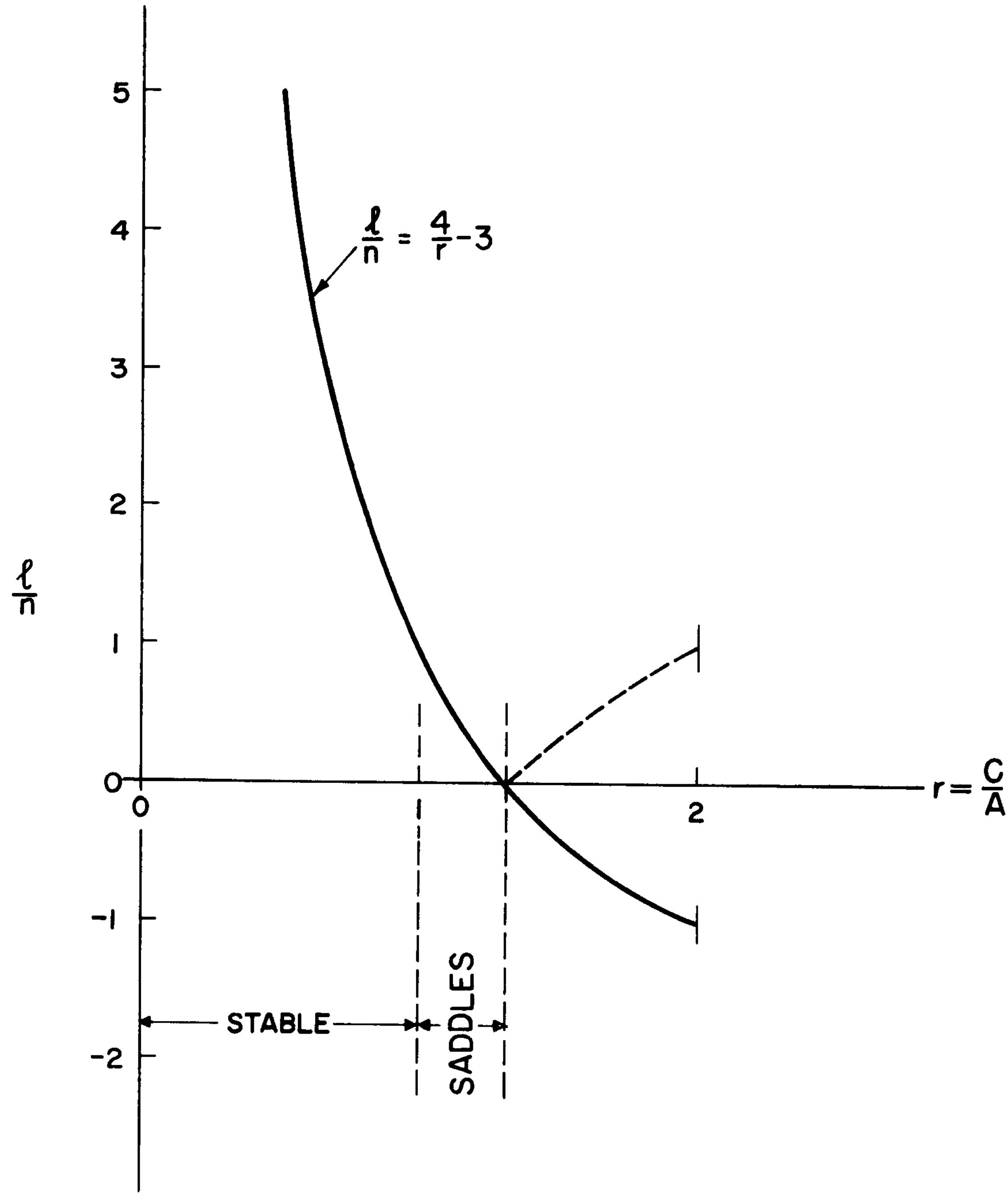


FIG. 5.4. CASE IIa: $\theta_o = 0, \pi$; $\sin\phi_o = \ell/n/(4/r-3)$

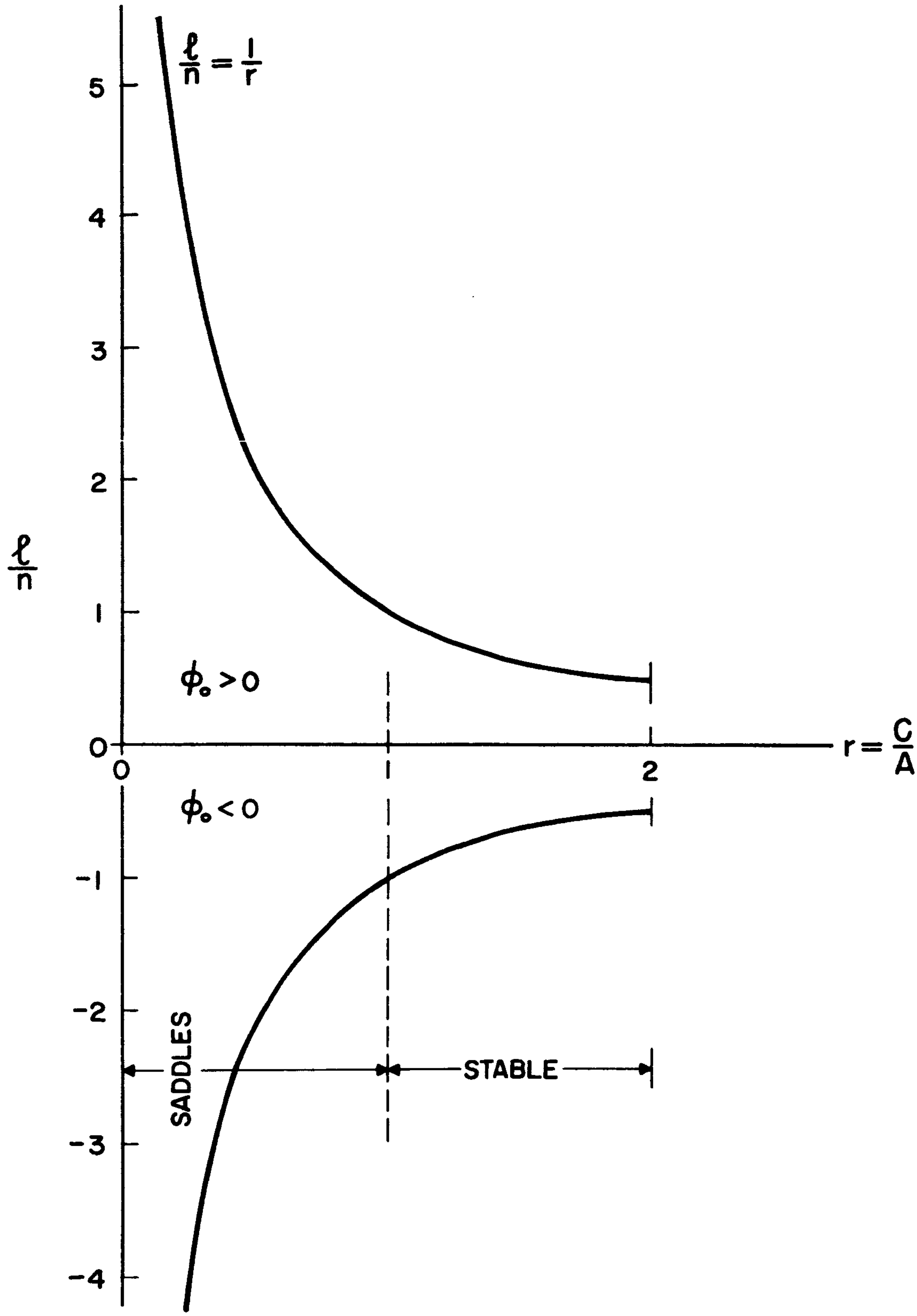


FIG. 5.5. CASE IIb: $\theta_o = \pm\ \pi/2$; $\sin\phi_o = \ell r/n$

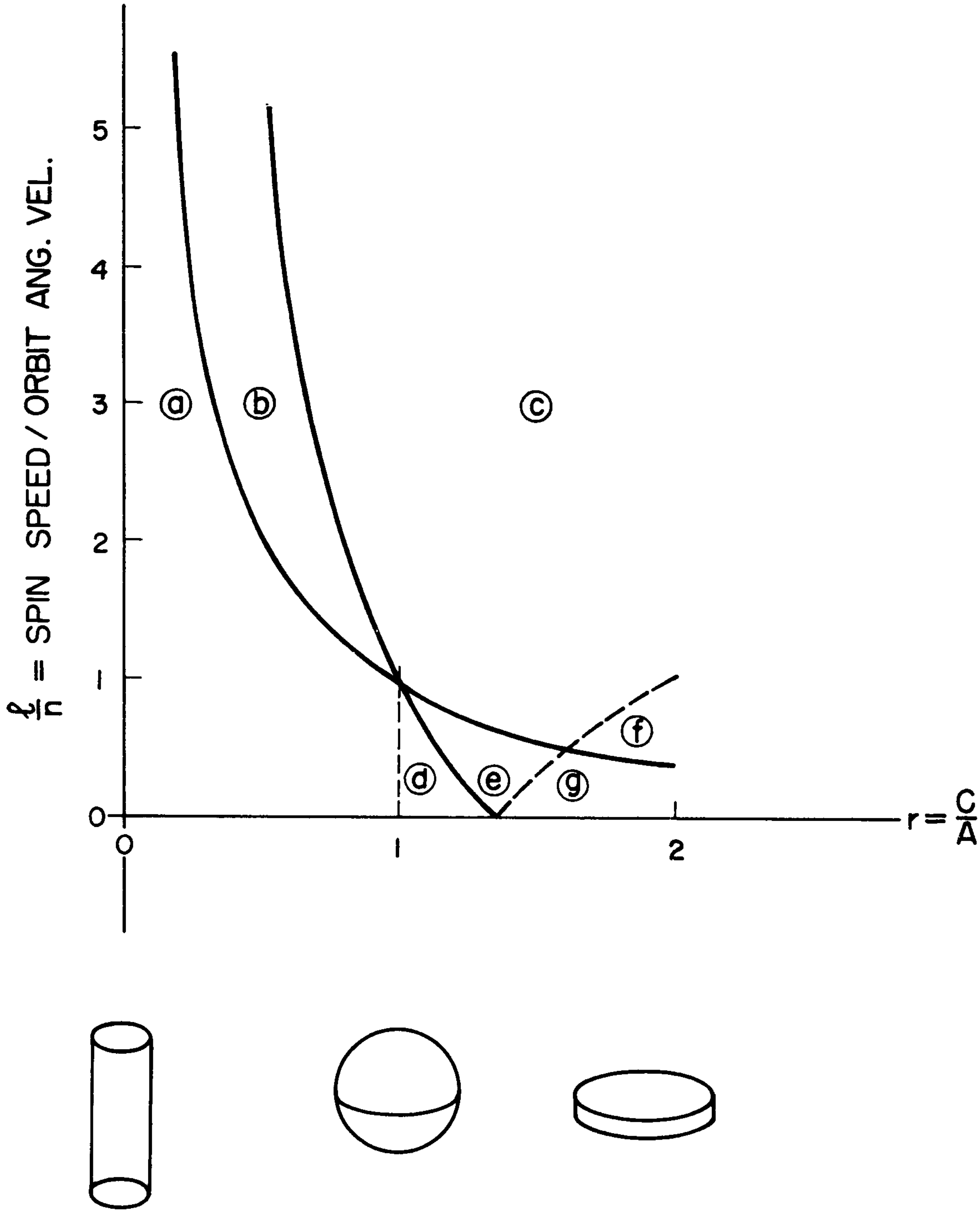


FIG. 5.6. CURVES OF BIFURCATION

These curves are plotted for the two cases in Figs. 5.3, 5.4, and 5.5. The solid curves in Fig. 5.6 are "curves of bifurcation." This means they divide areas (of the plot of ℓ/n versus r) with qualitatively different stability behavior. In Figs. 5.4 and 5.5 the solid lines are the curves beneath which the equilibrium point can exist. These are derived by noticing that $\sin \varphi_o \leq 1$ for existence of either equilibrium (IIa or IIb). Figure 5.6 is a composite plot showing all the curves of bifurcation on one plot. For the purpose of sketching the behavior in configuration space it is useful to know how many different kinds of qualitative behavior exist. The curves of bifurcation divide the ℓ/n, r plane in sectors (labeled a - g). If one moves across a curve of bifurcation then he must use a new letter, thus there are seven kinds of behavior in the quadrant $\ell > 0$ $(0 < \varphi_o < \pi/2)$ and likewise seven for $\ell < 0$ $(-\pi/2 < \varphi_o < 0)$.

E. CURVES OF CONSTANT U

It is now possible to plot contours of constant U in a space of φ, θ for Cases I and II. For the purpose of drawing them it is useful to project the points of unit vector $\hat{3}b$ on the plane of $\hat{1}$, $\hat{2}$. This is accomplished by the mapping

$$x = \cos \varphi \cos \theta$$

$$y = \cos \varphi \sin \theta$$

where x is in the $\hat{1}$ direction (radial) and y is in the $\hat{2}$ direction (tangential).

It is useful to have a relation between φ, θ for the slope $d\varphi/d\theta = 0$; this is of course the case of a contour parallel to the equator for which:

$$\frac{d\varphi}{d\theta} = - \frac{\frac{\partial U}{\partial \theta}}{\frac{\partial U}{\partial \varphi}} = 0$$

if $U_\varphi \neq 0$. From (5.8b) and (5.9b) $U_\theta = 0$ when $\theta = 0, \pm\pi/2, \pi$ and when $\varphi = \pm\pi/2$. The point $\varphi = \pm\pi/2$ is never a point of zero slope because $\varphi = \pm\pi/2$ are always equilibrium points. We then have $d\varphi/d\theta = 0$ when $\theta = 0, \pm\pi/2, \pi$ but $\theta = 0, \pm\pi/2, \pi$ are not equilibrium points. This is useful in plotting curves of U = constant.

The curves of constant U are shown in Figs. 5.7, 5.8, 5.9, . , and 5.10 for the cases (a) - (g) of Fig. 5.6. The points (+) are (stable) minima and the points (-) are maxima of U.

If initially the tip of the symmetry axis $(\hat{3}b)$ is within a closed contour, $U = H_o$, then, because $R_2 > 0$, we have $U \leq H_o$ for all subsequent motion. If a closed contour surrounds a minimum point, then a motion exceeding $U = H_o (U > H_o)$ would require $R_2 < 0$ which is impossible. This proves that if in Fig. 5.7, Case (a), we have an initial position P, and if the initial velocities give an $H = H_o$, <u>then the symmetry axis $(\hat{3}b)$ will always remain within the curve $C(U = H_o)$</u>.

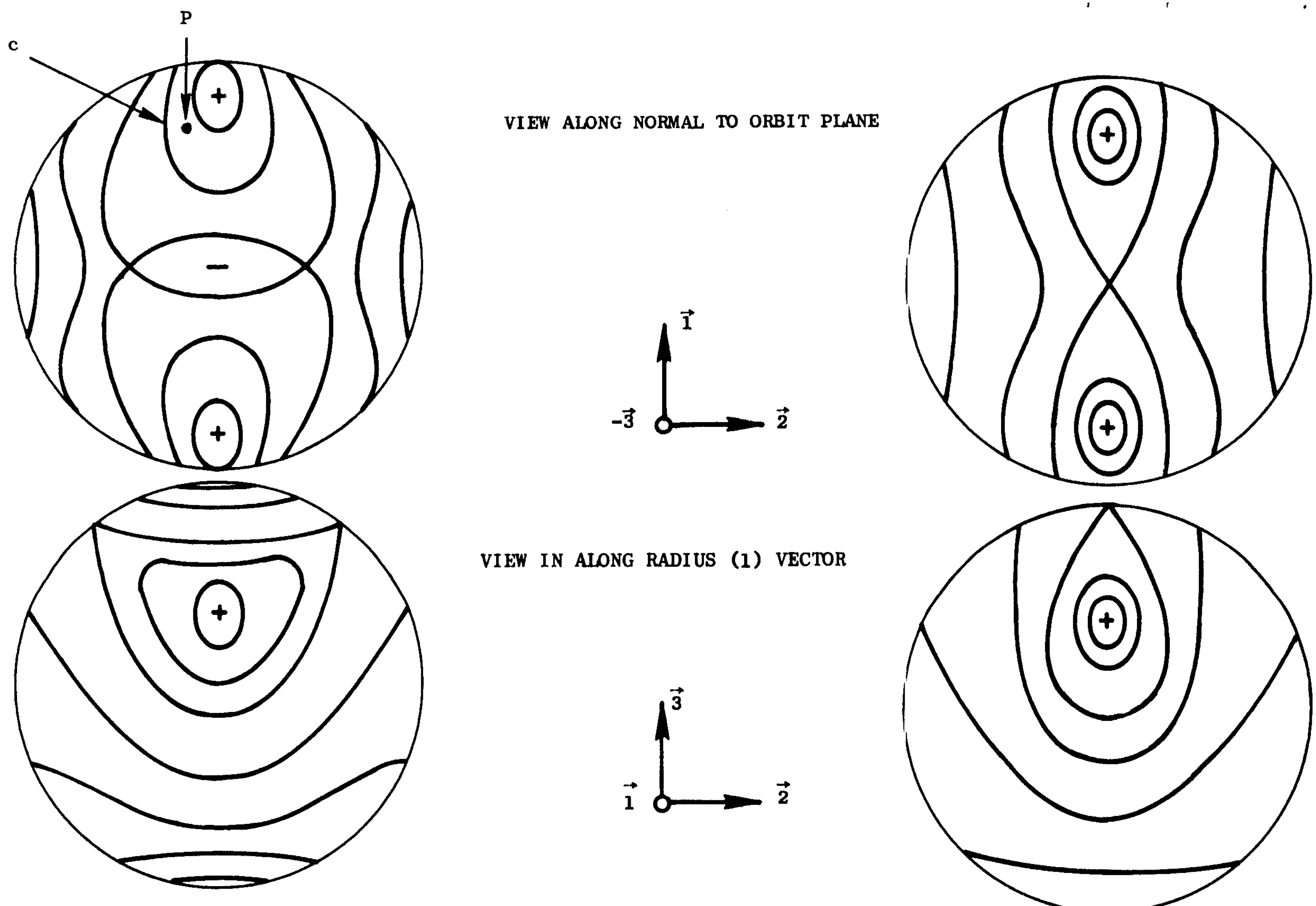
The motions of the $\hat{3}b$ axis within a contour C begin at a point P and never leave the interior of C. These motions will, however, reach C and at such times the relative angular velocity will vanish. At these instants the trajectories in the configuration space will either be tangent to C or form cusps along C.

The maximum regions of bounded motions about a minimum of U are surrounded by "separatrix" curves which always pass through saddle-points. If φ_s, θ_s is the position of a saddlepoint then separatrix curves are defined by:

$$\cos^2\theta = \frac{3(r-1)\cos^2\varphi_s \cos^2\theta_s + \cos^2\varphi - \cos^2\varphi_s + \sin\alpha(\sin\varphi - \sin\varphi_s)}{3(r-1)\cos^2\varphi}$$

where $\sin\alpha = \ell r/n$, $\sin\beta = \ell r/n(4 - 3r)$. For the cases where separatrices exist we have

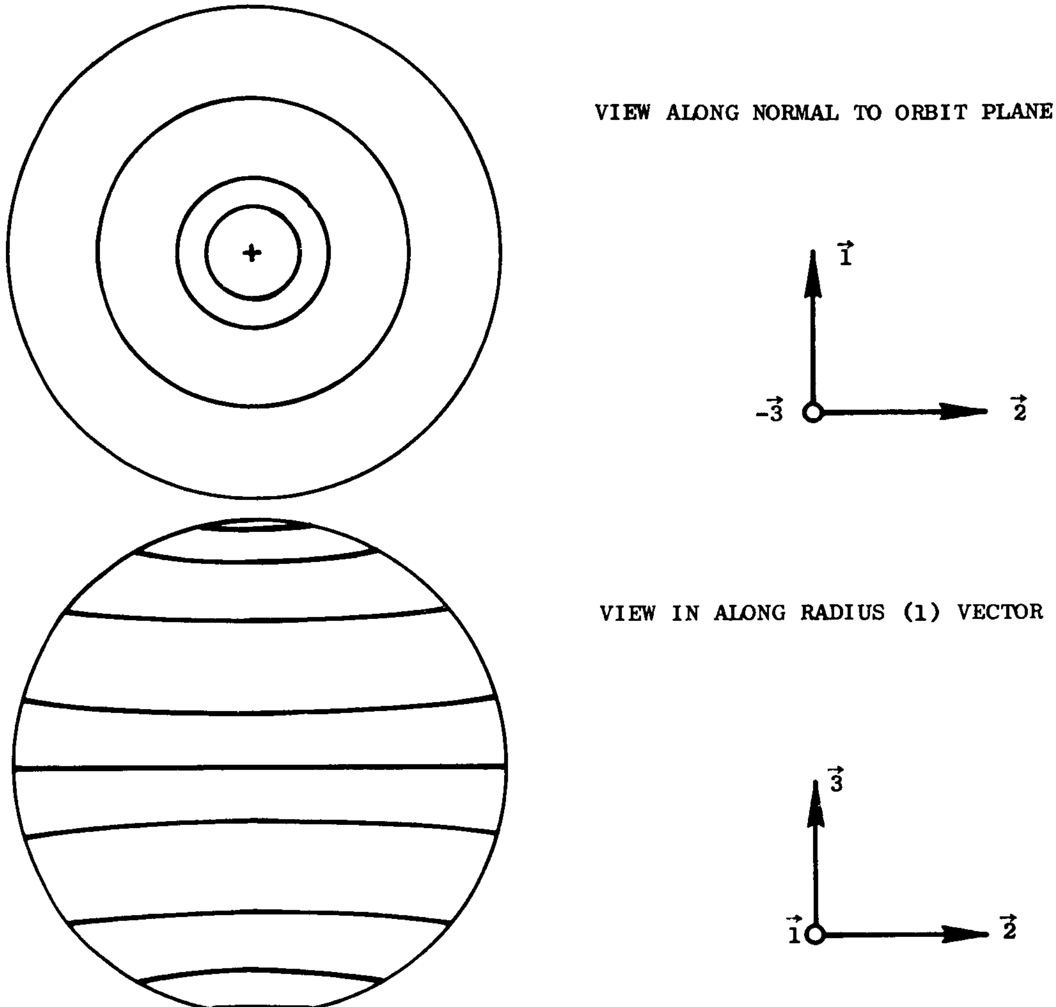
$$\text{Case a:} \quad \theta_s = \pm\pi/2; \qquad \varphi_s = \alpha$$



Case (a): Long, Slender Bodies

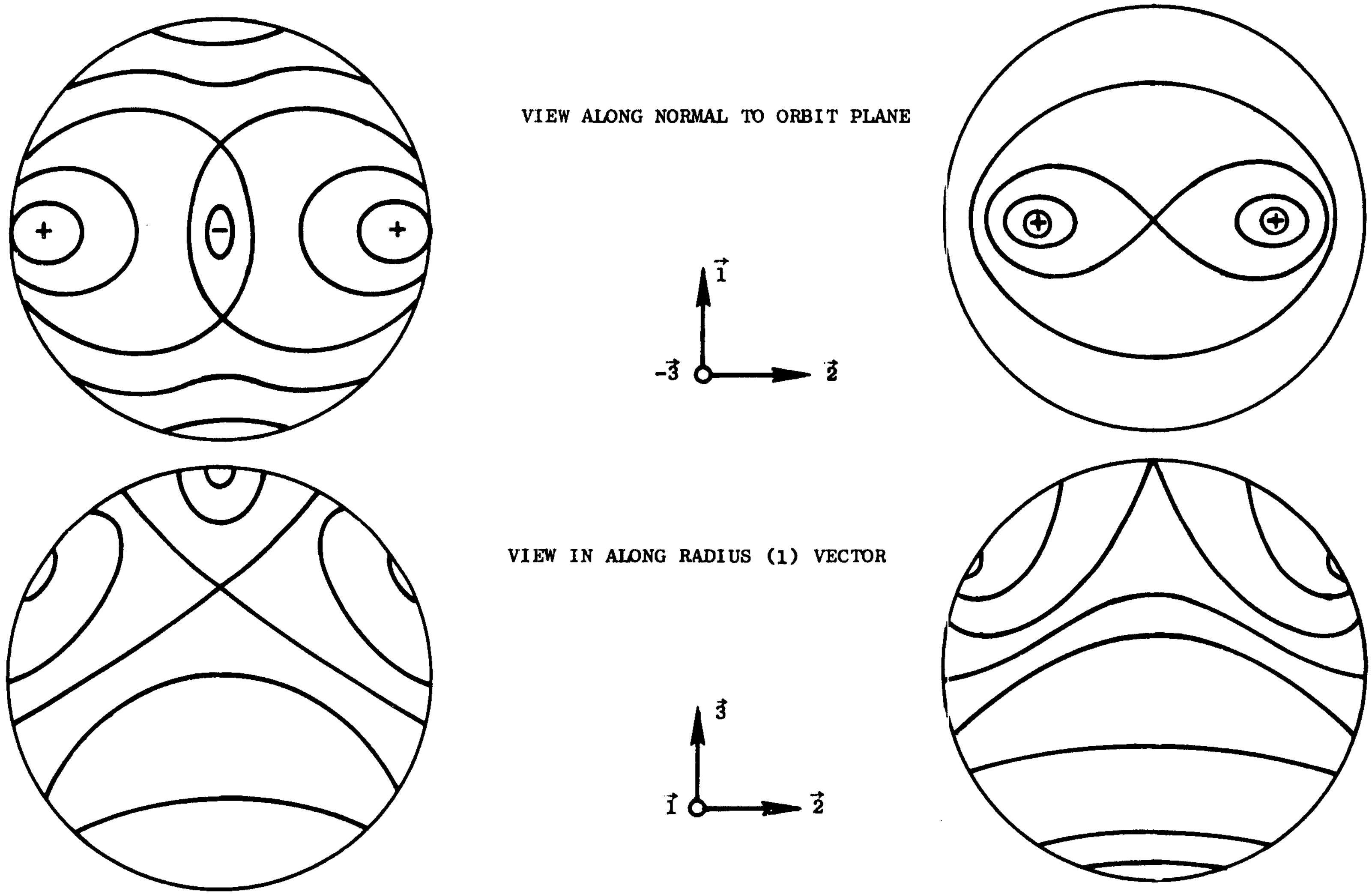
Case (b): Long, Slender Bodies

FIG. 5.7. CURVES OF CONSTANT U: CASES (a) AND (b)



Case (c): High Spin

FIG. 5.8. CURVES OF CONSTANT U: CASE (c)



Case (d): Flat Bodies, Low Spin

Case (e): Flat Bodies, Low Spin

FIG. 5.9. CURVES OF CONSTANT U: CASES (d) AND (e)

