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INTERIM REPORT ON METHODS OF DETERMINING THE TRANSONIC FLOW FIELD IN AN AXIALLY SYMMETRIC ROCKET NOZZLE

by R. S. MENDELSON
Propulsion and Vehicle Engineering Laboratory

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ABSTRACT

Starting data are provided for internal characteristic programs pertaining to axially symmetric rocket nozzles. Several approximate methods are examined before a complete analysis is attempted.

Using perfect frozen gas considerations, the potential function in the region of interest is approximated by a double power series in the space variables. The potential equation of motion, with the inviscid boundary condition, produces non-linear simultaneous equations.

The non-linear equations are handled uniquely, and the results are utilized to describe the flow field. Different methods of checking the validity of the results are applied, and comparisons are made with other analyses.

The remaining difficulties in the development of a production program that can handle arbitrary minimum section geometry and slightly varying mass flows are discussed.

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DEFINITION OF SYMBOLS

SYMBOL	DEFINITION
a	local speed of sound
a^*	local speed of sound where the velocity is sonic
a_j	coefficients in a series expansion of g(y), or constant in Method I approximate analysis
$a_{i,n}$	coefficients in the polynomial representation of $G_{i,p}$
$A_{i,2j}$	coefficients of the double power series expansion of the potential function,
$A(i,2j)$	same as $A_{i,2j}$
A_i	coefficient used in approximate analysis II
B_i	coefficients of the series expression for the boundary curve, or coefficient used in approximate analysis - Method II
b_i	coefficient used in approximate analyses in Method I for f_{2i} while in method II for a_i
D_i	the "unknown" of the i^{th} boundary condition equation, i.e., $A_{i+1,0}$
f_{2i}	coefficients of the Φ series in the approximate analyses
$g(y)$	$x = g(y)$ defines the surface of the sonic line in Method II
i, j	indices
$G(m, p)$	denotes any of the boundary condition equations (for $m \leq M$) containing p+1 columns
$G(i, p)$	denotes the i^{th} boundary equation, and contains p+1 columns

DEFINITION OF SYMBOLS (CONTINUED)

SYMBOL	DEFINITION
$G(i,\infty)$	a function equal to zero, denotes the i^{th} boundary equation and contains all of the columns
$\dot{m}$	mass flow (dimensional)
$\tilde{m}$	mass flow (non-dimensionalized)
M	denotes the number of D_i unknowns considered; it is one less than the number of rows considered
M*	Mach number referred to the speed of sound a*
m	index of the boundary condition equations
n	index
p	integer, one less than the total number of columns considered
s	subscript denoting boundary
u, v	perturbation velocities in the x and y directions respectively, non-dimensionalized on a*
U, V	total velocities in x and y directions respectively, dimensional
y_o	minimum throat radius
x, y	axial and radial directions in circular cylindrical coordinates
α	constant, equal to axial velocity gradient
ϵ	distance from minimum section to the sonic line along the x axis
ρ	throat radius of curvature at the minimum section
ξ	longitudinal coordinate, $\xi = x - g(y)$
γ	ratio of specific heats

DEFINITION OF SYMBOLS (Concluded)

Γ	equal to $\frac{\gamma - 1}{\gamma + 1}$
Φ	perturbation velocity potential
Φx, Φy	partial derivatives of Φ, equal to u and v, respectively
θ	flow angle
σ i, j	zero or one function $\begin{matrix} 1 \\ 0 \end{matrix} \begin{Bmatrix} i \neq j \\ i = j \end{Bmatrix}$
ρ_o	isentropic stagnation density

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INTERIM REPORT ON METHODS OF DETERMINING THE TRANSONIC FLOW FIELD IN AN AXIALLY SYMMETRIC ROCKET NOZZLE

SUMMARY

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This study will ultimately produce a working computer program to provide the initial data for the method of characteristics. The program must be more exact than those analytical methods now available.

Since the flow becomes supersonic in the vicinity of the throat of the nozzle, analysis of the transonic flow in the throat region is of interest. The physical problem was approximated by a steady-state axisymmetric system (co-ordinates x and y) with zero radial velocity along the axis. In this region, the fluid is considered to be a homogeneous, perfect gas in frozen equilibrium with no viscous forces. The system is also considered irrotational and isentropic with negligible energy losses through the nozzle surface. In the region of the throat, the nozzle surface is expressed as a power series in terms of the axial coordinate and is not limited to boundary surfaces of constant radius of curvature for the minimum section.

This method attempts to solve the full, non-linear potential equation of motion. A double power series in both x and y is assumed for the potential function. The original equation is satisfied by recurrence relationships for the general coefficients that are evaluated in terms of the coefficients of the velocity distribution along the axis. The remaining coefficients are determined by satisfying the inviscid boundary condition along the nozzle contour, where the contour is described by a power series in x.

The problem is that the analysis produces an infinite set of highly non-linear simultaneous boundary condition equations. The number of equations used was determined by the size of the region of convergence of the flow field. The minimum number was solved by subsequent iteration. The iterative method deliberately does not satisfy each equation. The error of the equation is used as a bias so that the effect of the iteration of the rest of the system will channel the forced error toward a correct solution. This method should be usable in other areas where non-linear simultaneous equations occur.

A library of solutions is needed to pick initial input values close to the actual solutions. The results are presented for constant radius of curvatures of 5 to 1.25 times the minimum section radius.

This analysis may be the first to allow small variation in mass flow in a given nozzle. The variation appears as a shift in the location of the point where the Mach one line intersects the axis.

The solution is used to compute input data for characteristics and to check the accuracy of the program. Although this method is a frozen flow analysis, it is used in conjunction with equilibrium programs by stipulating the value of the ratio of the specific heat from an equilibrium combustion and expansion program. The method of characteristics was used to confirm the validity of this method.

The obstacle to general use of this method is the amount of computer running time, but the program should prove to be valuable in nozzle design.

INTRODUCTION

The method of characteristics, as applied to gas dynamic problems, is restricted to certain regions in the flow field. These regions are those in which the flow is not only supersonic but also purely hyperbolic in the mathematical sense. Practically, it is restricted to those regions in which the local Mach angle is not large enough that the resulting characteristic mesh calls for excessive computer time. Therefore, it is necessary to generate, by some other analysis, certain calculated flow properties over a suitable surface from which the method of characteristics can be started.

Ultimately this study will produce a working computer program that will provide the starting data for the method of characteristics as applied to the flow field internal to a rocket exhaust nozzle. Since the flow becomes supersonic in the vicinity of the throat, the analysis of the transonic flow in the throat region is of interest. There are several approximate methods for treating this particular flow; however, none of these have an accuracy obtainable by characteristics (1)*. Some of these methods are discussed to show their salient features, regions of applicability, and limitations.

Also discussed in this report are the method developed to calculate the flow field to within an arbitrary degree of accuracy, the computer program utilizing that method, and a comparison of various methods.

This study was performed by R. S. Mendelson of the Structures and Mechanics Department of Chrysler Corporation Space Division, Huntsville, Alabama Operations under contract NAS 8-4016, Task Order M-P&VE-PA-M-6, Task Assignment M-P&VE-PA-6-63.

*Bracketed numbers that appear as supperscripts refer to the Bibliography.

ANALYSIS

1. GENERAL

There are several methods, other than the two presented in this paper, which give approximate solutions for an internal transonic flow. Before discussing the new method, which is arbitrarily exact, some of these methods should be studied because they serve to check the exact method.

In these analyses, the physical problem has been approximated by a steady state axisymmetric system. The fluid in the region investigated is assumed to be a homogeneous, inviscid, perfect gas of constant composition. Although there is energy lost to the nozzle wall, these losses are neglected, and the system is considered an irrotational, isentropic one. The boundary is required to be continuous and to have a continuous slope. The exact analysis also requires that the nozzle radius in the region of the throat be expressible as a power series in terms of the station variable x.

Combining the equations of continuity and momentum, and imposing the irrotationality condition results in the general equation of motion

$$(a^2 - U^2)\frac{\partial U}{\partial x} - 2UV\frac{\partial U}{\partial y} + (a^2 - V^2)\frac{\partial V}{\partial y} + a^2\frac{V}{y} = 0^{(2)} \qquad (1.1)$$

where: a is the local speed of sound.

U and V are the total velocity components in the x and y directions, respectively.

x and y are the axial and radial directions in circular, cylindrical coordinates.

The speed of sound, where the Mach number is unity, is defined as a* and is only a function of stagnation conditions. The velocities U and V are nondimensionalized with respect to a* and written in the form of perturbation velocities, u and v, so

$$\frac{U}{a^*} = 1 + u \qquad (1.2)$$

and

$$\frac{V}{a*} = v \qquad (1.3)$$

System dimensions, x and y, are nondimensionalized by the minimum nozzle radius y_0 (FIG 1).

The equation of state and the isentropic relationship and equations 1. 2 and 1. 3 are used to reduce equation 1. 1 to the form

$$- \left[u(2+u) + \Gamma\ v^2\right] \frac{\partial u}{\partial x} + \left[\frac{2}{(\gamma+1)} - \Gamma\ u(2+u) - v^2\right] \frac{\partial v}{\partial y} -$$

$$- 4v\ \frac{(1+u)}{(1+\gamma)} \frac{\partial u}{\partial y} + \left(\frac{2}{\gamma+1} - \Gamma\left[u(2+u) + v^2\right]\right) \frac{v}{y} = 0 \qquad (1.4)$$

where: γ is the ratio of specific heats, and

$$\Gamma = \frac{\gamma - 1}{\gamma + 1} \qquad (1.5)$$

The condition of irrotationality assures the existence of a velocity potential Φ , so

$$\left.\begin{aligned} \Phi_x &= u \\ \text{and} \\ \Phi_y &= v \end{aligned}\right\} \qquad (1.6)$$

where the subscript notation indicates partial differentiation with respect to x or y.

By virtue of equation 1. 6, equation 1. 4 is reduced to the form

$$\left[- (2+\Phi_x)\Phi_x - \Gamma\Phi^2_y\right] \Phi_{xx} + \left[\frac{2}{(\gamma+1)} - \Gamma\Phi_x (2+\Phi_x) - \Phi_y^2\right] \Phi_{yy} -$$

$$\frac{-4}{(\gamma+1)} (1+\Phi_x)\Phi_y \Phi_{xy} + \left(\frac{2}{(\gamma+1)} - \Gamma\left[\Phi_x (2+\Phi_x) + \Phi_y^2\right]\right)\frac{\Phi_y}{y} = 0 \qquad (1.7)$$

Equation 1. 7 is the basic equation from which all succeeding analyses originate.

2. APPROXIMATE ANALYSES

Deleting terms of the order Φ_x^2 and Φ_y^2 as small compared to unity, equation 1. 7 can be simplified to

$$(\gamma+1)\ \Phi_x \Phi_{xx} + 2\Phi_y \Phi_{xy} - \frac{1}{y} \frac{\partial}{\partial y} (y\Phi_y) = 0 \qquad (2.1)$$

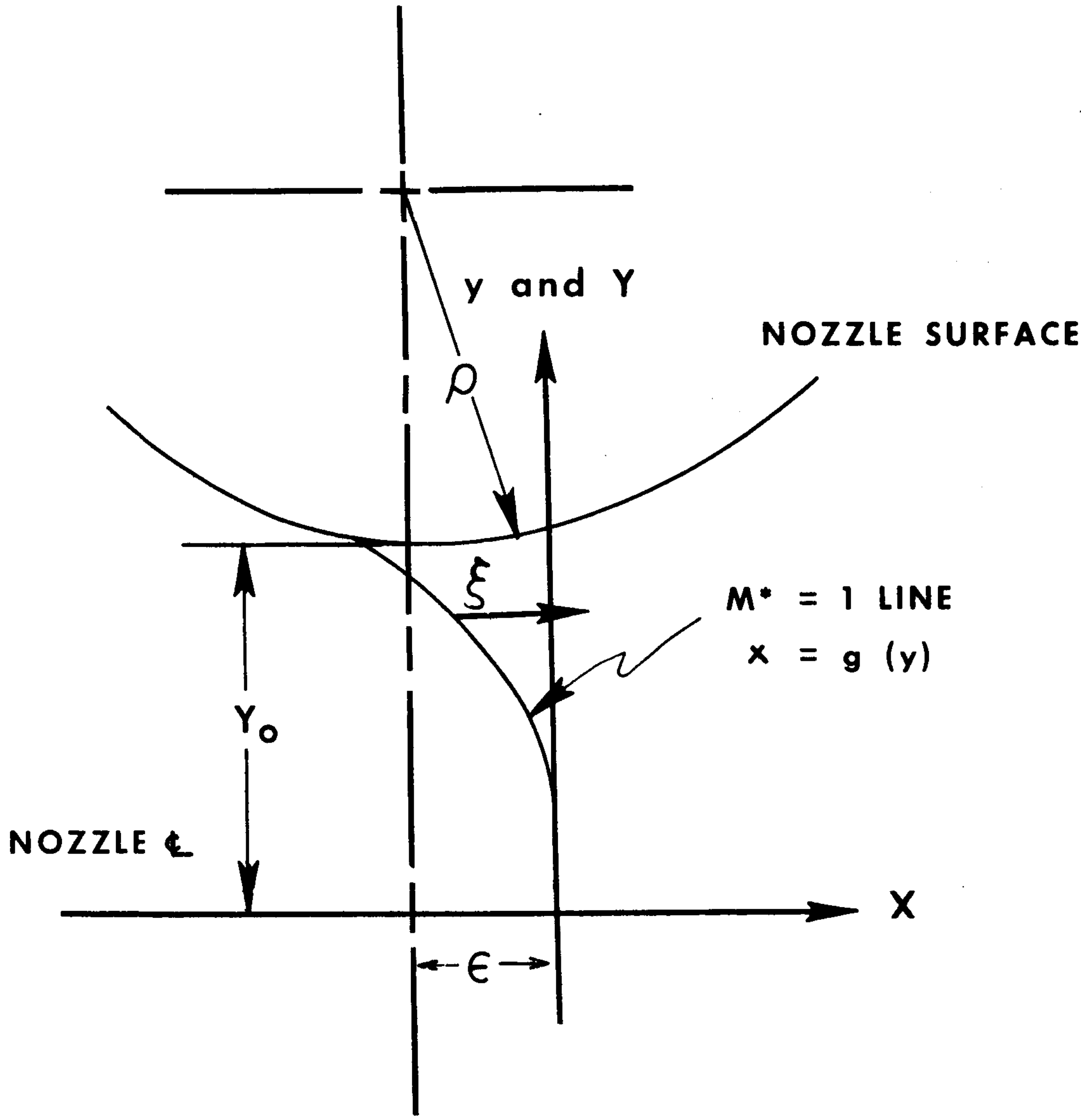

FIGURE 1 LOCATION OF COORDINATE SYSTEMS

a. <u>Sauer's Method</u> (3)

In Sauer's analysis, the origin of the coordinate system (FIG 1) is placed on the axis at a distance ε (to be calculated later) downstream of the minimum section so that it coincides with the intersection of the axis and the critical (sonic) line. Hence, from equations 1.2 and 1.3 and the definition of equation 1.6, it follows that $\Phi_x (0, 0) = \Phi_y (0, 0) = 0$. Furthermore, it is argued that both Φ_y and Φ_{xy} are small in the vicinity of the axis, y=0, and accordingly, the term $2\Phi_y \Phi_{xy}$ is deleted from equation 2.1, resulting in

$$(\gamma +1) \Phi_x \Phi_{xx} - \Phi_{yy} - \frac{1}{y} \Phi_y = 0 \tag{2.2}$$

Sauer assumes the existence of a solution to equation 2.2 in the form:

$$\Phi = \sum_{i=0}^{\infty} f_{2i} (x) \quad y^{2i} \tag{2.3}$$

where the coefficients $f_{2i}(x)$ are functions of x only.

The derivatives of Φ from equation 2.3 are substituted into equation 2.2. The resulting recurrence relationship and the approximation

$$f'_0 (x) = \alpha x \tag{2.4}$$

where: α = constant, results in

$$\left.\begin{aligned} f_0 (x) &= \alpha \frac{x^2}{2} \\ f_2 (x) &= \frac{(\gamma +1)}{4} \alpha^2 x \\ f_4 (x) &= \frac{(\gamma +1)^2}{64} \alpha^3 \end{aligned}\right\} \tag{2.5}$$

with $\quad f_6 (x) = f_8 (x) = ------f_n(x) = 0, \; n > 6$

The constant α is a first approximation to the longitudinal velocity gradient on the axis of symmetry.

Substituting equation 2.5 into equation 2.3 and differentiating gives

$$\left.\begin{aligned} \Phi_x &= u = \alpha x + \frac{(\gamma +1)}{4} \alpha^2 y^2 \\ &\text{and} \\ \Phi_y &= v = \frac{(\gamma + 1)}{2} \alpha^2 xy + \frac{(\gamma +1)^2}{16} \alpha^3 y^3 \end{aligned}\right\} \tag{2.6}$$

The two following boundary conditions are required to evaluate α and ε:

$$\left.\begin{aligned} &\text{at } x = -\varepsilon \text{ and } y = 1 \\ &\qquad\qquad\qquad v = 0 \\ &\text{and} \\ &\frac{1}{\rho} = \frac{1}{(1+u)} \frac{\partial v}{\partial x} \end{aligned}\right\} \tag{2.7}$$

where ρ is the radius of curvature at the minimum section.