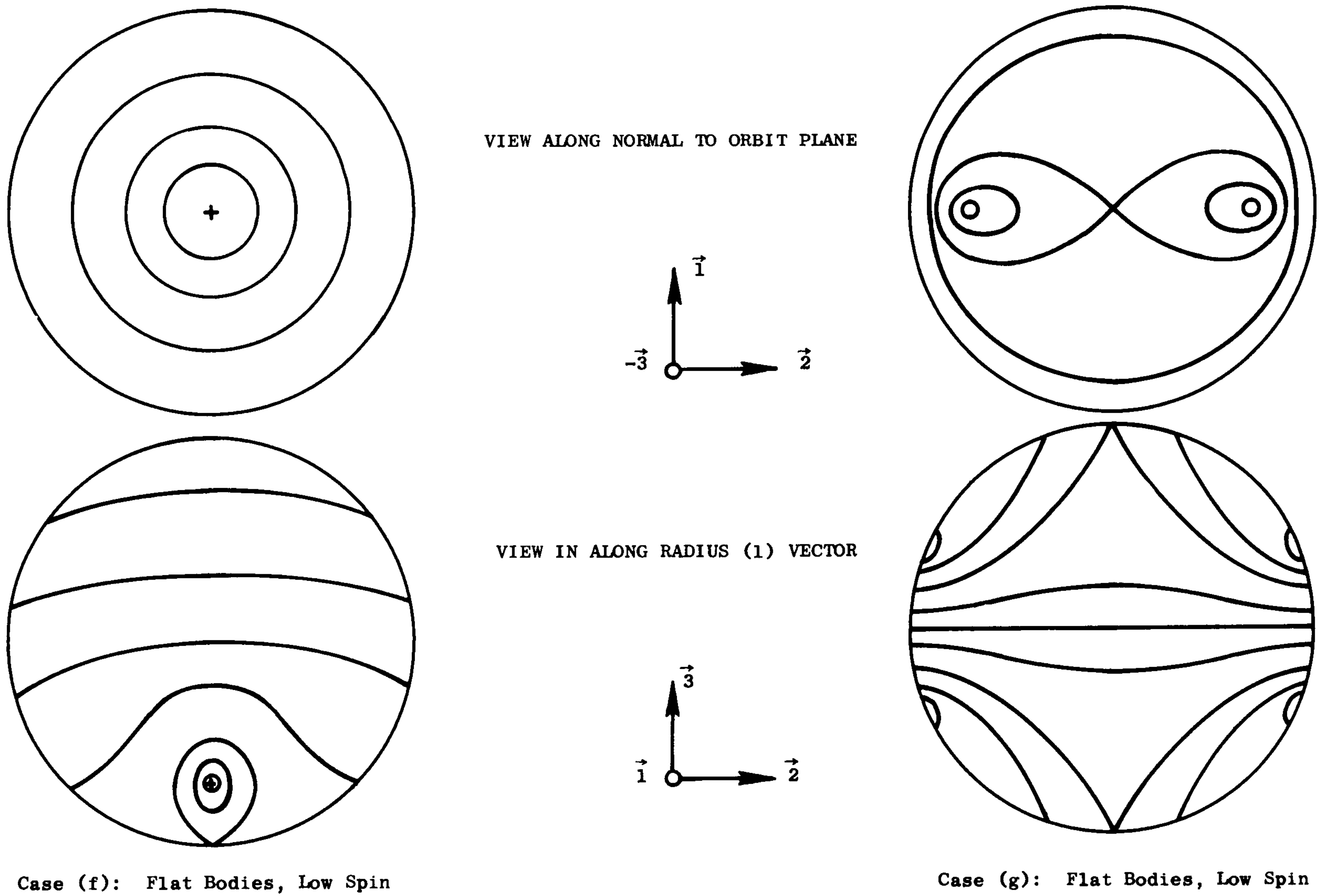


FIG. 5.10. CURVES OF CONSTANT U: CASES (f) AND (g)

Case b: $\varphi_s = \pi/2$

Case c: No Separatrix

Case d: $\theta_s = 0, \ \pi \ ; \varphi_s = \beta$

Case e: $\varphi_s = \pi/2$

Case f: $\varphi_s = -\pi/2$

Case g: $\varphi_s = \pm \pi/2$

The curves surrounding a maximum point of U tell us nothing about stability, but we can show, using small oscillation theory, that such a point may be stabilized if ℓ/n is large enough. Thomson and Kane (THOMSON 2 and KANE 1) have shown that there are points of region (a) of Fig. 5.6 that are stable. The condition that must hold for this to occur in region (a) is, for small oscillations,

$$\left(\frac{\ell r}{n}\right)^2 - 2\left(\frac{\ell r}{n}\right) + (3r - 1) > 0$$

An example is $\ell/n = 3$, $r = 0.1$ which is stable in the small and also lies in the "maximum of U" region of φ, θ. For large motions in region (a) the non-linear librations about $\varphi_o = \pi/2$ must be investigated. This has not been done.

It is well known, as reflected in (5.10), that gyroscopic forces can stabilize a system of two degrees of freedom with such a potential maximum as occurs in region (a). The curves surrounding a maximum point of U only provide bounds on the nearness of approach to equilibrium; this is not too helpful. One might be apprehensive about the maintenance of stability around a maximum of U in the presence of damping; this apprehension is justified by recourse to Lyapunov's stability theory.

It can be shown that if damping (energy dissipation) occurs due to forces which do not effect the mass distribution of the satellite (e.g., control jets or magnetic losses due to the earth's field) or to forces of internal damping involving elements with low mass and inertia, then the stability of the symmetry axis will occur if and only if at the equilibrium point U has a relative minimum (see Chapter III). This indicates that for engineering purposes stability only occurs in those situations where U has a relative minimum. It must, however, be stressed that each satellite system should be analyzed completely on its own. Results from the above analysis should only be applied to the system defined in Section B. The general approach of this chapter is applicable to a wide class of systems with and without damping; the extensions to these cases are given in Chapter III.

F. SUMMARY

The results of this chapter are presented in Figs. 5.6-5.10. Figure 5.6 is the parameter plane of spin velocity versus the moment of inertia ratio. The curves represent "curves of bifurcation"; if one of these curves is crossed, the qualitative nature of the stability of equilibrium points changes.

The seven regions separated by the curves of bifurcation are shown in Figs. 5.7-5.10. Cases (a) and (b) are for long, slim bodies, and in these cases the symmetry axis tends to point along the radius vector from the earth. It is displaced out of the orbit plane due to spin (gravity torque and gyroscopic torque is balanced) and its equilibrium (+) points therefore appear as in Fig. 5.7.

Case (c) is for a short, flat body; this case has an equilibrium solution with $\hat{3}b$ normal to the orbit plane, as in Fig. 5.8.

Cases (d) and (e) are very low-spin, short, flat bodies and they tend to point with their symmetry axes tangent to the orbit.

Cases (f) and (g) possess the property of having equilibriums in the lower hemisphere of the unit sphere at $\varphi_o \neq -\pi/2$; otherwise Case (f) is similar to Case (c) and Case (g) is similar to Case (e).

The technique used to find the stability regions can also be used for more complicated systems but the geometry is more difficult in several dimension spaces. To get the maximum-U boundary curve surrounding a particular minimum point, however, is fairly simple. If all the equilibriums are located (it is not always simple to solve the transcendental equations), then the value of U for these equilibriums can be found. Starting at a minimum point we simply seek the lowest-U saddle curve with energy greater than that of the minimum point in question; this is the lowest energy separatrix curve for the equilibrium point in question.

CHAPTER VI. A METHOD FOR CALCULATING THE NATURAL MOTIONS OF AN OSCILLATORY SYSTEM WITH LIGHT NON-LINEAR DAMPING

A. INTRODUCTION

In the application of dynamics to many engineering systems with mechanical elements the effect of energy losses, called damping, is of critical importance. There are a variety of physical processes which may provide this damping; often the damping law is a non-linear function of the coordinates of the system. Frequently, especially in space mechanics and vibration applications, the forces providing the damping are weak compared to the other forces of the motion. This means that over one period of oscillation the system response is not greatly effected by the damping forces. In a linear system this would be called lightly-damped behavior.

This chapter presents a method, similar to the Krylov-Bogoliubov method (KRYLOV 1 and BOGOLIUBOV 1), for attacking the problem of lightly-damped oscillation in non-linear systems with several degrees of freedom. This method handles a large number of damping laws and requires only that the average power loss from damping be specified. Thus we need not know the damping forces explicitly to get a solution accurate to first order. The approach taken is simple in that very little algebra is necessary to get the expressions for damped behavior. The method, in effect, extends the classical method of Lindstedt-Poincaré (MINORSKY 1 and 2) to the case of lightly-damped behavior. These characteristics make this a natural technique for handling damping by force hysteresis, e.g., elastic or magnetic hysteresis.

B. METHOD

The discussion which follows will center around the basic equations of a mechanical system as conveniently derived using the Lagrange equations. The system is described in terms of generalized coordinates, $q_j (j = 1,2,\ldots,N)$, and generalized damping forces, Q_j, as follows

(summing on repeated indices)

$$a_{ij}\ddot{q}_j + b_{ij}\dot{q}_j + c_{ij}q_j = Q_i \qquad (6.1)$$

where $a_{ij} = a_{ji}$, $b_{ij} = -b_{ji}$, $c_{ij} = c_{ji}$. The b_{ij} are the so-called "gyroscopic" terms. The gyroscopic terms as defined do not include a b_{ii} term..

These equations of motion are given with constant coefficients for simplicity of exposition. However, each step in the subsequent analysis can be made for the general case by using the Lagrange equations and their energy relation directly. This is not included, but is an easy exercise. after identifying H as the Hamiltonian function.

The energy relation is found by multiplying (6.1) by $\dot{q}_i$, summing on i and j, and integrating with respect to time (a_{ij}, c_{ij} are independent of time and q_i, $\dot{q}_j$). This gives:

$$H = \frac{1}{2} a_{ij}\dot{q}_i\dot{q}_j + \frac{1}{2} c_{ij}q_i\, q_j = H_o + \int_o^t Q_i\dot{q}_i dt \qquad (6.2)$$

where H_o is the value of H at $t = 0$. If $Q_i = 0$ for all i then $\dot{H} = 0$ and $H = H_o$, a constant (this is the case without damping forces Q_i). By differentiation of (6.2) we get the important relation

$$\boxed{P = \dot{H} = \sum_{i=1}^{N} Q_i\dot{q}_i = \left\{ \begin{matrix} \text{Power into system} \\ \text{by external forces} \end{matrix} \right\}} \qquad (6.3)$$

There is implicit in (6.1), (6.2), and (6.3) the fact that the equations of motion are linear with constant coefficients (see Chapter III for a generalization of the above results).

1. Analysis for $Q_i = 0$

If the elements a_{ij}, b_{ij}, c_{ij} are not dependent on $q_i, \dot{q}_i$ then the system is linear and we can write down the linear solution to the case $Q_i = 0$ in the well-known manner. Define $\bar{q}_j$ a complex number by $q_j \triangleq \mathrm{Re}\ \bar{q}_j$. (Re stands for "the real part of") and $\bar{q}_j = A_j e^{\lambda t}$. Substitution of $\bar{q}_j$ into (6.1) for q_j gives for $Q_j = 0$ (all j):

$$(\lambda^2 a_{ij} + \lambda b_{ij} + c_{ij})A_j = 0 \tag{6.4}$$

which can be written,

$$M_{ij}(\lambda)A_j = 0$$

where the matrix M_{ij} is defined to be

$$M_{ij}(\lambda) \triangleq \lambda^2 a_{ij} + \lambda b_{ij} + c_{ij} .$$

Note that by using the properties of a_{ij}, b_{ij}, c_{ij} we can write

$$|M_{ji}(\lambda)| = |M_{ij}(\lambda)| = |M_{ij}(-\lambda)|$$

where $|M_{ij}|$ stands for the determinant of M_{ij}. This proves that the characteristic determinant is an even function of λ and thus the roots are $\pm \lambda_j$, where λ_j is either purely real or purely imaginary. We are interested in the "stable, oscillatory" case where the λ_j are all purely imaginary and, therefore, $\lambda_j = \pm i\omega_j$, ω_j real and positive. The reality of $\omega_j (j = 1,2,\ldots,N)$ is a necessary but not sufficient condition for asymptotic stability (see Theorem V of Chapter III). We assume in what follows that the motions are asymptotically stable.

The solution may be written as

$$\bar{q}_j = A_{jk}\Lambda_k e^{i(\omega_k t+\psi_k)} \qquad (j = 1,2,\ldots,N)$$

$$\dot{\bar{q}}_j = i\omega_k A_{jk}\Lambda_k e^{i(\omega_k t+\psi_k)} \tag{6.5}$$

where Λ_k is the (real) amplitude and ψ_k is the (real) phase, and the A_{jk} relate to the mode shape. There are N^2 of the A_{jk}, but we know they are related by the fact that there are only 2N arbitrary constants in the solution (these are the Λ_k, ψ_k). Therefore, there are only N independent A_{jk} each having a real and imaginary part. These are chosen arbitrarily. The relations between the A_{jk} are found from equations (6.4) and

$$M_{ij}A_{jk} = (-\omega_k^2 a_{ij} + i\omega_k b_{ij} + c_{ij})A_{jk} = 0 \tag{6.6}$$

where

$$|M_{ij}(i\omega_k)| = 0$$

for a solution of (6.4) and (6.6) to exist.

The procedure is to (a) find the ω_k, (b) solve (6.6) for A_{jk} in terms of one A_{jk}, say A_{1k}, which is arbitrary, and (c) take the real part of $\bar{q}_j$ to find the motions. If the equations (6.1) are non-linear then steps (a) and (b) above are replaced by any appropriate approximations (MINORSKY 1 and 2).

2. Modification of the Solution for Q_i Small (Light Damping)

The next step, which is crucial, is to admit small damping forces, Q_i, such that the oscillation envelope decays in a time which is long compared to one period of oscillation. The motion of each "normal mode" is a fast oscillation within a slowly varying envelope. For approximation of the "envelope" we shall allow the "arbitrary constants," Λ_k, to vary with time in order to admit small Q_i. The $\Lambda_k(t)$ will be determined by use of the energy relation (6.3) on an average basis.

3. Approximation for the Decay of Λ_k

The crux of the present method is to separate the smoothed (averaged) envelope motion from the vibratory motion. To do this we employ an averaging argument which has asymptotic validity as $Q_i \to 0$ (see Appendix B or BOGOLIUBOV 1, KRYLOFF 1, MINORSKY 1 and 2 for similar reasoning).

Define the average of a function of time as

$$\left\{\overline{f(q_i,\dot{q}_i)}\right\}_{\Lambda_k,\psi_k} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(q_i,\dot{q}_i)dt \qquad (6.7)$$

where Λ_k, ψ_k are held constant (denoted by the subscript). The average energy is, using (6.2), (6.5), (6.7)

$$\overline{H} = \frac{1}{4}\left[a_{ij}\omega_k^2 + c_{ij}\right] Re(A_{ik}A_{jk}^*)\Lambda_k^2 = \frac{1}{2}H_k\Lambda_k^2 \qquad (6.8)$$

where A^* is the complex conjugate of A, and H_k is a constant defined by (6.8). This result is easily modified for non-constant a_{ij}, b_{ij}, c_{ij}. Now the average of the function $\dot{H} = Q_i\dot{q}_i = P(q_j,\dot{q}_j)$ is calculated to be

$$\boxed{\overline{\dot{H}} = \overline{P}(q,\dot{q}) = \dot{\overline{H}}} \qquad (6.9)$$

where $\overline{\dot{H}} = \dot{\overline{H}}$ if the function H is well behaved (which it is, as is clearly seen from (6.8)). The differentiation $\dot{\overline{H}}$ refers to the fact that the Λ_k vary with time to make (6.9) hold.

Now postulate that the power expressions is a polynomial in $q_i,\dot{q}_i$.

$$\sum_{j=1}^{N} Q_j\dot{q}_j = P(q,\dot{q}) = -\alpha_{ij}\dot{q}_i\dot{q}_j - \beta_{ijk}q_i\dot{q}_j\dot{q}_k$$
$$- \gamma_{ijk}\dot{q}_i\dot{q}_j\dot{q}_k - \ldots\ldots \qquad (6.10)$$

+ FOURTH AND HIGHER DEGREE FORMS IN q_i, $\dot{q}_i$

To average such an expression we substitute q_j, $\dot{q}_j$ into (6.10) and, holding Λ_k, ψ_k constant perform the averaging operation of (6.7). It is easy to see that this operation reduces all the terms with an odd number of indices to zero and gives a form in Λ_k (dropping terms higher than third degree).

$$\bar{P} = -F_k \Lambda_k^2 - G_{jk} \Lambda_k^2 \Lambda_j^2 - J_{jk\ell} \Lambda_j^2 \Lambda_k^2 \Lambda_\ell^2 \tag{6.11}$$

The coefficients in (6.11) stand for functions of frequency and can be derived from an expression such as (6.10) or from experimental considerations directly.

Now we use (6.8), (6.9), and (6.11) to calculate the power balance requirement. This is one condition to determine the $N \Lambda_k$'s. This leads to (defining a useful function W)

$$W \triangleq \sum_k \Lambda_k \left\{ H_k \dot{\Lambda}_k + F_k \Lambda_k + \sum_{j=1}^{N} G_{jk} \Lambda_j^2 \Lambda_k \right. \tag{6.12}$$
$$\left. + \sum_{j=1}^{N} \sum_{\ell=1}^{N} J_{jk\ell} \Lambda_j^2 \Lambda_\ell^2 \Lambda_k \right\} = 0$$

We need N-1 conditions on Λ_k to fully determine the envelope motion. These are provided by letting $\Lambda_k = \Lambda_k \left[\Lambda_j(o); t\right]$ $(j = 1,2,\ldots,N)$; that is, each Λ_k motion depends only on N <u>arbitrary</u> initial motions and time. Note that, $\Lambda_k(o) = \Lambda_k \left[\Lambda_j(o), o\right]$, by definition (see (3.2)).

This requirement is a direct consequence of our desire to describe envelope motion by N differential equations of first order. Note that the expression (6.12) for W is to vanish <u>identically</u> for all motions Λ_k, What if all the terms in curly brackets in (6.12) are zero except one term which is not? If we express W as

$$W = \sum_{k=1}^{N} \Lambda_k g_k(\Lambda, t) = 0 \tag{6.13}$$

where (not summing on k)

$$g_k(\Lambda, t) = H_k \dot{\Lambda}_k + F_k \Lambda_k$$

$$+ \sum_{j=1}^{N} G_{jk} \Lambda_j^2 \Lambda_k + \sum_{j,\ell} J_{jk\ell} \Lambda_j^2 \Lambda_\ell^2 \Lambda_k$$

The assumption is

$$\left\{ \begin{array}{lll} g_k = 0 & \text{for} & k \neq n \\ g_k \neq 0 & \text{for} & k = n \end{array} \right\} \tag{6.14}$$

But from (6.13) ($W \equiv 0$)

$$\Lambda_n(t) \; g_n(\Lambda, t) = 0 \qquad \text{(no sum)}$$

which requires $\Lambda_n(t) = 0$ all t. This contradicts our assumption that the $\Lambda_j(o)$ are arbitrary. Clearly the assumption of which number n takes is arbitrary; so the following hold for all t and k:

$$\boxed{\begin{aligned} g_k = H_k \dot{\Lambda}_k + F_k \Lambda_k + \sum_{j=1}^{N} G_{jk} \Lambda_j^2 \Lambda_k \\ + \sum_{\substack{j=1 \\ \ell=1}}^{N} J_{jk\ell} \Lambda_j^2 \Lambda_\ell^2 \Lambda_k = 0 \end{aligned}} \tag{6.15}$$

This is the fundamental equation of the method. From this equation we can deduce the motion, Λ_k, of the envelope of oscillations for sufficiently small power loss functions. There are N first order non-linear differential equations with N initial conditions. The solution to these equations is not possible in general because of the non-linear coupling; but for $N = 2$ it is possible to discuss the equations using phase plane methods.

The equations (6.15) represent an extension of Rayleigh's observation that for viscous-type damping $(P = \alpha_{ij}\, \dot{q}_i\, \dot{q}_j)$ for forces and thus the motions can be derived from a single function, the power (STRUTT 1). He named this the "dissipation function," an apt expression because Rayleigh excluded from P the part of Q_i derivable from a potential; the remaining part, of course, defines the damping. Our function $\overline{P}$ is sufficient to derive a first approximation in the cases where (6.11) holds; thus the assertion that $\overline{P}$ represents an extended Rayleigh dissipation function.

Summary of the Method: Given an expression for $\overline{P}$, either from an expression like (6.10) or empirically, one uses the expressions for the energy (6.2) in the following manner.

(a) Obtain the solution to the linear differential equations (6.1) with the damping forces $Q_i = 0$. This involves finding the matrix A_{jk} by solving the set of equations (6.4) and also finding the natural frequencies ω_k from the determinant of (6.6).

(b) Calculate the average energy (Hamiltonian) using (6.5) and (6.2). This gives equation (6.8) and thus the coefficients H_k.

(c) Use the coefficients of $\overline{P}$ (given) and the H_k to complete specification of (6.15).

(d) Solve (6.15) by approximate or analytical means or by using a digital computer. A computer will be saved many integration points, and therefore computations, by using (6.15) in lieu of (6.1). Of course, if $\overline{P}$ is only known experimentally, then (6.15) represents the only solution because the Q_i are not known ahead of time (before integration of (6.15)).

In the cases where the coefficients of equations (6.1) are not constant but vary with q_i, $\dot{q}_i$, we must make a further observation. If we set $Q_i = 0$ and employ the Lindstedt-Poincare procedure (MINORSKY 1 and 2) to develop an asymptotic series for q_i, $\dot{q}_i$, then we can use these approximations in place of (6.5) as "generating" solutions. The equations (6.15) will be modified only to the extent that their coefficients become functions of Λ_k. It turns out that only perturbation in the phase ψ_k in the Lindstedt-Poincare procedure will effect the damping motion to first order. This addition involving non-linear equations (6.1) is easy to apply upon mastering the arguments leading up to (6.15) but cumbersome to write down in the general case.

To demonstrate the above procedure two examples will be given.

C. TWO EXAMPLES

As examples, a one-degree-of-freedom harmonic oscillator and a two-degree-of-freedom gyroscopic system are chosen. Each of these systems is linear except for the damping law, which is non-linear.

1. Single Degree-Of-Freedom Harmonic Oscillator

This example treats the case of a system described by the differential equation

$$\ddot{q} + \omega_o^2 q = Q(q, \dot{q}) \tag{6.16}$$

for the case where the average power loss law is given by

$$\overline{P} = - b\, q_{max}^{\alpha} \quad (\alpha \geq 1,\ q_{max} \geq 0) \tag{6.17}$$

where b and α are some constants, possibly determined empirically. This power loss could also be derived from a force law, e.g., $Q = - C\, |\dot{q}|^{\alpha-1} \mathrm{sgn}\, \dot{q}$ (where C is simply related to b).*

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* $\mathrm{sgn}\, \dot{q} = \dot{q}/|\dot{q}|$.

The linear solution for $Q = 0$ is given by (see Step (a))

$$q = q_{max} \cos(\omega_o t + \psi) \tag{6.18}$$

$$\dot{q} = -\omega_o\, q_{max} \sin(\omega_o t + \psi)$$

These can be verified by substitution into (6.16) holding q_{max}, ψ constant. If q_{max}, ψ are allowed to vary slowly (which they will if b is sufficiently small) then the average energy and power are (see Steps b and c), since $E = H$,

$$\overline{E} = \frac{\omega_o^2}{2}\, q_{max}^2$$

$$\dot{\overline{E}} = \omega_o^2\, q_{max}\dot{q}_{max} = \overline{P} = -b q_{max}^{\alpha} \tag{6.19}$$

These equations (6.19) are derived by analogy with (6.12). The motion of $q_{max}(t)$ by analogy with Λ of (6.15) is easily seen to be (Step d)

$$\dot{q}_m + \frac{b}{\omega_o^2}\, q_{max}^{\alpha-1} = 0 \tag{6.20}$$

The solution of (6.20) is, by direct integration,

$$q_m(t) = \left\{ q_m^{2-\alpha}(0) - (2-\alpha)\,\frac{b}{\omega_o^2}\, t \right\}^{\frac{1}{2-\alpha}} \qquad (a \neq 2)$$

$$q_m(t) = q_m(0) \exp\left[-\frac{b}{\omega_o^2}\, t\right] \qquad (a = 2) \tag{6.21}$$

The case $\alpha = 2$ corresponds to the ordinary linear damping case; the case $\alpha = 1$ corresponds to an ideal type of coulomb friction ($Q = -C\,\mathrm{sgn}\,\dot{q}$); the other cases may possibly follow some empirical laws for damping.

2. Two-Degree-of-Freedom, Gyroscopically-Coupled Systems

The system to be treated may be based on a model of a two-degree-of-freedom gyroscope with restraining gimbal springs, on the roll-yaw equations of a satellite librating in the gravity field of the earth (DEBRA 1 and 2), or on many other such systems involving cyclic coordinates. The equations of motion are given as

$$\begin{aligned} \ddot{q}_1 + \omega_1^2 q_1 + h\dot{q}_2 &= Q_1 \\ \ddot{q}_2 + \omega_2^2 q_2 - h\dot{q}_1 &= Q_2 \end{aligned} \tag{6.22}$$

The roots of the characteristic equation are found from the factors of the polynomial

$$\Delta(\lambda) = (\lambda^2 + \omega_1^2)(\lambda^2 + \omega_2^2) + h^2\lambda^2 \tag{6.23}$$

We denote these roots by Ω_1, Ω_2 and calculate A_{ij} as in (6.6) letting $A_{1j} = 1$. This gives

$$A_{ij} = \begin{bmatrix} 1 & 1 \\ \dfrac{i\,h\,\Omega_1}{\Omega_1^2 - \omega_1^2} & \dfrac{-i\,h\,\Omega_2}{\Omega_2^2 - \omega_2^2} \end{bmatrix} \tag{6.24}$$

The response can be written down as in (6.5)

$$\begin{aligned} \bar{q}_1 &= \Lambda_1 e^{i(\Omega_1 t+\psi_1)} + \Lambda_2 e^{i(\Omega_2 t+\psi_2)} \\ \dot{\bar{q}}_1 &= i\left\{\Omega_1 \Lambda_1 e^{i(\Omega_1 t+\psi_1)} + \Omega_2 \Lambda_2 e^{i(\Omega_2 t+\psi_2)}\right\} \end{aligned} \tag{6.25}$$

$$\bar{q}_2 = ih \left\{ \frac{\Omega_1 \Lambda_1 e^{i(\Omega_1 t+\psi_1)}}{\Omega_1^2 - \omega_1^2} - \frac{\Omega_1 \Lambda_2 e^{i(\Omega_2 t+\psi_2)}}{\Omega_2^2 - \omega_2^2} \right\}$$

$$\dot{\bar{q}}_2 = -h \left\{ \frac{\Omega_1^2 \Lambda_1 e^{i(\Omega_1 t+\psi_1)}}{\Omega_1^2 - \omega_1^2} - \frac{\Omega_1 \Omega_2 \Lambda_2 e^{i(\Omega_2 t+\psi_2)}}{\Omega_2^2 - \omega_2^2} \right\}$$

The energy is found from (6.22) to be (as in (6.2)), since $E = H$,

$$E = \frac{1}{2} (\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2} (\omega_1^2 q_1^2 + \omega_2^2 q_2^2) \tag{6.26}$$

The average energy is computed, using (6.25) and (6.26)

$$\begin{aligned} \bar{E} = {} & \frac{1}{4} (\Omega_1^2 \Lambda_1^2 + \Omega_2^2 \Lambda_2^2) + \frac{1}{4} \omega_1^2 (\Lambda_1^2 + \Lambda_2^2) \\ & + \frac{1}{4} (\Omega_1^2 + \omega_2^2) \left(\frac{\Omega_1 h \Lambda_1}{\Omega_1^2 - \omega_2^2} \right)^2 \\ & + \frac{1}{4} (\Omega_2^2 + \omega_2^2) \left(\frac{\Omega_2 h \Lambda_2}{\Omega_2^2 - \omega_2^2} \right)^2 \end{aligned} \tag{6.27}$$

This energy expression is actually exact $(E = \bar{E})$ since the solution (6.25) is exact and $\dot{E} = 0$ if Λ_1, Λ_2 are constant. This is true in all linear cases of (6.8).