Equation 2.7 insures that the streamline at the wall has the slope and curvature of the wall at the minimum section. Solution of equation 2.7 gives

$$\epsilon = \frac{\gamma + 1}{8} \alpha \tag{2.8}$$

and

$$\frac{1}{\rho} = \frac{1}{\left[1 + \left(\frac{\gamma + 1}{8}\right) \alpha^2\right]} \frac{(\gamma + 1)}{2} \alpha^2 \tag{2.9}$$

Sauer assumes that in equation 2.9

$$\left(\frac{\gamma + 1}{8}\right)\alpha^2 << 1$$

and drops that term to get

$$\alpha = \sqrt{\frac{2}{(\gamma + 1)} \frac{1}{\rho}} \tag{2.10}$$

But closer examination shows that terms of the order of α^2 have not been dropped any place else in the method. If the term is not dropped from equation 2.9 then equation 2.10 becomes

$$\alpha = \sqrt{\frac{2}{(\gamma + 1)} \left[\frac{4}{4\rho - 1}\right]} \tag{2.11}$$

Obviously, unless $4\rho >> 1$, the $\frac{\gamma + 1}{8} \alpha^2$ term cannot be legitimately dropped in equation 2.9.

The relationship

$$(1 + u)^2 + v^2 = 1 \tag{2.12}$$

defines the critical curve $M^* = 1$, where $M^* = \sqrt{(1+u)^2 + v^2}$.

Sauer used for a first approximation of the curve the condition

$$u = 0. \tag{2.13}$$

Sauer's method was never intended to be an exact procedure for obtaining the solution of the transonic flow field. Consequently, Sauer's method should not be expected to produce results having accuracy comparable to the method of characteristics. In the discussion of the other simplified analyses, various approximations that Sauer used are eliminated.

b. Other Methods

In addition to Sauer's approximation, there are other approximate analyses which give useful comparisons. One of the more notable analyses is that of Oswatitsch and Rothstein [2]. It, along with that of Sauer, is discussed briefly by Shapiro [4]. The former method, while it uses the full equation of motion (Equation 1.7), nevertheless, requires some apriori knowledge of the velocity distribution on the axis. In addition, the resulting flow field does not necessarily satisfy the initial equation in the middle regions.

Shapiro also discusses a relaxation of finite-difference technique. Since this is a purely numerical method that does not lend itself to generalized parameter studies, it is not discussed in this document.

c. Other Methods Developed in Conjunction with the Present Method

Two approximate methods that originated in the present study are of interest because they can be used to check the arbitrarily exact solution.

An investigation of Sauer's method, including the linear coefficient Sauer dropped, was undertaken. It was concluded that for a large nondimensional radius of curvature (ρ), Sauer's method would be quite close to an exact solution. However, as ρ approaches one or less than one, Sauer's method does not produce a good approximation.

In the vicinity of the nozzle wall, the term $2\Phi_y \Phi_{xy}$, which Sauer drops, is not of negligible magnitude. By writing

$$2\Phi_y \Phi_{xy} = \frac{\partial (\Phi_y)^2}{\partial x} \tag{2.14}$$

and noting that near the wall

$$\Phi_y \approx (1+\Phi_x)\frac{dy_s}{dx} \tag{2.15}$$

it becomes apparent that in the proximity of the boundary

$$2\Phi_y \Phi_{xy} \approx \frac{\partial}{\partial x}\left[(1+\Phi_x)^2 \left(\frac{dy_s}{dx}\right)^2\right] \tag{2.16}$$

where $y_s = y_s(x)$ denotes the expression for the boundary.

Thus, by observation of equation 2.16 it is seen that the condition $2\Phi_y\Phi_{xy} << 1$ is met only for small nozzle slopes (large radius of curvatures), particularly on the supersonic side of the throat where $(1 + \Phi_x) > 1$. Accordingly, it should be expected that Sauer's method is invalid for highly curved walls, though it should give excellent results for a slowly varying nozzle cross-section area. An analogy between the present case and external, axisymmetric flow indicates that this is precisely what should be expected. In external flow problems, linearized theory is valid only for small longitudinal area gradients of the body; therefore, it gives good results only when such is the case.

Method I

In this method, the procedure used by Sauer is followed with the $2\Phi_y\Phi_{xy}$ term retained (equation 2.1). This results in a non-terminating series of terms $f_{2n}(x)\, y^{2n}$, whereas, in Sauer's case all terms are zero beyond $n = 2$.

The coefficients, assuming $f_0{}' = \alpha x$, are of the form

$$f_{2n} = \frac{(\gamma+1)^n}{2n^2}\,\alpha^{2n-1}\,(a_n + b_n\,\alpha x) \tag{2.17}$$

$$n \neq 0$$

where a_n and b_n are evaluated from the recursion formulae.

These formulae can be simplified to a recursion involving a_n and b_n rather than f_{2n}, i.e.,

$a_1 = 0$ and $b_1 = \frac{1}{2}$

while

$$a_n = \frac{b_{n-1}}{4(n-1)^2} + \sum_{i=1}^{i=n-2} \frac{a_{i+1}\ b_{n-i-1}}{(i+1)\ (n-i-1)}$$

and

$$b_n = \sum_{i=0}^{i=n-2} \frac{b_{i+1}\ b_{n-i-1}}{(i+1)\ (n-i-1)} \qquad n \geq 2 \tag{2.18}$$

Evaluation of equation 2. 3, using the first five terms of the above equations, produces, after differentiation, the following perturbation velocity components:

$$u = \alpha x + \frac{(\gamma + 1)\,\alpha^2\, y^2}{4} + \frac{[(\gamma + 1)\,\alpha^2\, y^2]^2}{32} + \frac{[(\gamma + 1)\alpha^2\, y^2]^3}{144} + \frac{11\,[(\gamma + 1)\,\alpha^2\, y^2]^4}{6144}$$

and

$$v = \frac{(\gamma + 1)\alpha^2}{2}\, xy + \frac{(\gamma + 1)^2\alpha^3}{16}\,(1 + 2\alpha x)\, y^3 + \frac{(\gamma + 1)^3\alpha^5}{24}\left(\frac{3}{8} + \alpha x\right) y^5 + \frac{11}{768}\,(\gamma + 1)^4\,\alpha^7\left(\frac{1}{3} + \alpha x\right) y^7 \tag{2.19}$$

The parameter ϵ can be evaluated by applying the boundary equation that the velocity component in the y direction (v) equals zero at $y = 1.0$ and $x = -\epsilon$, i.e:

$$v = \Phi_v = \sum_{i=1}^{i=\infty} (2i)\ f_{2i}\ (-\epsilon)\ (1.00)^{2i-1} = 0 \tag{2.20}$$

Consequently,

$$\epsilon_i = \frac{\left[\frac{(\gamma+1)}{8}\ \alpha_i + \sum_{j=3}^{j=n} 2\ \frac{(\gamma+1)}{j}^{j-1}\ \alpha_i^{2j-3}\ a_j\right]}{\left[1 + \frac{(\gamma+1)}{4}\ \alpha_i^2 + \sum_{j=3}^{j=n} \frac{2(\gamma+1)^{j-1}}{j}\ \alpha_i^{2(j-1)} b_j\right]} \tag{2.21}$$

At any point along the nozzle surface, defined by $y = y_s(x)$, the following should be true:

$$y_s' = \frac{v}{1+u}$$

or (2.22)

$$v - (1+u)\ y_s' = 0$$

where u is the non-dimensional perturbation velocity in the x direction, and y_s' is the slope of the nozzle contour.

Using

$\Phi_y = v$ and $\Phi_x = u$ and substituting equations 2.3, 2.17, 2.18, and 2.21 in equation 2.22 results in

$$\alpha_i = \frac{-B_i + \sqrt{B_i^2 - 4A_i C_i}}{2A_i} \tag{2.23}$$

where α_i is the α value of the i^{th} iteration attempt, and the functions A_i, B_i, and C_i are:

$$
\left.
\begin{aligned}
A_i &= \frac{(\gamma+1)^2}{32}\left[\frac{(\gamma+1)}{2}\, y_{i-1} - \beta_{i-1}\right] y^4_{i-1} \\
B_i &= \frac{(\gamma+1)}{4}\left[\frac{(\gamma+1)}{4}\, y_{i-1} - \beta_{i-1}\right] y^2_{i-1} \\
C_i &= -\beta_{i-1} + \sum_{j=3}^{j=n-1}\left\{(\gamma+1)^j\, \alpha^{2j}_{i-1}\left[\frac{(\gamma+1)}{(j+1)}\, a_{j+1}\, y_{i-1} - \frac{b_j}{2j^2}\, \beta_{i-1}\right] y^{2j}_{i-1} - \frac{(\gamma+1)^n\, \alpha^{2n}_{i-1}\, b_n}{2n^2}\, \beta_{i-1}\, y^{2n}_{i-1}\right\} \\
\text{where: } \beta_i &= \frac{1}{\alpha_i}\left(\frac{\epsilon_i}{\rho}\right)\Big/\sqrt{1-\left(\frac{\epsilon_i}{\rho}\right)^2} \\
y_i &= 1+\rho\left[1-\sqrt{1-\left(\frac{\epsilon_i}{\rho}\right)^2}\,\right]
\end{aligned}
\right\}\quad (2.24)
$$

These are solved by starting with an α value from Sauer's analysis and then cycling between the equations until a satisfactory solution is achieved.

In attempting to construct a constant M* line, it should be noted from an examination of u and v, as given by equation 2. 19, that Sauer's approximation, equation 2.13, is only good when three terms are used to describe the potential function.

A better approximation for an M*= 1 curve is

$$u = -\frac{v^2}{2}$$

But, in this analysis (and all further analyses) a curve of an arbitrary constant Mach number will be given by

$$2\Phi_x + \Phi_x^2 + \Phi_y^2 = d(2+d) \tag{2.25}$$

where d (a constant) is defined by M* = 1 + d

The curve is defined as $x \approx x(y)$, where

$$x = -\frac{L1 + \sqrt{(L1)^2 - 4(L2)(L0)}}{2(L2)} \tag{2.26}$$

with

$$L2 = \alpha^2 + \sum_{\substack{i=1 \\ i=n+1}}^{\substack{i=2n-1 \\ i=n}} \sum_{\substack{j=0 \\ j=i-n}}^{\substack{j=n-1 \\ j=i-1}} \frac{(\gamma+1)^{i+1} \alpha^{2(i+1)} b_{j+1} b_{i-j}}{(j+1)(i-j)} y^{2i}$$

$$L1 = 2\alpha + \sum_{\substack{i=1 \\ i=n+1}}^{\substack{i=2n-1 \\ i=n}} \sum_{\substack{j=0 \\ j=i-1}}^{\substack{j=n-1 \\ j=i-1}} \frac{(\gamma+1)^{i+1} \alpha^{2i+1}}{(j+1)(i-j)} (b_{j+1} a_{i-j} +$$

$$+ b_{i-j} a_{j+1}) y^{2i} + \sum_{i=1}^{i=n} \frac{(\gamma+1)^i \alpha^{2i+1}}{i^2} b_i y^{2i} \tag{2.27a}$$

and

$$
\begin{aligned}
LO = {} & -d\,(2+d) \; + \\
& + \sum_{\substack{i=1 \\ i=n+1}}^{\substack{i=2n-1 \\ i=n}} \; \sum_{\substack{j=0 \\ j=i-n}}^{\substack{j=n-1 \\ j=i-1}} \frac{(\gamma+1)^{i+1} \; \alpha^{2i}}{(j+1) \quad (i-j)} \; a_{j+1} \, a_{i-j} \, y^{2i} \; + \\
& + \sum_{\substack{i=2 \\ i=n+1}}^{\substack{i=2n \\ i=n}} \; \sum_{\substack{j=1 \\ j=i-n}}^{\substack{j=n \\ j=i-1}} \frac{(\gamma+1)^{i} \; \alpha^{2i} \; b_j \; b_{i-j} \; y^{2i}}{4j^2 \; (i-j)^2} \; + \\
& + \sum_{i=1}^{i=n} \frac{(\gamma+1)^{i} \; \alpha^{2i} \; b_i \; y^{2i}}{i^2}
\end{aligned}
\qquad (2.27b)
$$

(NOTE: In using equation 2.27 when the summation has two sets of values, use all the values closest to the summation sign at the same time and then use the set of values furthest away.)

Method II

Since the interest is in the flow about Mach one, it was theorized that a quicker converging series might be obtained if the coordinate system was changed so that the axes were the nozzle centerline and the Mach one surface. Consequently, this method involves a transformation of the coordinate x.

Let the variable x be replaced by the variable ξ , defined so that $\xi = x - g(y)$, where g(y) denotes the curve representing the critical (sonic) line (FIG. 1). The differential equation (including the $2 \Phi y \Phi xy$ term) is transformed from x, y coordinates to the ξ , Y frame of reference.

The g(y) curve is unknown; however, the assumption is made that it can be represented as an even polynomial in y, i.e.:

$$g(y) = \sum_{j=0}^{j=\infty} a_j \ y^{2j} \tag{2.28}$$

where the a_j's are, at this point, yet to be resolved.

Let the x-y origin of the coordinates be located on the axis at the Mach one point, in which case $a_o = 0$. Due to the location of the x-y axis, epsilon is still defined as in Sauer's analysis (FIG. 1). The potential function is assumed to exist in the form

$$\Phi = \sum_{i=0}^{i=\infty} f_{2i} (\xi) \ Y^{2i} \tag{2.29}$$

The change of coordinate system produces

$$\left.\begin{aligned} f_x \Big|_y &= f_\xi \Big|_Y = f' \\ \text{and} \qquad & \\ f_y \Big|_x &= -\frac{dg}{dy} \ f_\xi = -g'f' \end{aligned}\right\} \tag{2.30}$$

As in prior analyses, recurrence formulae can be calculated by substituting Φ and the polynomial form of g(y) into the transformed differential equation. This gives $f_{2i}\ (\xi)$ in terms of $f_o\ (\xi)$ and the a_j coefficients. The assumption is made, similar to Sauer's analysis, that

$$f_0'\ (\xi) = \alpha \ \xi$$

In addition, it is assumed that the coefficients of the Mach one polynomial are in the form

$$a_i = b_i \ (\gamma+1)^i \ \alpha^{2i-1} \tag{2.31}$$

where b_i is not a function of α, x, y, or ξ. It becomes convenient to represent f_{2n} as

$$f_{2n} = \frac{(\gamma+1)\,\alpha^{2n-1}}{2n^2} \left[A_n + B_n \ \alpha \ \xi\right] \quad n \neq 0 \tag{2.32}$$

where A_n and B_n are not functions of x, ξ, or α but may be functions of Y or any one of the set of b_n's.

With the proper manipulation of equation 2.31 and 2.32 in the simplified equation of motion given by 2.1, the recurrence formulae for A_n and B_n are obtained as

$$B_i = \beta_i + 2i^2 b_i \text{, when } i \geq 1 \tag{2.33}$$

and

$$A_i = \frac{B_{i-i}}{4(i-1)^2} +$$

$$+ \sum_{j=1}^{j=i-1} \left\{ \frac{\beta_j A_{i-j}}{j(i-j)} + jb_j \left[\frac{(2i-j)}{(i-j)^2} \beta_{i-j} + 2ib_{i-j} \right] - \right.$$

$$\left. - \frac{\beta_j}{j} \sigma_{j,i-1} \sum_{k=0}^{k=i-j-2} \frac{(k+1)\, b_{k+1}\, B_{i-j-k-1}}{(i-j-k-1)^2} \right\} \tag{2.34}$$

when $i \geq 2$

where

$$A_1 = 0, \beta_1 = \tfrac{1}{2} \tag{2.35}$$

where

$$\beta_i = \sum_{j=1}^{j=i-1} \frac{\beta_j \beta_{i-j}}{j(i-j)} \text{, when } i \geq 2 \tag{2.36}$$

and

$$\sigma_{j,i-1} = \left\{ \begin{matrix} 0 & j=i-1 \\ 1 & j \neq i-1 \end{matrix} \right\}$$

Before the usual boundary conditions can be applied, the values of the b_i's must be found. The restriction to find the necessary extra boundary condition arises from the definition of $x = g(y)$ as the curve of M^* equal one.