The coefficients in (6.8) are

$$\begin{aligned} 2E_1 &= \omega_1^2 + 2\Omega_1^2 + \omega_2^2 \left(\frac{\Omega_1^2 h^2}{(\Omega_1^2 - \omega_1^2)^2} \right) \\ 2E_2 &= \omega_2^2 + 2\Omega_2^2 + \omega_2^2 \left(\frac{\Omega_2^2 h^2}{(\Omega_2^2 - \omega_2^2)^2} \right) \end{aligned} \tag{6.28}$$

As an interesting example of the method, the dissipation forces are assumed to be of the non-linear form,

$$Q_1 = -D_1\dot{q}_1 - D_2 q_1^2 \dot{q}_1 \tag{6.29}$$

$$Q_2 = 0$$

Thus the power is, applying (6.2)

$$P = Q_1\dot{q}_1 = -D_1\dot{q}_1^2 - D_2 q_1^2 \dot{q}_1^2 \tag{6.30}$$

and the average of the second term of (6.30) is

$$\lim_{t\to\infty} \frac{D_2}{T}\int_0^T \left\{\Lambda_1\cos\phi_1 + \Lambda_2\cos\phi_2\right\}^2 \left\{\Omega_1\Lambda_1\sin\phi_1 + \Omega_2\Lambda_2\sin\phi_2\right\}^2 dt \tag{6.31}$$

where the integration of (6.31) is carried out holding Λ_1, Λ_2, ψ_1, ψ_2 constant and where $\phi_1 = \Omega_1 t + \psi_1$, $\phi_2 = \Omega_2 t + \psi_2$. The average of (6.31) is, of course, equivalent to the first term in the Fourier series P. Thus we have

$$\begin{aligned}\overline{P} = -\frac{D_1}{2}(\Omega_1^2\Lambda_1^2 + \Omega_2^2\Lambda_2^2) - \frac{D_2}{4}\Big\{&(\Lambda_1^2 + \Lambda_2^2)(\Lambda_1^2\Omega_1^2 + \Lambda_1^2\Omega_2^2) \\ &- \tfrac{1}{2}(\Lambda_1^4\Omega_1^2 + \Lambda_2^4\Omega_2^2)\Big\}\end{aligned} \tag{6.32}$$

By applying (6.15) we get two differential equations for the envelopes Λ_1 and Λ_2.

$$\begin{aligned} E_1\dot{\Lambda}_1 + \frac{D_1}{2}\Omega_1^2\Lambda_1 + \frac{D_2}{4}\Lambda_1\left\{\frac{1}{2}\Lambda_1^2\Omega_1^2 + \Lambda_2^2\Omega_2^2\right\} = 0 \\ E_2\dot{\Lambda}_2 + \frac{D_1}{2}\Omega_2^2\Lambda_2 + \frac{D_2}{4}\Lambda_2\left\{\frac{1}{2}\Lambda_2^2\Omega_2^2 + \Lambda_1^2\Omega_1^2\right\} = 0 \end{aligned} \tag{6.33}$$

Let S_1, S_2, R_1, R_2, μ be defined as

$$S_1 = \frac{D_1 \Omega_1^2}{2E_1}$$

$$S_2 = \frac{D_1 \Omega_2^2}{2E_2} \tag{6.34}$$

$$\mu R_1 = \frac{D_2}{4E_1}$$

$$\mu R_2 = \frac{D_2}{4E_2}$$

The μ parameter is assumed small so as to approximate the equations

$$\dot{\Lambda}_1 + S_1\Lambda_1 + \mu R_1 \left\{ \frac{1}{2}\Lambda_1^2\Omega_1^2 + \Lambda_2^2\Omega_2^2 \right\}\Lambda_1 = 0$$

$$\dot{\Lambda}_2 + S_2\Lambda_2 + \mu R_2 \left\{ \frac{1}{2}\Lambda_2^2\Omega_2^2 + \Lambda_1^2\Omega_1^2 \right\}\Lambda_2 = 0 \tag{6.35}$$

by the expression

$$\Lambda_1 = \Lambda_1^o + \mu\Lambda_1^1 + \ldots..$$

$$\Lambda_2 = \Lambda_2^o + \mu\Lambda_2^1 + \ldots.. \tag{6.36}$$

Using (6.36) in (6.35) we get by identifying coefficients of powers of μ up to order μ^2,

$$\dot{\Lambda}_1^o + S_1 \Lambda_1^o = 0$$

$$\dot{\Lambda}_2^o + S_2 \Lambda_2^o = 0$$

$$\dot{\Lambda}_1^1 + S_1 \Lambda_1^1 + R_1 \left\{ \frac{1}{2} (\Lambda_1^o)^2 \Omega_1^2 + (\Lambda_2^o)^2 \Omega_2^2 \right\} = 0 \tag{6.37}$$

$$\dot{\Lambda}_2^1 + S_2 \Lambda_2^1 + R_2 \left\{ \frac{1}{2} (\Lambda_2^o)^2 \Omega_2^2 + (\Lambda_1^o)^2 \Omega_1^2 \right\} = 0$$

It is assumed that initially (t = 0)

$$\Lambda_1(o) = \Lambda_1^o(o) \quad ; \quad \Lambda_1^1(o) = 0$$

$$\Lambda_2(o) = \Lambda_2^o(o) \quad ; \quad \Lambda_2^1(o) = 0 \tag{6.38}$$

Solving recursively (6.37) subject to (6.38) we obtain in a straightforward manner a first-order approximation in μ.

$$\Lambda_1(t) = e^{-S_1 t} \left\{ \Lambda_1(o) - \mu R_1 \left[\frac{\Omega_1^2 \Lambda_1^2(o)}{2S_1} (1 - e^{-S_1 t}) + \frac{\Omega_2^2 \Lambda_2^2(o)}{2S_2 - S_1} (1 - e^{-(2S_2 - S_1)t}) \right] \right\}$$

$$\Lambda_2(t) = e^{-S_2 t} \left\{ \Lambda_2(o) - \mu R_2 \left[\frac{\Omega_2^2 \Lambda_2^2(o)}{2S_2} (1 - e^{-S_2 t}) + \frac{\Omega_1^2 \Lambda_1^2(o)}{2S_1 - S_2} (1 - e^{-(2S_1 - S_2)t}) \right] \right\} \tag{6.39}$$

We observe that if $\mu R_1 > 0$ the perturbation decreases the envelope response time and if $\mu R_1 < 0$ the envelope response is slowed down. This is as we might expect from power considerations since $\mu R_1 > 0$ means more power loss at a given Λ_1, Λ_2 and vice versa. This can be seen analytically by calculating for $\Lambda_2(o) = 0$, the "time constant," τ, defined as the time for the value of $\Lambda_1(t)/\Lambda_1(o)$ to reach $1/e$. To first order in μ this is easily found to be

$$\tau = \frac{1}{S_1} \left\{ 1 - 0.316. \frac{\mu R_1}{S_1} \Omega_1^2 \Lambda_1(o) \right\}$$

This shows that to first order in μ, τ decreases linearly with increasing $\Lambda_1(o)$ for $\mu R_1 > 0$.

D. CONCLUDING REMARKS

In this chapter the author has attempted to show how, by applying a power balance relation, one can arrive at the "envelope motions" of a lightly-damped oscillatory system. The method depends on knowledge of the undamped differential equations plus the expression for average power loss. The expression for average power loss is sufficient for a first-order approximation in a large number of cases involving non-linear damping.

The above method may be especially important in the analysis of vibrations involving empirically known damping laws. The results are derived more directly and give more physical insight than similar equations derived for one-degree-of-freedom systems by Krylov and Bogoliubov (KRYLOV 1 and BOGOLIUBOV 1). These results will be used extensively in Chapter VII to approximate the damped librations of various satellites.

CHAPTER VII. THE ANALYSIS OF CONNECTED SATELLITES

A. INTRODUCTION

Until about 1960 nearly all work* on gravity gradient stabilization schemes was confined to consideration of a single rigid body with perhaps gyroscopes (CANNON 1) or a "damping sphere" (DEBRA 1) inside for damping. In the last three years several vehicles have been proposed using connected, external moving parts for damping and to magnify the gravity effect. These vehicles seem to be the most efficient purely-passive attitude control systems because they are capable of large relative motion between bodies, and because they can increase stability about any given axis.

This chapter presents the results of an analysis of three particularly interesting connected satellites, interesting because they possess important engineering advantages while at the same time creating certain unusual dynamical problems. The analysis of each satellite is given in essentially similar steps. These are: (1) equations of motion, (2) stability of equilibrium, (3) response to disturbances due to an eccentric orbit, (4) damping of the transient motions.

The analysis of these examples is preceded by a brief discussion of a single rigid body in order to display clearly with a simple example some features of the more complicated systems. The techniques developed in Chapters II, III, and VI are used freely in this chapter.

The three satellite designs to be discussed are:

(a) Vertistat - This vehicle, shown in Fig. 7.1, consists of a long "boom," rigidly attached to the satellite, and two hinged "rods" at the end of the boom. The rods are at equilibrium in a plane normal to the axis of the boom. The boom and the mass on the end provide increased gravity torque stabilization for the body, while the rods are coupled to the boom via a dissipation mechanism to damp the vehicle motions.

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* A notable exception is Reference BREAKWELL 1.

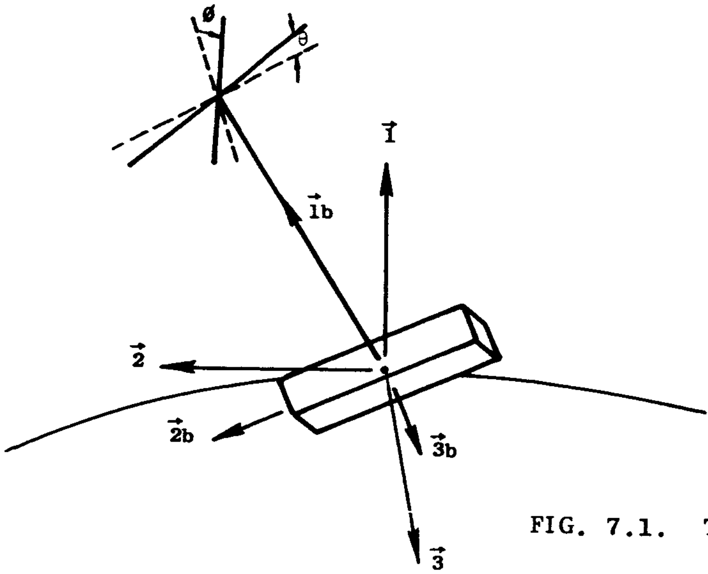

FIG. 7.1. THE VERTISTAT SATELLITE

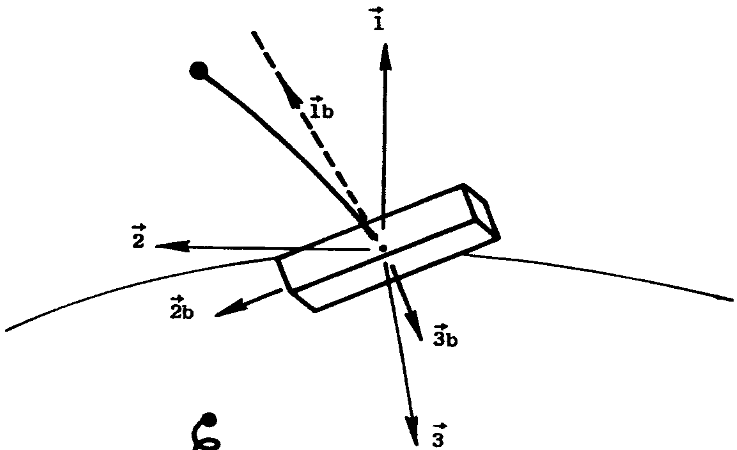

FIG. 7.2. THE BEAM SATELLITE

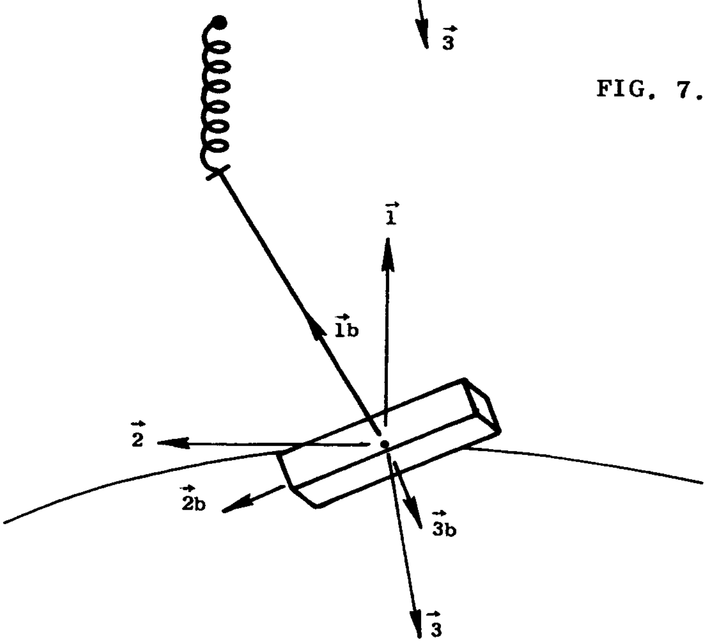

FIG. 7.3. THE TRAAC SATELLITE

(b) Beam - This design (Fig. 7.2) uses a small mass attached to the main satellite by a thin beam; during satellite librations the beam oscillates and thereby dissipates energy (elastic hysteresis), attenuating the vehicle librations.

(c) TRAAC - This satellite design (Fig. 7.3) consists of a small mass attached to the main satellite body via a "lossy spring" and a rod rigidly attached to the body. The spring damps (by elastic hysteresis) the motions in the orbit plane, and ferromagnetic rods interact with the earth's magnetic field to damp out-of-orbit-plane motions. This satellite has been orbited twice.

These three examples of damping and stabilization systems will be discussed in sequence in the sections that follow.

B. THE SINGLE RIGID BODY

It is well known (see Chapter II, Section B) that the kinetic and potential energy expressions for the attitude motion* of a single rigid body in a gravity field are

$$\begin{aligned} T^a_{RB} &= \frac{1}{2}\ \underline{\omega}^a \cdot \Pi^a \cdot \underline{\omega}^a \\ V^a_{RB} &= \frac{3}{2}\ \frac{k}{|\underline{R}^c|^3}\ \hat{R}^c \cdot \Pi^a \cdot \hat{R}^c \end{aligned} \qquad (7.1)$$

where $\underline{\omega}^a$ is the angular velocity of body "a" relative to inertial space axes, Π^a is the body moment of inertia dyadic given in (2.9), $\underline{R}^c$ is the radius vector from the center of attraction to the center of mass of "a", $\hat{R}^c$ is the unit vector in the direction $\underline{R}^c$, and k is the gravitation constant for the particular attracting body in question.

---

* That is, for rotation relative to the mass center of the body.

In order to describe the energy expressions in terms of generalized coordinates we choose a set of Euler angles, γ_1^a, γ_2^a, γ_3^a, described by successive rotations about the $\hat{1}$ axis (γ_1^a), the new $\hat{2}$ axis (γ_2^a), and then the new $\hat{3}$ axis (γ_3^a) (Fig. 7.4). The basis unit vectors are a $\hat{1}$ axis along the radius vector, $\underline{R}^c$, a $\hat{3}$ axis along the orbit angular momentum vector, and a $\hat{2}$ axis normal to the other axes such that $\hat{3} \times \hat{1} = \hat{2}$. The $\hat{2}$ axis points roughly along the velocity vector of the vehicle in orbit. The new set of unit vectors attached to the body along principal axes are $\hat{1}a$, $\hat{2}a$, $\hat{3}a$ and are described by the Euler angles γ_1^a, γ_2^a, γ_3^a. The rotation scheme is given in Fig. 7.4 and the transformation equations are given by

$$
\begin{aligned}
\hat{1}a &= \hat{1}\cos\gamma_2^a \cos\gamma_3^a + \hat{2}[\cos\gamma_1^a \sin\gamma_3^a + \sin\gamma_1^a \sin\gamma_2^a \cos\gamma_3^a] \\
&\quad + \hat{3}[\sin\gamma_1^a \sin\gamma_3^a - \cos\gamma_1^a \sin\gamma_2^a \cos\gamma_3^a] \\
\hat{2}a &= -\hat{1}\cos\gamma_2^a \sin\gamma_3^a + \hat{2}[\cos\gamma_1^a \cos\gamma_3^a - \sin\gamma_1^a \sin\gamma_2^a \sin\gamma_3^a] \\
&\quad + \hat{3}[\cos\gamma_1^a \sin\gamma_2^a \sin\gamma_3^a + \sin\gamma_1^a \cos\gamma_3^a] \\
\hat{3}a &= \hat{1}\sin\gamma_2^a - \hat{2}\sin\gamma_1^a \cos\gamma_2^a + \hat{3}\cos\gamma_1^a \cos\gamma_2^a
\end{aligned}
\tag{7.2}
$$

From geometry based on Fig. 7.4 we can derive the angular rates about the $\hat{1}a$, $\hat{2}a$, $\hat{3}a$ axes as

$$
\begin{aligned}
\omega_1^a &= [\dot{\gamma}_1^a \cos\gamma_2^a \cos\gamma_3^a + \dot{\gamma}_2^a \sin\gamma_3^a + \dot{\eta}(\hat{3} \cdot \hat{1}a)] \\
\omega_2^a &= [\dot{\gamma}_2^a \cos\gamma_3^a - \dot{\gamma}_1^a \cos\gamma_2^a \sin\gamma_3^a + \dot{\eta}(\hat{3} \cdot \hat{2}a)] \\
\omega_3^a &= [\dot{\gamma}_3^a + \dot{\gamma}_1^a \sin\gamma_2^a + \dot{\eta}(\hat{3} \cdot \hat{3}a)]
\end{aligned}
\tag{7.3}
$$

where η is the true anomaly of the satellite's orbit.

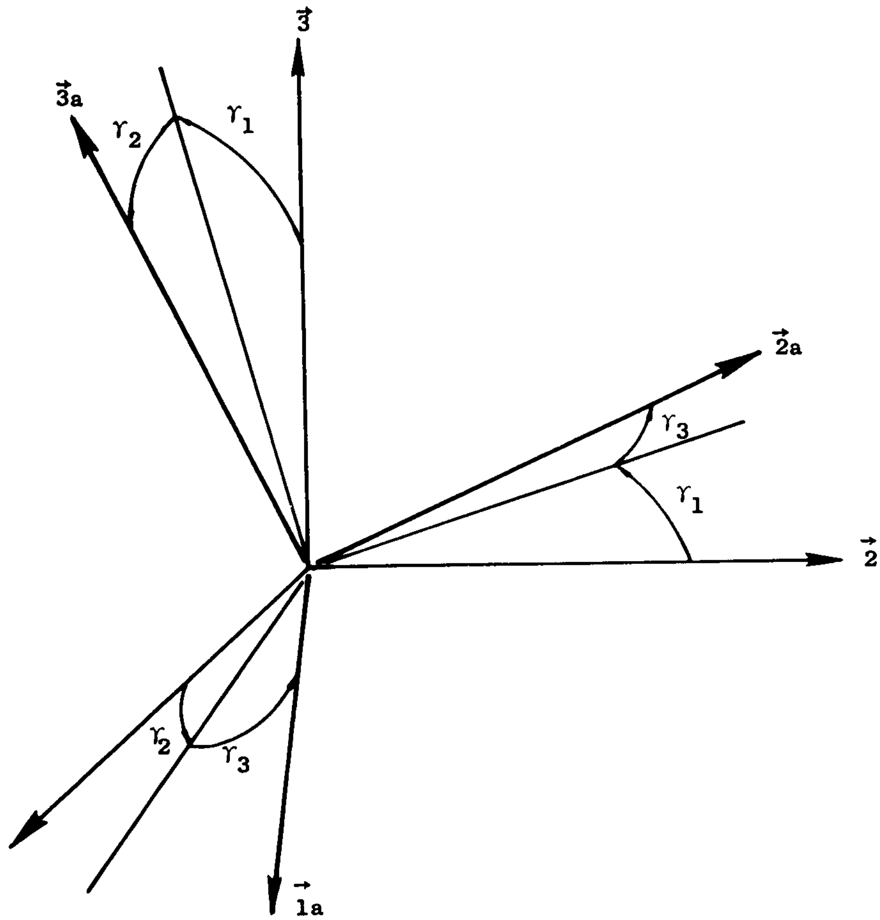

FIG. 7.4. BODY EULER ANGLES

Using (7.2) and (7.3) in (7.1), noticing that $\hat{1} = \hat{R}^c$, and assuming that the $\hat{1}a$, $\hat{2}a$, $\hat{3}a$ axes diagonalize Π^a because they are its principal axes, we can derive the full expressions for the kinetic and potential energy of a single rigid body in the gravity field of a particle.

$$T^a_{RB} = \frac{1}{2}\left[I^a_1(\omega^a_1)^2 + I^a_2(\omega^a_2)^2 + I^a_3(\omega^a_3)^2\right]$$

$$V^a_{RB} = \frac{3}{2}\,\frac{k}{|\underline{R}^c|^3}\left[(I^a_1 - I^a_2)\cos^2\gamma^a_2\cos^2\gamma^a_3 + (I^a_3 - I^a_2)\sin^2\gamma^a_2\right] \tag{7.4}$$

1. Stability of Motion

The case $I^a_2 = I^a_3$ (the "symmetrical" case) has been treated exhaustively in Chapter V from the point of view of the methods of stability analysis discussed there. The case of a general rigid body will be treated here for the small oscillation case, although the methods of Chapter V are available for large-angle analysis. In particular we can obtain bounds on the libration motions using the stability methods of Lyapunov as applied to mechanical systems.

It can be shown easily for the single rigid body that the system is in equilibrium any time the body principal axes are aligned with the rotating axes $\hat{1}$, $\hat{2}$, $\hat{3}$. To investigate this equilibrium, which holds for a circular orbit, we may use the relation $k/|\underline{R}^c|^2 = n^2|\underline{R}^c|$ for circular orbits where n is the angular rate of the satellite orbit. The rate of change of the true anomaly, $\dot{\eta}$, is equal to the mean angular rate, n. Using a small-angle approximation we may write the Hamiltonian for the single rigid body as (see 3.8),

$$H = \frac{1}{2}\, I^a_1(\dot{\gamma}^a_1)^2 + I^a_2(\dot{\gamma}^a_2)^2 + I^a_3(\dot{\gamma}^a_3)^2 + \frac{n^2}{2}\left\{3(I^a_2 - I^a_1)(\gamma^a_3)^2 + 4(I^a_3 - I^a_1)(\gamma^a_2)^2 + (I^a_3 - I^a_2)(\gamma^a_1)^2\right\} \tag{7.5}$$

The above expression for the Hamiltonian gives us the criteria for stability. Referring to the stability method of Chapter III, we see that for H to be positive definite the following inequalities must hold:

$$I_3^a > I_2^a > I_1^a \tag{7.6}$$

With this information we may state that (7.6) is a sufficient condition for the stability of motion of the single rigid body. Furthermore, if there is any dissipation of energy (for example by means of gas-jets and feedback) the condition (7.6) is both necessary and sufficient for asymptotic stability providing there is damping in both pitch (γ_3^a) and yaw-roll (γ_1^a, γ_2^a) motions (see Theorem V, Chapter III, and its corollary).

2. Linear Equations of Motion

From the energy expressions (7.1) we may also derive the equations of motion by using Lagrange's Equations. The linearized* Lagrangian for the single rigid body is

$$L = \frac{1}{2}\left\{ I_1^a(\dot\gamma_1^a - n\gamma_2^a)^2 + I_2^a(\dot\gamma_2^a + n\gamma_1^a)^2 + I_3^a(\dot\gamma_3^a + \dot\eta)^2 \right\} - \frac{n^2}{2}\left\{ 3(I_2^a - I_1^a)(\gamma_2^{a2} + \gamma_3^{a2}) + (I_3^a - I_2^a)(\gamma_2^a)^2 \right\} \tag{7.7}$$

Using this in Lagrange's Equation for the angle γ_i^a gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\gamma_i^a}\right) - \frac{\partial L}{\partial \gamma_i^a} = Q_{\gamma_i^a} \tag{7.8}$$

---

* "Linearizing," refers to the retaining of all linear terms in the equations of motion. This necessitates retaining second-order terms in the energy.

$$\begin{bmatrix} p^2 + k_{32}^a & -h_1^a p & 0 \\ h_2^a p & p^2 + 4k_{31}^a & 0 \\ 0 & 0 & p^2 + 3k_{21}^a \end{bmatrix} \begin{bmatrix} \gamma_1^a \\ \gamma_2^a \\ \gamma_3^a \end{bmatrix} = \begin{bmatrix} Q_{\gamma_1^a}/I_1 n^2 \\ Q_{\gamma_2^a}/I_2 n^2 \\ Q_{\gamma_3^a}/I_3 n^2 \\ + 2e \sin\tau \end{bmatrix}$$

In the above equations the products of angles and eccentricity have been neglected on the assumption of small eccentricity. The relations for expansions of the coordinates in terms of the elliptic elements have been used, (SMART 1). The time was normalized to the orbit angular velocity n by the relation, $\tau = nt$. The nomenclature for the equations of motion in terms of normalized parameters is:

p = s/n = normalized Laplace transform variable

s = Laplace transform variable

τ = nt = normalized time variable

e = orbit eccentricity

$Q_{\gamma_i^a}$ = generalized force -- torques on the body

$k_{21}^a = \dfrac{I_2^a - I_1^a}{I_3^a}$ = body shape parameter

$k_{31}^a = \dfrac{I_3^a - I_1^a}{I_2^a}$ = body shape parameter

$k_{32}^a = \dfrac{I_3^a - I_2^a}{I_1^a}$ = body shape parameter

$$h_1^a = \frac{I_1^a + I_2^a - I_3^a}{I_1^a}$$

$$h_2^a = \frac{I_1^a + I_2^a - I_3^a}{I_2^a}$$

It is very important to notice that the pitch motions (γ_3^a) are uncoupled from the roll-yaw motions to first order in the angles and the eccentricity. This property means that any method used for damping must either damp both motions separately or manage to couple the motions while damping either pitch or yaw-roll.