The $M^* = 1.0$ curve is necessarily that along which

$$2\Phi_x + \Phi_x^2 + \Phi_y^2 = 0$$

(equation 2.25). The above equations produce a recurrence relationship for the b_i's, i.e.:

$$\left.\begin{aligned}
b_1 &= -\tfrac{1}{4} \\
b_2 &= -\frac{1}{32} \\
&\text{and} \\
b_i &= -\tfrac{1}{2}\left(\frac{\beta_i}{i^2} + \sum_{j=1}^{j=i-2} \left[\frac{B_j\, B_{i-j}}{j^2\, (i-j)^2} + \right.\right. \\
&\quad + (\gamma+1) \left[\frac{A_{j+1}\, A_{i-j}}{(j+1)\,(i-j)} + \sum_{k=0}^{k=i-j-2} (k+1)\; b_{k+1} \left(- \frac{A_{j+1}\, B_{i-j-k-1}}{(j+1)\,(i-j-k-1)^2} + \right.\right. \\
&\quad \left.\left.\left.\left. + \sum_{\ell=0}^{\ell=i-j-k-2} \frac{(\ell+1)\, b_{\ell+1}\, B_j\, B_{i-j-k-\ell-1}}{j^2\, (i-j-k-\ell-1)^2} \right)\right]\right]\right)
\end{aligned}\right\} \quad 2.37$$

when

$$i \geq 3$$

The first five terms of the potential equation, when differentiated, produce velocity perturbation components of

$$
\left.
\begin{aligned}
u &= \alpha\,\xi - \frac{(\gamma+1)^4\,\alpha^6}{512}\,y^6 - \frac{(\gamma+1)^5\,\alpha^8}{512}\,y^8 \\
&\text{and} \\
v &= \frac{(\gamma+1)\,\alpha^2}{2}\,\xi\,y - \frac{(\gamma+1)^2\,\alpha^3}{16}\,y^3 - \\
&\quad - \frac{(\gamma+1)^3\,\alpha^5}{8}\left[\frac{1}{4} - \frac{1}{3}\,\alpha\,\xi\right]y^5 - \\
&\quad - \frac{(\gamma+1)^4\,\alpha^7}{32}\left[\frac{1}{9} + \frac{(\gamma+1)}{64} - \frac{11\,\alpha\,\xi}{24}\right]y^7
\end{aligned}
\right\} \quad (2.38)
$$

while, the first four terms used to describe the Mach one surface are

$$
\begin{aligned}
g(y) = -\frac{(\gamma+1)\ \alpha}{4}\,y^2\Bigg[1 &+ \frac{(\gamma+1)}{8}\,\alpha^2\,y^2 + \\
&+ \frac{(\gamma+1)^2}{4}\,\alpha^4\left(\frac{\gamma+1}{32} + \frac{1}{9}\right)y^4 + \\
&+ \frac{(\gamma+1)^3\,\alpha^6}{128}\left(\frac{23}{12} + \gamma\right)y^6\Bigg]
\end{aligned}
\quad (2.39)
$$

The boundary conditions that are applied are the same as used in Sauer's analysis (equation 2.7). The noticeable difference, however, is that while Sauer applies them at $x = -\epsilon$ and $y=1.0$, in this analysis the point is $\xi = -\left[\epsilon + g(1.0)\right]$ and $y = 1.0$. The resulting equations are

$$
\epsilon + g(1.0) = \frac{\displaystyle\sum_{i=1}^{\infty} 2(\gamma+1)^i\,\alpha^{2j-1}\left[\frac{A_{i+1}}{(i+1)} - \sum_{j=0}^{j=i-1}\frac{(j+1)\ b_{j+1}\,B_{i-j}}{(i-j)^2}\right]}{\displaystyle 2\sum_{i=0}^{\infty}\left[(\gamma+1)^i\,\alpha^{2i}\ \beta_{i+1}\ /\ (i+1)\right]} \quad (2.40)
$$

and

$$\alpha_k = +\sqrt{\frac{-L_1 + \sqrt{L_1^2 - 4L_k L_2}}{2L_2}} \tag{2.41}$$

where

α_k is the k th iteration attempt to find α

$$\left.\begin{aligned}
L_1 &= \frac{(\gamma+1)}{2} \quad (\rho - 1/4) \\
L_2 &= \frac{(\gamma+1)^2}{8} \quad (\rho - 1/4) \\
L_k &= -1 - \sum_{i=2}^{i=n} \frac{(\gamma+1)^{i+1} \alpha_{k-1}^{2(i+1)}}{(i+1)} \left[\frac{B_{i+1}}{2(i+1)^2} - \rho\, \beta_{i+1}\right] + \\
&+ \frac{(\gamma+1)}{8} \alpha_{k-1}^2 \left[1 + (\gamma+1)\, \alpha_{k-1}^2 /4\right] + \\
&+ \frac{\sum_{i=1}^{i=n} (\gamma+1)^i \alpha_{k-1}^{2i} \left[\frac{A_{i+1}}{(i+1)} - (\gamma+1)\, \alpha_{k-1}^2 \sum_{j=0}^{j=i} \frac{(j+1)\, b_{j+1}\, B_{i-j+1}}{(i-j+1)^2}\right]}{\sum_{i=0}^{i=n} \left[(\gamma+1)^i\, \alpha_{k-1}^{2i}\, \beta_{i+1} /(i+1)\right]}
\end{aligned}\right\} \; 2.42$$

$$\alpha_o = \sqrt{\frac{2}{(\gamma+1) \quad (\rho-1/4)} \left[-(\rho-1/4) + \sqrt{\rho^2 + \frac{3}{2}\rho - \frac{7}{16}}\right]} \tag{2.43}$$

and n is the number of terms considered in the series.

Once the α and ϵ values are known, it is relatively easy to establish constant Mach lines. Along constant $M^*=1 + d$ lines the ξ value for any y value is given by:

$$\xi = -\lambda_1 + \sqrt{\lambda_1^2 + \lambda_2^2}, \tag{2.44}$$

where

$$\lambda_1 = \frac{1}{\alpha}\left\{1 + \sum_{i=1}^{i=n} (\gamma+1)^{i+1}\alpha^{2(i+1)}y^{2(i+1)}\left[\frac{\beta_i+1}{2(i+1)^2} + (\gamma+1)\sum_{k=o}^{k=i-1}\frac{\beta k+1}{(k+1)}\right.\right.$$

$$\left.\left.\left(\frac{A_i + 1-k}{i+1-k} - \sum_{j=o}^{j=i-k-1}\frac{(j+1)bj+1 \quad \beta i-j-k}{(i-j-k)^2}\right]\right\}$$

$$\left\{1 + \sum_{i=0}^{i=n}\sum_{j=0}^{j=i}(\gamma+1)^{i+2}\alpha^{2(i+1)}\; y^{2(i+1)}\frac{\beta j+1 \quad \beta i+1-j}{(j+1) \quad (i+1-j)}\right\}^{-1} \qquad (2.45)$$

and

$$\lambda_2 = \frac{d(2+d)}{\alpha^2}\left[1 + \sum_{i=0}^{\infty}\sum_{j=0}^{j=i}(\gamma+1)^{i+2}\alpha^{2(i+1)}y^{2(i+1)}\frac{\beta j+1 \quad \beta i-j+1}{(j+1) \quad (i-j+1)}\right]^{-1}$$

Methods I and II have a built-in error because they have not taken into consideration the Φ_x^2 and Φ_y^2 terms in equation 1.7, but they have included similar order of magnitude terms in the series expansion of Φ.

Neither methods I nor II have yet been programmed or substituted to replace Sauer's analysis. But this will be done later, and further comparison and analysis will be attempted.

3. PRESENT METHOD

In view of the requirements for an analysis giving exact results, it was considered advantageous to proceed directly to a solution of the exact equation of motion. For convenience, that equation is repeated here.

$$\left[(-2+\Phi_x)\Phi_x - \Gamma\;\Phi_y^2\right]\Phi_{xx} + \left[\frac{2}{(\gamma+1)} - \Gamma\;\Phi_x(2+\Phi_x) - \Phi_y^2\right]\Phi_{yy} -$$

$$-\frac{4}{(\gamma+1)}(1+\Phi_x)\;\Phi_y\Phi_{xy} + \left(\frac{2}{(\gamma+1)} - \Gamma\;\left[\Phi_x(2+\Phi_x)+\Phi_y^2\right]\frac{\Phi_y}{y} = 0 \qquad (3.1)$$

The potential function is assumed to exist in the form

$$\Phi = \sum_{i=0}^{i=\infty}\sum_{j=0}^{j=\infty} A(i,\ 2j)\,x^i\,y^{2j} \qquad (3.2)$$

where the even exponent of the variable y is caused by the symmetry of the flow field. The velocity distribution along the x axis would be

$$\sum_{i=1}^{\infty} i \ A(i, 0)x^{i-1}$$

and not just the first approximation used by Sauer (αx)

Upon differentiation and substitution of equation 3.2 into 3.1, two algebraic relationships result. These relations, which are the necessary and sufficient conditions for the assumed potential function to satisfy the equation of motion, are the recurrence formulae defining the A(i, 2j) coefficients in terms of the set of A(i, o) coefficients. The recurrence formulae result from the existence condition that the coefficient of each x and y term in the series must vanish separately in order to satisfy the equation of motion 3.1, for an arbitrary x and y value.

These recurrence formulae are given below. Their formidable appearance is the direct result of the extreme nonlinearity of the equation of motion. With the definition of

$$\sigma_{i,j} = \begin{Bmatrix} 1 & i \neq j \\ 0 & i = j \end{Bmatrix} \tag{3.3}$$

$$\begin{aligned} A(i, 2) = \left(\frac{\gamma+1}{8}\right) \sum_{k=0}^{k=i-1} \Big\{ & 2(k+2)(k+1)(i+1-k)A(k+2, 0)A(i+1-k, 0) + \\ & + 8 \ \Gamma \ (i+1-k)A(k, 2)A(i+1-k, 0) + \\ & + \sum_{a=0}^{a=i-k-1} \sigma_{a,0} \ \big[(k+2)(k+1)(a+1)(i+1-a-k) A(k+2, 0) A(a+1, 0) A(i+1-a-k, 0) + \\ & + 4 \ \Gamma \ (a+1)(i+1-a-k) A(k, 2) A(a+1, 0) A(i+1-a-k, 0)\big] \Big\} \end{aligned} \tag{3.4}$$

and

$$A(i,2(j+2)) = \frac{(\gamma+1)}{8(j+2)^2}\Bigg[\sigma i,0\Bigg\{\sum_{k=0}^{k=i-1} 8(i+1-k)(j+2)^2\,\Gamma\,A\big(k,2(j+2)\big)A(i+1-k,0)+$$

$$+\ 2(k+2)(k+1)(i+1-k)\,A(k+2,2(j+1))A(i+1-k,0)+$$

$$+\sum_{a=0}^{a=i-k-1}\Big[(a+1)(i+1-a-k)(k+2)(k+1)\,A(k+2,2(j+1))A(a+1,0)A(i+1-a-k,0)\ +$$

$$+\ (a+1)(i+1-a-k)4(j+2)^2\,\Gamma\,A(k,2(j+2))A(a+1,0)A(i+1-a-k,0)\Big]\Bigg\}\ +$$

$$+\sum_{k=0}^{k=i}\ \sum_{n=0}^{n=j}\Big\{\,2(i+1-k)(k+2)(k+1)A(k+2,2n)A(i+1-k,2(j+1-n)\ +$$

$$+\ (i+1-k)8(n+1)^2\,\Gamma\,A(k,2(n+1))A(i+1-k,2(j+1-n))\ +$$

$$+\left(\frac{16}{\gamma+1}\right)(n+1)(k+1)(j+1-n)A(k+1,2(n+1))A(i-k,2(j+1-n))\ \bullet$$

$$+\sum_{a=0}^{a=i-k}\Big[\,(a+1)(i+1-a-k)4(n+1)^2\,\Gamma\,A(k,2(n+1))A(a+1,2(j+1-n))A(i+1-a-k,0)\ +$$

$$+\ (a+1)(i+1-a-k)(k+2)(k+1)A(k+2,2n)A(a+1,2(j+1-n))A(i+1-a-k,0)\ + \tag{3$$

$$+\sum_{b=0}^{b=j-n}\Big\{(a+1)(i+1-a-k)4(n+1)^2\,\Gamma\,A(k,2(n+1))A(a+1,2b)A(i+1-a-k,2(j+1-b-n))\ +$$

$$+\ (a+1)(i+1-a-k)(k+2)(k+1)A(k+2,2n)A(a+1,2b)A(i+1-a-k,2(j+1-b-n))\ +$$

$$+\ (j+1-n-b)(k+2)(k+1)(b+1)4\,\Gamma\,A(k+2,2n)A(a,2(b+1))A(i-a-k,2(j+1-b-n)+$$

$$+\ (j+1-n-b)(n+1)(k+1)(a+1)\left(\frac{16}{\gamma+1}\right)A(k+1,2(n+1))A(a+1,2b)A(i-k-a,2(j+1-n-b))\ +$$

$$+\ (j+1-n-b)(n+1)(b+1)(16)\left(n+1-\frac{1}{\gamma+1}\right)\times$$

$$\times\ A(k,2(n+1))A(a,2(b+1))A(i-a-k,2(j+1-b-n))\Big\}\Big]\Big\}\Bigg]$$

The original written outline of a method to handle the programming of the two recurrence relations (equations 3.4 and 3.5) had the $A_{i,2j}$'s resolved into polynomials in the unknowns $A_{i,0}$'s. All advice obtained suggested that the $A_{i,2j}$'s be evaluated as pure numbers instead of functional representations of γ and the unknowns, this suggestion was followed, but one third of the computer running time will be saved if the program is revised to its original form.

For convenience in future discussions, let the unknowns be represented by D_i's , where

$$D_i = A_{i+1,0} \tag{3.6}$$

when

$$i \geq 1$$

The equations are now being re-programmed to appear in the form of polynomials of the type

$$A_{i,2j} = \sum_{i=1}^{i=k} e_i \, D_1^{d_{1,i}} \, D_2^{d_{2,i}} \cdots D_n^{d_{n,i}} \, \theta^{d_{\theta,i}} \, \Gamma^{d_{\Gamma,i}}, \tag{3.7}$$

where

e_i is a pure number, $d_{n,i}$ is the power to which the D_n unknown is raised in the i^{th} term

and

$$\theta = \gamma+1$$

As equation 3.4 is already programmed in the above form, some of the $A_{i,2}$'s are presented in Table I.

Here, as in Sauer's analysis, the origin of coordinates is chosen on the axis a distance ϵ downstream of the minimum section.

At this point in the analysis the unknown functions have been reduced to a system of unknown coefficients of the type D_i and also the unknown ϵ.

The general inviscid flow boundary condition is that the velocity at the boundary surface must be tangent to that boundary. The boundary surface y_s is represented as a power series in x

$$y_s = \sum_{i=0}^{i=\infty} B_i x^i \tag{3.8}$$

While, the inviscid boundary condition can be written as

$$(1+u) \frac{dy_s}{dx} - v = 0$$

where both u and v are evaluated along the boundary surface, i.e:

$$\left.\begin{array}{l} u = u(y_s, x) = u(\Sigma B_i x^i, x) = u_s(x \text{ only}) \\ \text{and} \\ v = v(y_s, x) = v(\Sigma B_i x^i, x) = v_s(x \text{ only}) \end{array}\right\} \tag{3.10}$$

Substituting equations 3.10 and 3.8 into 3.9 and collecting the resulting terms as a power series in x, results in the boundary condition equation in the form of

$$\sum_{i=1}^{i=\infty} G(i,\infty)\, x^{i-1} = 0 \tag{3.11}$$

where

$$G(i,\infty) = iB_i + \sum_{j=0}^{j=\infty} \sum_{q=0}^{q=i} \left\{ \left[\sum_{a=0}^{a=i-q} (a+1)q\, B(a+1)\, C(i-q-a, 2j) \right] - 2\, \sigma_{q,i}\, (j) C(i-1-q, 2j-1) \right\} A(q, 2j) \tag{3.12}$$

with C (i, n) given by

$$C(i, 0) = \begin{Bmatrix} +1 \text{ for } i = 0 \\ 0 \text{ for } i \neq 0 \end{Bmatrix} \tag{3.13}$$

$$
\left.
\begin{aligned}
&C(i,1) = B_i \\
&\text{and} \\
&C(i,n) = \sum_{b=0}^{b=i} B_b\, C_{i-b,\,n-1} \\
&\text{with} \qquad\qquad n > 2
\end{aligned}
\right\}
\tag{3.14}
$$

In equation 3.11, the $G(i,\infty)$ designation simply denotes the total collection of terms constituting the coefficient of x^{i-1}. $G(i,\infty)$ contains an infinite number of columns of terms nonlinear in $A_{p,0}$ and hence the ∞ notation.

The only possible solution of equation (3.11) that would be valid for any arbitrary value of x, would be that

$$G(i,\infty) = 0 \tag{3.15}$$

for all values of i $(i=1,2,3---\infty)$.
Equation (3.15) thus denotes an infinite set of nonlinear simultaneous equations in the unknowns D_i $(i=1,2,3---\infty)$, each one containing an infinite number of terms.

In computing numerical solutions the infinite set of equations need not be considered, but only those equations that will insure convergence to the required accuracy in the domain of interest. Furthermore, only the number of terms in each equation need be considered that are necessary to produce the required accuracy.

To better picture this, the $A_{i,2j}$ coefficients are arranged in the form of an array

$$
\begin{array}{c|cccccc}
i \backslash 2j & 0 & 2 & 4 & 6 - - - - - - - - & 2j - - - - - - \\
\hline
0 & A_{0,0} & A_{0,2} & A_{0,4} & A_{0,6} - - - - - - - - & A_{0,2j} - - - - - - \\
1 & A_{1,0} & A_{1,2} & A_{1,4} & A_{1,6} - - - - - - - - & A_{1,2j} - - - - - - \\
2 & A_{2,0} & A_{2,2} & A_{2,4} & A_{2,6} - - - - - - - - & A_{2,2j} - - - - - - \\
- & - - - - & - - - - & - - - - & - - - - - - - - & - - - - - - \\
i & A_{i,0} & A_{i,2} & A_{i,4} & A_{i,6} - - - - - - - - & A_{i,2j} - - - - - - \\
- & - - - - & - - - - & - - - - & - - - - - - - - & - - - - - -
\end{array}
\tag{3.16}
$$

Remembering that the first column constitutes the unknowns with all the other coefficients expressible in terms of the first column, (through application of equations 3.4 and 3.5) a maximum number of rows, M+1, can be considered, together with a maximum number of columns, p+1. All rows and columns beyond M+1 and p+1 can be considered negligible.