3. Forced Motion Due to Eccentricity

The forced motion that arises from the term $2e \sin\tau$ on the right of (7.8) disturbs only the pitch motion, to first order. The forced motion is

$$\gamma_3^a = \frac{2e}{3k_{21}^a - 1} \sin \tau \quad . \tag{7.9}$$

This motion will hold always unless the parameter k_{21} is set near a resonance or sub-resonance point of the orbit frequency, or unless the roll-yaw equations possess a natural frequency that is close to the orbit angular velocity. These eventualities have been investigated by DeBra (DEBRA 1) for the non-linear equations of motion. The possible elimination of large motions and instabilities due to resonances is one of the objects of any design procedure for actual damping systems.

4. Damping of the Motions

There is, of course, no damping of the motions of the single rigid body. To damp the motions passively we attach other bodies to the main rigid body and utilize the relative motion between bodies for damping. The *raison d'être* for the systems that follow is to achieve damping and simultaneously to provide greater stability of the main body against disturbances.

C. THE VERTISTAT SATELLITE

This satellite design, similar to Breakwell's "hinged satellite," was first proposed by Kamm (KAMM 1). The satellite (Fig. 7.1) consists of a main body with a long extensible boom rigidly attached;

on the end of the boom, and at right angles to the boom and the orbit plane, is mounted a "pitch rod"; and at right angles to the pitch rod, in a plane normal to the boom, is mounted a "roll rod." These rods are suspended in a pair of torsion bearings from their centers of mass; their motions are kinematically independent. Damping is supplied in the bearings either by the viscous action of a fluid, by magnetic hysteresis between a permanent magnet attached to the boom and ferromagnetic pieces attached to the rods, or by magnetic eddy current effects. The pitch rod provides damping for the body's pitch motion and the roll rod for its (coupled) roll-yaw motion. (Pitch and roll-yaw motions are, to first order, independent.)

This system has been investigated by workers at the Bell Telephone Laboratories (FLETCHER 1, PAUL 1) and by Tinling and Merrick of NASA (TINLING 1). These publications show that such a system could be built and would have dynamical advantages over other passive devices. The Tinling and Merrick paper describes a system with only one bar placed across the orbit at an angle of less than 90 degrees to the orbit plane. This couples all motions and thus provides damping of all modes with just one rod.

1. Energy Expressions

Consider a rigid body described by the Euler angles γ_1, γ_2, γ_3 in the usual sense of rotations (see Sec. B, Part 1). The body is centered at the origin of the $\hat{1}$, $\hat{2}$, $\hat{3}$ system of unit vectors. The kinetic energy can be easily written using (2.7-8) and defining the position of the "pitch rod" (Fig. 7.1) as an angular deflection, θ, from the equilibrium position (which is perpendicular to $\hat{1}b$ and parallel to $\hat{2}b$) in a counterclockwise rotation about an axis parallel to the $\hat{3}b$ axis. The position of the "roll rod" is defined as a counterclockwise rotation of the rod, ϕ, about an axis parallel to the $\hat{2}b$ axis and passing through the axis of connection of the rods.

Let us define the mass of the roll and pitch rods m_R and m_p, respectively. The rods are taken to be long and thin and thus are specified by one moment of inertia per rod; let I_p and I_r be the moments of inertia about the centers of mass for the pitch and roll rods, respectively. The rods are suspended in the bearings from their centers of mass.

Using (2.7) and (2.8) the kinetic energy is written as

$$T = T_{RB} + \mu\ell^2(\omega_2^{b2} + \omega_3^{b2}) + \frac{I_p}{2}(\omega_3^b + \dot{\theta})^2$$
$$+ \frac{I_R}{2}(\dot{\phi} + \omega_2^b)^2 + \frac{I_p}{2}(\omega_1^b \cos\theta + \omega_2^b \sin\theta)^2 \qquad (7.10)$$
$$+ \frac{I_R}{2}(\omega_1^b \cos\phi + \omega_3^b \sin\phi)^2$$

where ℓ is the distance from the center of mass of the rigid satellite-plus-boom to the point of attachment of the rods. The reduced mass is defined as

$$\mu = \frac{M_B(m_R + m_p)}{M_B + m_R + m_p} .$$

The potential energy is written using (2.8) and a linear law for the torsion springs that form the rod bearings. The spring constants for these torsion bearings are k_{sp} and k_{sr} for pitch and roll springs, respectively.

$$V = V_{RB} + \frac{1}{2} k_{sp}\theta^2 + \frac{1}{2} k_{SR}\phi^2$$
$$+ \frac{3}{2}\frac{k}{|\underline{R}^c|^3}\Big\{\mu\ell^2[(\hat{1}\cdot\hat{2}b)^2 + (\hat{1}\cdot\hat{3}b)^2]$$
$$+ I_p[(\hat{1}\cdot\hat{3}b)^2 + (\hat{1}\cdot\hat{1}b)^2\cos^2\theta + (\hat{1}\cdot\hat{2}b)^2\sin^2\theta + (\hat{1}\cdot\hat{1}b)(\hat{1}\cdot\hat{2}b)\sin 2\theta] \qquad (7.11)$$
$$+ I_R[(\hat{1}\cdot\hat{2}b)^2 + (\hat{1}\cdot\hat{1}b)^2\cos^2\phi + (\hat{1}\cdot\hat{3}b)^2\sin^2\phi + (\hat{1}\cdot\hat{1}b)(\hat{1}\cdot\hat{3}b)\sin 2\phi]\Big\}$$

2. Linear Model of the Vertistat

It is desirable to have an analytically tractable model for the small librations of the satellite, while still maintaining sufficient accuracy for engineering purposes. We form a linear model by noticing that the point $\gamma_1 = \gamma_2 = \gamma_3 = \phi = \theta = 0$ is an equilibrium point for the system and expanding in power series about this point. Assuming small eccentricity this gives, using Lagrange's equations, the following:

$$\begin{bmatrix} p^2 + 3g_{21} & -r_3(p^2 - 3) \\ -(p^2 - 3) & p^2 + b_p' \, p + \omega_p^2 \end{bmatrix} \begin{bmatrix} \gamma_3 \\ \theta \end{bmatrix} = \begin{bmatrix} Q_{\gamma_3}/n^2(I_p + C) + 2e \sin\tau \\ Q\theta/n^2 I_p - 2e \sin\tau \end{bmatrix} \tag{7.12}$$

$$\begin{bmatrix} p^2 + g_{32} & h_1 p & -2r_1 p \\ -h_2 p & p^2 + 4g_{31} & r_2(p^2 - 4) \\ 2p & p^2 - 4 & p^2 + b_R' p + \omega_R^2 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \phi \end{bmatrix} = \begin{bmatrix} Q_{\gamma_1}/n^2 I_1 \\ \dfrac{Q_{\gamma_2}/n^2}{I_R + B} \\ \dfrac{Q_\phi}{I_R n^2} \end{bmatrix} \tag{7.13}$$

In these equations viscous damping has been assumed in the bearings of the form $Q_\theta = -b_p\dot{\theta}$, $Q_\phi = -b_R\dot{\phi}$.

The nomenclature used in the above expressions is the following:

p	$= s/n$	= Laplace Transform variable for normalized time
τ	$= nt$	= normalized time
I_1, I_2, I_3		= body and boom moments of inertia
I_p, I_r		= rod moments of inertia

$B \quad = \quad I_2 + \mu \ell^2$

$C \quad = \quad I_3 + \mu \ell^2$

$\ell \quad =$ distance from body C.M. to point of rod connection

$$\mu = \frac{M_B(m_R + m_p)}{M_B + m_R + m_p} = \text{reduced mass}$$

M_B, m_r, m_p = masses of the main body and boom, roll rod, pitch rod, respectively

$$g_{21} = \frac{B - I_1 - I_p}{C + I_p}$$

$$g_{32} = \frac{I_3 - I_2 - I_R + I_p}{I_1}$$

$$g_{31} = \frac{C - I_1 + 1/4\ I_p - 3/4\ I_R}{B + I_R}$$

$$h_1 = \frac{C + I_p - B - I_1 - I_R}{I_1}$$

$$h_2 = \frac{C - B - I_1 + I_p - I_R}{B + I_R}$$

$$r_1 = \frac{I_R}{I_1}$$

$$r_2 = \frac{I_R}{I_R + B}$$

$$r_3 = \frac{I_p}{I_p + C}$$

$$b'_p = \frac{b_p}{nI_p} \qquad \omega_p^2 = \frac{k_{sp}}{n^2\ I_p} - 3$$

$$b'_R = \frac{b_R}{nI_R} \qquad \omega_R^2 = \frac{k_{sR}}{n^2\ I_R} - 4$$

The stability conditions are obtained from the requirement that H be positive definite. This is equivalent to requiring the principal minors of the matrices (7.12) and (7.13) to be positive with $p = 0$. This computation gives the following inequalities:

$$
\begin{aligned}
&g_{21} > 0 \\
&\omega_p^2 > 0 \\
&g_{21}\omega_p^2 - 3r_3 > 0 \\
&g_{32} > 0 \\
&g_{31} > 0 \\
&\omega_R^2 > 0 \\
&g_{31}\omega_R^2 - 4r_2 > 0
\end{aligned}
\tag{7.14}
$$

These inequalities relax the condition on the shape parameters of the body if ω_p^2, ω_R^2 are large enough. Notice that none of the stability criteria in this chapter depend on the damping parameters if $\dot{H} \leq 0$; this property follows from Theorem IV in Chapter III since the testing condition for stability is independent of P and dependent only on U. This is stated as a Corollary to Theorem V of Chapter III.

3. Forced Motions

The forced motions of (7.12) due to eccentricity of the orbit can be determined by the usual methods. The motions are given by solving

$$
\begin{bmatrix} 3g_{21} - 1 & 2r_3 \\ 2 & \omega_p^2 - 1 + jb_p' \end{bmatrix} \begin{bmatrix} Y_3 \\ \theta \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} 2ej \tag{7.15}
$$

where $\theta = \Theta e^{j\tau}$, $\gamma_3 = Y_3 e^{j\tau}$. The solution of (7.15) gives the motions for small eccentricity. Let

$$\alpha = (3g_{21} - 1)(\omega_p^2 - 1) - 4r_3$$

$$\beta = b_p' \, (3g_{21} - 1)$$

Then, the forced response is

$$\gamma_3 = \left[\frac{-2e}{\alpha^2 + \beta^2}\right] \left\{ [\alpha(\omega_p^2 - 1 + 2r_3) + b_p'\beta] \sin\tau \right.$$

$$\left. + \; [b_p'\alpha - \beta \; (\omega_p^2 - 1 + 2r_3)] \cos\tau \right\} \tag{7.16}$$

$$\theta = \frac{2e(3g_{21} + 1)}{\alpha^2 + \beta^2} [- \beta \cos\tau + \alpha \sin\tau]$$

Since for good attitude control we would like γ_3 small, it may well be that there are choices of parameters ω_p, r_3, b_p' such that γ_3 is minimized. It will be shown below that there exist sets of parameters for which $\gamma_3 \equiv 0$. These are the so-called "vibration absorber" designs so useful in design of machines. Clearly the condition for such designs is that the coefficients of both $\sin\tau$ and $\cos\tau$ in (7.16) for γ_3 must be zero. This leads to the equations

$$\begin{bmatrix} (\omega_p^2 - 1 + 2r_3) & b_p' \\ b_p' & -(\omega_p^2 - 1 + 2r_3) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

These require that, quite independent of α and β, the determinant vanish.

$$(\omega_p^2 - 1 + 2r_3)^2 + b_p'^2 = 0$$

This is impossible except for the undamped case $(b_p' \equiv 0)$ for which, in addition,

$$\omega_p^2 = 1 - 2r_3 \tag{7.17}$$

for vibration absorption. In this case the motion is

$$\begin{aligned} \gamma_3 &\equiv 0 \\ \theta &= -\frac{e}{r_3} \sin \tau \end{aligned} \tag{7.18}$$

For stability $\omega_p^2 > 0$, so

$$\begin{aligned} &0 < r_3 < 1/2 \\ &g_{21} > \frac{3r_3}{1 - 2r_3} < 0 \end{aligned} \tag{7.19}$$

must hold for simultaneous stability and vibration absorption. Notice that, since $r_3 < 1/2$, trouble with damping will be encountered for vibration absorber designs. This is true because the coupling between the γ_3 motion and the rod motion (θ) will be very weak and therefore the γ_3 motion will converge much too slowly. This will be seen more clearly in the next section.

4. Damping of the Motions

The vertistat motions in response to initial disturbances can be calculated for arbitrary, light damping forces by the method of Chapter VI. In this section the methods of approximation developed in Chapter VI will be applied to the motions of a typical vertistat design with magnetic hysteresis damping.

Yaw-Roll

The characteristic determinant for the yaw-roll case is found from (7.13) to be

$$\Delta_{yR}(p^2) = [p^2 + g_{32}][p^4(1 - r_2) + p^2(4g_{31} + \omega_R^2 + 8r_2) + 4\omega_R^2 g_{31} - 16r_2] \tag{7.20}$$

$$+ h_2 p^2[(h_1 + 2r_1)p^2 + (h_1\omega_R^2 - 8r_1)]$$

$$+ 2p^2[(h_1 r_2 + 2r_1)p^2 + (8r_1 g_{31} - 4h_1 r_2)]$$

with $b_R' = 0$. The solution for the response envelope can be written, following Chapter VI, Section B, as:

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \emptyset \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \Lambda_1 e^{j(\omega_1\tau + \psi_1)} \\ \Lambda_2 e^{j(\omega_2\tau + \psi_2)} \\ \Lambda_3 e^{j(\omega_3\tau + \psi_3)} \end{bmatrix} \tag{7.21}$$

where the "mode shape factors," A_{ij}, are

$$A_{2k} = \frac{j\omega k}{D_k} N_k$$

$$A_{3k} = \frac{j\omega k}{D_k} M_k$$

after defining N_k, M_k, and D_k as:

$$N_k = h_2(\omega_R^2 - \omega_k^2) - 2r_2(4 + \omega_k^2)$$

$$M_k = 2(\omega_k^2 - 4g_{31}) + h_2(4 + \omega_k^2)$$

$$D_k = (4g_{31} - \omega_k^2)(\omega_R^2 - \omega_k^2) - r_2(4 + \omega_k^2)^2$$

The method of calculating the response of the system to initial disturbances follows that discussed in Chapter VI. It consists of requiring that the amplitudes Λ_k vary with time in such a manner that the rate of change of energy equals the average power lost in damping. The Hamiltonian can be written upon averaging as

$$\bar{H} = \frac{1}{2} \sum_{k=1}^{3} H_k \Lambda_k^2 \tag{7.22}$$

The power balance relation is

$$\dot{\bar{H}} = \bar{\dot{H}} = \bar{P} = \sum_{k=1}^{3} H_k \Lambda_k \dot{\Lambda}_k \tag{7.23}$$

This equation must be supplemented by a relation between the average power and the amplitudes Λ_k. This relation is derived or taken from experiment by defining the average to be

$$\bar{P} = \lim_{T\to\infty} \frac{1}{T} \int_0^T P(t)\, dt$$

In this time integration the variables ψ_k and Λ_k are held constant. At this point it is assumed (for lack of experimental evidence as to the analytical character of $\bar{P}$) that the energy dissipation per cycle of oscillation depends only on the amplitude $A_{3k} \Lambda_k$ and, furthermore, that the energy dissipated in each modal oscillation is independent of that dissipated in the other modes. This means that in traversing the magnetic hysteresis loops the form of the area swept out is the same for each oscillation frequency. These considerations lead to the following functional form for the average power dissipated.

$$\bar{P} = -\left\{ \lambda\, I_R\, \omega_R^3 \sum_{k=1}^{3} F_k(A_{3k}\Lambda_k) \right\} \tag{7.24}$$

where λ is a "damping parameter."

If the function $F_k(A_{3k}\Lambda_k)$ is quadratic (the linear case) then we obtain the following solution to the transient problem:

$$\bar{P} = -\sum_{k=1}^{3} P_k\Lambda_k^2 = \sum_{k=1}^{3} H_k\Lambda_k\dot{\Lambda}_k$$

$$P_k = \lambda\, I_R\, \omega_R^2\, |A_{3k}|^2 \tag{7.25}$$

$$H_k\dot{\Lambda}_k + P_k\Lambda_k = 0$$

In the present problem the energy coefficients are calculated using (6.21) and the Hamiltonian, H, as

$$H_k = \frac{I_R + B}{2}\left\{\frac{r_2}{r_1} + |A_{2k}|^2 + r_2\,|A_{3k}|^2 + 2r_2\mathrm{Re}(A_{2k}A_{3k}^*)\right\}\omega_k^2$$

$$+ \frac{I_R + B}{2}\left\{\frac{r_2}{r_1}\, g_{32} + 4g_{31}|A_{2k}|^2 + r_2\omega_R^2\,|A_{3k}|^2 - 8r_2\,\mathrm{Re}(A_{2k}\,A_{3k}^*)\right\}$$

The solution to the equations of (7.25) are simple exponentials given by

$$\Lambda_k(t) = \Lambda_k(o)e^{-t/T_k} \quad ; \quad T_k = \frac{E_k}{P_k} \tag{7.26}$$

where T_k is the "time constant" of the envelope decay.

As a specific numerical design example assuming the "linear" damping of (7.25), the following vehicle parameters are chosen; they have been chosen by trial and error and are not in any sense optimum.

$I_1 = I_2 = I_3 = 120$ slug ft^2 = body moments of inertia

$B = C = 1020$ slug ft^2

μ	$= 1$ slug	= mass of rods, etc.
ℓ	$= 30$ ft	= length of boom
I_r	$= 50$ slug ft^2	= inertia of roll rod
I_p	$= 112.5$ slug ft^2	= inertia of pitch rod
g_{21}	$= 0.75$	
g_{31}	$= 0.83$	
g_{32}	$= 0.52$	
h_1	$= -0.48$	
h_2	$= -0.054$	
r_1	$= 0.42$	
r_2	$= 0.05$	
r_3	$= 0.1$	
ω_r^2	$= 4.0$	
ω_p^2	$= 2.22$	

Using the above parameters and solving for the roots of Δ_{yR}, the shape parameters, A_{ij}, and the time constants gives the results (see Fig. 7.5) of Table 2.

TABLE 2

Mode Number k	ω_k	A_{2k}	A_{3k}	λT_k
1	0.565	$-j0.034$	$-j0.35$	1.35
2	1.69	$j1.18$	$+j1.23$	8.5
3	2.66	$-j0.44$	$j3.28$	1.07

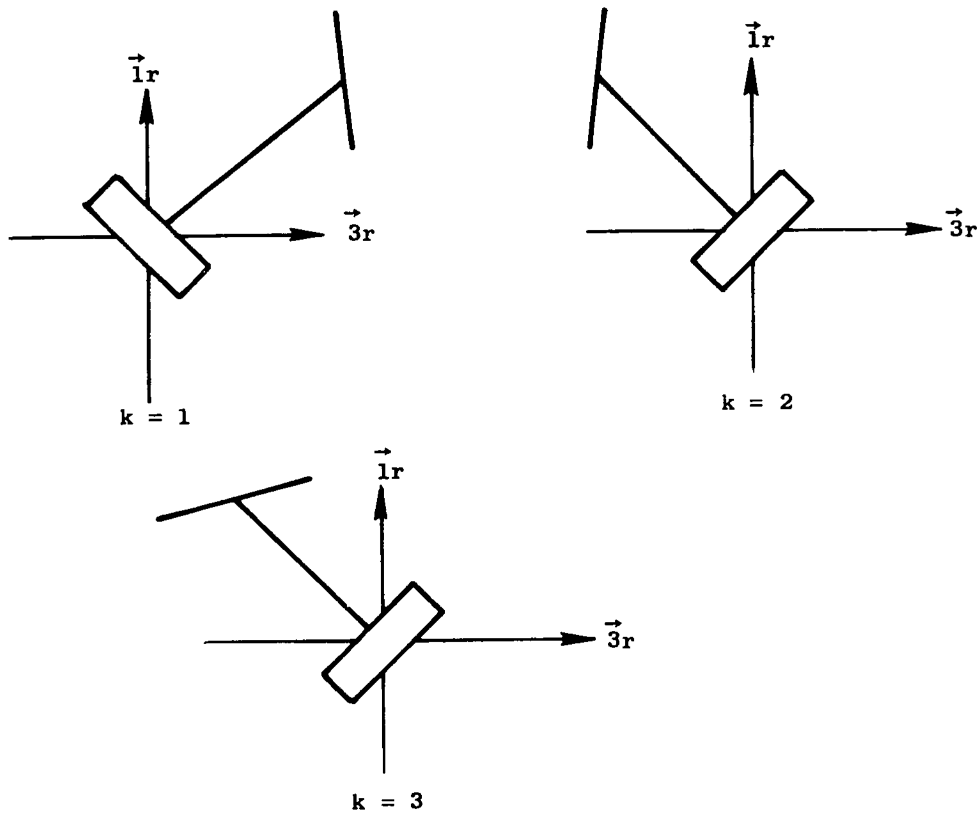

FIG. 7.5. MODAL MOTION FOR YAW-ROLL OF VERTISTAT (YAW MOTION; sin(o) IN ALL MODES)

For the case of $\lambda = 1/8\pi$ the longest time constant is $T_2 \doteq 213$ (or 34 orbit periods). This is a realistic but possibly conservative estimate of λ for the magnetic hysteresis case and corresponds to a situation in a one-degree-of-freedom system where one-half of the peak stored energy is dissipated per cycle of oscillation. The actual magnetic materials will behave in a non-linear fashion which will now be shown.

Since we do not have data on $\overline{P}$ we may use the single-degree-of-freedom result given in Ref. FISCHELL 3 for ferromagnetic materials in large oscillations. Generalizing the result for one degree of freedom in an obvious way gives a power law behavior for the dissipation in each mode (still assuming that the dissipation doesn't couple the mode response).

$$\overline{p} = -\lambda I_R \omega_R^3 \sum_{k=1}^{3} |A_{3k} \Lambda_k|^{\ell} \tag{7.26}$$

where ℓ is a coefficient that is not equal to two (linear case) but is not necessarily an integer. The damping law of (7.26) gives an envelope response differential equation of the form

$$H_k \dot{\Lambda}_k + \lambda I_R \omega_R^3 |A_{3k}|^{\ell} \Lambda_k^{\ell-1} = 0 \tag{7.27}$$

These three differential equations may be separately integrated to give the time response of the envelopes.

$$\frac{\Lambda_k(\tau)}{\Lambda_k(o)} = \left[1 + \frac{(\ell-2)}{(\Lambda_k(o))^{2-\ell}} \quad \frac{\tau}{T_k} \right]^{\frac{1}{2-\ell}} \quad (\ell > 2) \tag{7.28}$$

The concept of "time constant" is generalized by the definition

$$T_k = \frac{H_k}{\lambda I_R \omega_R^3 |A_{3k}|^{\ell}}$$

It can be seen that, for the value often given for magnetic materials, (FISCHELL 3) $\ell = 3$, we get a response which depends on the initial conditions but is a well-behaved hyperbolic shape:

$$\frac{\Lambda_k(\tau)}{\Lambda_k(o)} = \left[1 + \Lambda_k(o) \quad \frac{\tau}{T_k} \right]^{-1} \tag{7.29}$$

For the cases $\ell > 3$ we run into trouble and the response takes a long time to damp out for small initial disturbances. It goes without saying that in these considerations the constant λ takes on different meanings for each value of ℓ.

Pitch Case

The characteristic determinant for the pitch motion is found from (7.12) to be

$$\Delta_p(P^2) = (P^2 + 3g_{21})(P^2 + \omega_p^2) - r_3 \; (P^2 - 3)^2 \tag{7.30}$$

with $\eta_p = 0$. Let us assume that $3g_{21} \cong \omega_p^2$ or that $3g_{21} = \omega_p^2 + J\epsilon$. Here $\epsilon = I_p/c \ll 1$ and $J \leq 1$. We can now factor the biquadratic (7.30) and approximately determine ω_1, ω_2 for small ϵ.

$$\omega_1^2 = \omega_p^2 \; (1 + \epsilon) + (3 + \frac{J}{2})\epsilon - \epsilon^{1/2}(\omega_p^2 + 3) \left[1 + \frac{\epsilon}{2} \left(\frac{\omega_p^2 + 3 + J/2}{\omega_p^2 + 3} \right)^2 \right]$$

$$\omega_2^2 = \omega_p^2 \; (1 + \epsilon) + (3 + \frac{J}{2})\epsilon + \epsilon^{1/2}(\omega_p^2 + 3) \left[1 + \frac{\epsilon}{2} \left(\frac{\omega_p^2 + 3 + J/2}{\omega_p^2 + 3} \right)^2 \right]$$

The mode shape factors are, for this problem:

$$A_{21} = \frac{\omega_1^2 + 3}{\omega_1^2 - \omega_p^2}$$

$$A_{22} = \frac{\omega_2^2 + 3}{\omega_2^2 - \omega_p^2}$$

The solutions to the equations of motion are, then,

$$\begin{aligned} \gamma_3 &= \Lambda_1 e^{j(\omega_1\tau+\psi_1)} + \Lambda_2 e^{j(\omega_2\tau+\psi_2)} \\ \theta &= A_{21}\Lambda_1 e^{j(\omega_1\tau+\psi_1)} + A_{22}\Lambda_2 e^{j(\omega_2\tau+\psi_2)} \end{aligned} \tag{7.31}$$

The energy coefficients are found to be

$$\begin{aligned} E_k = \frac{1}{2}(C+I_p)\Big\{(\omega_k^2+\omega_p^2) + \epsilon\Big[\omega_k^2(|A_{2k}|^2 - 2A_{2k}) \\ + J + A_{2k} + \omega_p^2|A_{2k}|^2\Big]\Big\} \end{aligned} \tag{7.32}$$

and the power coefficients are (for the "linear" case of (7.25)):

$$P_k = \lambda I_p\ \omega_p^3|A_{2k}|^2 \tag{7.33}$$

for magnetic hysteresis damping. The time constant of the transient decay is

$$T_k = \frac{H_k}{\lambda I_p\ \omega_p^3|A_{2k}|^2} = \frac{H_k}{P_k} \tag{7.34}$$

where $\dot{\Lambda}_k + (P_k/H_k)\Lambda_k = 0$ and $\Lambda_k = \Lambda_k(o)e^{-\tau/T_k}$

As an example of a typical design, let:

$$\omega_p^2 = 2.22$$

$$g_{21} = 0.75$$

$$J = 0$$

$$\epsilon = 0.11$$

This gives, using (7.24-28):

$$A_{21} = -3.14 \quad ; \quad \omega_1 = 0.975 \quad ; \quad \lambda T_1 = 0.73$$

$$A_{22} = 3.18 \quad ; \quad \omega_2 = 2.14 \quad ; \quad \lambda T_2 = 1.18$$

The forced response of this example is:

$$\gamma_3 = -1.8e \sin \tau$$

$$\theta = 3.2e \sin \tau$$

Using the results for the damped motion we can estimate the time constant of the transient decay. The largest time constant becomes $T_2 = 33.3$ (5.3 orbit periods) if $\lambda = 1/8\pi$. The forced response may be unacceptably large, since for $e = 0.05$ the excursions exceed eight degrees. In all cases of passive gravity attitude control the eccentricity must be small; this is achieved by proper injection into orbit.

5. Tumbling of the Vertistat in Pitch

In the design of a particular satellite system (e.g., the Vertistat) it is desirable to know the effectiveness of the passive damper in attenuating the tumbling rates that might occur in the course of operation. The methods of Chapter IV produce a differential equation relating the average pitch tumbling acceleration, N', to the average pitch tumbling rate, N. This equation can be solved for the rate, N, as a function of time. In this section the Vertistat motions in tumbling are studied and the results are related to those for the "resonant" case of Chapter IV.

The equations of motion of the Vertistat in the pitch plane with $\theta \ll 1$, γ arbitrary, are

$$\gamma' = \lambda\omega$$

$$\lambda\omega' = \frac{r_3}{1 - r_3}\left[(\omega_p^2 + 3)\theta + b_p'\,\theta'\right] - \frac{3}{2}\,\frac{g_{21}}{1 - r_3}\sin 2\gamma \tag{7.35}$$

$$\theta'' + \frac{b_p'}{1 - r_3}\,\theta' + \frac{\omega_p^2 + 3}{1 - r_3}\,\theta = \frac{3}{2}\sin 2(\gamma + \theta) + \frac{3}{2}\,\frac{g_{21}}{1 - r_3}\sin 2\gamma$$

where $d/d\tau(\ ) = (\ )'$ and $\tau = nt$. These equations assume that the pitch tumbling motion is stable and that rod motion is small ($\theta \ll 1$).

The motion of the system can be approximated by the following,

$$\theta = \bar{\theta} + \alpha_1 \cos 2\bar{\gamma} + \beta_1 \sin 2\bar{\gamma}$$

$$\lambda\omega = \lambda\bar{\omega} + \alpha_2 \cos 2\bar{\gamma} + \beta_2 \sin 2\bar{\gamma} \tag{7.36}$$

$$\gamma = \bar{\gamma} + \alpha_3 \cos 2\bar{\gamma} + \beta_3 \sin 2\bar{\gamma}$$

where α_i, β_i are functions of $\lambda\bar{\omega}$ as in Chapter IV. Using (7.36) in (7.35) gives the following equations for determining the α_i, β_i and $\bar{\theta}$, $\bar{\omega}$, $\bar{\gamma}$.

$$\alpha_1 = -\frac{3}{2}\left(1 + \frac{g_{21}}{1 - r_3}\right)\left(\frac{2b_p'N}{1 - r_3}\right)\frac{1}{\Delta}$$

$$\beta_1 = \frac{3}{2}\left(1 + \frac{g_{21}}{1 - r_3}\right)\left(K - 4N^2 - \frac{9}{4K}\right)\frac{1}{\Delta}$$

$$\alpha_2 = 2N\,\beta_3$$

$$\beta_2 = -2N\,\alpha_3$$

$$\alpha_3 = \frac{1}{2N} \left(\frac{r_3 b_p'}{1 - r_3}\right)\left(\frac{3/2(1 + \frac{g_{21}}{1 - r_3})}{\Delta}\right)\left(4N^2 + \frac{9}{4\mathcal{K}}\right)$$

$$\beta_3 = \frac{-1}{4N^2} \left(\frac{r_3}{1 - r_3}\right)\left(\frac{3/2(1 + \frac{g_{21}}{1 - r_3})}{\Delta}\right)\left[(\omega_p^2 + 3)\left(-4N^2 - \frac{9}{4\mathcal{K}}\right) + \left(\frac{4N^2 b_p'^2}{1 - r_3}\right)\right]$$

where

$$\Delta = \left[(\mathcal{K} - \frac{9}{4\mathcal{K}} - 4N^2)(\mathcal{K} - 4N^2) + \frac{4N^2 b_p'^2}{(1-r_3)^2}\right]$$

$$N = \lambda\bar{\omega} \qquad \mathcal{K} = \frac{\omega_p^2 + 3}{1 - r_3}$$

Now the barred variables are solutions to the equations (in steady state $\bar{\theta}$ motion).

$$\boxed{\begin{aligned} \bar{\theta} &= \frac{3}{2}\frac{\alpha_1}{\mathcal{K}} \\ N' &= -\left\{2K\, b_p' \frac{N}{\Delta} + \frac{L}{N\Delta}\right\} \end{aligned}} \tag{7.37}$$

where,

$$K = -\frac{9}{4}\left(\frac{r_3}{1 - r_3}\right)\left(1 + \frac{g_{21}}{1 - r_3}\right)^2$$

$$L = -\frac{81}{32}\left(\frac{r_3\, \eta_p}{1 - r_3}\right)\left(1 + \frac{g_{21}}{1 - r_3}\right)\left(\frac{g_{21}/\mathcal{K}}{1 - r_3}\right)$$

are constants.

The solution to the second of (7.37) gives the secular decay of N with time. The term $L/N\Delta$ is smaller of order $1/N^2$ than the first term in the N' equation. Thus an estimate of the motion if given by

$$N' \leq -2\, b_p' K \frac{N}{\Delta} \tag{7.38}$$

where the error is of order $1/N^2$ and this is small for $N \ll 1$. The strict equality in equation (7.38) yields the same results as equation (4.17) and the results conform to Table 4.1 for the resonant case, if the parameters K of the two problems are identified with one another.* The results using (7.38) to estimate the response give, of course, slower response than the results using (7.37), but they are useful because of the existence of Table 4.1.

An example of the use of Table 4.1 can be given using the design of Section C Part 4. The relevant parameters are: $g_{21} = 0.75$, $r_3 = 0.1$, $\omega_p^2 = 2.22$. The result of Table 1 of Chapter IV for $N_0 = 4$ and $N_{final} = \sqrt{3}$ is that the optimum time to decay is $(\tau_c)_{opt} = 19$ or about three orbit periods. The time to decay using an arbitrary value of the damping parameter, η, is (see 4.21)

$$\frac{\tau_c}{(\tau_c)_{opt}} = \frac{1}{2}\left[\frac{\eta}{3.6} + \frac{3.6}{\eta} + \frac{21}{3.6\eta}\right]$$

For a value of η which is 5 percent of critical ($\eta = 0.15$) the time to decay is 473 (75 orbit periods).

D. THE BEAM SATELLITE

This satellite has not yet been discussed in the literature but is a natural simplification of the TRAAC design using elastic hysteresis damping. The system consists of a main body and an extensible "beam" which is flexible and cantelevered to the main body (Fig. 7.2). The beam must be flexible enough to allow fairly large excursions relative to the main body and thus significant stresses in the beam material; this is accomplished by adding a mass to the end of the beam. This mass also "amplifies" the gravity stabilization effect. The beam is coated with Cadmium or some other "dissipative" material to enhance damping by the material.

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* We must also identify $\Delta \approx D$, $\eta = \dfrac{b_p'}{1 - r_3}$, $\omega_0^2 = 3 + \omega_p^2$.

This system will be feasible if the beam can be made to be limber. The system is elegant and simple and, furthermore, provides damping to first order in both the pitch and roll-yaw motions. This obviates the need for magnetic rods or other means of achieving roll-yaw damping.

Other systems similar to the beam system can be devised. One might consider a series of short rods joined by some kind of viscoelastic material which would provide damping and spring restoring forces. This system would look quite like the beam dynamically but would have advantages in deployment and in the fact that the rods could be made more rigid.

1. Energy Expressions

Consider a rigid body with its center of mass at the origin of unit vectors $\hat{1}, \hat{2}, \hat{3}$ as defined in Section B. A vector $\underline{R}^b$ defines the position of the particles along the length of the beam with respect to the center of mass of the rigid body. This vector is expressed in coordinates along the principal axis system of the rigid body ($\hat{1}b$, $\hat{2}b$, $\hat{3}b$). Since the beam is connected to the satellite at a point a distance a along the $\hat{1}b$ axis (Fig. 7.2), the vector $\underline{R}^b$ can be expressed as

$$\underline{R}^b = \hat{1}b(a + x) + \hat{2}b\ y + \hat{3}b\ z$$

The functions x, y, z are functions of the arc length, s, along the beam in the deflected position. These quantities are, of course, the rectangular coordinates of points along the deflected beam with center of the coordinates at the point $\hat{1}b\ a$. It shall be assumed that all the mass is concentrated in the end of the beam -- the beam is massless. This is a commonly made approximation for first bending mode analysis of actual beams; the mass at the tip is increased to account for the beam mass. If the reduced mass is given by μ, the kinetic energy of the system can be written as follows ($\dot{x} = 0$) using (2.17) and (2.8):

$$T = T_{RB} + \frac{1}{2}\mu[\overset{\circ}{\underline{R}}{}^b_b + \underline{\omega}^b \times \overset{\bullet}{\underline{R}}{}^b\ ]^2$$

$$T = T_{RB} + \frac{\mu}{2}\Big\{[\dot{y} + \omega_2^b z - \omega_3^b y]^2 \qquad (7.35)$$

$$+ [\dot{z} + \omega_3^b (a + L) - \omega_1^b z]^2 + [\omega_1^b y - \omega_2^b (a + L)]^2 \Big\}$$

where $\gamma_1, \gamma_2, \gamma_3$ are the Euler angles of the rigid body as defined in Section B and where $\omega_1^b, \omega_2^b, \omega_3^b$ are given by (7.3)

$$\omega_1^b = \dot{\gamma}_1 \cos\gamma_2 \cos\gamma_3 + \dot{\gamma}_2 \sin\gamma_3 + \dot{\eta}\ (\hat{3} \cdot \hat{1}b)$$

$$\omega_2^b = -\dot{\gamma}_1 \cos\gamma_2 \sin\gamma_3 + \dot{\gamma}_2 \cos\gamma_3 + \dot{\eta}\ (\hat{3} \cdot \hat{2}b)$$

$$\omega_3^b = \dot{\gamma}_3 + \dot{\gamma}_1 \sin\gamma_2 + \dot{\eta}\ (\hat{3} \cdot \hat{3}b)$$

where the direction cosines are given in (7.2).
These equations use x, y, z to represent those functions of arc length, s, evaluated at the end point of the beam, s = L. In a similar manner the potential energy of the system due to gravity can be derived using (2.8). If we define the proper tensor according to (2.19), we get the following:

$$\mathbb{R}_b^b \quad \begin{bmatrix} y^2 + z^2 & -(a + L)y & -(a + L)z \\ -(a + L)y & z^2 + (a + L)^2 & -yz \\ -(a + L)z & -yz & y^2 + (a + L)^2 \end{bmatrix}$$

$$V_g = V_{RB} + \frac{k\mu}{|\underline{R}^c|^3}\left\{ -\frac{1}{2}\, \mathrm{tr}\mathcal{R}^b_b + \frac{3}{2}\hat{R}^c \cdot \mathcal{R}^b_b \cdot \hat{R}^c \right\}$$

$$V_g = V_{RB} + \frac{k\mu}{|\underline{R}^c|^3}\left\{ -[y^2 + z^2 + (a+L)^2] \right.$$

$$+ \frac{3}{2}[(\hat{1}\cdot\hat{1}b)^2(y^2+z^2) + (\hat{1}\cdot\hat{2}b)^2((a+L)^2 + z^2)$$

$$+ (1\cdot 3b)^2(y^2 + (a+L)^2) \tag{7.36}$$

$$- 2(\hat{1}\cdot\hat{1}b)(\hat{1}\cdot\hat{2}b)(y)(a+L) - 2\,(\hat{1}\cdot\hat{1}b)(\hat{1}\cdot\hat{3}b)(z)(a+L)$$

$$\left. - 2(\hat{1}\cdot\hat{2}b)(\hat{1}\cdot\hat{3}b)yz] \right\}$$

The potential energy of elastic deflection of the beam is given by using the Lagrangian density

$$\mathcal{L} = -\frac{EI}{2}[(y'')^2 + (z'')^2] - \frac{1}{2}P(s)[(y')^2 + (z')^2]$$

and the following equations of motion for the beam and the end condition $y(L,t)$ and $z(L,t)$. (see Appendix C.)

$$\frac{\partial^2}{\partial s^2}\left(\frac{\partial\mathcal{L}}{\partial y''}\right) - \frac{\partial}{\partial s}\left(\frac{\partial\mathcal{L}}{\partial y'}\right) = 0$$

$$\text{all } s \tag{7.37}$$

$$\frac{\partial^2}{\partial s^2}\left(\frac{\partial\mathcal{L}}{\partial z''}\right) - \frac{\partial}{\partial s}\left(\frac{\partial\mathcal{L}}{\partial z'}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial(T-V_g)}{\partial\dot{y}}\right) - \frac{\partial(T-V_g)}{\partial y} + \left\{\frac{\partial}{\partial s}\left(\frac{\partial\mathcal{L}}{\partial y''}\right) - \frac{\partial\mathcal{L}}{\partial y'}\right\}_{s=L} = Q_y$$

$$\frac{d}{dt}\left(\frac{\partial(T-V_g)}{\partial\dot{z}}\right) - \frac{\partial(T-V_g)}{\partial z} + \left\{\frac{\partial}{\partial s}\left(\frac{\partial\mathcal{L}}{\partial z''}\right) - \frac{\partial\mathcal{L}}{\partial z'}\right\}_{s=L} = Q_z$$

where $P = 3\mu n^2(a+L)$. The third and fourth of (7.37) are boundary conditions at the end ($s = L$) of the beam. The other boundary conditions are given by

$$y' = z' = y = z = 0 \qquad (s = 0)$$

$$y'' = z'' = 0 \qquad (s = L)$$

The equations of beam bending are from (7.37)

$$\left.\begin{aligned} EIy''''(s) - Py''(s) &= 0 \\ ELz''''(s) - Pz''(s) &= 0 \end{aligned}\right\} \quad \text{all } s \tag{7.38}$$

The equations of motion for the beam satellite are formed by using the ordinary Lagrange equations for γ_1, γ_2, γ_3. The equations are completed by using the solutions to the bending equations (7.38) in the end conditions given in the third and fourth equations of (7.37). These five equations of motion of the beam satellite are used in what follows to analyze the dynamics of the vehicle.

2. Linear Model of the Beam Satellite

It is desirable to investigate the small librations of the beam satellite about the equilibrium solution $\gamma_1 = \gamma_2 = \gamma_3 = y = z = 0$. This can be done by writing the expressions for T and V out to second order in the coordinates and velocities. Defining "angle" variables $\psi_y = -\ y(L,t)/(a+L)$ and $\psi_z = z(L,t)/(a+L)$, the following equations of motion are obtained using Lagrange's equations:

$$\begin{bmatrix} p^2 + 3g_{21} & K_3(p^2+3) \\ (p^2+3) & p^2+\Omega_B^2 \end{bmatrix} \begin{bmatrix} \gamma_3 \\ \psi_y \end{bmatrix} = \begin{bmatrix} Q_{\gamma_3}/(I'+I_3)h^2 \\ +2e\sin\tau \\ Q_y/\mu(a+L)h^2 \\ +2e\sin\tau \end{bmatrix} \tag{7.39}$$

$$\begin{bmatrix} p^2+k_{32} & -h_1p & 0 \\ h_2p & p^2+4g_{31} & -K_2(p^2+4) \\ 0 & -(p^2+4) & p^2+1+\Omega_B^2 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \psi_z \end{bmatrix} = \begin{bmatrix} Q_{\gamma_1}/I_1n^2 \\ Q_{\gamma_2}/(I'+I_2)n^2 \\ Q_z/\mu(a+L)^2n^2 \end{bmatrix} \qquad (7.40)$$

These equations represent the pitch (7.39) and roll-yaw (7.40) motions.

The Hamiltonian may be derived in the usual way to be

$$\begin{aligned} H = {} & \frac{1}{2}\Big\{(I_3+I')\dot{\gamma}_3^2 + I'(\dot{\psi}_y^2 + \dot{\psi}_z^2) + I_1\dot{\gamma}_1^2 \\ & - 2I'\dot{\psi}_z\dot{\gamma}_2 \qquad +(I'+I_2)\dot{\gamma}_2^2\Big\} \\ & + \frac{1}{2}\Big\{3g_{21}(I'+I_3)\gamma_3^2 + 6I'\psi_y\gamma_3 \\ & + \Omega_B^2 I'\psi_y^2 + (1+\Omega_B^2)I'\psi_z^2 + k_{32}I_1\gamma_1^2 \\ & + 4g_{32}(I'+I_2)\gamma_2^2 - 8I'\psi_z\gamma_2\Big\} \end{aligned} \qquad (7.41)$$

The normalized variables are:

$\tau = nt =$ Normalized Time

$p = s/n$

$$\Omega_B^2 = \frac{3\left(\frac{a+L}{L}\right)\alpha L\,(1+e^{-2\alpha L})}{(\alpha L-1)+(\alpha L+1)e^{-2\alpha L}}$$

$$P = 3\mu n^2(a+L)$$

$$\alpha^2 = P/EI$$

$$g_{31} = \frac{I_3 - I_1 + I'}{I_2 + I'} \qquad K_1 = I'/I_1$$

$$g_{21} = \frac{I_2 + I' - I_1}{I' + I_3} \qquad K_2 = I'/(I' + I_2)$$

$$k_{32} = \frac{I_3 - I_2}{I_1} \qquad K_3 = I'/(I' + I_3)$$

$$y = (a + L)\psi_y \qquad h_1 = \frac{I_1 + I_2 - I_3}{I_1}$$

$$z = (a + L)\psi_z \qquad h_2 = \frac{I_1 + I_2 - I_3}{I_2 + I'}$$

The physical variables are:

$$\mu = \frac{mM}{M + m}$$

I_1, I_2, I_3 = moments of inertia of main body about $\hat{1}b$, $\hat{2}b$, $\hat{3}b$ axes.

EI = beam structural rigidity

$I' = \mu(a+L)^2$

L = length of the beam

a = distance of beam mounting from C.M. of main body

e = orbit eccentricity

Q_j = torques and forces acting on the system (generalized forces)

The stability inequalities for $H > 0$ (Theorem IV, Chapter III) are:

$$g_{21} > \frac{3K_3}{\Omega_B^2} > 0 \quad ; \quad g_{31} > \frac{4K_2}{1+\Omega_B} > 0$$
$$k_{32} > 0 \tag{7.42}$$

These requirements for stability are, as in the Vertistat satellite, weaker than those of the rigid body; they grow weaker monotonically as Ω_B increases (beam stiffens). Note that for zero elasticity, $\Omega_B^2 = 3(1+a/L)$.

3. Forced Motions

The pitch plane motions are forced sinusoidally by the accelerations of the reference system caused by orbit eccentricity. For small damping and small eccentricity the linear model of (7.39) gives approximately:

$$\begin{bmatrix} 3g_{21} - 1 & 2K_3 \\ 2 & \Omega_B^2 - 1 \end{bmatrix} \begin{bmatrix} \Upsilon_3 \\ \Psi_y \end{bmatrix} = -2ej \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{7.43}$$

Here we assumed $\gamma_3 = \Upsilon_3 e^{j\tau}$, $\psi_y = \Psi_y e^{j\tau}$. The inversion of (7.43) gives the forced response.

$$\gamma_3 = \frac{2e[\Omega_B^2 - 1 - 2K_3]}{\left\{(3g_{21}-1)(\Omega_B^2-1) - 4K_3\right\}} \sin\tau$$

$$\psi_y = \frac{6e(g_{21} - 1)}{\left\{(3g_{21}-1)(\Omega_B^2-1) - 4K_3\right\}} \sin\tau \tag{7.44}$$

For vibration absorption in γ_3, $\Omega_B^2 = 1+2K_3$. This requires $1 \leq \Omega_B^2 \leq 3$. We can get a minimum $\Omega_B^2 = 3(1+a/L) > 3$, so vibration absorption of the eccentricity disturbance is not possible in the beam satellite but, of course, can be approached quite closely.

It should be emphasized that the foregoing has considered only disturbance due to eccentricity of the orbit, and therefore, the vibration absorption results are for this case. One must in practical cases include other periodic disturbances; it may be possible to do some filtering of these by adjusting parameters.

4. Damping of the Motions

It is desirable to apply the methods of Chapter VI to the pitch motions of the beam system with hysteresis damping in the beam. The damping is not expected to be heavy and therefore asymptotic methods are useful.

Consider the pitch equations (4.39) with zero forcing and no damping. The solution to (7.39) is given by

$$\gamma_3 = \Lambda_1 e^{j(\omega_1 \tau + \psi_1)} + \Lambda_2 e^{j(\omega_2 \tau + \psi_2)}$$

$$\psi_y = A_{21} \Lambda_1 e^{j(\omega_1 \tau + \psi_1)} + A_{22} \Lambda_2 e^{j(\omega_2 \tau + \psi_2)} \tag{7.45}$$

where the "mode shapes" are written

$$A_{21} = \frac{3g_{21} - \omega_1^2}{K_3(3 - \omega_1^2)}$$

$$A_{22} = \frac{3g_{21} - \omega_2^2}{K_3(3 - \omega_2^2)}$$

It is understood that we take the real part of (7.45) to find the solution. The energy can be written

$$H = \frac{I' + I_3}{2} [\dot{\gamma}_3^2 + 2K_3 \dot{\psi}_y \dot{\gamma}_3 + K_3 \dot{\psi}_y^2]$$

$$+ \frac{I' + I_3}{2} [3g_{21} \gamma_3^2 + 6K_3 \psi_y \gamma_3 + K_3 \Omega_B^2 \psi_y^2] \tag{7.46}$$

At this point an assumption must be made as to the behavior of the beam under elastic hysteresis damping. In the absence of any good evidence about the behavior of materials excited by almost periodic functions, it will be assumed that each frequency component traverses a hysteresis loop with the same phase lag, ϕ, as every other. The average power in the present example, computed on this basis, is then

$$\bar{P} = \frac{-\Omega_B^2 I' \phi \omega_k}{2} \sum_{k=1}^{2} |A_{2k}|^2 \Lambda_k^2 = - \sum_{k=1}^{2} P_k \Lambda_k^2 \tag{7.47}$$

The coefficients H_k in the energy expression can be written

$$\begin{aligned} H_k &= \frac{I'+I_3}{2} [1 + K_3 |A_{2k}|^2 + 2K_3 A_{2k}] \omega_k^2 \\ &+ \frac{I'+I_3}{2} [3g_{21} + \Omega_B^2 K_3 |A_{2k}|^2 + 3K_3 A_{2k}] \end{aligned} \tag{7.48}$$

The differential equation of energy balance is

$$H_k \dot{\Lambda}_k + P_k \Lambda_k = 0$$

and its solution is

$$\Lambda_k(\tau) = \Lambda_k(o) e^{-\tau/T_k}$$

where

$$T_k = \frac{H_k}{P_k}$$

It remains to factor the characteristic determinant and find ω_k; then the A_{2k} must be computed along with the T_k. The characteristic determinant is

$$D(p) = (p^2 + \Omega_B^2)(p^2 + 3g_{21}) - K_3(p^2 + 3)^2 \tag{7.49}$$

It is interesting to consider the special case $k_{21} = 0$, a <u>neutrally stable body</u>. The analysis is restricted to the case of a long boom (relative to body dimensions). A long boom is needed for static stabilization and because length in the boom decreases Ω_B to levels where the system is able to damp sufficiently. (Note that Ω_B^2 varies in proportion to $1/L^3$.) Since $K_3 = 1/\{1+I_3/I'\}$, if $I_3/I' = \epsilon << 1$, then $K_3 = 1 - \epsilon$ approximately. Using this fact one can approximately factor (7.49) obtaining as natural frequencies

$$\omega_1^2 = 3 - 3\epsilon - 9\epsilon^2/(\Omega_B^2 - 3)$$

$$\omega_2^2 = \frac{\Omega_B^2 - 3}{\epsilon}$$

This gives, using (7.48), the values for A_{2k} and T_k.

$$A_{21} = \frac{3\epsilon}{\Omega_B^2 - 3}$$

$$A_{22} = \frac{1}{1 - \epsilon}$$

$$\not{0}T_1 = \frac{1}{3\Omega_B^2}\left\{\frac{(\Omega_b^2 - 3)^2}{\epsilon^2} + 9 + \Omega_B^2 + \frac{7(\Omega_B^2 - 3)}{\epsilon}\right\}$$

$$\not{0}T_2 = \frac{\sqrt{\epsilon}\,[\Omega_B^2 + 3 + \frac{4(\Omega_B^2 - 3)}{\epsilon}]}{\Omega_B^2\sqrt{\Omega_B^2 - 3}}$$

Figure 7.6 shows a plot of $\not{0}T_1$ vs. ϵ for various Ω_B^2. It can be seen that the beam must be flexible and still not too long. In the case shown $T_2 << T_1$.