The expression $G(i,\infty) = 0$ (equation 3.15) contains all terms occurring in the first i+1 rows of 3.16 and an infinite number of its columns. For the M+1, p+1 system the expression $G(i,\infty)$ is changed in notation to read $G(m, p)$, thus,

$$\left.\begin{aligned} & G(m, p) = 0 \\ \text{where} \quad & m = 1,\ 2,\ 3 \text{ --- } M \end{aligned}\right\} \tag{3.17}$$

The expression for $G(m, p)$ is the same as equation 3.12 with i replaced by m and ∞ replaced by p. This system (equation 3.17) has M equations and contains the terms of the first M+1 rows and p+1 columns of the $A_{i,\,2j}$ array, 3.16.

In using the recurrence equations, 3.4 and 3.5, to evaluate the $A_{i,\,2j}$ coefficients $(i \leq M, j \leq p)$ considered in the boundary condition equations ($G_{m,\,p}$'s), certain $A_{k,\,o}$ terms for which $k > M+1$ appear. This is due to diagonal propagation characteristics of the unknowns throughout the array of 3.16. These unknowns must be considered zero in the array of coefficients $i \leq M$, $j \leq p$. This consideration is justifiable on the grounds of the expected small magnitudes of these unknowns.

A judicious initial choice for M and p values must necessarily result from experience. This choice shall be discussed in Section **VI** b.

The system is now reduced to one containing M equations and M number of unknown $A_{i,\,o}$'s. The quantity ϵ, which locates the origin of coordinates, is an additional unknown, giving a total of M+1 unknowns. The M+1st constraint, necessary to evaluate ϵ, stems from mass flow considerations and will be discussed later.

At this point in the analysis the major task remaining is the evaluation of the unknowns ϵ & D_i ----- D_m. Calculation of these quantities allows the calculation of the velocities, u and v, and hence the calculation of all other flow properties. The D_i terms are to be calculated for various prescribed ϵ values. An iterative scheme necessary to compute the D_i terms is discussed below.

The procedure used in the present computer program is as follows: The i^{th} equation will be used to solve for the i^{th} unknown by arranging $G(i,p)$ in the form (note: there is no difference between $G_{i,p}$ or $G(i,p)$)

$$G_{i,p} = a_{i,0}(D_1, D_2 \text{ --- } D_{i-1}, D_{i+1}\text{--}, D_M) + a_{i,1}(D_1, D_2, \text{--}D_{i-1}, D_{i+1}\text{--}, D_M)D_i +$$

$$+ a_{i,2}(D_1, D_2, \text{--}D_{i-1}, D_{i+1}\text{---}, D_M)D_i^2 + \text{----------}$$

$$+ a_{i,n}(D_1, D_2, \text{--}D_{i-1}, D_{i+1}\text{--}, D_M)\, D_i^n + \text{etc.} \tag{3.18}$$

where D_i represents the unknown of $G(i,p) = 0$. Equation 3.18 is simply an expression for $G(i,p)$ written as a polynominal in $D(i)$ where the coefficients are functions of D_1, D_2--- D_M exclusive of D_1.

The present program used three values of D_i (holding all others constant) to fit a cubic formula to D_i vs $G_{i,p}$. Using the cubic formula, a $G_{i,p} = 0$ point is predicted for a certain D_i value. After that D_i value is tried, a more accurate cubic is fitted; this continued until an actual $G_{i,p} = 0$ point is found or 50 tries are made.

The process is repeated for all values of i in equation 3.18 ($i = 1, 2$ --- M). A suitable root for each $G_{i,p}$ is used as new input in the $G_{i+1,p}$ equation to generate a second set of coefficients and so on - until the final equation ($G_{M,p}$). After the process is performed for the M^{th} equation, all of the $G(i,p)$'s are evaluated with the on hand values of the D's. If the absolute values of each and every one of the boundary condition equations is less than or equal to a specified tolerance (α), then the on-hand D values are considered a good solution.

Let a cycle be defined as a set of equation solutions - from i=1 to i=M of $G(i,p)$ - and let k be the number of cycles performed. After the initial cycle (k=0), $G_{m,p}$ is apriori non-zeroed. To explain, let $G_{m,p}$ be equal to minus a number (n) times the value of the $G_{m,p}$ from the k-1 iteration cycle, i.e.,

$$G_{m,p}\Big|_k = -n\, G_{m,p}\Big|_{k-1} \tag{3.19}$$

The problem is to determine n.

On the last equation (M^{th}), it was necessary to leave n equal to zero, as no perturbation occurred to this equation after a D_M is found. In all the other equations, on the k=1 and k=2 cycle it was determined best to let n=.75. After the k=2nd cycle, n could become a function of the equation as well as the cycle, i.e.

$$G_{m,p}\Big|_{k} = -n_{m,k}\, G_{m,p}\Big|_{k-1} \qquad (3.20)$$

where

$$n_{m,k} = n_{m,k-1} + \beta q \qquad (3.21)$$

On the m^{th} equation and the k^{th} cycle a q is calculated from

$$q = \frac{G_{m,p}\big|_{k-1}}{G_{m,p}\big|_{k-2}} \qquad (3.22)$$

With the restrictions on equation 3.21 as follows

$$\text{If } 3 \leq |q| < 50, \text{ then } n_{m,k} = n_{m,k-1} \pm 0.3\beta \qquad (3.23)$$

depending on the sign of q

$$\text{Or if } |q| \geq 50, \text{ then } n_{m,k} = n_{m,k-1} \qquad (3.24)$$

While β is an input parameter to the program, it was found best to let it equal 0.1 or zero.

The a priori non-zeroing procedure is repeated until every

$$|G_{m,p}| \leq \alpha \qquad m = 1, 2 \text{ --- } M \qquad (3.25)$$

It is obvious from the method just described that the closer the initial estimated D values are to the actual solution D values, the quicker the computer will obtain the solution. Consequently, it becomes advantageous to prepare a library of solutions that can be consulted before a new problem is started. The library obtained appears in the Appendix.

The remaining variable is ϵ . Neither a one dimensional analysis nor Sauer's analysis allows a variation in mass flow in the nozzle. As it is physically possible to have a slightly changing nondimensionalized mass flow through a rocket nozzle, it should be possible, in theoretical calculations, to have varying mass flow conditions without violating the theoretical restrictions on the problem. Consequently, it should be possible to allow a variation of ϵ , so as to allow a variation in the nozzle mass flow.

The variation in ϵ will be physically restricted to a range of values between choking conditions and no supersonic flow. Violation of the physical restrictions in ϵ might show up in the numerical analysis.

There are two different ways in which too high, or too low an value will appear.

Physically, too high an ϵ value can show up as a supersonic flow stream with a subsonic pocket (FIG 2). What seems to happen is that as ϵ increases the mass flow also increases to the point where it is impossible to have a completely subsonic cross-section. Consequently, the Mach one line falls back to the axis instead of proceeding to the nozzle wall. With too low an ϵ value it is expected that the Mach one line will intersect the nozzle downstream of the minimum section. A downstream intersection would probably violate the entropy law.

One of the mathematic assumptions made quite early in the analysis was that

$$\left|D_{i+1}\right| \ll \left|D_i\right| \tag{3.26}$$

for any i value. Consequently, when the absolute value of D_2 approaches the value of D_1, it is necessary to question the mathematical validity of the resulting solution (this occurs when ϵ is too large or too small.) Applying a rule of thumb: A mathematically valid solution shall be considered one where

$$\left|D_2\right| \leq .20\ D_1 \tag{3.27}$$

The above statement, necessarily, restricts the range of ϵ . In the problems observed so far, the lower limit of ϵ has always been stipulated by the mathematical limitations, while the upper limit has been

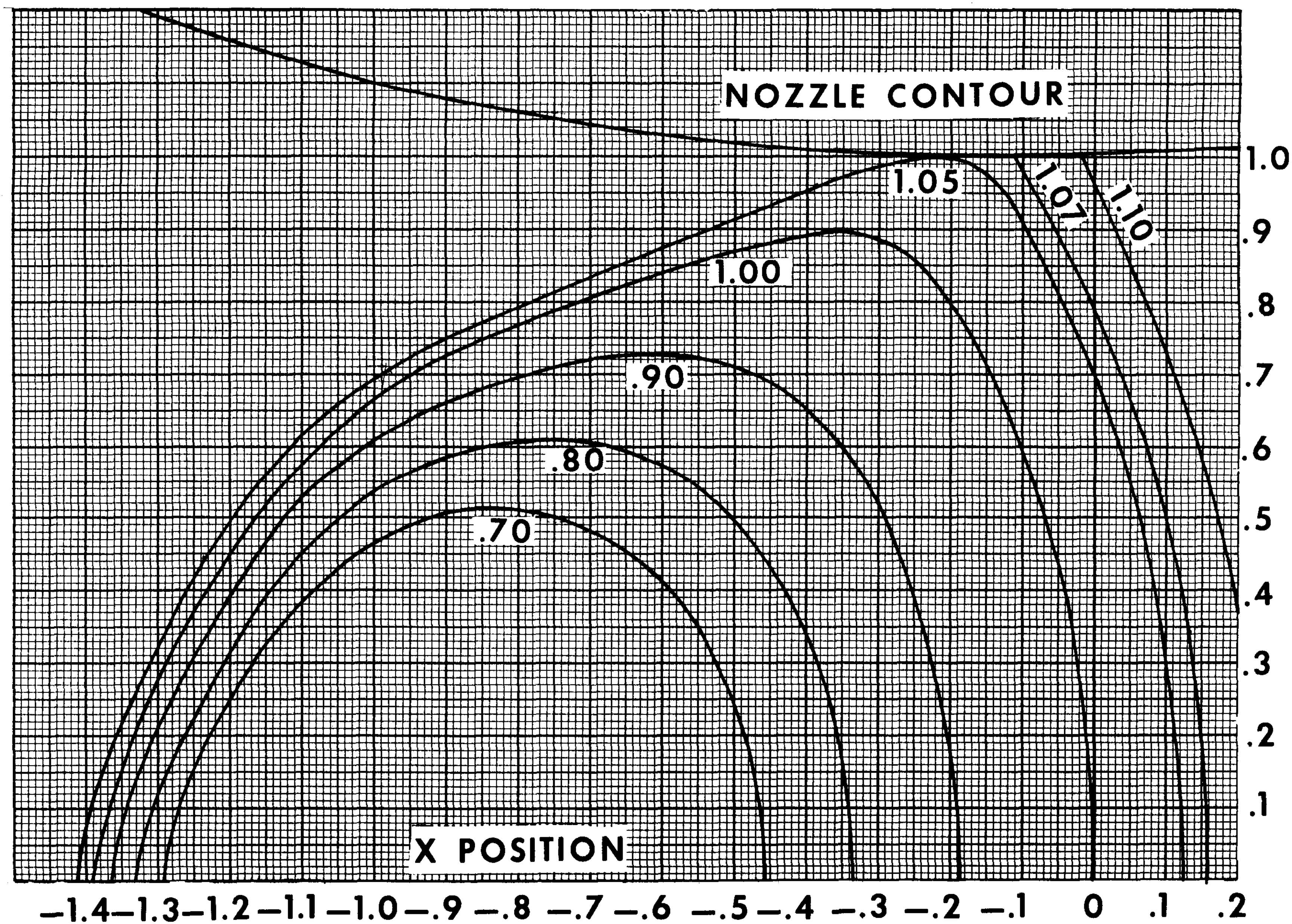

FIGURE 2 SUPERSONIC FLOW WITH SUBSONICPOCKET

stipulated either by the mathematic limitation or by the physical limitation.

Since ϵ is defined implicitly in terms of mass flow, it is necessary to solve the problem for various values of ϵ and then calculate the mass flow for each of these solutions. The ϵ values are then plotted versus mass flow. If the mass flow is prescribed, the corresponding ϵ is read from the plot, and a final solution can be obtained on that basis. A tentative plot of ϵ versus non dimensionzlied mass flow appears in the Appendix as FIG A-9.

4. ANALYSIS AND USE OF THE DATA FROM THE PRESENT METHOD

Once a solution is found, the problem has just begun. From the definition of the potential function, equation 3. 2, the perturbation velocity components u and v are:

$$u = u(x, y) = \sum_{i=0}^{i=M} \sum_{j=0}^{j=p} (i+1)A(i+1, 2j)\ x^i\ y^{2j} \qquad (4.1)$$

and

$$v = v(x, y) = \sum_{i=0}^{i=M} \sum_{j=0}^{j=p} 2(j+1)A(i, 2(j+1)\ x^i\ y^{2j+1} \qquad (4.2)$$

All of the $A_{i,\,2j}$'s are evaluated as pure numbers for the particular flow field for which a solution exists. Consequently, at any point in the flow field the Mach number, velocity components, and non-dimensionalized pressure and density can be evaluated.

If M^* is defined in terms of a parameter d_1 as

$$M^* = 1+d_1 \qquad (4.3)$$

then the equation

$$2u + u^2 + v^2 - d_1(2+d_1) = 0 \qquad (4.4)$$

describes a constant M* curve (where M* is given by equation 4. 3). Along the curve the velocity components and the flow angle (θ) are evaluated, the velocity components by equations 4. 1 and 4. 2 and the flow angle by

$$\theta = \tan^{-1} \frac{v}{1+u} \tag{4.5}$$

The initial equation to be satisfied was equation 3. 1. An idea can be obtained as to whether a point in the flow field is within the region of convergence of the solution or not by evaluating equation 3. 1 to see how close it comes to zero. Define that number as zero.

At the intersection of the constant M* curve and the nozzle, a check is made as to how close the solution satisfies the imposed boundary condition. Define the error of the difference between the stream slope and the nozzle wall slope as er, where

$$er = \frac{\left(y_s' - \frac{v}{1+u}\right)}{y_s'} \tag{4.6}$$

The constant M* line is used as input in the characteristics nozzle design program. Which M* line is chosen depends on the error printout values (zero and er).

Another type of output available is a straight line and is defined as starting on the axis at a given position and extending to x=0 and the nozzle wall.

At each increment in position along such a line the flow angle, Mach number and M* value are obtained.

Along the nozzle surface the M*, y_s', tan θ, Zero, and er values are calculated.

With all the available printouts, it is not too difficult a problem to analyze the resulting flow field and stipulate the region of applicability of the solution.

5. INPUT PARAMETERS

The present method assumes that the minimum section of the nozzle's surface can be expressed as a power series in x, i.e.:

$$y_s = \sum_{i=o}^{i=\infty} B_i \, x^i \tag{5.1}$$

(a repeat of equation 3.8).

Before a problem can be started for a given nozzle contour, a power series must be fitted to the nozzle surface. If the minimum section can be represented as an arc of a circle, one can use the equation

$$(x+\epsilon)^2 + \left[y - (1+\rho)\right]^2 = \rho^2 \tag{5.2}$$

where ρ and ϵ are defined as shown on FIG 1.

The above equation can be written as

$$y = f(x) = 1+\rho \left[1 - \sqrt{1 - \left(\frac{x+\epsilon}{\rho}\right)^2}\right] \tag{5.3}$$

A MacLaurin series can then be applied in the form

$$y = f(0) + f'(0)\, x + \left[\frac{f''(o)}{2!}\right] x^2 + \left[\frac{f'''(0)}{3!}\right] x^3 \quad \text{etc,} \tag{5.4}$$

where f(0), f'(0), and so on, can be evaluated from equation 5.3.

After differentiating equation 5.3 the necessary number of times, the B_i values are found as

$$B_0 = 1 + \rho \;\; (1 - g) \tag{5.5}$$

$$B_1 = \frac{1}{g}\left(\frac{\epsilon}{\rho}\right) \tag{5.6}$$

$$B_2 = \frac{1}{2}\left(\frac{1}{g}\right)\left(\frac{1}{\rho g^2}\right) \tag{5.7}$$

$$B_3 = \frac{1}{2g}\left(\frac{1}{\rho g^2}\right)^2\left(\frac{\epsilon}{\rho}\right) \tag{5.8}$$

$$B_4 = \frac{1}{8g}\left(\frac{1}{\rho g^2}\right)^3\left[1 + 4\left(\frac{\epsilon}{\rho}\right)^2\right] \tag{5.9}$$

$$B_5 = \frac{15}{8g}\left(\frac{1}{\rho g^2}\right)^4\left(\frac{\epsilon}{\rho}\right)\left[1 + \frac{4}{3}\left(\frac{\epsilon}{\rho}\right)^2\right] \tag{5.10}$$

$$B_6 = \frac{1}{16g}\left(\frac{1}{\rho g^2}\right)^5\left[1 + 12\left(\frac{\epsilon}{\rho}\right)^2 + 8\left(\frac{\epsilon}{\rho}\right)^4\right] \tag{5.11}$$

$$B_7 = \frac{5}{16g}\left(\frac{1}{\rho g^2}\right)^6\left(\frac{\epsilon}{\rho}\right)\left[1 + 4\left(\frac{\epsilon}{\rho}\right)^2 + \left(\frac{8}{5}\right)\left(\frac{\epsilon}{\rho}\right)^4\right] \tag{5.12}$$

$$B_8 = \frac{5}{128g}\left(\frac{1}{\rho g^2}\right)^7\left[1 + 24\left(\frac{\epsilon}{\rho}\right)^2 + 48\left(\frac{\epsilon}{\rho}\right)^4 + \frac{64}{5}\left(\frac{\epsilon}{\rho}\right)^6\right] \tag{5.13}$$

$$B_9 = \frac{35}{128g}\left(\frac{1}{\rho g^2}\right)^8\left(\frac{\epsilon}{\rho}\right)\left[1 + 8\left(\frac{\epsilon}{\rho}\right)^2 + \frac{48}{5}\left(\frac{\epsilon}{\rho}\right)^4 + \frac{64}{35}\left(\frac{\epsilon}{\rho}\right)^6\right] \tag{5.14}$$

and

$$B_{10} = \frac{7}{256g}\left(\frac{1}{\rho g^2}\right)^9\left[1 + 40\left(\frac{\epsilon}{\rho}\right)^2 + 160\left(\frac{\epsilon}{\rho}\right)^4 + 128\left(\frac{\epsilon}{\rho}\right)^6 + \frac{128}{7}\left(\frac{\epsilon}{\rho}\right)^8\right] \tag{5.15}$$

where

$$g = \sqrt{1 - \left(\frac{\epsilon}{\rho}\right)^2} \tag{5.16}$$

One of the purposes of this analysis is to be able to handle a non-constant radius of curvature solution. A subprogram to evaluate any nozzle minimum section in terms of a power series is in process.

DISCUSSION

1. MATHEMATICAL PROBLEM AREAS OF THE PROGRAM

Some mathematical areas to be clarified from the preliminary report (Ref 5) are:

a. Non-Unique Solutions

The form of the boundary condition equations (equations 3.12 and 3.17) is such that more than one solution for D_i is possible in the iteration scheme. While a certain number of them will be complex conjugates, more than one might be real. There are two possible physical explanations for more than one real solution. If a particular solution of a particular boundary equation would not iterate successfully, one might be investigating an unstable solution that could not exist physically for more than an instant of time. Needless to say, in the transonic flow region, unstable solutions have become infamous. Even if more than one complete problem solution set is found, it is suspected that all except one would represent a diverging series. A diverging series could quite possibly represent another unstable solution or possibly be the result of extraneous roots. A diverging solution, mathematically, cannot be considered a valid solution.

In most cases, the starting data represents a solution similar to the one desired. Consequently, it is felt that the iteration scheme solves for the closest one which is the right one. Also, if the program tries to come up with an extraneous solution, the iteration scheme might oscillate until the program found itself back at the valid case.

While the constant M^* lines are being constructed, the multi-value problem arises again. For any given y, there is more than one x value that lies on the constant M^* curve because of the nature of u and v, and conversely.

Experience with the program has shown that the transonic range is repeating itself, just as a sine or cosine wave does every 360°. The region of interest, however, is the vicinity of the throat of the nozzle.

Consequently, the program is looking for the particular curve that starts at the x axis, close to the origin, and propagates upward, toward y greater than one.

The program for calculating the M* lines is written so it will reject any point it finds that is further away from the previous point than four times the step size but will find the right point instead.

b. Region of Applicability

Let the region of applicability be defined as that portion of the flow field where the boundary condition and initial equation are satisfied to a desired degree of accuracy. The number of unknowns that need be considered is directly a function of the region of interest. If one was interested only in the region about the y axis (x quite small), then a small number of unknowns need be considered. On the other hand, the larger the highest value of x becomes, the more unknowns that are required to satisfy a given tolerance.

It is usual in transonic analyses to always include, inside the region of applicability, the Mach one line. But, in this analysis, to include the Mach one line necessitates use of quite a few more unknowns than would otherwise be needed. Consequently, with the proper restrictions, the analysis can be limited to four unknowns.

The constant M* line that would become the output of the program must be chosen prudently by analyzing the various error functions. Fortunately, it turns out that as the radius of curvature becomes smaller the preferred M* value becomes higher, producing better accuracy in the next analysis (characteristics).

In the case of a radius of curvature of 4.0, a plot of various M* lines for a M, p system of 4, 7 and 7, 9 (FIG 3) shows that the M*=1.0 lines vary considerably. But, the higher valued M* lines for the 4 and 7 unknown systems are similar (M*=1.10 through 1.15), with the lines closest to x=0, $y=y_s$ (M*=1.13) practically identical. Consequently, if the M*=1.13 line was used as the output data, only a 4 unknown system would be required. Along the M* = 1.13 line for

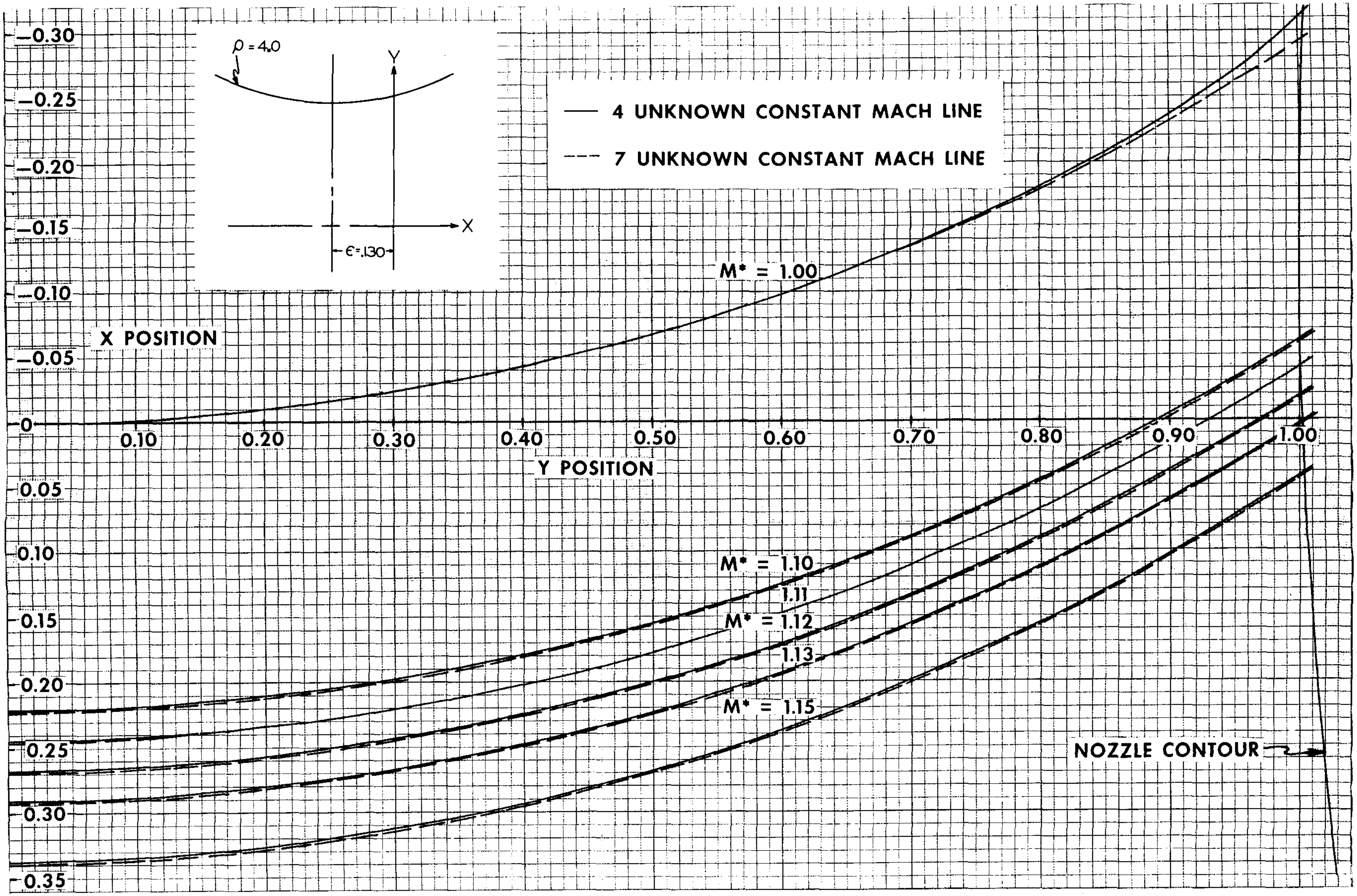

FIGURE 3 COMPARISON OF A FOUR UNKNOWN SYSTEM WITH A SEVEN UNKNOWN SYSTEM FOR A MINIMUM SECTION RADIUS OF CURVATURE OF FOUR

the 4 unknown case, the maximum zero value is .001 with er = +.00005. Therefore, $M^* = 1.13$ is a good choice of data to be used in the characteristic program.

The choice of the number of columns to use is more difficult than the choice of the number of unknowns. The main restriction to be satisfied is that the last column should have no effect on the region of interest. The problem complicates itself if too many columns are included because too many possible solutions are allowed. At a certain point the addition of another column introduces a solution that is slightly different than the solution required, leaving the computer with no way of choosing which solution is better. Experience has shown that 9 columns is more than sufficient but yet not too numerous to cause extraneous solutions.

The array of $A_{i,2j}$'s for a $\rho = 2.00$ and $\epsilon = .165$ (Table II) shows that the effect of the ninth column for an x value less than $\pm .2$ would be small compared to the other columns.

One of the initial assumptions made about the $A_{i,2j}$ series is that

$$\left.\begin{aligned} &\lim_{i\to\infty} A_{i,2j}X^i \leq \alpha \qquad j=1,2-----\infty \\ &\text{and} \\ &\lim_{j\to\infty} A_{i-2j} \leq \alpha \qquad i=1,2-----\infty; \end{aligned}\right\} \qquad (6.1)$$

in all cases checked the above has been true.

One of the minor problems in handling the program was the discovery of unbalanced arrays of $A_{i,2j}$. An unbalanced array can be defined as that array where the overemphasis on one of the variables produces a solution that has no physical significance. With the case of M=4 and p=3 the x variable is overemphasized. This occurs when the number of columns (p+1) is smaller than the number of rows (M+1). There exists the implicit assumption that the x variable exerts a greater effect than the y variable. However in the actual physical system, the converse is true; the boundary condition is applied where y>1.0 and x<1.0. Consequently, on physical grounds alone, one is justified in not trusting any solution where M>p.

2. UTILIZATION OF THE PRESENT PROGRAM

This analysis is just one of three parts of a procedure to design a given rocket nozzle. Initially, the propellants, chamber pressure, fuel to oxidizer ratio, minimum section contour radius, nozzle area ratio, and the rocket nozzle length are specified. A chemical equilibrium program is used to evaluate the chemical reactions and find the pressure and ratio of specific heats (γ) at various Mach numbers. The present analysis utilizes the equilibrium γ value to produce input data for a method of characteristics equilibrium program that then designs the diverging part of the nozzle for various exit pressures.

Using equilibrium data as an input parameter for a frozen flow program (Transonic Analysis) and then using these results in an equilibrium program (Characteristics Program) seems contradictory.

Although the total flow field is to be treated as an equilibrium problem, one part of that field (under certain conditions) can be treated as frozen flow if

$$\frac{\delta}{V} << R_t$$

where δ is the significant length parameter of the frozen flow region, (FIG 4), V is the lowest velocity through that region, and R_t is the fastest reaction rate of the chemical processes that might occur in the region. As V and R_t cannot be varied in the programs, the only choice is to reduce, as much as possible the size of δ. The region defined by δ is reduced by using, in the transonic analysis, the γ value from the chemical equilibrium program which corresponds, as closely as possible, to the Mach number of the most accurate constant Mach number line.

3. COMPARISON WITH OTHER ANALYSIS

It was predicted previously that this analysis and Sauer's would tend to agree more with one another as the radius of curvature increases. For the case of a radius of curvature of five, there is already a noticeable difference as can be seen in FIG 5.

There is a band of Mach numbers where a transonic analysis and the method of characteristics are both applicable. If the transonic data agreed with the characteristic data in this band one could draw the conclusion that the transonic method used was a valid one.

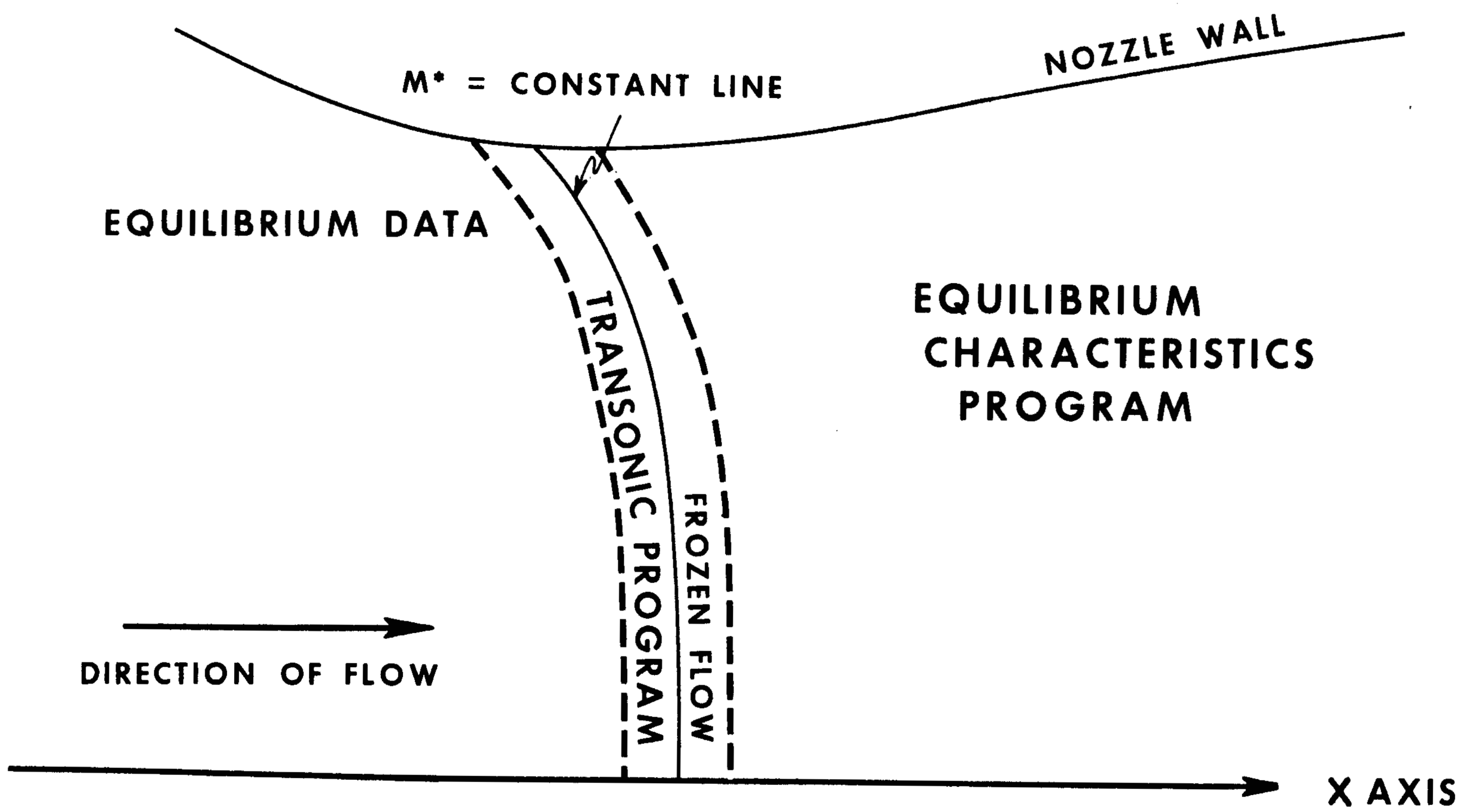

FIGURE 4 FLOW FIELD OF THE THREE PROGRAMS

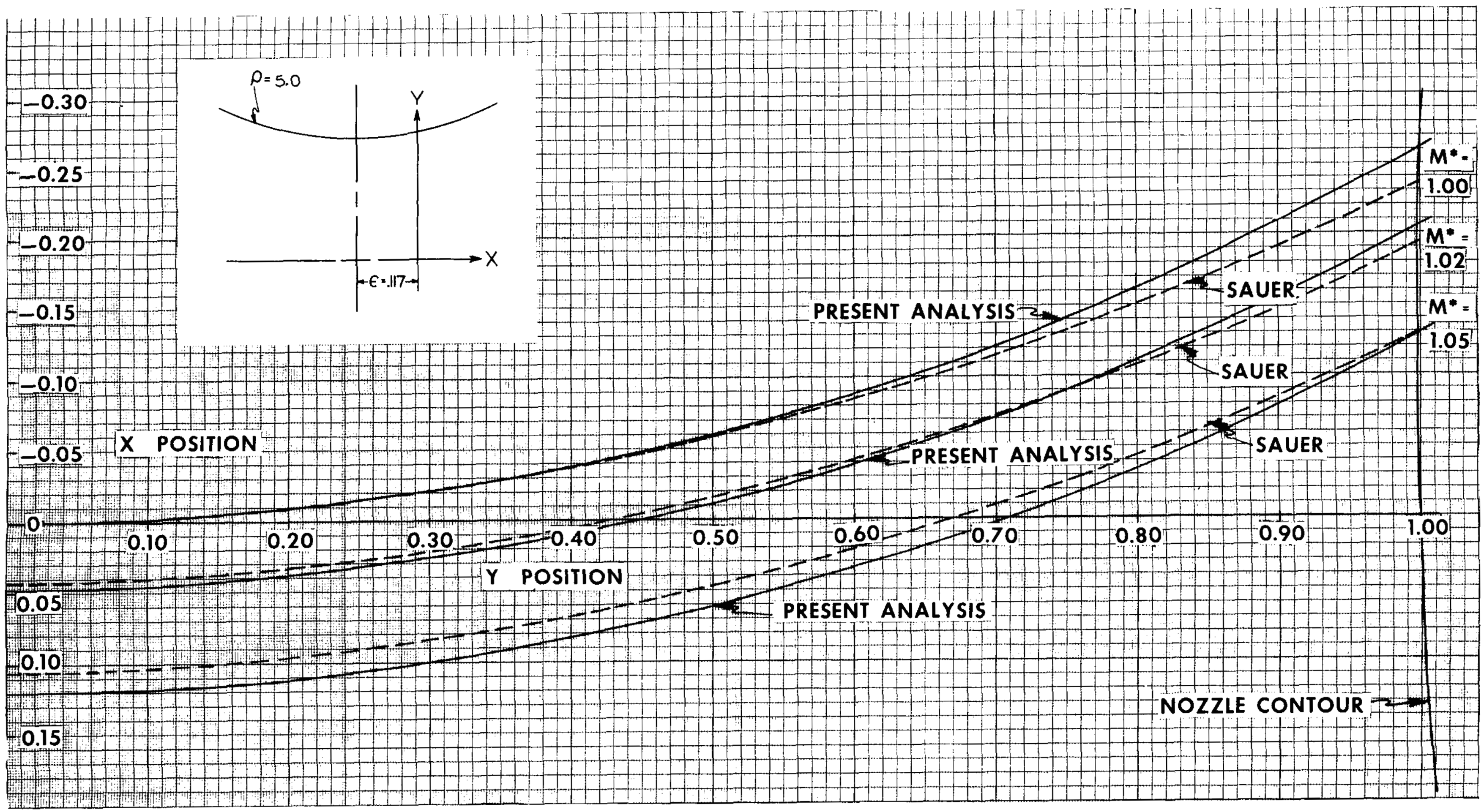

FIGURE 5 COMPARISON OF CONSTANT MACH LINES COMPUTED BY SAUER'S METHOD AND THE PRESENT ANALYSIS

A comparison was made utilizing a nozzle of radius of curvature of two. First, two different M* lines were calculated by Sauer's method as inputs for the characteristic program. The characteristic program proceeded, in each case, to calculate the Mach number distribution along the nozzle wall (FIG 6). The two different curves are symbolized by a ⊡ and a △ . From Sauer's equations, the hypothesized Mach number distribution was calculated - it is represented by ⌂ on FIG 6. The characteristic curves and the predicted curve do not agree too well with each other.