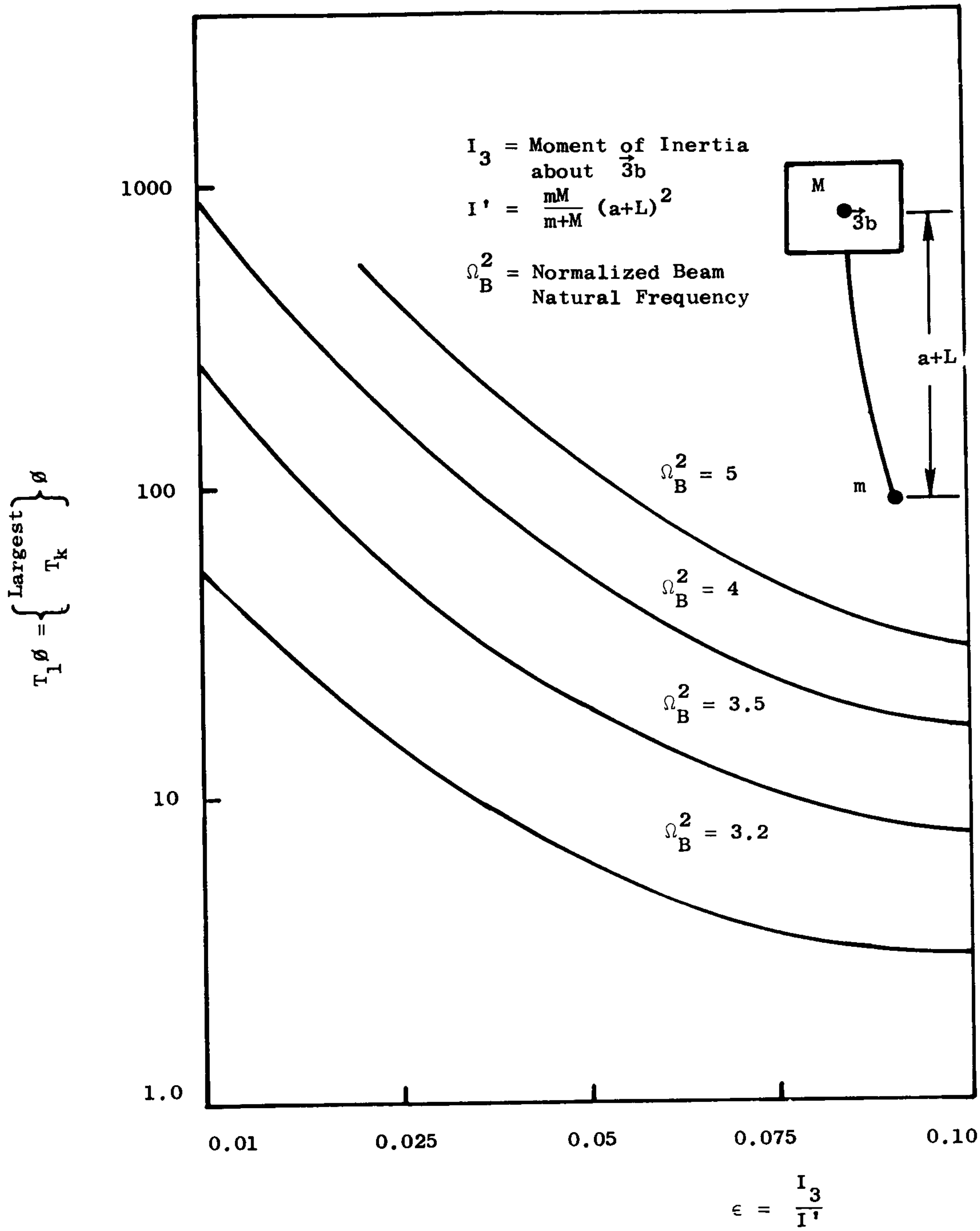

FIG. 7.6. PLOT OF THE TIME OF TRANSIENT DECAY IN PITCH VS. THE PARAMETER ϵ FOR VARIOUS BEAM NATURAL FREQUENCIES Ω_B

Consider the following example. Let $I_3 = 120$ slug ft^2, $I' = 1200$ slug ft^2, $\mu = 1$ slug, $L = 34.5$ ft, $\Omega_B^2 = 3.5$, with a berrylium-copper beam ($E = 18 \times 10^2$ psi), be the parameters of the system in pitch. Using a hollow cylindrical shell for a beam gives a stiffness for the beam of $EI = \pi E \delta R_o^3$.* A typical realizable thickness, δ, for the cylinder wall is 0.002 inches and, along with the radius of the cylinder R_o, defines the system completely. From the body parameters we see that for this design $\phi T_1 = 6.9$. Using the formula for Ω_B^2, given under "nomenclature," we can calculate the radius of the cylindrical beam as $R_o = 0.076$ in. This means that with a reasonable size tubular beam we may realize a time response $\phi T_1 = 6.9$ and with the cadmium coating available for TRAAC we might well assume that $\phi = 1/8\pi$. This gives $T_1 = 173$ (or 27.6 orbit periods). Thus with "reasonable" dimensions for the beam this body was damped in pitch in the order of the time needed to damp TRAAC (Section E, Part 5). This compares with an approximate result of 5.3 orbit periods for vertistat pitch motions.

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*The stiffness of the beam (EI) assumes a hollow, closed cylindrical shell. If the beam is constructed by rolling up a prestressed sheet of metal (as in the STEM unit used in connection with the TRAAC satellite) then the stiffness will be decreased by a factor depending upon the details of construction. It is hard to imagine that the factor EI would be decreased more than a factor of ten; in this case the cylinder's radius would increase by a factor of $\sqrt[3]{10} = 2.15$. Thus it can be seen that the radius of the cylinder is relatively insensitive to the value of EI for a given beam frequency, Ω_B.

E. THE TRAAC SATELLITE

This satellite design, due to Fischell, et al. (FISCHELL 1, FISCHELL 2), has been orbited twice; the second time it was evidently successful. This is the only design of a purely passive system that has been orbited by the West. The satellite consists of a main body with a 100-foot-long extensible boom (Fig. 7.3) at the end of which is attached a 20-foot-length of very limber helical spring (7.6 inches in diameter). The spring carries an end mass weighing 3.85 pounds. The spring is beryllium-copper wire (0.008 inches diameter) coated with Cadmium (0.0002 thick), to obtain good damping characteristics due to stresses in the Cadmium. By a torsion pendulum experiment on the spring wire in the laboratory it was determined that for oscillations of 55 minute periods approximately half of the maximum energy stored in the spring was dissipated per cycle of oscillation. Thus in the TRAAC satellite the spring's elastic dissipation provides the damping and the long boom-spring-mass system provides the position restoring force.

The spring, however, only provides damping (to first order) in the orbit plane (pitch). For the roll-yaw motions the system uses ferromagnetic rods which interact with the earth's magnetic field and produce damping torques by means of magnetic hysteresis. The combination of spring damping and magnetic hysteresis rods succeeded in damping the peak pitch oscillations from 44 degrees to 6 degrees off of equilibrium in about 8 days or 115 orbital revolutions (100 minute orbit periods).

This transient response is rather slow. One would like a decay of the order of a day so that large disturbances (e.g., meteoroids) would not disrupt operation of the system for many days. This is particularly important for communication satellites.

1. Energy Expressions

The TRAAC satellite system (Figs. 7.3 and 7.7) consists of two rigid bodies (labeled "a" and "b") connected together by a very limber, helical spring. The model to be considered assumes no torsion in the

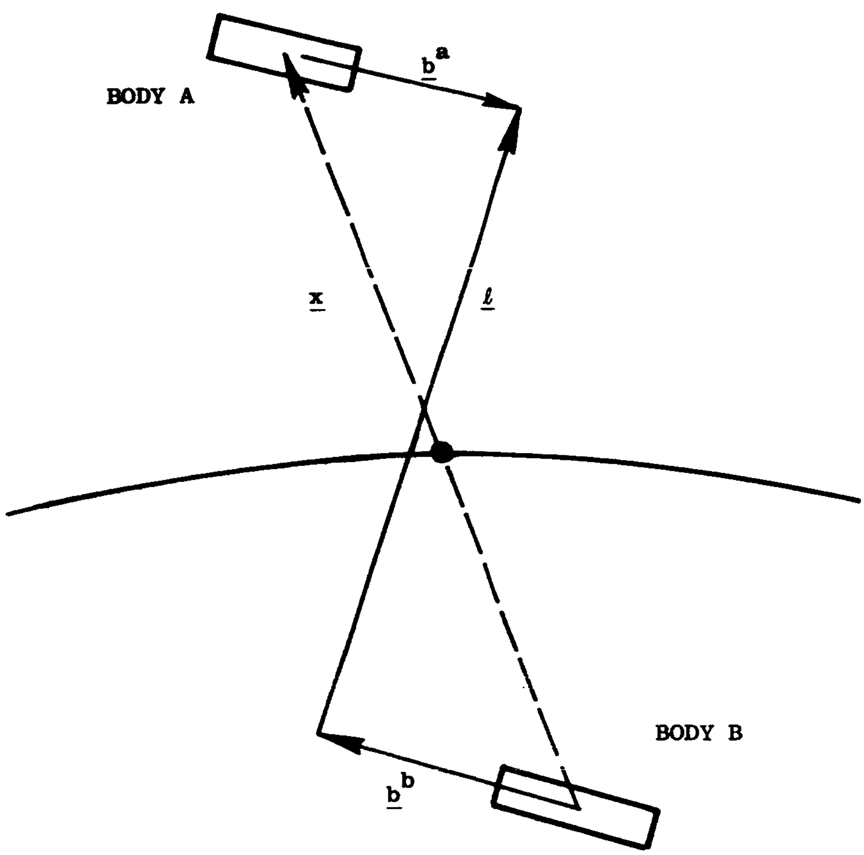

FIG. 7.7. TWO CONNECTED BODIES

spring; after deriving more general equations for the energies we will assume that one of the rigid bodies consists of a point mass (no rotary inertia). This model will portray the salient features of the system and still lead to a manageable analytical problem.

The inertia dyadics and masses of bodies "a" and "b" are $\mathbb{I}^a$, $\mathbb{I}^b$ and m_a, m_b, respectively. The Euler angles of the body axes with respect to the rotating axes of the reference frame are γ_1^a, γ_2^a, γ_3^a and γ_1^b, γ_2^b, γ_3^b. The distance between the body centers of mass is called x and the unit vector from the C.M. of "b" to the C.M. of "a" is $\hat{1}x$. The direction of $\hat{1}x$ with respect to the reference axes $(\hat{1}, \hat{2}, \hat{3})$ is given in Fig. 7.8 by Euler angles ϕ, θ.

The reference coordinate system to be used is a system $\hat{1}$, $\hat{2}$, $\hat{3}$ whose $\hat{1}$ axis unit vector points along the radius vector $(\hat{1} = \hat{R}^c)$, whose $\hat{3}$ points along the orbit angular momentum vector and whose $\hat{2}$ is such that $\hat{2} = \hat{3} \times \hat{1}$. This coordinate system is centered at the center of mass of the entire system of particles.

The body systems for "a" and "b" are described by Euler angles γ_1, γ_2, γ_3 (Fig. 7.4) which are the angles of successive rotations about the $\hat{1}$, the new $\hat{2}$, and the $\hat{3}a$ and $\hat{3}b$ axes where the $\hat{1}a$, $\hat{2}a$, $\hat{3}a$, $\hat{1}b$, $\hat{2}b$, $\hat{3}b$, are body unit vectors (aligned with the principal axes of the bodies) and are centered in bodies "a" and "b", respectively.

The Euler angles ϕ, θ are shown in Fig. 7.8. These angles describe the direction of $\hat{1}x$ with reference to $\hat{1}$, $\hat{2}$, $\hat{3}$.

The expressions given in (2.7) and (2.8) are, for this problem,

$$T = \frac{1}{2}\mu(\dot{\underline{x}})^2 + T_{RB}^a + T_{RB}^b \tag{7.50}$$

$$V_g = V_{RB}^a + V_{RB}^b + \frac{\mu k}{|\underline{R}^c|^3}\left\{-\frac{1}{2}\,\mathrm{tr}\,\mathbb{R} + \frac{3}{2}\hat{R}^c\cdot\mathbb{R}\cdot\hat{R}^c\right\} \tag{7.51}$$

where T_{RB}^a, T_{RB}^b, V_{RB}^a, V_{RB}^b are in (Chapter VII, Sec. B, Part 1) and involve "rigid body terms." The reduced mass μ and the quantities $\hat{R}^c$, $\mathbb{R}$, $\underline{x}^2$ are given by

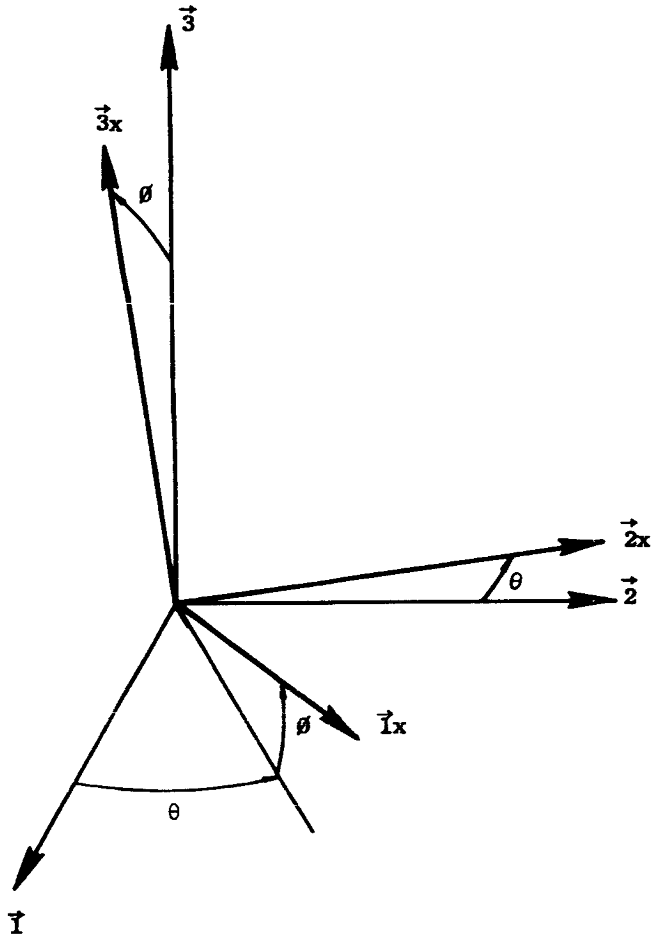

FIG. 7.8. EULER ANGLES OF $\hat{1}x$

$$\dot{\underline{x}}^2 = \dot{x}^2 + x^2[\dot{\phi}^2 + (\dot{\theta} + \dot{\eta})^2 \cos^2\phi]$$

$$\mathbb{R} = x^2[\hat{2}x\ \hat{2}x + \hat{3}x\ \hat{3}x]$$

$$\hat{R}^c = \hat{1} = \hat{1}x \cos\theta\cos\phi - \hat{2}x \sin\theta - \hat{3}x \cos\theta\sin\phi$$

$$\mu = \frac{m_a m_b}{m_a + m_b}$$

Here we have used η to indicate the true anomaly of the orbit. Using the above in (7.50) and (7.51) gives for the kinetic energy and the potential energy due to gravity:

$$T = T^a_{RB} + T^b_{RB} + \frac{\mu}{2}\left\{\dot{x}^2 + x^2\ [\dot{\phi}^2 + (\dot{\theta} + \dot{\eta})^2 \cos^2\phi]\right\} \tag{7.52}$$

$$V_g = V^a_{RB} + V^b_{RB} + \frac{\mu x^2}{2}\ \frac{k}{|\underline{R}^c|^3}\left\{1 - 3\cos^2\phi\ \cos^2\theta\right\} \tag{7.53}$$

The potential energy of the spring (linear) is given by the expression

$$V_s = \frac{1}{2}\alpha(|\underline{\ell}| - \ell_s)^2 \tag{7.54}$$

where α is the spring constant and $|\underline{\ell}|$ and ℓ_s are the lengths of the spring in the stretched and unstretched positions, respectively. The distances from the body centers of mass to the points of connection of the spring are given by b^a and b^b for bodies "a" and "b", respectively. The vector expressions are as follows, from geometry (Fig. 7.6).

$$\underline{\ell} = \underline{x} + \underline{b}^a - \underline{b}^b$$

$$\underline{x} = \hat{1}x \; x$$

$$\underline{b}^a = -\hat{1}a \; b^a$$

$$\underline{b}^b = \hat{1}b \; b^b$$

Using these relations one finds that

$$\begin{aligned} |\underline{\ell}| &= x^2 + (b^a)^2 + (b^b)^2 + 2b^a b^b(\hat{1}a\cdot\hat{1}b) \\ &\quad - 2x(b^a \; \hat{1}x\cdot\hat{1}a + b^b \; \hat{1}x\cdot\hat{1}b) \end{aligned} \tag{7.55}$$

where the direction cosines are given by

$$\begin{aligned} \hat{1}x\cdot\hat{1}a &= \cos\emptyset \cos\theta \cos\gamma_2^a \cos\gamma_3^a \\ &\quad + \sin\theta \cos\emptyset(\cos\gamma_1^a \sin\gamma_3^a + \sin\gamma_1^a \sin\gamma_2^a \cos\gamma_3^a) \\ &\quad + \sin\emptyset(\sin\gamma_1^a \sin\gamma_3^a - \cos\gamma_1^a \sin\gamma_2^a \cos\gamma_3^a) \\ \hat{1}x\cdot\hat{1}b &= \cos\emptyset \cos\theta \cos\gamma_2^b \cos\gamma_3^b \\ &\quad + \sin\theta \cos\emptyset(\cos\gamma_1^b \sin\gamma_3^b + \sin\gamma_1^b \sin\gamma_2^b \cos\gamma_3^b) \\ &\quad + \sin\emptyset(\sin\gamma_1^b \sin\gamma_3^b - \cos\gamma_1^b \sin\gamma_2^b \cos\gamma_3^b) \\ \hat{1}a\cdot\hat{1}b &= \cos\gamma_2^a \cos\gamma_2^b \cos\gamma_3^a \cos\gamma_3^b \\ &\quad + (\cos\gamma_1^a \sin\gamma_3^a + \sin\gamma_1^a \sin\gamma_2^a \cos\gamma_3^a)(\cos\gamma_1^b \sin\gamma_3^b+\sin\gamma_1^b\sin\gamma_2^b\cos\gamma_3^b) \\ &\quad + (\sin\gamma_1^a \sin\gamma_3^a - \cos\gamma_1^a \sin\gamma_2^a \cos\gamma_3^a)(\sin\gamma_1^b \sin\gamma_3^b-\cos\gamma_1^b\sin\gamma_2^b\cos\gamma_3^b) \end{aligned} \tag{7.56}$$

Using (7.54)-(7.56) in (7.52) and (7.53) we get the complete energy expressions except for the rigid body terms.

2. Linear Model of TRAAC

The equations of motion are derived by energy considerations and turn out to be (in the linear approximation to first order in eccentricity) with body "b" considered as a particle ($\Pi^b = 0$).

$$\begin{bmatrix} p^2 + 3k_{21} + 3K_3 r & -3rK_3 & 0 \\ -3r & p^2 + 3(1+r) & 2p \\ 0 & -2p & p^2 + \Omega_\alpha^2 - 3 \end{bmatrix} \begin{bmatrix} \gamma_3 \\ \theta \\ \xi \end{bmatrix} = \begin{bmatrix} Q_{\gamma_3}/I_3 n^2 \\ + 2e\sin\tau \\ Q_\theta / I' n^2 \\ + 2e\sin\tau \\ Q_\xi / I' n^2 \\ + 10e\cos\tau \end{bmatrix} \quad (7.57)$$

$$\begin{bmatrix} p^2 + k_{32} & -h_1 p & 0 \\ h_2 p & p^2 + 4k_{31} + 3K_2 r & +3r\,K_2 \\ 0 & +3r & p^2 + 4 + 3r \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \phi \end{bmatrix} = \begin{bmatrix} Q_{\gamma_1}/I_1 n^2 \\ Q_{\gamma_2}/I_2 n^2 \\ Q_\phi / I' n^2 \end{bmatrix} \quad (7.58)$$

where the normalized parameters are

$$r = \frac{b}{x_o - b} \doteq \frac{b}{\ell_o} \qquad \Omega_\alpha^2 = \frac{\alpha}{\mu n^2} = 3\left(\frac{x_o}{x_o - x_s}\right)$$

$$p = s/n \qquad x = x_o(1 + \xi)$$

$$\tau = nt = \text{normalized time} \qquad h_1 = \frac{I_1 + I_2 - I_3}{I_1}$$

$$I' = \mu x_o^2 \qquad h_2 = \frac{I_1 + I_2 - I_3}{I_2}$$

$$\mu = \frac{m_a m_b}{m_a + m_b} \qquad K_1 = I'/I_1$$

$$k_{21} = \frac{I_2 - I_1}{I_3} \qquad K_2 = I'/I_2$$

$$k_{31} = \frac{I_3 - I_1}{I_2} \qquad K_3 = I'/I_3$$

$$k_{32} = \frac{I_3 - I_2}{I_1}$$

The physical parameters are:

Q_j = torques and forces acting on the system

s = Laplace transform variable (angular frequency)

b = boom length

n = mean angular rate of the satellite in orbit

e = orbit eccentricity

x_o = equilibrium distance between the C.M. of main body and m

m = mass of body "b"

I_1, I_2, I_3 = moments of inertia of body about $\hat{1}a$, $\hat{2}a$, $\hat{3}a$

α = spring constant of helical spring

x_s = static spring deflection

Notice that the TRAAC linearized equations imply a decoupling of the orbit plane motions, (7.57), from the out-of-plane motions, (7.58), (pitch and roll-yaw are analogously decoupled in the case of a single rigid body).

The Hamiltonian for the system of linearized equations can be derived by referring to (3.:8) where it is shown that

$$H = T_2 + V - T_o$$

This relation applied to the TRAAC system gives

$$\begin{aligned} H = \; & \frac{1}{2}\left\{I_1\dot{\gamma}_1^2 + I_2\dot{\gamma}_2^2 + I_3\dot{\gamma}_3^2 + I'(\dot{\xi}^2 + \dot{\phi}^2 + \dot{\theta}^2)\right\} \\ + \; & \frac{1}{2}\left\{3I_3(k_{21} + rK_3)\gamma_3^2 - 3(rK_3I_3 + r\, I')\gamma_3\theta \right. \\ + \; & \left. 3I'(1+r)\theta^2 + (\Omega_\alpha^2 - 3)I'\xi^2 \right\} \\ + \; & \frac{1}{2}\left\{k_{32}\, I_1\gamma_1^2 + (4k_{31} + 3K_2r)I'\gamma_2^2 \right. \\ + \; & \left. 3(rK_2I_2 + rI')\phi\gamma_2 + (4 + 3r)I'\phi^2 \right\} \end{aligned} \qquad (7.59)$$

By the theorem given in Chapter III (Theorem IV) we know that H must be positive definite while $\dot{H}$ is negative semi-definite (Theorem V) for the system to be asymptotically stable. The power, $P = \dot{H}$, is negative semi-definite since energy always leaves the system due to elastic hysteresis action. This requires that $U = V - T_o$ be positive definite in the angles and in ξ. The stability inequalities are, therefore,

$$\Omega_\alpha^2 > 3$$

$$k_{21} > -\frac{rK_3}{1-r} < 0 \tag{7.60}$$

$$k_{32} > 0$$

$$k_{31} > -\frac{9}{16}\left(\frac{r\,K_2}{1 + 3/4\,r}\right) < 0$$

For the stability of a single rigid body with damping k_{21}, k_{31}, k_{32} must all be positive. It is seen that (7.60) relax these requirements for k_{31} and k_{21}. This is reasonable since the system with $m \neq 0$ has greater inertia about the $\hat{2}a$ and $\hat{3}a$ axes. The requirement $\Omega_\alpha^2 > 3$ means that the spring must be stiff enough to overcome the gravity and centrifugal separation forces.

3. Forced Motions

Consider the TRAAC system in the orbit plane (equation (7.57)) with $e \ll 1$, $Q_{\gamma_3} = Q_\theta = Q_\xi \equiv 0$, i.e., the damping is very small. Using the usual techniques of steady-state sinusoidal analysis, let $\cos\tau = \mathrm{Re}[e^{j\tau}]$, $\sin\tau = -\mathrm{Re}[je^{j\tau}]$ and employ exponentials to analyze equation (7.57). The result is

$$\begin{bmatrix} 3k_{21} + 3rK_3 - 1 & -3rK_3 & 0 \\ -3r & 2 + 3r & 2j \\ 0 & -2j & \Omega_\alpha^2 - 4 \end{bmatrix} \begin{bmatrix} \Upsilon_3 \\ \Theta \\ \Xi \end{bmatrix} = \begin{bmatrix} -j \\ -j \\ 5 \end{bmatrix} e \tag{7.61}$$

where $\gamma_3 = \Upsilon_3 e^{j\tau}$, $\theta = \Theta e^{j\tau}$, and $\xi = \Xi\, e^{j\tau}$. These simultaneous algebraic equations can be inverted to find γ_3, θ, ξ as functions of time. Let us consider this accomplished and require that $\gamma_3 \equiv 0$. Thus we choose the parameters which give $\gamma_3 \equiv 0$ according to the condition that

$$\det \begin{bmatrix} -j & -3K_3 r & 0 \\ -j & 3r + 2 & 2j \\ 5 & -2j & \Omega_\alpha^2 - 4 \end{bmatrix} = 0 \tag{7.62}$$

This gives a condition on r:

$$r = \frac{2(6 - \Omega_\alpha^2)}{3(K_3 + 1)\left[\Omega_\alpha^2 - 4 + \dfrac{10K_3}{1 + K_3}\right]} \tag{7.63}$$

The response of the system for the condition (7.63) is

$$\begin{aligned} \gamma_3 &= 0 \\ \theta &= -\frac{2e}{3rK_3} \sin \tau \\ \xi &= \frac{2e(1 + 6K_3)}{K_3(\Omega_\alpha^2 - 6)} \cos \tau \end{aligned} \tag{7.64}$$

Notice that the vibration absorption condition (7.63) does not depend on the body's shape but only on r, K_3, Ω_α^2. This can be seen by realizing that with $\gamma_3 \equiv 0$ the parameter k_{21} has no effect in (7.61). One would expect that as K_3 gets large (body gets small in dimension compared to the length of the satellite (x_o)) the motions would approach a flexible dumbell's motion; this is indeed the case as can be verified using (7.64) and (7.57) (letting $K_3 \to \infty$ in (7.57)).

Notice that unless $\Omega_\alpha^2 > 3$ the system is unstable. Equation (7.64) indicates that for $\Omega_\alpha^2 > 4$ the motions may get rather large if e, the eccentricity, is not very small. The motion of θ, ξ in the orbit axes makes an elliptical figure assuming $\theta = \sin \theta$.

As an example of a system with vibration absorption consider the following parameters:

$$I_1 = 22 \text{ slug ft}^2$$

$$I_2 = I_3 = 720 \text{ slug ft}^2$$

$$m = 0.12 \text{ slugs}$$

$$x_o = 120 \text{ ft}$$

$$I' = 1730 \text{ slug ft}^2$$

$$k_{21} = 0.973$$

$$K_3 = I'/I_3 = 2.4$$

$$\Omega_\alpha^2 = 4$$

Using (7.63), the vibration absorption condition, gives $r = 0.055$, and the boom length, $b = 6.7$ ft. This system has a property typical of systems following (7.63); it has a very short boom compared to the total length of the system ($r \ll 1$). If the eccentricity is less than 0.05, the excursions will be less than 39 ft. The design based upon the vibration absorption condition has the property of very poor transient response because the coupling between the γ_3 motion and the θ, ξ motions is small of the order of r^2.

4. Effect of Damping on TRAAC

Of the utmost interest in passive attitude control is the time it takes an oscillation to go from a given initial condition to an acceptable operating level. If the law of damping were a linear function of the attitude rates, there would be no problem that could not be solved by the well-known transient methods. However, in practice the damping law usually does not even approximate this type behavior; new methods are required to analyze the damping process. One such method has been described in Chapter VI and will be used in the following analysis.

Once we have established a method of calculating the damping response, we must ask if, on hueristic grounds, there will be enough damping force to warrant consideration using quantitative methods. In the TRAAC system it can be shown that, neglecting torsion of the spring, there is only very slow convergence in motions out of the orbit plane (see Section E, Part 5). In practice ferromagnetic rods were used (FISCHELL 1, FISCHELL 2) to damp motion out of the orbit plane.

As an example of the calculation of the damping response and of the expected numerical results, we consider the TRAAC orbit plane motions. We can solve equations (7.57) for arbitrary initial conditions and obtain the result that

$$\begin{aligned}
\gamma_3 &= \Lambda_1 e^{j(\omega_1\tau+\psi_1)} + \Lambda_2 e^{j(\omega_2\tau+\psi_2)} + \Lambda_3 e^{j(\omega_3\tau+\psi_3)} \\
\theta &= A_{21}\Lambda_1 e^{j(\omega_1\tau+\psi_1)} + A_{22}\Lambda_2 e^{j(\omega_2\tau+\psi_2)} + A_{23}\Lambda_3 e^{j(\omega_3\tau+\psi_3)} \\
\xi &= A_{31}\Lambda_1 e^{j(\omega_1\tau+\psi_1)} + A_{32}\Lambda_2 e^{j(\omega_2\tau+\psi_2)} + A_{33}\Lambda_3 e^{j(\omega_3\tau+\psi_3)}
\end{aligned} \tag{7.65}$$

where the ω_k are the natural frequencies of (7.57), the A_{ik} are the "mode shapes," and the Λ_k, ψ_k depend only on the initial conditions. The A_{ik} are found to be

$$
\begin{bmatrix} A_{1k} \\ \\ A_{2k} \\ \\ A_{3k} \end{bmatrix} = \begin{bmatrix} 1 \\ \\ \dfrac{3k_{21} + 3K_3 r - \omega_k^2}{3rK_3} \\ \\ \dfrac{2j\omega_k(3k_{21} + 3K_3 r - \omega_k^2)}{(\Omega_\alpha^2 - 3 - \omega_k^2)(3r\ K_3)} \end{bmatrix} \tag{7.66}
$$

The average energy (or Hamiltonian) in an undamped oscillation can be calculated from (7.59) as

$$
\bar{H} = \frac{1}{2} \sum_{k=1}^{3} H_k \Lambda_k^2
$$

$$
H_k = \frac{1}{2} \left\{ \omega_k^2 I_3 + I' \omega_k^2 \left(|A_{2k}|^2 + |A_{3k}|^2 \right) \right\} \tag{7.67}
$$

$$
+ \frac{1}{2} \left\{ 3I_3(k_{21} + rK_3) - 6rI' A_{2k} + (\Omega_\alpha^2 - 3) I' |A_{3k}|^2 \right.
$$

$$
\left. + 3I'(1 + r)|A_{2k}|^2 \right\}
$$

The method of calculating the damping response consists of requiring that the amplitudes Λ_k vary with time in such a manner that the rate of change of energy equals the power lost in damping. In equation form this is

$$
\dot{\bar{H}} = \bar{\dot{H}} = \bar{P} = \sum_{k=1}^{3} H_k \Lambda_k \dot{\Lambda}_k \tag{7.68}
$$

By the same assumption used in the elastic hysteresis analysis of the beam satellite (see 7.47) we can write the expression for the average power loss in damping of the TRAAC as:

$$\overline{P} = - \frac{\Omega_\alpha^2 \, I'\phi}{2} \, [\omega_1 |A_{31}|^2 \Lambda_1^2 + \omega_2 |A_{32}|^2 \Lambda_2^2 + \omega_3 |A_{33}|^2 \Lambda_3^2] \tag{7.69}$$

Let us now combine (7.68) and (7.69) and equate the coefficients of Λ_k to get differential equations of the envelope response. These are (k = 1,2,3)

$$\dot{\Lambda}_k + \frac{\Omega_\alpha^2 \, I'\phi}{2E_k} \omega_k |A_{3k}|^2 \Lambda_k = 0 \tag{7.70}$$

This equation, because of the quadratic nature of $\overline{P}$, is linear in the Λ_k

$$\Lambda_k(\tau) = \Lambda_k(o) e^{-\tau/T_k}$$

where the "time constant," T_k, is given by

$$T_k = \frac{2H_k}{\phi\Omega_\alpha^2 \, I' \, \omega_k |A_{3k}|^2} \tag{7.71}$$

The line of reasoning employed here is given in Chapter VI in greater detail and with more explanation.

As a numerical example of significance the numbers used in the example on vibration absorption will be used with a different boom length parameter. These numbers were chosen to approximate those given for the TRAAC satellite in (FISCHELL 2). The parameters are

$b \;=\; 100\text{ ft} \;=\;$ boom length

$\ell_o \;=\; 20\text{ ft} \;=\;$ equilibrium spring length

$\ell_s \;=\; 0.125\text{ ft} =$ static spring length

$m \;=\; 0.12$ slugs = mass at the end of the spring

$I_1 \;=\; 22\text{ slug ft}^2 =$ yaw moment of inertia

$\bar{I}_2 \;=\; \bar{I}_3 \;=\; 720\text{ slug ft}^2 =$ roll and pitch moments of inertia

$r \;=\; 5$

$I' \;=\; 1730\text{ slug ft}^2$

$k_{21} \;=\; 0.973$

$\Omega_\alpha^2 \;=\; 18$

$K_3 \;=\; 2.4$

Upon factoring the characteristic equation of (7.57) to find ω_k we get, using (7.66) and (7.71), Table 3.

TABLE 3

MODE	ω_k	$T_k\phi$
1	1.55	4.3
2	4.23	0.52
3	7.45	11.0

The mode shape factors are:

$$A_{11} = A_{12} = A_{13} = 1$$

$$A_{21} = 1.0$$

$$A_{31} = 0.25\ j$$

$$A_{22} = 0.58$$

$$A_{32} = -1.64\ j$$

$$A_{23} = -0.46$$

$$A_{33} = 0.17\ j$$

The response to orbit eccentricity forces is given for this example by

$$\gamma_3 = 1.7e \sin \tau$$

$$\theta = 1.7e \sin \tau$$

$$\xi = 4.65e \cos \tau$$

The tests that Fischell, et al. ran on the spring material indicated that half the stored energy in an oscillation was dissipated in one cycle of oscillation. This implies for a simple oscillator a phase lag $\phi = 1/8\pi$. From the table above the longest time constant is, for $\phi = 1/8\pi$, $T_3 = 11 \times 8\pi$. This is equivalent to a time constant of 44 orbit periods. This response seems a bit slow and is characteristic of systems using elastic hysteresis damping. The response can probably be optimized by trying various parameter combinations and comparing responses but the upper limit of performance would probably be a time constant of about 10 orbits. Any further improvement in performance would have to come from an improvement in material damping properties.

5. Ineffectiveness of Roll-Yaw Damping

The question of whether the elastic hysteresis effect, which damps pitch oscillations, will also damp roll-yaw oscillations is important. As it turns out the damping of roll-yaw oscillations by the spring is of third order in the angles and thus has little effect for small angle motions. One expects the large oscillations to damp out to a certain level and then persist without appreciable attenuation for a long period of time.

Consider viscous damping (to simplify the analysis) which exerts a force $-D\, d/dt|\ell|$ along the spring. This force would do work $\overline{\delta W} = -D(d/dt|\ell|)(\delta|\underline{\ell}|)$. This distance $|\underline{\ell}|$ is expressed in terms of the generalized coordinates as

$$|\ell|^2 = x^2 + b^2 - 2bx\cos\beta \tag{7.72}$$

where

$$\cos\beta = \hat{1}a\cdot\hat{1}x = \cos\phi\cos\theta\cos\gamma_2\cos\gamma_3$$

$$+ \sin\theta\cos\phi(\cos\gamma_1\sin\gamma_3 + \sin\gamma_1\sin\gamma_2\cos\gamma_3)$$

$$+ \sin\phi(\sin\gamma_1\sin\gamma_3 - \cos\gamma_1\sin\gamma_2\cos\gamma_3)$$

Keeping terms up to second order in $\cos\beta$ and letting $|\ell| = \ell_o + \bar{\ell}$ where ℓ_o is the equilibrium spring length we get

$$\cos\beta = 1 - \frac{1}{2}\beta^2$$

$$\beta^2 = (\phi + \gamma_2)^2 + (\theta + \gamma_3)^2$$

$$\bar{\ell} = x_o\xi + \frac{x_o^2}{2\ell_o}\xi^2 + \frac{1}{2}r\,x_o\beta^2$$

$$r = b/\ell_o$$

If it is assumed that the pitch angles and ξ have decayed to zero, the power loss function for viscous damping becomes for small angles

$$P = - x_o^2 r^2 D(\phi + \gamma_2)^2 (\dot{\phi} + \dot{\gamma}_2)^2 \tag{7.73}$$

where the above geometrical relations have been used to calculate ℓ. If it is further assumed that the motion of the angle γ_1, the yaw angle, is arbitrary and relatively uncoupled from the motion of ϕ and γ_2, the motion of the roll angles can be considered separately. The yaw angle will decay slowly or perhaps there will be a constant yaw rate that persists for a long time but this eventuality, similar to the phenomena of Chapter VI, will not seriously alter the deductions from the results to be given. Consider the response in (7.58) with $\gamma_1 \equiv 0$ to be of the form

$$\begin{bmatrix} \gamma_2 \\ \phi \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \Lambda_1 e^{j(\omega_1 \tau + \psi_1)} \\ \Lambda_2 e^{j(\omega_2 \tau + \psi_1)} \end{bmatrix} \tag{7.74}$$

Using (7.73) and (7.74) we compute the envelope response and the average power.

$$\dot{\Lambda}_k + \frac{D}{\mu} r^2 \left(\sum_{j=1,2} \bar{G}_{jk} \Lambda_j^2 \right) \Lambda_k = 0 \qquad (k = 1,2) \tag{7.75}$$

$$\bar{P} = - \sum_{j,k=1,2} G_{jk} \Lambda_j^2 \Lambda_k^2 \tag{7.76}$$

where $G_{jk} = H_k (D/\mu) r^2 \bar{G}_{jk}$ and the constants $\bar{G}_{jk}$ and A_{jk} are given by

$$\overline{G} = \frac{I'}{8} \begin{bmatrix} \dfrac{\omega_1^2 \alpha^4}{E_1} & \dfrac{(\omega_1^2 + \omega_2^2)\alpha^2\beta^2}{E_2} \\ \\ \dfrac{(\omega_1^2 + \omega_2^2)\alpha^2\beta^2}{E_1} & \dfrac{\omega_2^2\, \beta^4}{E_2} \end{bmatrix} \tag{7.77}$$

$$\hat{A}_{2k} = \frac{-3r}{4 + 3r - \omega_k^2} \qquad \begin{aligned} \alpha &= 1 + A_{21} \\ \beta &= 1 + A_{22} \end{aligned} \tag{7.78}$$

The natural frequencies ω_k are given after factoring the characteristic determinant.

$$\Delta = (p^2 + 4 + 3_r)(p^2 + 4k_{31} + 3K_2 r) - 9r^2 K_2 \tag{7.79}$$

The energy coefficients can be computed based on the energy expression of (7.59). They are

$$\begin{aligned} H_k = {} & \frac{\omega_k^2}{2} \left\{ I_2 + I' \left| A_{2k} \right| \right\} \\ & + \frac{1}{2} \Big\{ I_2 (4k_{31} + 3rK_2) + 6r\, I'\, A_{2k} \\ & \qquad + I'(4 + 3r) \left| A_{2k} \right|^2 \Big\} \end{aligned} \tag{7.80}$$

Based upon the above relations and the parameters (which approximate those given in FISCHELL 2) the mode shapes and the natural frequencies can be computed by factoring (7.79) and using (7.78).

$$\omega_1^2 = 4.1 \qquad A_{21} = -0.997$$

$$\omega_2^2 = 54.9 \qquad A_{22} = 0.42$$

where,

$$k_{31} = 0.96 \quad ; \quad K_2 = 2.4 \quad ; \quad r = 5$$

The parameter α given in (7.78) is therefore 0.003. This small value means that the envelope function Λ_1 will decay very slowly compared to the decay of Λ_2. But the response of Λ_2 is slow because of the non-linearity and, therefore, the TRAAC system has sensibly no damping in yaw-roll for small angles. By decreasing the parameter r (ratio of length of boom to length of spring) we may increase the parameter α while decreasing the power dissipated in yaw-roll, but the situation also depends upon the parameters K_2 and k_{31} in a very interesting manner. Let us write the characteristic equation (7.79) with a root locus of r in the p-plane in mind.

$$D = (p^2 + 4)(p^2 + 4\,k_{31}) + 3r(1 + K_2)(p^2 + \omega_R^2) = 0 \tag{7.81}$$

where $\omega_R^2 = 4(K_2+k_{31}/1+K_2)$. The root locus with r as a parameter appears in Fig. 7.9 and a plot of ω_R^2 vs. K_2 appears in Fig. 7.10. It can be seen from the expression for A_{21} (which we want to be as far away from -1 as we can make it) that if the zero at ω_R^2 is near the point $p^2 = 4$ then the denominator of A_{21} will be close to $3r$ and, therefore, A_{21} will be near to -1. It is clear that in order to produce good damping we must have k_{31} as far away from one as possible and we must not let K_2 be too large. Under these circumstances ω_R^2 will be about three and decreasing r will have a strong effect of producing a value of A_{21} sufficiently far from minus one to provide some damping for the first mode envelope, Λ_1.

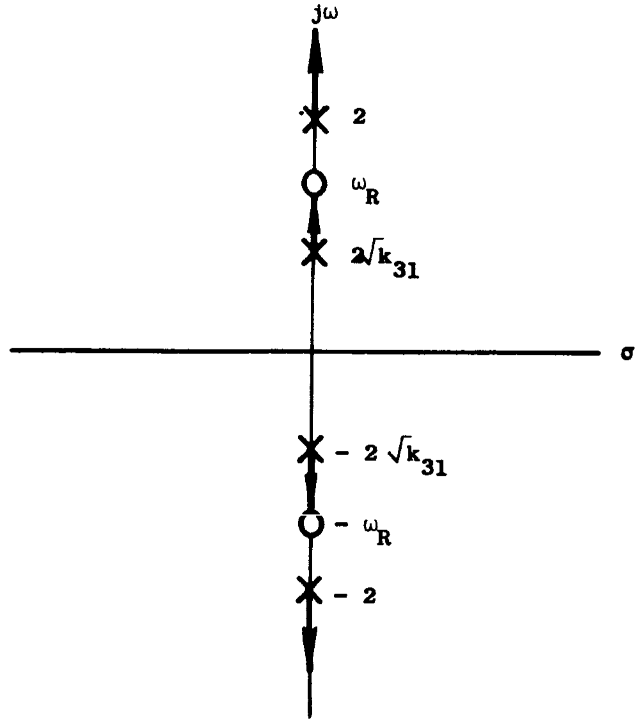

FIG. 7.9. ROOT LOCUS OF ROLL MOTIONS IN THE p-PLANE VS. r

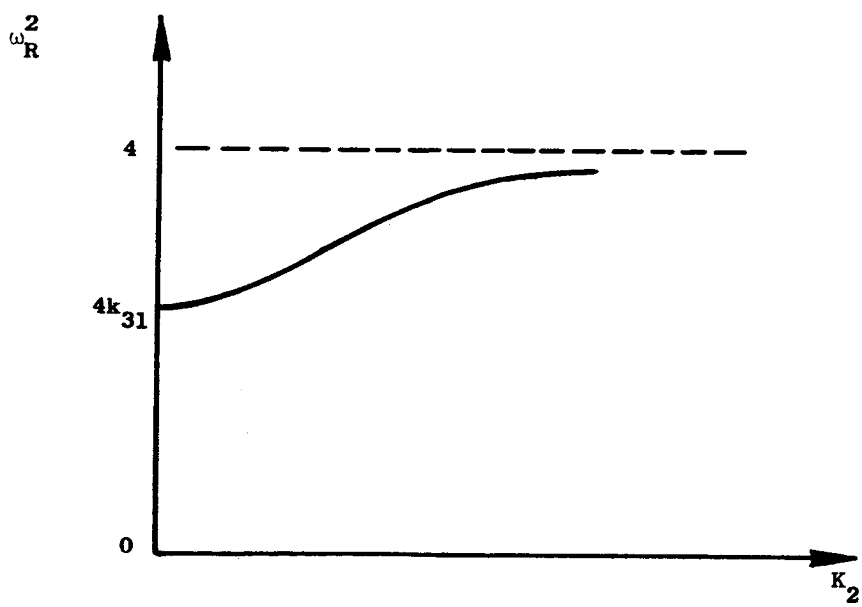

FIG. 7.10. PLOT OF ω_R^2 VS. K_2 (QUALITATIVE ONLY)

In order to illustrate the above statements let us take a system with $k_{31} = \frac{1}{2}$, $K_2 = 1$, $r = 1$. For this system the natural frequencies are $\omega_1^2 = 2.82$, $\omega_2^2 = 9.18$ and the value of α is 0.28. This is a much better design for yaw-roll damping but it has two serious drawbacks. First, the choice of $r = 1$ does not provide good coupling for the pitch oscillations and the performance of pitch will suffer measurably. Second, the body shape $k_{31} = \frac{1}{2}$ is impractical because for the type of vehicles we are considering the boom inertia is included with the body and therefore we will always have bodies with $I_2 \approx I_3$ and with $I_1 \ll I_2$. The influence of k_{31} on the dynamics of the roll motions indicates that the mode of motion near the natural frequency two is essentially a "dumbell" mode which will always be important for long, thin satellites of the type we are discussing.

The above non-linear analysis indicates that practically no damping due to elastic hysteresis will occur in the roll motions of the TRAAC system or other systems built on the same pattern. In the TRAAC system there was an additional mechanism provided for damping the yaw-roll oscillations. This mechanism was the use of the magnetic hysteresis effect caused by the interaction of the earth's magnetic field with ferromagnetic "damping rods." This magnetic damping system damped the roll-yaw oscillations in about the order of the time for damping pitch oscillations.

F. CONCLUSIONS

After consideration of the three examples of connected vehicles, some general conclusions as to their comparative dynamical properties are in order. It can be demonstrated that each of the three vehicles is practical for <u>some</u> realistic mission requirements; it seems that they possess individual properties that may be exploited to advantage in particular cases. The advantages and disadvantages of the three examples will be discussed in the succeeding paragraphs.

The TRAAC and Beam systems, which rely upon hysteresis damping of elastic vibrations, tend to damp out natural librations more slowly than the Vertistat system which uses magnetic hysteresis damping. This is largely because the magnetic damping forces can be made effectively larger than those of the elastic material.

The Vertistat has the outstanding feature that it provides considerable yaw stabilization due to its horizontal rod structure. This problem of yaw damping arises in the vehicle designs with long, pencil-like shapes because the roll and pitch moments of inertia are nearly the same ($I_2 \approx I_3$), effectively. The TRAAC design has the drawback that the spring damper is effective only in pitch; the yaw-roll motions must be damped by using other means. The Beam design, however, will damp oscillations in both pitch and roll-yaw.

It is clear that the Vertistat design is the most effective of the systems considered for high performance (damping in the order of ten orbit periods); however, the Beam system looks promising as a simple damping system for medium performance (damping of the order of fifty orbit periods). The TRAAC system is the only proven passive attitude control system; the other systems must be viewed as "second generation" systems.

In the damping of the motions by any scheme the most important design feature, for a given damping device, is the coupling of motion through proper geometrical and mechanical parameter choice. It is important to choose structural members which are short enough to properly couple modal motions, and long enough to provide stiffness against disturbances.

Disturbances due to the orbit eccentricity were found to be of the order of the eccentricity parameter for the design examples given. In all cases the roots of the characteristic equations of both pitch and yaw-roll motions must not be near the orbit frequency. This is true not only because of eccentricity disturbances but, also because many disturbances occur at orbit frequency and its harmonics, e.g., solar radiation effects, magnetic disturbances, aerodynamic torques, etc.

The vibration absorption designs, which are possible in Vertistat and TRAAC satellites when they are excited in pitch by orbit motions, do not seem practical since they imply configurations that have inherently poor coupling between the modes; this leads to poor damping characteristics in these cases. The vibration absorption idea does, however, indicate the possibilities for filtering the sinusoidal disturbances by proper design.

The analysis of the Beam satellite points the way toward flexible-body analysis for all the designs of passive systems. The assumption of rigidity must be examined quite closely when designing actual systems. It would be especially important to examine the lateral vibrations of the TRAAC spring, and also the higher modes of the Beam satellite. The effect of higher modes tends to diminish in the TRAAC and Beam systems as the size of the mass at the tip of the spring increases. The analysis of non-rigid motions of these satellites can be carried out using the techniques given for analyzing the Beam motion combined with the Rayleigh-Ritz method (STRUTT 1).

It has been shown by example that the methods of Chapters II, III, VI are extremely useful in the analysis of connected satellites with passive damping. The examples used shed some light on the properties of the passive satellite designs and provide a framework for judging and discussing future designs.

CHAPTER VIII. SUMMARY OF IMPORTANT RESULTS

This dissertation is an attempt to illuminate the problems of passive gravity-stabilized satellites by devising appropriate analytical methods, and by applying these methods to gain insight and quantitative information upon which to base designs of future satellites.

The results of the study have been given in Chapters II through VI (analysis) and Chapter VII (application). The contributions can be classified under the following four headings.

1. Stability Methods

The chapter on stability methods (Chapter III) gives the results of an application of Lyapunov's direct method to mechanical systems with damping. The distinction is made between the total energy and the Hamiltonian so that the choice of the Hamiltonian as a Lyapunov function will be clearly understood; this distinction is important in the gyroscopically-coupled systems studied in later chapters.

Using the Hamiltonian as a Lyapunov function we can prove a theorem giving the necessary and sufficient conditions for stability of a damped mechanical system.

A corollary to this important theorem is the result that the stability or instability of a mechanical system is independent of the functional form of the power dissipation function as long as the power into the system is negative. This means that in the attitude control systems discussed in Chapter VII the functional form of the power dissipation function is not a determining factor for stability. The stability of such systems depends only on the sign definite nature of the Hamiltonian function about the equilibrium point in question.

The results on the boundedness of the motions of damped systems and the capture of satellites ("global" questions in contrast to the local nature of the stability theorems) are useful also in the problem of the librations of a symmetrical satellite (Chapter V) and in the discussion of tumbling and capture (Chapters IV and VII).

2. Motion of a Tumbling Satellite

The problem of a satellite tumbling in the gravity field and under the influence of damping is a new problem which has attracted some attention of late as a computer problem, but which, to the author's knowledge, has not been attacked analytically. Chapter IV presents approximate expressions for the motions of a simple tumbling satellite; these expressions were derived using the asymptotic methods of non-linear mechanics. Chapter VII, Section C presents an application of the method of Chapter IV to the Vertistat satellite tumbling in the pitch plane. These results show that the time to damp the tumbling motion increases rapidly with initial tumbling angular velocity.

3. Approximate Time Response of Systems with Light, Non-Linear Damping

Chapter VI presents an asymptotic method for finding the envelope motions of underdamped systems with non-linear damping forces. The main results are a set of first-order, non-linear differential equations for the envelope motions, and a theorem which compliments Rayleigh's theorem on the dissipation function. The non-linear envelope equations have the advantages that the oscillation envelope functions are much smoother and more slowly-varying than the physical variables, and that the damping forces need not be specified explicitly. This latter observation is the non-linear compliment to Rayleigh's theorem that the damping forces can be derived from a single "dissipation function." Our envelope equations of motion depend only on the knowledge of a single function, the average power dissipation; this function may be derived from empirical data and assumptions or from the knowledge of the actual damping forces.

The above envelope response method is applied to a number of non-linear problems in Chapter VII as well as to two examples in Chapter VI. The method has the outstanding virtue that it can be applied with little effort to systems up to three or four envelope variables (sixth or eighth order).

4. Application of the Methods of Analysis to Three Systems of Interest

Chapter VII consists of an analytical attack, using the methods developed for the purpose, upon three interesting satellite designs. These systems are the Vertistat, Beam, and TRAAC satellites.

The TRAAC and Beam satellites definitely have slower time response capabilities than the Vertistat. The Beam has the virtue of getting both pitch and yaw-roll damping from the same damping device. The TRAAC design gets pitch damping from an elastic hysteresis device and yaw and roll damping from magnetic damping rods.

The Vertistat has several advantages over the TRAAC and Beam designs, they are: (a) increased yaw stiffness and damping, due to the horizontal bars, (b) higher coupling of damper motions with body motions, due to the horizontal bars and the fact that the coupling torques due to gravity and inertia are in phase, (c) greater capability for damping because the device is not dependent upon "weak" damping forces such as elastic hysteresis.

The Beam system should be a very simple, reliable system for low-performance missions.

APPENDIX A: THE NATURE OF THE HAMILTONIAN IN MECHANICS

In mechanics H is defined in terms of the kinetic and potential energy expressions. This is true by analogy in electromechanics and electrodynamics but the Hamiltonian form may be present in systems with no easily identified physical significance. It is useful to trace the physical significance of H in mechanical systems by deriving the energy expressions directly from physical reasoning. The kinetic energy is defined for a system of mechanical particles (non-relativistic) as

$$T = \sum_{k=1}^{M} m_k \underline{\dot{R}}^k \cdot \underline{\dot{R}}^k \tag{A.1}$$

for M particles (for continua we replace the sum by a Riemann-Stieltjes integral).

The introduction of P holonomic constraints brings the number of independent variables to $N = 3M - P$. Thus, $\underline{R}^k$, the position of a particle, can be expressed with respect to an inertial reference frame as

$$\underline{R}^k = \underline{R}^k \,(q_1, q_2, \ldots, q_N, t) \tag{A.2}$$

in terms of the N generalized coordinates q_i and the time, t. The explicit dependence of the $\underline{R}^k$ on time may arise from rotation of coordinates or time-varying constraints. We see that:

$$\begin{aligned} \underline{\dot{R}}^k &= \frac{\underline{R}^k}{\partial t} + \sum_{j=1}^{N} \frac{\partial R^k}{\partial q_j} \dot{q}_j \\ T &= \frac{1}{2} \left\{ \sum_{j,\ell=1}^{N} \alpha_{j\ell} \dot{q}_j \dot{q}_\ell + \sum_{j=1}^{N} 2\beta_j \dot{q}_j + \gamma(q) \right\} \end{aligned} \tag{A.3}$$

where:

$$\alpha_{j\ell} = \sum_{k=1}^{M} m_k \frac{\partial \underline{R}^k}{\partial q_j} \cdot \frac{\partial \underline{R}^k}{\partial q_\ell}$$

$$\beta_j = \sum_{k=1}^{M} m_k \frac{\partial \underline{R}^k}{\partial t} \cdot \frac{\partial \underline{R}^k}{\partial q_j}$$

$$\gamma = \sum_{k=1}^{M} m_k \left(\frac{\partial \underline{R}^k}{\partial t} \right)^2$$

We now form H and get the expression for H in terms of (A.3).

$$H = \sum_{k=1}^{N} \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L(q,\dot{q})$$

$$H = \frac{1}{2} \sum_{j,\ell=1}^{N} \alpha_{j\ell}\dot{q}_j\dot{q}_\ell - \frac{1}{2}\gamma(q) + V(q) \tag{A.4}$$

If we call the homogeneous part of the kinetic energy of degree n in the velocities, T_n, we have defined:

$$T_2 = \frac{1}{2} \sum_{j,\ell}^{N} \alpha_{j\ell}\dot{q}_j\dot{q}_\ell$$

$$T_1 = \sum_{j}^{N} \beta_j \dot{q}_j$$

$$T_0 = \frac{1}{2}\gamma(q)$$

This leads to the important expression for H in terms of q_i, $\dot{q}_i$:

$$H = T_2 - T_0 + V$$
$$H = T_2 + U \qquad (A.5)$$

where $U = V - T_0$. This expression may be derived by inspection using Euler's theorem on homogeneous forms (noticing $T = T_2 + T_1 + T_0$). T_0 is the same term that in vector mechanics gives the centripetal acceleration on a particle while rounding a curve. The terms T_1 and T_0 arise due to the expression of the equations of motion in a non-inertial reference frame, e.g., a rotating frame.