The procedure was repeated with the present transonic analysis calculating the input M* line for the characteristic program. The resulting Mach number distribution is symbolized by ⊙ on FIG 6. It can be seen on the figure that the Mach number distribution along the wall calculated using the present method (⊕) lies quite close to the one calculated by the method of characteristics (⊙). Consequently, the present transonic analysis seems to be accurate.

RECOMMENDATIONS

Before the conclusion of this study, the following has yet to be completed:

1. SIMPLIFIED ANALYSES

The two simplified analyses (Methods I and II) must be tried for various cases. Most important is deciding how many terms are to be used in their potential functions infinite series. After numerical cases are tried, both methods will be used in place of Sauer's method in the Allison program (6). Of course, a comparison between Sauer's analysis, Method I, Method II, and the present method will be made.

2. LITERATURE SEARCH

The literature search, now in progress, will be continued until all available report, books, and periodicals have been reviewed for useful methods and experimental data.

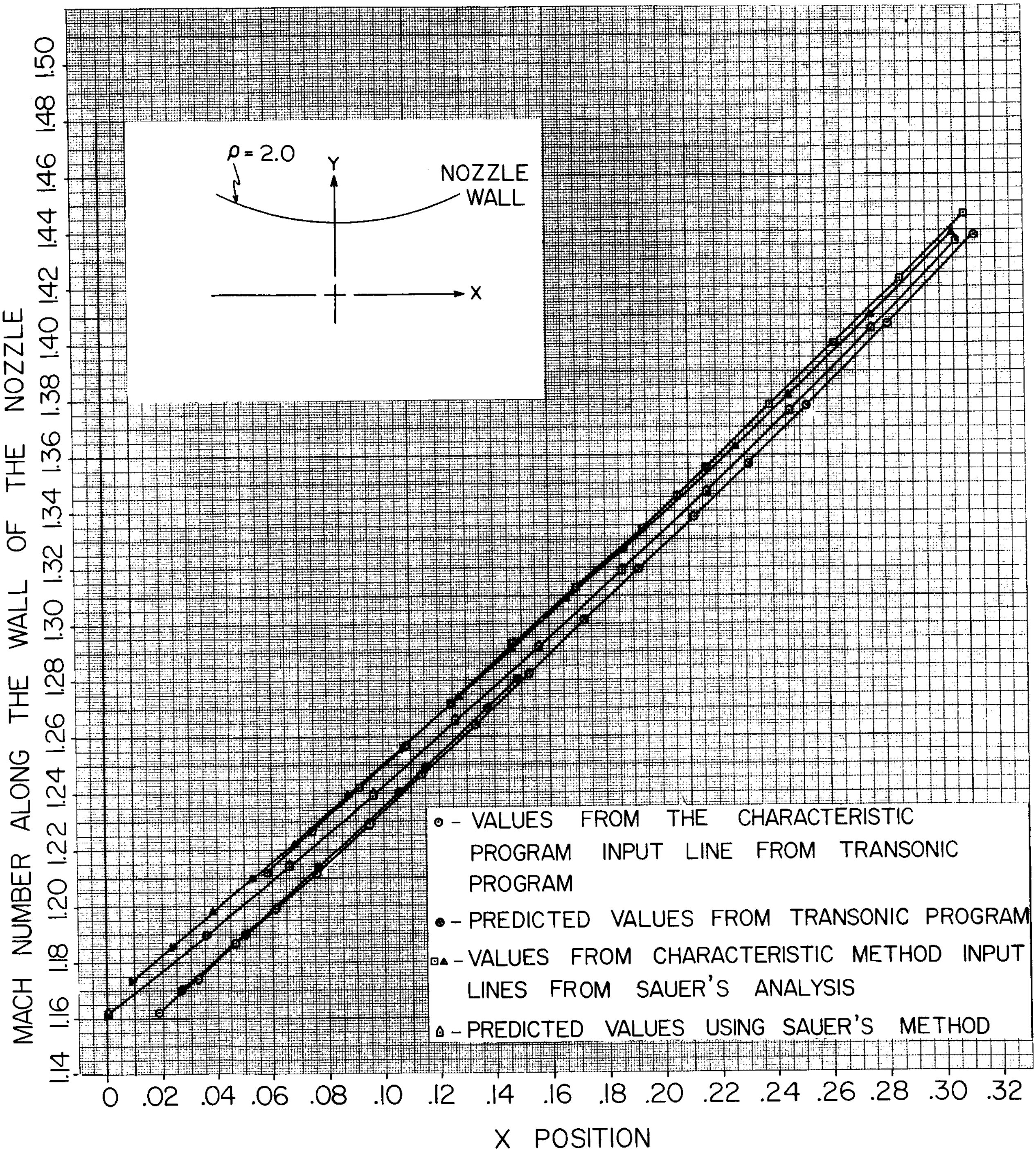

FIGURE 6 COMPARISON OF PREDICTED MACH NUMBER DISTRIBUTION WITH ACTUAL DISTRIBUTION

3. PRESENT METHOD

The $A_{i,2j}$ sub-routine must be re-programmed as mentioned in Section III. Afterwards, it is expected that other changes will become apparent that can further shorten the running time of the program.

After these changes are made, the ρ value will be pushed as low as the program will go. The resulting listing of all the unknowns, for particular ρ and ϵ values, will become part of the program for use with a routine for choosing initial values of the unknowns to start the program. Most of the solutions obtained are based on a γ of 1.20; to efficiently handle other γ values further work will have to be done. Increments of the unknowns, (not necessarily small) versus increments in γ values, must be obtained for various ρ and ϵ values.

The nozzle parameter sub-routine will soon be revised so it can handle non-constant radius of curvature cases. After no more program changes are necessary the programmer will streamline and polish it up, so that anyone can utilize it with a minimum of instructions. Such instructions will appear in the final report.

4. FINAL REPORT

The final report will contain all of the material presented herein, including revisions and changes. Examples and trends discovered in the course of the evaluation of the analysis will be included. Derivations of the equations used, as well as the operational methods, will appear as appendixes.

The report will contain a write-up of all the programs, with a section written by the programmer presenting symbolic write-ups, instructions for use, and flow charts of all the sub-programs in use.

APPENDIX

LIBRARY OF SOLUTIONS

The solutions achieved to date appear in two forms, the unknowns versus the radius of curvature of the minimum section, ρ, and the unknowns versus the distance, ϵ, of the Mach one line from the minimum section.

The last curve (FIG A-9) represents the various non dimensionalized mass flows (for constant ρ values) versus ϵ values. The relationship between the mass flow ($\mathring{m}$) and the non dimensionalized mass flow ($\widetilde{\dot{m}}$) is

$$\mathring{m} = y_o^2 \; \rho_o a^* \; \widetilde{\dot{m}},$$

where ρ_o is the isentropic stagnation density.

TABLE I

$A_{i,2}$ CONSIDERING ONLY 4 UNKNOWNS

Coeff.	Power of:					
	D_1	D_2	D_3	D_4	$\gamma+1$	Γ
$A_{1,2}$						
1	2	0	0	0	1	0
$A_{2,2}$						
4.5	1	1	0	0	1	0
1.0	3	0	0	0	1	0
2.0	3	0	0	0	2	1
$A_{3,2}$						
8.0	1	0	1	0	1	0
6.0	2	1	0	0	1	0
4.5	0	2	0	0	1	0
12.0	2	1	0	0	2	1
4.0	4	0	0	0	2	1
4.0	4	0	0	0	3	2
$A_{4,2}$						
12.5	1	0	0	1	1	0
10.0	2	0	1	0	1	0
11.25	1	2	0	0	1	0
15.0	0	1	1	0	1	0
20.0	2	0	1	0	2	1
30.0	3	1	0	0	2	1
22.5	1	2	0	0	2	1
30.0	3	1	0	0	3	2
2.0	5	0	0	0	2	1
12.0	5	0	0	0	3	2
8.0	5	0	0	0	4	3
$A_{5,2}$						
15.0	2	0	0	1	1	0
36.0	1	1	1	0	1	0
22.5	0	1	0	1	1	0
30.0	2	0	0	1	2	1
48.0	3	0	1	0	2	1

TABLE I (CONTINUED)
$A_{i,2}$ CONSIDERING ONLY 4 UNKNOWNS

Coeff. (Cont.)	Power of:					
	D_1	D_2	D_3	D_4	γ	Γ
$A_{5,2}$						
6.75	0	3	0	0	1	0
81.0	2	2	0	0	2	1
12.0	0	0	2	0	1	0
72.0	1	1	1	0	2	1
48.0	3	0	1	0	3	2
18.0	4	1	0	0	2	1
108.0	4	1	0	0	3	2
13.5	0	3	0	0	2	1
81.0	2	2	0	0	3	2
72.0	4	1	0	0	4	3
12.0	6	0	0	0	3	2
32.0	6	0	0	0	4	3
16.0	6	0	0	0	5	4
$A_{6,2}$						
52.5	1	1	0	1	1	0
28.0	1	0	2	0	1	0
70.0	3	0	0	1	2	1
31.5	0	2	1	0	1	0
252.0	2	1	1	0	2	1
35.0	0	0	1	1	1	0
105.0	1	1	0	1	2	1
70.0	3	0	0	1	3	2
28.0	4	0	1	0	2	1
168.0	4	0	1	0	3	2
94.5	1	3	0	0	2	1
63.0	3	2	0	0	2	1
378.0	3	2	0	0	3	2
56.0	1	0	2	0	2	1
63.0	0	2	1	0	2	1
252.0	2	1	1	0	3	2
112.0	4	0	1	0	4	3
126.0	5	1	0	0	3	2

TABLE I (CONTINUED)
$A_{i,2}$ CONSIDERING ONLY 4 UNKNOWNS

Coeff. (Cont.)	Power of: D_1	D_2	D_3	D_4	$\gamma+1$	Γ
$A_{6,2}$						
336.0	5	1	0	0	4	3
94.5	1	3	0	0	3	2
252.0	3	2	0	0	4	3
168.0	5	1	0	0	5	4
4.0	7	0	0	0	3	2
48.0	7	0	0	0	4	3
80.0	7	0	0	0	5	4
32.0	7	0	0	0	6	5
$A_{7,2}$						
80.0	1	0	1	1	1	0
45.0	0	2	0	1	1	0
36.0	2	1	0	1	2	1
48.0	0	1	2	0	1	0
192.0	2	0	2	0	2	1
40.0	4	0	0	1	2	1
240.0	4	0	0	1	3	2
432.0	1	2	1	0	2	1
192.0	3	1	1	0	2	1
1152.0	3	1	1	0	3	2
25.0	0	0	0	2	1	0
160.0	1	0	1	1	2	1
90.0	0	2	0	1	2	1
360.0	2	1	0	1	3	2
160.0	4	0	0	1	4	3
192.0	5	0	1	0	3	2
512.0	5	0	1	0	4	3
108.0	2	3	0	0	2	1
40.5	0	4	0	0	2	1
648.0	2	3	0	0	3	2
540.0	4	2	0	0	3	2
1440.0	4	2	0	0	4	3

TABLE I (CONCLUDED)
$A_{i,2}$ CONSIDERING ONLY 4 UNKNOWNS

Coeff. (Conc.)	Power of: D_1	D_2	D_3	D_4	$\gamma + 1$	Γ
$A_{7,2}$						
96.0	0	1	2	0	2	1
192.0	2	0	2	0	3	2
432.0	1	2	1	0	3	2
768.0	3	1	1	0	4	3
256.0	5	0	1	0	5	4
48.0	6	1	0	0	3	2
576.0	6	1	0	0	4	3
960.0	6	1	0	0	5	4
40.5	0	4	0	0	3	2
432.0	2	3	0	0	4	3
720.0	4	2	0	0	5	4
384.0	6	1	0	0	6	5
32.0	8	0	0	0	4	3
160.0	8	0	0	0	5	4
192.0	8	0	0	0	6	5
64.0	8	0	0	0	7	6

TABLE II

$A_{i,\ 2j}$ for the case of $\rho = 2.00$ and $\epsilon = .165$ with the tolerance (α) equal to 10^{-4}

i \ 2j	0	2	4	6	8	10	12	14	16	18
0	0	0	+.0133	+.00416	+.00135	+.000510	+.000211	+.0000917	+.0000408	+.0000183
1	0	+.172	+.0397	+.0108	+.00392	+.00162	+.000707	+.000305	+.000126	+.0000465
2	+.279	+.0489	+.00527	+.00200	+.00854	+.000131	-.000223	-.000343	-.000341	-.000287
3	-.00719	-.0371	-.0183	-.0105	-.00730	-.00544	-.00409	-.00303	-.00218	-.00151
4	-.00828	-.00603	-.00742	-.00901	-.00897	-.00785	-.00630	-.00465	-.00312	-.00182

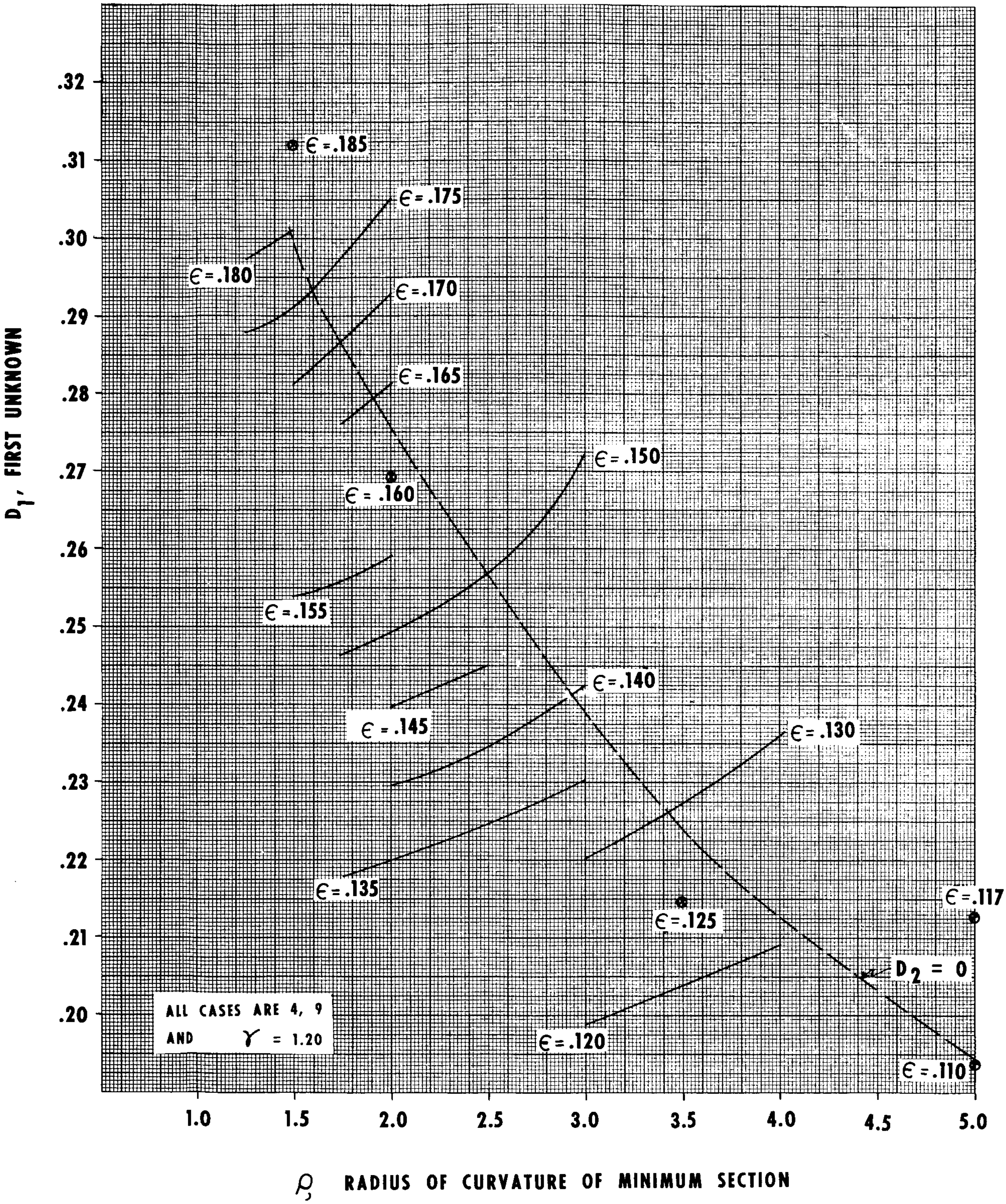

FIGURE A1 FIRST UNKNOWN VERSUS RADIUS OF CURVATURE OF MINIMUM SECTION FOR CONSTANT VALUES OF THE DISTANCE BETWEEN MINIMUM SECTION AND MACH ONE LINE

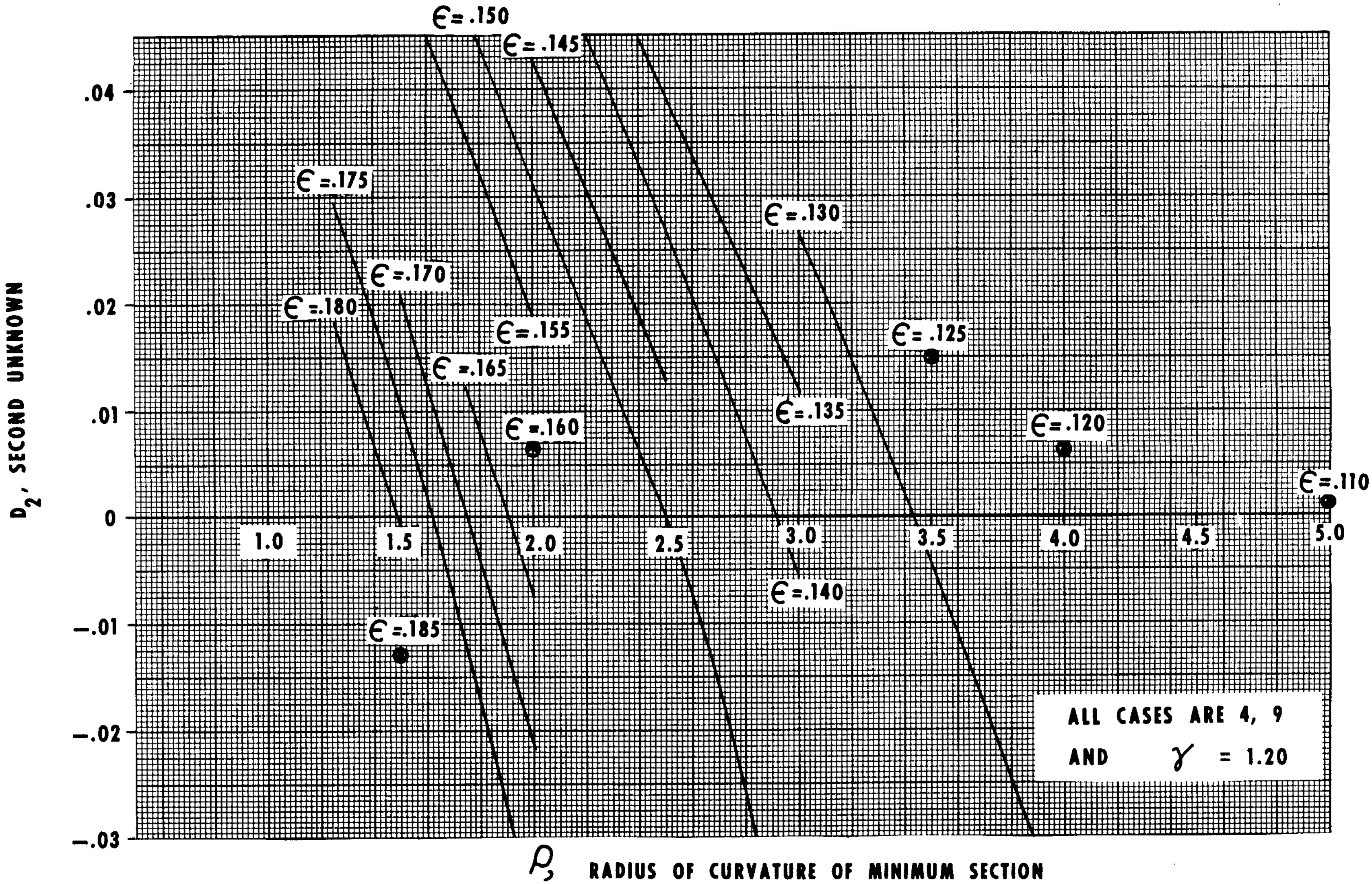

FIGURE A2 SECOND UNKNOWN VERSUS RADIUS OF CURVATURE OF MINIMUM SECTION FOR CONSTANT VALUES OF THE DISTANCE BETWEEN MACH ONE LINE AND MINIMUM SECTION

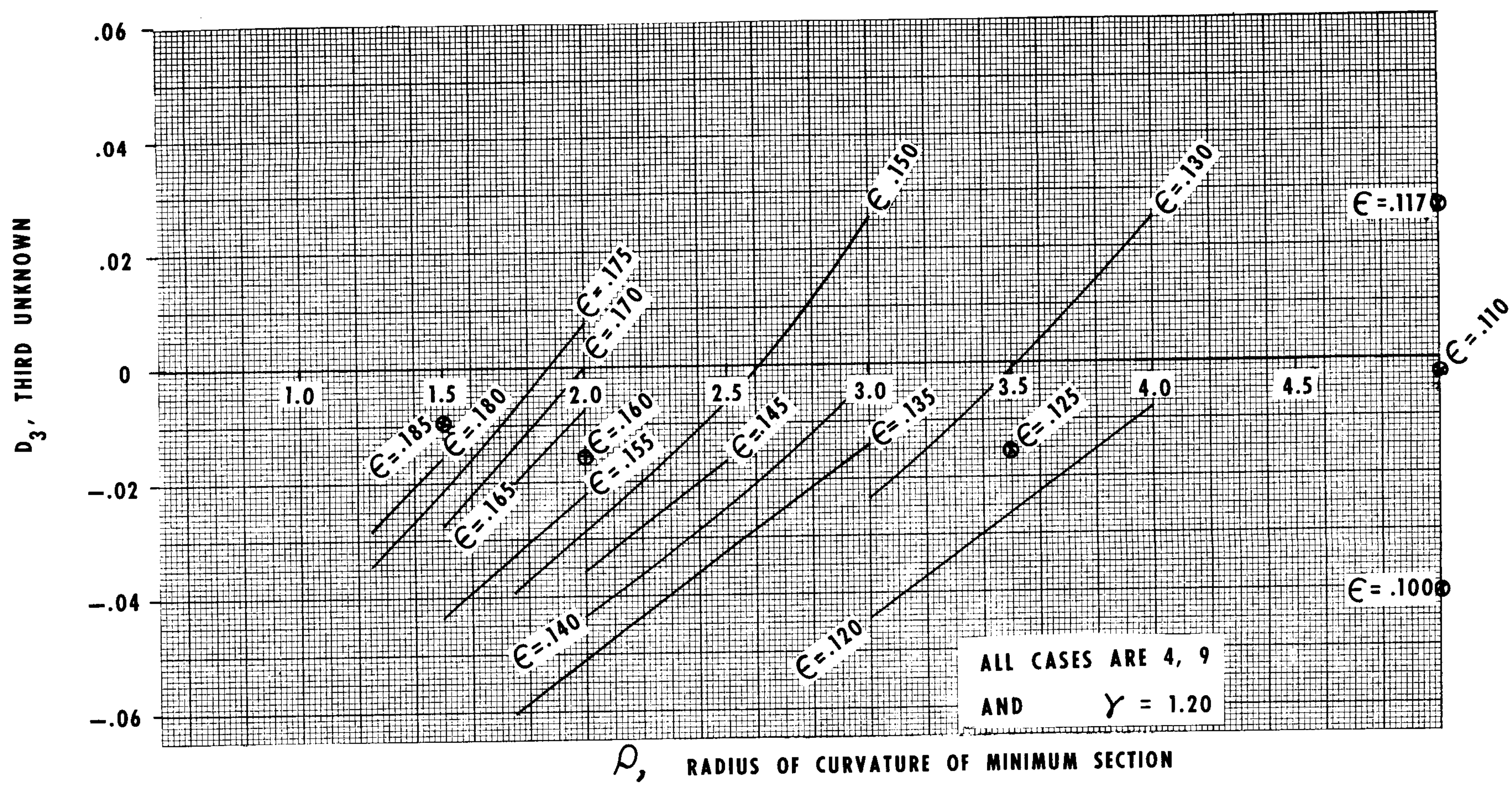

FIGURE A3 THIRD UNKNOWN VERSUS RADIUS OF CURVATURE OF MINIMUM SECTION FOR CONSTANT VALUES OF THE DISTANCE BETWEEN MACH ONE LINE AND MINIMUM SECTION

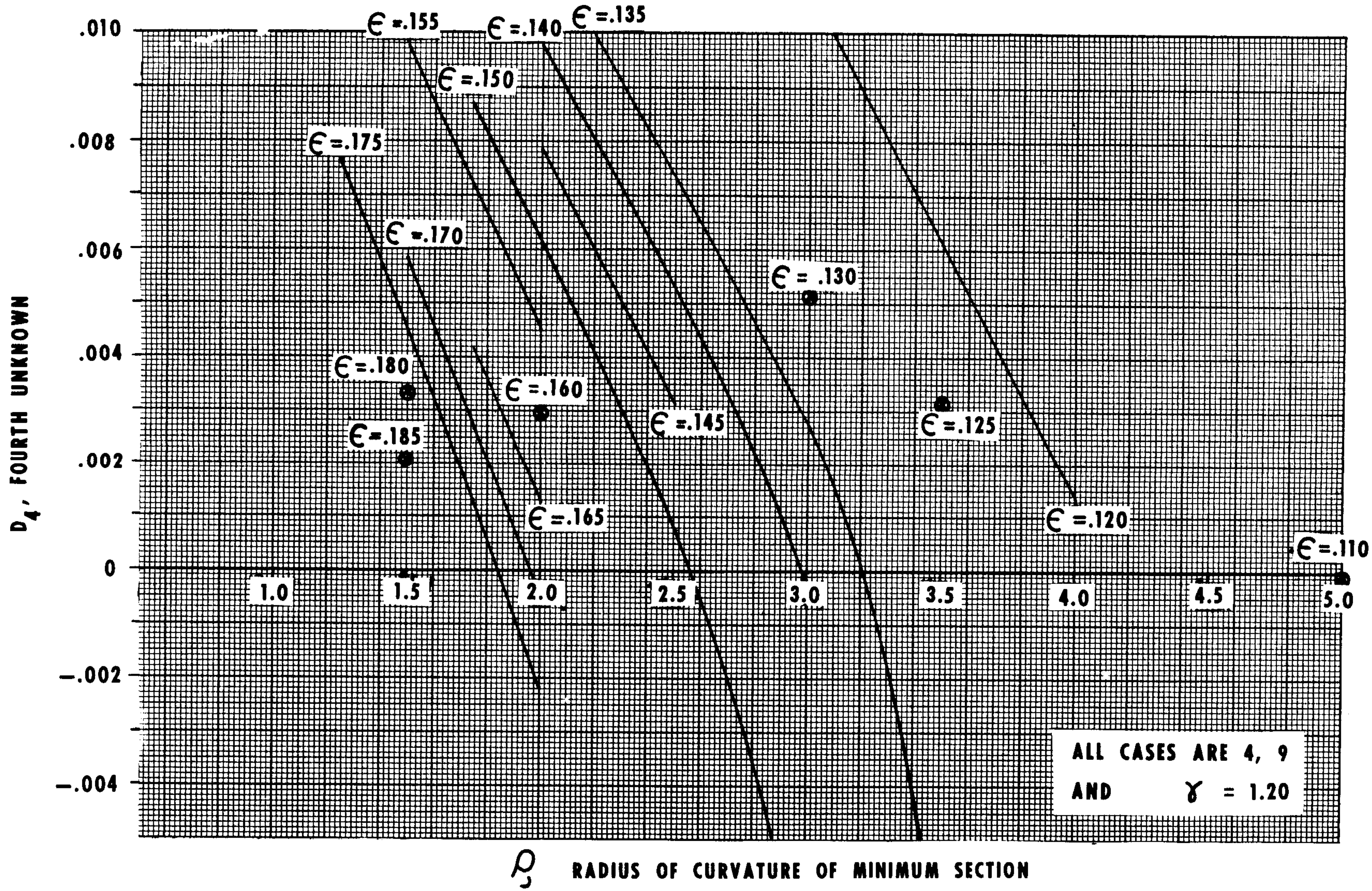

FIGURE A4 FOURTH UNKNOWN VERSUS RADIUS OF CURVATURE OF MINIMUM SECTION FOR CONSTANT VALUES OF THE DISTANCE BETWEEN MACH ONE LINE AND MINIMUM SECTION

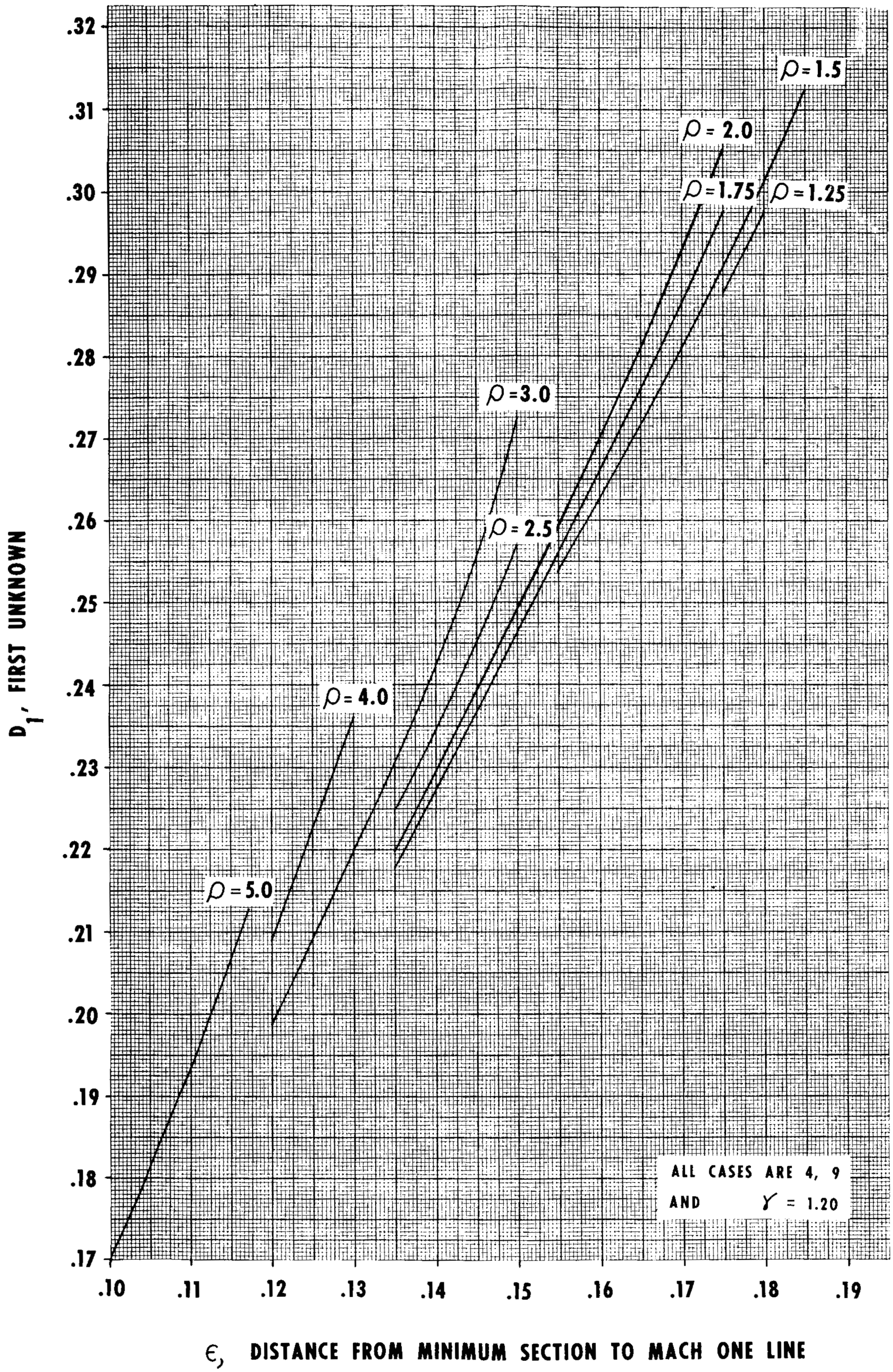

FIGURE A5 FIRST UNKNOWN VERSUS DISTANCE FROM MINIMUM SECTION TO MACH ONE LINE FOR CONSTANT VALUES OF RADIUS OF CURVATURE OF MINIMUM SECTION