Let us notice that the total energy $T + V$ is in general different from H. This difference is

$$E - H = T_1 + 2\,T_0 \qquad (A.6)$$

and is in general not a constant. Thus, there is a fundamental distinction that must be made between E, the total energy, and H, the Hamiltonian. If $\partial \underline{R}^k/\partial t = 0$ then $\underline{R}^k = \underline{R}^k(q)$ (no rotations or time dependent constraints), and $T = T_2$. In this case $E = H$ and the Hamiltonian is the total energy. It is possible for $\partial \underline{R}^k/\partial t \neq 0$ and $\partial H/\partial t = 0$; this occurs often in practice in cases of rotations of coordinate frames. It is just these cases that are of interest in space mechanics of satellites; here the total energy is never conserved — even in the absence of damping.

Let us add an additional coordinate onto the q-space such that this coordinate, ψ, does not appear explicitly in the expression for L. We may therefore write, quite generally:

$$T = T_2 + T_1 + T_0 + \left\{ \sum_{j=1}^{N} a_j \dot{q}_j + \frac{1}{2} b\dot{\psi} + c \right\} \dot{\psi} \qquad (A.7)$$

where T_2, T_1, and T_0 are as before and where

$$\underline{R}^k = \underline{R}^k(q_1, q_2, \ldots, q_N, \psi, t)$$

$$a_j = \sum_{k=1}^{M} m_k \frac{\partial \underline{R}^k}{\partial \psi} \cdot \frac{\partial \underline{R}^k}{\partial q_j}$$

$$b = \sum_{k=1}^{M} m_k \left(\frac{\partial \underline{R}^k}{\partial \psi} \right)^2$$

$$c = \sum_{k=1}^{M} m_k \frac{\partial \underline{R}^k}{\partial \psi} \cdot \frac{\partial \underline{R}^k}{\partial t}$$

Using the Routhian procedure (LANCZOS 1) to eliminate the cyclic coordinate, ψ, we get the Routhian, R.

$$R = T_2 + T_1 + T_0 - \frac{c}{2} \dot{\psi}^2 \tag{A.8}$$

Using the notation of subscripts to denote the degrees of homogeneity of the forms in $\dot{q}_i$ we get

$$R = R_2 + R_1 + R_0$$

where

$$R_2 = T_2 - \frac{1}{2c} \sum_{j,k=1}^{N} a_j a_k \dot{q}_j \dot{q}_k$$

$$R_1 = T_1 - \frac{\sum_{j=1}^{N} a_j \dot{q}_j (p_\psi - b)}{2c}$$

$$R_0 = T_0 - \frac{(p_\psi - b)^2}{2c}$$

The symbol p_ψ denotes a constant of the motion, the momentum about the "ψ axis." Having eliminated the variable $\dot{\psi}$ from the expression for the Routhian, R, which is the new "kinetic energy" we get the equations of motion, given as

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_j}\right) - \frac{\partial R}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j \qquad \text{(A.9)}$$

with ψ determined by the expression $p_\psi = \partial T/\partial\dot{\psi}$. We see that although $\dot{\psi}$ has been eliminated from consideration we still have the equations of motion in Lagrangian (and thus Hamiltonian) form. We may operate with R in place of T and derive H. This gives us expressions corresponding to (A.5) and (A.6).

$$H = R_2 + U = R_2 + V - R_0$$

$$U = V - T_0 + \frac{(p_\psi - b)^2}{2c}$$

$$E - H = T_1 + 2T_0 + (p_\psi - b)\dot{\psi}$$

These equations show that there is a difference between the total energy and H in the case of cyclic coordinates even if $T_1 = T_0 = 0$. These systems in which there are cyclic coordinates or rotating coordinates may be properly called "gyroscopic" because they are mathematically analogous to the equations of a gyroscope.

APPENDIX B: THE METHOD OF AVERAGING

The method of averaging has been used in various forms for many years, notably in celestial mechanics. In recent years the Russian school of "differential equationists" has used the method and developed its rigorous foundations. The modern version of the method of averaging is largely due to the work of N. Krylov, N. Bogoliubov, and Y. A. Mitropolsky (KRYLOV 1, BOGOLIUBOV 1); for this reason the method of averaging is often called the KBM method.

Consider a differential equation with a small parameter, μ,

$$\frac{dx}{dt} = f(x,t;\mu) \tag{B.1}$$

where x is an n-vector, f is an n-vector function of x and t (time) with a parameter μ. If the function $\phi(t,p)$, where p is an arbitrary vector constant of integration, can be found such that it satisfies the equation

$$\frac{\partial\phi}{\partial t} = f(\phi,t;0) \tag{B.2}$$

then ϕ is called a "generating solution" for equation (B.1). From ϕ we will hope to form power series approximations in μ to the equation (B.1).

If we allow the constant vector p to vary with time (thus accounting for the perturbations due to $\mu \neq 0$) for $\mu \neq 0$ we can arrive at a differential equation in p as follows.

$$\frac{dx}{dt} = \frac{d\phi}{dt} = \frac{\partial\phi(t;p)}{\partial t} + \frac{\partial\phi(t;p)}{\partial p} \cdot \dot{p}$$

$$= f(\phi,t;\mu) = f(\phi,t;0) + \mu \frac{\partial f}{\partial\mu}(\phi,t;0) + \mu^2 \ldots\ldots$$

where the symbolism $\partial\phi/\partial p$ stands for the hessian matrix $\partial\phi_j/\partial p_i$. Using (B.2) in the above gives the resulting differential equation in p.

$$\frac{dp}{dt} = \left[\frac{\partial\phi}{\partial p}\right]^{-1} \cdot \left[\mu \frac{\partial f(\phi, t;0)}{\partial\mu} + \mu^2(\ )\ldots\right] \tag{B.3}$$

This equation is of the form

$$\frac{dp}{dt} = \mu F(p,t;\mu) \tag{B.4}$$

which is the "standard form" of Bogoliubov and Mitropolsky. Because μ is small we expect that dp/dt will be small and therefore p varies slowly with time. Assume that $F(p,t;\mu)$ can be expanded in a trigonometric series in the following manner.

$$F(p,t;\mu) = F_o(p;\mu) + \sum_{\substack{k=-N \\ k\neq 0}}^{N} F_k(p;\mu)e^{j\omega_k t} \tag{B.5}$$

If we define the notation

$$\begin{aligned} F_o(p;\mu) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T F(p,t;\mu)dt \\ \tilde{F}(p,t;\mu) &= \sum_{\substack{k=-N \\ k\neq 0}}^{N} \frac{F_k(p;\mu)}{j\omega_k} e^{j\omega_k t} \end{aligned} \tag{B.6}$$

where F_o is the time average of $F(p,t;\mu)$ with p held constant, then the first approximation to the solution to the differential equation (B.4) is

$$P = \pi(t) + \mu\,\tilde{F}(\pi,t;\mu) \tag{B.7}$$

where the "secular equation" for $\pi(t)$, the "average p," is

$$\frac{d\pi}{dt} = \mu F_o(\pi;\mu) \tag{B.8}$$

Notice that $\pi(t)$ is the result of (B.8), the differential equation of the standard form with its right-hand side averaged with respect to time.

To determine the accuracy of the above first approximation, substitute (B.7) and (B.8) into (B.4) and evaluate the remaining "error terms," ϵ.

$$\frac{dp}{dt} = \frac{d\pi}{dt} + \mu \frac{\partial\tilde{F}}{\partial\pi} \cdot \frac{d\pi}{dt} + \mu \sum_{k=-N}^{N} F_k(\pi;\mu) e^{j\omega_k t}$$

$$= \mu F_o(\pi;\mu) + \mu \sum_{k=-N}^{N} F_k(\pi;\mu) e^{j\omega_k t}$$

$$+ \mu^2 \left[\frac{\partial F(\pi,t;0)}{\partial\pi} \cdot \tilde{F}(\pi,t;0) + \frac{\partial F}{\partial\mu}(\pi,t;0) \right]$$

$$\epsilon = \mu^2 \left[\frac{\partial F(\pi,t;0)}{\partial\pi} \cdot \tilde{F}(\pi,t;0) + \frac{\partial F(\pi,t;0)}{\partial\mu} \right.$$

$$\left. - \frac{\partial\tilde{F}(\pi,t;0)}{\partial\pi} \cdot F_o(\pi;0) \right] + \ldots\ldots\ldots$$

This shows that the error in satisfaction of the differential equations (B.4) by the approximation (B.7) and (B.8) is of order in μ^2.

The method of averaging forms the basis of the approximations used in Chapters IV and VI. The basic first approximation given above can be extended to solutions which asymptotically converge to the correct solution (BOGOLIUBOV 1, Chapters V and VI).

APPENDIX C: HAMILTON'S PRINCIPLE FOR CONTINUOUS AND DISCRETE SYSTEMS

For certain mechanical problems involving the interaction of rigid bodies (finite number of degrees of freedom) with continuous elastic members (infinite number of degrees of freedom) it is convenient to use Hamilton's principle and the Calculus of Variations to derive the differential equations of motion (mixed set of partial and ordinary differential equations). This will be done in this appendix.

Let us consider Hamilton's principle stated in the form of a theorem in which the following notation is useful.

$L = L(q_i, \dot{q}_i, a_j, \dot{a}_j, b_j, \dot{b}_j, \alpha_j, \dot{\alpha}_j, \beta_j, \dot{\beta}_j, t)$

= the part of the Lagrangian involving only the generalized coordinates of the discrete system

$\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i, a_j, \dot{a}_j, b_j, \dot{b}_j, \alpha_j, \dot{\alpha}_j, \beta_j, \dot{\beta}_j, \phi_j, \dot{\phi}_j, \phi_j', \phi_j'', \dot{\phi}_j', s, t)$

= the Lagrangian density, involving the generalized coordinates of the discrete system and those of the continuous system as well.

$q_i = q_i(t)$ = generalized coordinates of the discrete system where $(i = 1,2,\ldots,N)$

$\phi_j = \phi_j(s,t)$ = generalized coordinates of the continuous part of the system as functions of time and the space variable, s. $(j = 1,2,\ldots,M.)$

$t =$ time $(t_1 \leq t \leq t_2)$

$s =$ space coordinate (independent variable) $(0 \leq s \leq \ell)$

$a_j = \phi_j(0,t)$ = generalized coordinate at the boundary point

$b_j = \phi_j(\ell,t)$

$\alpha_j = \phi_j'(0,t)$

$\beta_j = \phi_j'(\ell,t)$

$$\phi'_j = \frac{\partial \phi_j}{\partial s}$$

$$\dot{\phi}_j = \frac{\partial \phi_j}{\partial t}$$

$$\dot{\phi}'_j = \frac{\partial^2 \phi_j}{\partial s \partial t}$$

Theorem (Hamilton's Principle): For the dynamical system described by the generalized coordinates $q_i, a_j, b_j, \alpha_j, \beta_j, \phi_j$ $(i = 1,2,\ldots,N)(j=1,2,\ldots,M)$ and the generalized forces $Q_i, A_j, B_j, \mathcal{A}_j, \mathcal{B}_j, \Phi_j$, the motion is described by the "action integral," I, given by

$$I = \int_{t_1}^{t_2} L\, dt + \int_{o}^{\ell} \int_{t_1}^{t_2} \mathcal{L}\, ds\, dt \tag{C.1}$$

in such a manner as to satisfy the "principle of virtual work" according to the expression

$$\delta I = -\sum_{i,j} \int_{t_1}^{t_2} \left[Q_i \delta q_i + A_j \delta a_j + B_j \delta b_j + \mathcal{A}_j \delta \alpha_j + \mathcal{B}_j \delta \beta_j + \int_{o}^{\ell} \Phi_j \delta \phi_j ds \right] dt \tag{C.2}$$

where the variations (see LANCZOS 1) are taken under the assumption of fixed end points in time but variable end points in space (s) and where the second term of (C.2) represents the virtual work done by external forces.

The use of the theorem in arriving at the differential equations of motion is of interest here. Let us take variations of the first integral in (C.1) and integrate by parts (using the fact that the variations of q, a, b, α, β, Φ vanish at the points $t = t_1$ and $t = t_2$.

$$\delta \int_{t_1}^{t_2} L\, dt = \sum_i \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right] \delta q_i\, dt \qquad (C.3)$$

$$+ \sum_j \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial a_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{a}_j} \right] \delta a_j dt + \sum_j \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial b_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{b}_j} \right] \delta b_j \delta t$$

$$+ \sum_j \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial \alpha_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\alpha}_j} \right] \delta \alpha_j dt + \sum_j \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial \beta_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\beta}_j} \right] \delta \beta_j dt$$

The variation of the second term of (C.1) is

$$\delta \int_{t_1}^{t_2} \int_{o}^{\ell} \mathcal{L}\, d\, s\, dt = \int_{t_1}^{t_2} dt \int_{o}^{\ell} ds \left\{ \frac{\partial \mathcal{L}}{\partial \phi_j} \delta \phi_j + \frac{\partial \mathcal{L}}{\partial \phi'_j} \delta \phi'_j + \frac{\partial \mathcal{L}}{\partial \phi''_j} \delta \phi''_j \right.$$

$$\left. + \frac{\partial \mathcal{L}}{\partial \dot{\phi}_j} \delta \dot{\phi}_j + \frac{\partial \mathcal{L}}{\partial \dot{\phi}'_j} \delta \dot{\phi}'_j \right\} \qquad (C.4)$$

+ [terms identical to (C.3) with $\int_{o}^{\ell} \mathcal{L}\, dt$ as the Lagrangian.]

Integrating the first five terms of (C.4) by parts in both s and t and collecting terms to form expression (C.2) gives the principle of virtual work in coordinate form.

$$\sum_{i=1}^{N} \int_{t_1}^{t_2} \left[\frac{\partial \Lambda}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial \Lambda}{\partial \dot{q}_i} \right) + Q_i \right] \delta q_i dt \quad +$$

$$+\sum_{j=1}^{M}\int_{t_1}^{t_2}dt\int_{0}^{\ell}ds\left[\frac{\partial\mathcal{L}}{\partial\phi_j}-\frac{\partial}{\partial s}\left(\frac{\partial\mathcal{L}}{\partial\phi'_j}\right)+\frac{\partial^2}{\partial s^2}\left(\frac{\partial\mathcal{L}}{\partial\phi''_j}\right)-\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}_j}\right)\right.$$

$$\left.+\frac{\partial^2}{\partial t\partial s}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}'_j}\right)+\Phi_j\Bigg\}\right]\delta\phi_j$$

$$+\sum_{j=1}^{M}\int_{t_1}^{t_2}\left\{\left[\frac{\partial\mathcal{L}}{\partial\phi'_j}-\frac{\partial}{\partial s}\left(\frac{\partial\mathcal{L}}{\partial\phi''_j}\right)-\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}'_j}\right)\right]_{s=0}\right. \tag{C.5}$$

$$\left.+\left[\frac{\partial\Lambda}{\partial a_j}-\frac{d}{dt}\frac{\partial\Lambda}{\partial\dot{a}_j}\right]+A_j\right\}\delta a_j dt$$

$$+\sum_{j=1}^{M}\int_{t_1}^{t_2}\left\{\left[\frac{\partial\mathcal{L}}{\partial\phi'_j}-\frac{\partial}{\partial s}\left(\frac{\partial\mathcal{L}}{\partial\phi''_j}\right)-\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}'_j}\right)\right]_{s=\ell}\right.$$

$$\left.+\left[\frac{\partial\Lambda}{\partial b_j}-\frac{d}{dt}\left(\frac{\partial\Lambda}{\partial\dot{b}_j}\right)\right]+B_j\right\}\delta b_j dt$$

$$+\sum_{j=1}^{M}\int_{t_1}^{t_2}\left\{\left[\frac{\partial\mathcal{L}}{\partial\phi''_j}\right]_{s=0}+\frac{\partial\Lambda}{\partial\alpha_j}-\frac{d}{dt}\frac{\partial\Lambda}{\partial\dot{\alpha}_j}+\mathcal{A}_j\right\}\delta\alpha_j dt$$

$$+\sum_{j=1}^{M}\int_{t_1}^{t_2}\left\{\left[\frac{\partial\mathcal{L}}{\partial\phi''_j}\right]_{s=\ell}+\frac{\partial\Lambda}{\partial\beta_j}-\frac{d}{dt}\left(\frac{\partial\Lambda}{\partial\dot{\beta}_j}\right)+\mathcal{B}_j\right\}\delta\beta_j dt$$

$$=\quad 0$$

where we have defined $\Lambda = L + \int_0^{\ell} \mathfrak{L}\, ds.$

Equating the terms multiplying the variations (independent) δb, $\delta\alpha$, δq, δa, $\delta\beta$, $\delta\phi$ to zero by the fundamental lemma of the Calculus of Variations (LANCZOS 1) gives us the differential equations of the continuous system, the discrete system, and the boundary conditions.

$$\frac{d}{dt}\frac{\partial\Lambda}{\partial\dot{q}_i} - \frac{\partial\Lambda}{\partial q_i} = Q_i \tag{C.6}$$

$$\frac{d}{dt}\left(\frac{\partial\mathfrak{L}}{\partial\dot{\phi}_j}\right) - \frac{\partial\mathfrak{L}}{\partial\phi_j} - \frac{\partial^2}{\partial s^2}\left(\frac{\partial\mathfrak{L}}{\partial\phi''_j}\right) + \frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{L}}{\partial\phi'_j}\right)$$

$$- \frac{\partial^2}{\partial t\partial s}\left(\frac{\partial\mathfrak{L}}{\partial\dot{\phi}'_j}\right) = \Phi_j(s,t) \tag{C.7}$$

$$\left\{\left[\frac{\partial\mathfrak{L}}{\partial\phi'_j} - \frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{L}}{\partial\phi''_j}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\mathfrak{L}}{\partial\dot{\phi}'_j}\right)\right]_{s=0}\right.$$

$$\left. + \frac{\partial\Lambda}{\partial a_j} - \frac{d}{dt}\frac{\partial\Lambda}{\partial\dot{a}_j} + A_j\right\}\delta a_j = 0 \tag{C.8}$$

$$\left\{\left[\frac{\partial\mathfrak{L}}{\partial\phi'_j} - \frac{\partial}{\partial s}\left(\frac{\partial\mathfrak{L}}{\partial\phi''_j}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\mathfrak{L}}{\partial\dot{\phi}'_j}\right)\right]_{s=\ell}\right.$$

$$\left. + \frac{\partial\Lambda}{\partial b_j} - \frac{d}{dt}\left(\frac{\partial\Lambda}{\partial\dot{b}_j}\right) + B_j\right\}\delta b_j = 0 \tag{C.9}$$

$$\left\{ \left[\frac{\partial}{\partial \phi''_j} \right]_{s=0} + \frac{\partial \Lambda}{\partial \alpha_j} - \frac{d}{dt} \left(\frac{\partial \Lambda}{\partial \dot{\alpha}_j} \right) + \mathcal{A}_j \right\} \delta\alpha_j = 0 \qquad (C.10)$$

$$\left\{ \left[\frac{\partial}{\partial \phi''_j} \right]_{s=\ell} + \frac{\partial \Lambda}{\partial \beta_j} - \frac{d}{dt} \left(\frac{\partial \Lambda}{\partial \dot{\beta}_j} \right) + \mathcal{B}_j \right\} \delta\beta_j = 0 \qquad (C.11)$$

Equation (C.6) are the equations governing the discrete degrees of freedom. Equation (C.7) are the partial differential equations governing the continuous system. Equations (C.8), (C.9), (C.10), (C.11) are boundary conditions on the space variables ϕ_j. These boundary conditions are either simple algebraic relations, if the variables a, b, α, β are constants, or differential equations (natural end conditions), if the variations δa, δb, $\delta\alpha$, $\delta\beta$ do not vanish.

In the problem of Chapter VII of the equations of the beam satellite the equations (C.6) govern the rigid body and Q_i are rigid body torques, (C.7) governs the beam motion, and the end conditions are "built-in" at one end $(\delta\alpha, \delta\beta = 0)$ and are "natural" at the other end. (C.10) and (C.11) govern the motion of the end mass.

REFERENCES

AUELMANN 1 — Auelmann, R. R., AIAA Journal, 1, pp. 1445-1447, Jun. 1963.

BELETSKII 1 — Beletskii, V. V., "Motion of an Artificial Earth Satellite About its Center of Mass," Artificial Earth Satellites, (Ed: L. V. Kurnosova), Vol. 1 and 2, Plenum Press, N. Y., 1960.

BELLMAN 1 — Bellman, R., Matrix Analysis, McGraw-Hill, 1960.

BREAKWELL 1 — Breakwell, J. V., "The Hinged Satellite, A Working Memorandum," Draft dated 16 Feb. 1954. Personal communication from the author.

BOGOLIUBOV 1 — Bogoliubov, N. N., Mitropolsky, Y. A., Asymptotic Methods in the Theory of Non-Linear Oscillations, Hindustan Publishing Corp., Delhi, 1961.

BROUWER 1 — Brouwer, D., Clemence, G. M., Methods of Celestial Mechanics, Academic Press, 1961.

CANNON 1 — Cannon, R. H., Jr., "Basic Response Relations for Space Vehicle Attitude Control Using Gyros," presented at the Joint Automatic Control Conference, Minneapolis, Minn., June 1963, and at the Second Congress of the International Federation of Automatic Control, Basle, Switzerland, Sept. 1963.

CHETAYEV 1 — Chetayev, N. G., The Stability of Motion, Pergamon Press, 1961.

DEBRA 1 — DeBra, D. B., The Large Attitude Motions and Stability, Due to Gravity, of a Satellite with Passive Damping in an Orbit of Arbitrary Eccentricity About an Oblate Body, Ph.D. Dissertation, Stanford University, 1962.

DEBRA 2 — DeBra, D. B., Delp, R. H., "Satellite Stability and Natural Frequencies in a Circular Orbit," Journal of Astronautical Sciences, Vol. VIII, No. 1, Spring, 1961.

FISCHELL 1 — Fischell, R. E., "The TRAAC Satellite," APL Technical Digest, Vol. 1, No. 3, Jan.-Feb. 1962, pp. 2-9.

FISCHELL 2 — Fischell, R. E., Mobley, F. F., "A System for Passive Gravity-Gradient Stabilization of Earth Satellites," paper presented at the AIAA Guidance and Control Conference, Cambridge, Mass., Aug. 1963, (Paper No. 63-326).

FISCHELL 3 Fischell, R. E., "Magnetic Damping of the Angular Motions of Earth Satellites," ARS Journal, Vol. 31, No. 9. Sept. 1961, pp. 911-913.

FLETCHER 1 Fletcher, H. J., et al., "Dynamics Analysis of a Two-Body Gravitationally Oriented Satellite," Bell System Technical Journal, Vol. 42, No. 5, Sept. 1963.

GIBBS 1 Gibbs, J. W., Wilson, E. B., Vector Analysis, Dover Publications, 1960.

HILL 1 Hill, G. W., "Researches in the Lunar Theory," Am. Journal Math., Vol. 1, pp. 8-26, 1878.

KALMAN 1 Kalman, R. E., Bertram, J. E., "Control System Analysis and Design Via the 'Second Method' of Lyapunov: Continuous Time Systems," ASME Paper No. 59-NAC-2.

KAMM 1 Kamm, L. J., "Vertistat -- An Improved Satellite Orientation Device," ARS Journal, Vol. 32, No. 6, June 1962, pp. 20-21.

KANE 1 Kane, T. R., et al., "Letter to the Editor," Journal of the Astronautical Sciences, Vol. 9, No. 4, pp. 108-109, 1962. (Correction to THOMSON 1.)

KRASOVSKII 1 Krasovskii, N. N., Stability of Motion, Stanford University Press, 1963.

KRYLOFF 1 Kryloff, N., Bogoliuboff, N., Introduction to Non-Linear Mechanics, Princeton University Press, 1947.

LAGRANGE 1 Lagrange, J. L., Méchanique Analytique, Gauthier-Villars, Paris, 1788, 1888.

LAGRANGE 2 Lagrange, J. L., "Theorie de la Libration de la Lune," Oeuvres, Tome V, p. 1.

LANCZOS 1 Lanczos, C., The Variational Principles of Mechanics, University of Toronto Press, Toronto, 1949.

LIAPUNOV 1 Liapunov, M. A., Problem General de la Stabilité du Mouvement, Princeton University Press, 1949.

LEFSCHETZ 1 Lefschetz, S., La Salle, J., Stability by Liapunov's Direct Method, Academic Press, 1961.

MALKIN 1 Malkin, I. G., Theory of Stability of Motion, AEC-tr-3352, (U.S. Atomic Energy Commission Translation).

MALKIN 2 — Malkin, I. G., Some Problems in the Theory of Non-linear Oscillations, Books I and II, AEC-tr-3766, (U.S. Atomic Energy Commission Translation).

MINORSKY 1 — Minorsky, N., Introduction to Non-Linear Mechanics, J. W. Edwards, 1947, Ann Arbor.

MINORSKY 2 — Minorsky, N., Nonlinear Oscillations, D. Van Nostrand, 1962.

PAUL 1 — Paul, B., et al., "A Passive Gravitational Attitude Control System for Satellites," Bell System Technical Journal, Vol. 42, No. 5, Sept. 1963.

PLUMMER 1 — Plummer, H. C., An Introductory Treatise on Dynamical Astronomy, Dover, 1960.

POINCARÉ 1 — Poincaré, H., "Sur l'Equilibrium d'une Masse Fluid Animée d'un Mouvement de Rotation," Acta Math., Vol. 7, p. 259, 1885.

ROBERSON 1 — Roberson, R. E., WADD Tech. Report 60-643, Methods for the Control of Satellite and Space Vehicles, 31 July, 1960.

ROUTH 1 — Routh, E. J., The Advanced Part of the Treatise on the Dynamics of a System of Rigid Bodies, Dover, 1955.

SCOTT 1 — Scott, E. D., "Control Moment Gyro Gravity Stabilization," Paper No. 63-324, AIAA Guidance and Control Conference, Aug. 1963.

SMART 1 — Smart, W. M., Celestial Mechanics, Longmans, 1953.

STRUTT 1 — Strutt, J. W., (Lord Rayleigh), Theory of Sound, Vol. 1, Dover, 1945.

THOMSON 1 — Thomson, W., (Lord Kelvin), Tait, P. G., Treatise on Natural Philosophy, Cambridge University Press, 1879.

THOMSON 2 — Thomson, W. T., "Spin Stabilization of Attitude Against Gravity Torque," Journal of the Astronautical Sciences, Vol. 9, No. 2, pp. 31-33, 1962. (See KANE 1.)

TINLING 1 — Tinling, B. E., Merrick, V. K., "The Exploitation of Inertial Coupling in Passive Gravity-Gradient Stabilized Satellites," Paper No. 63-342, AIAA Guidance and Control Conference, Aug. 1963.

TISSERAND 1 — Tisserand, F., Traité de Mécanique Céleste, Tome II, Gauthier-Villars, Paris, 1891.