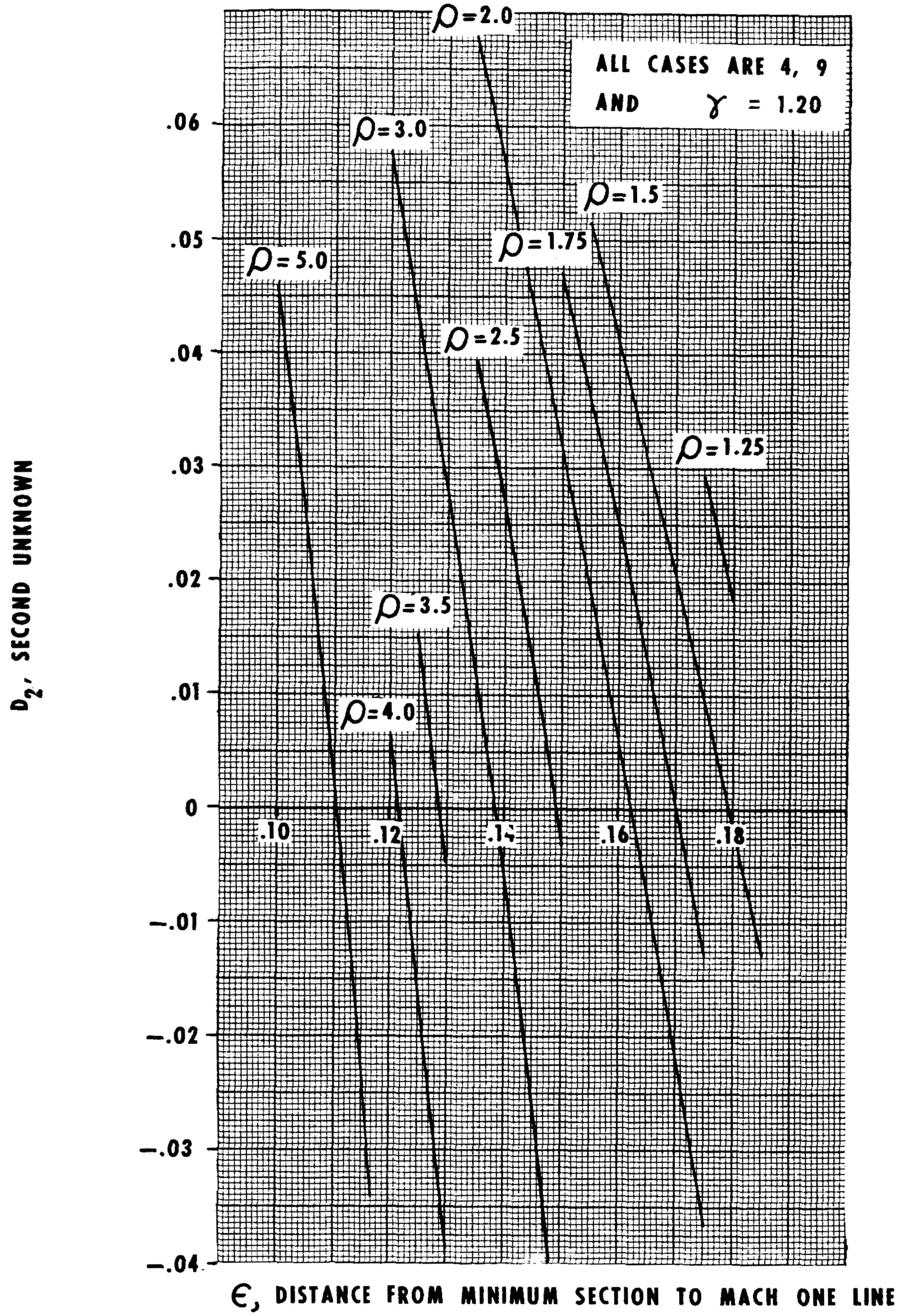

FIGURE A6 SECOND UNKNOWN VERSUS DISTANCE FROM MINIMUM SECTION TO MACH ONE LINE FOR CONSTANT VALUES OF RADIUS OF CURVATURE OF MINIMUM SECTION

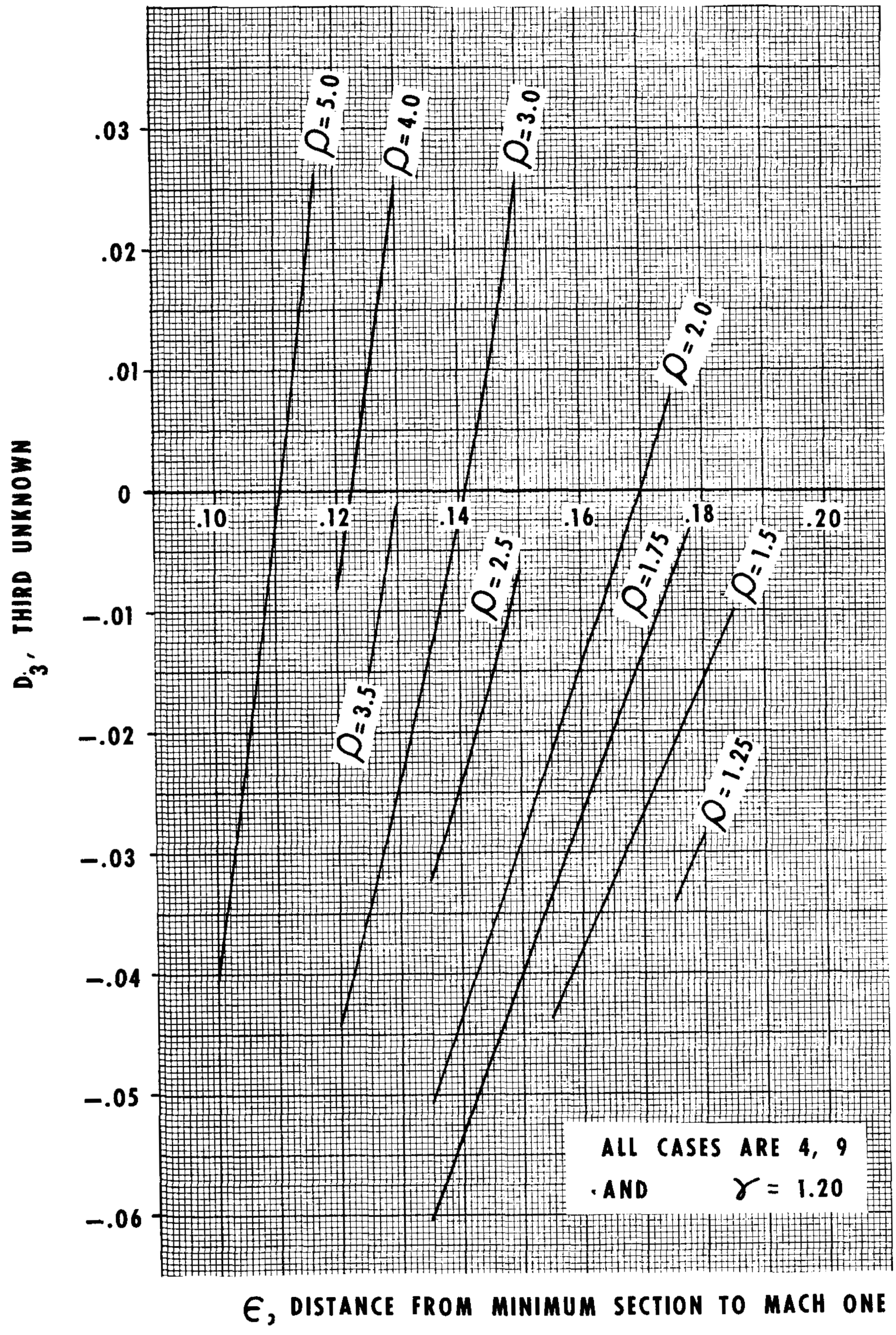

FIGURE A7 THIRD UNKNOWN VERSUS DISTANCE FROM MINIMUM SECTION TO MACH ONE LINE FOR CONSTANT VALUES OF RADIUS OF CURVATURE OF MINIMUM SECTION

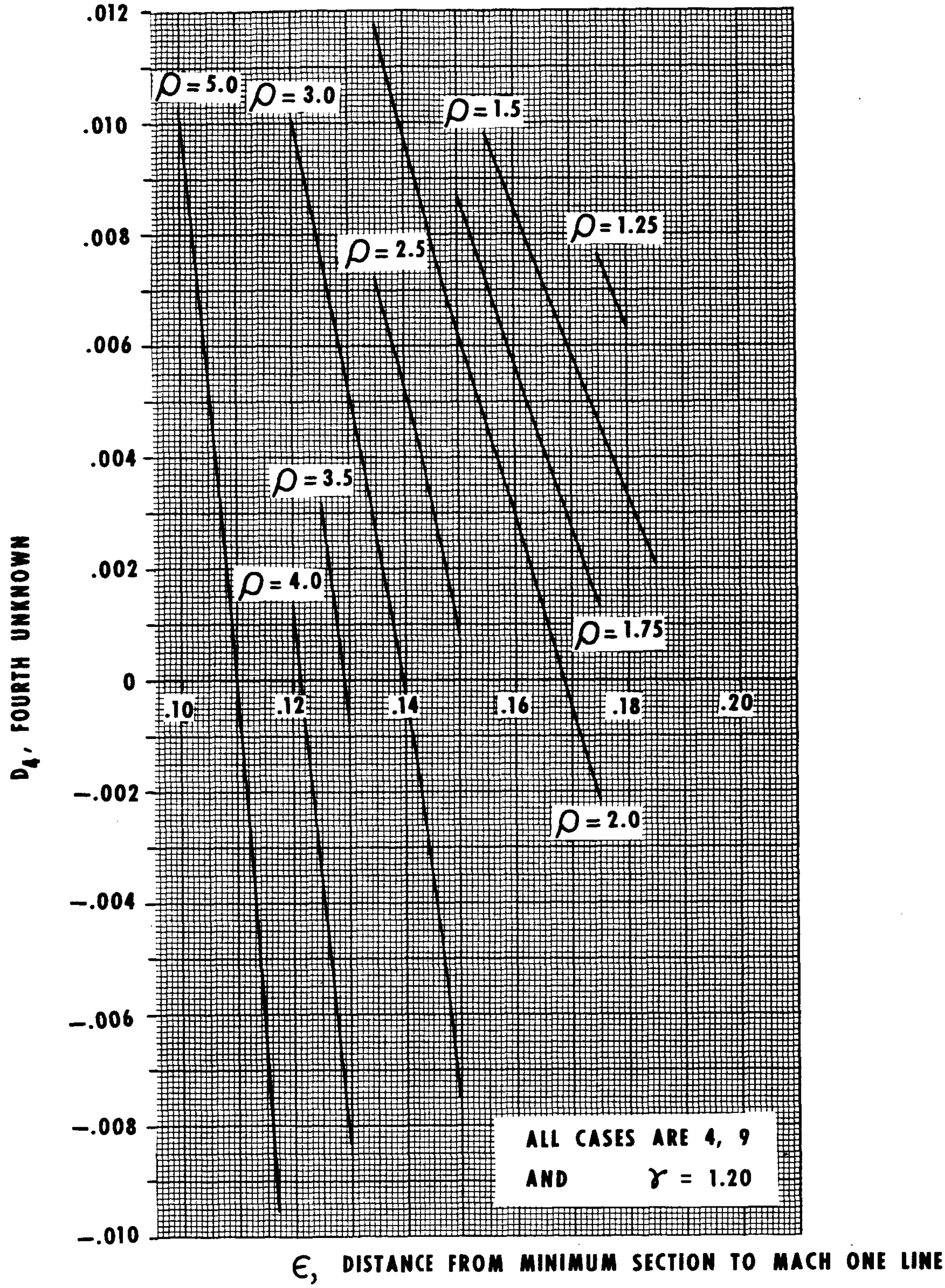

FIGURE A8 FOURTH UNKNOWN VERSUS DISTANCE FROM MINIMUM SECTION TO MACH ONE LINE FOR CONSTANT VALUES OF RADIUS OF CURVATURE OF MINIMUM SECTION

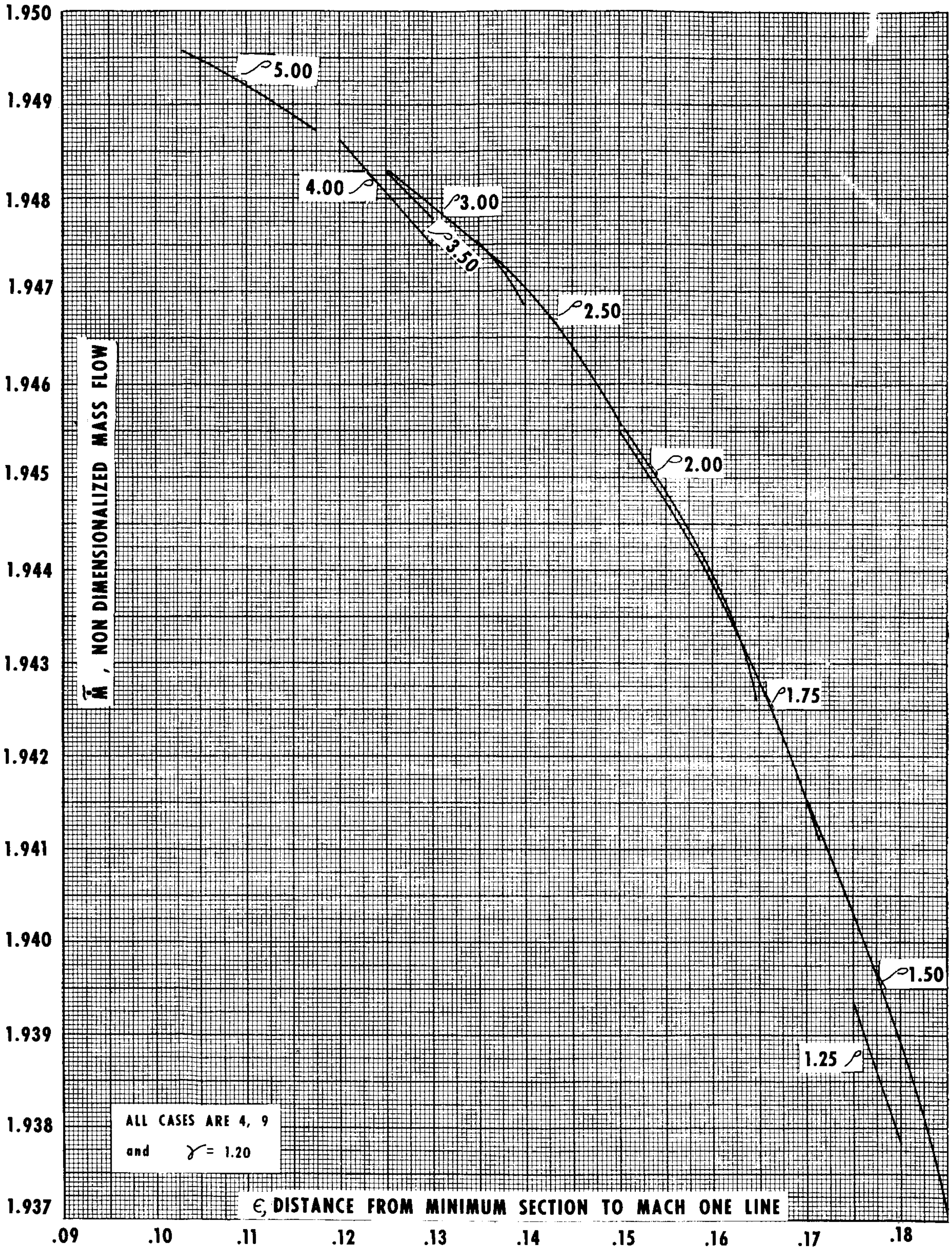

FIGURE A-9 VARIATION IN NON DIMENSIONALIZED MASS FLOW VERSUS DISTANCE BETWEEN MINIMUM SECTION AND MACH ONE LINE FOR VARIOUS CONSTANT VALUES OF THE RADIUS OF CURVATURE OF THE MINIMUM SECTION

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INTERIM REPORT ON METHODS OF DETERMINING THE TRANSONIC FLOW FIELD IN AN AXIALLY SYMMETRIC ROCKET NOZZLE

By R. S. Mendelson

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J. Thomson

J. THOMSON
Chief, Engine Systems Branch

H. G. Paul

H. G. PAUL
Chief, Propulsion Division

F. B. Cline

F. B. CLINE
Acting Director, Propulsion and Vehicle Engineering Laboratory

TECHNICAL MEMORANDUM X-53084

INTERIM REPORT ON METHODS OF DETERMINING THE TRANSONIC FLOW FIELD IN AN AXIALLY SYMMETRIC ROCKET NOZZLE

By

R. S. Mendelson

George C. Marshall Space Flight Center

Huntsville, Alabama

ABSTRACT

Starting data are provided for internal characteristic programs pertaining to axially symmetric rocket nozzles. Several approximate methods are examined before a complete analysis is attempted.

Using perfect frozen gas considerations, the potential function in the region of interest is approximated by a double power series in the space variables. The potential equation of motion, with the inviscid boundary condition, produces non-linear simultaneous equations.

The non-linear equations are handled uniquely, and the results are utilized to describe the flow field. Different methods of checking the validity of the results are applied, and comparisons are made with other analyses.

The remaining difficulties in the development of a production program that can handle arbitrary minimum section geometry and slightly varying mass flows are discussed.

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