On Generalized Dynamical Systems Defined by Contingent Equations.

by

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1. Introduction.

Already in 1934 and 1936, A. Marchaud [3], [4] and S. C. Zaremba [8] generalized the notion of ordinary differential equations, considering at any point x in euclidean n-space, not one but a whole set of possible tangent directions defining a family of trajectories. This is related to the control systems defined, for example, by a differential equation $x' = f(x, t, u)$, where u is a control parameter which can be chosen more or less arbitrarily. The above mentioned set of possible tangent directions was called "contingent", and the corresponding equation, "contingent equation".

Dynamical systems as a generalization of solutions of ordinary differential equations are already a classical subject in the mathematical literature. Its systematic generalization to systems with no unique solutions ("generalized dynamical systems") was developed by Barbashin [1] and the author [5]. Related is also the work of Bushaw about polysystems [2].

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Already Marchaud and Zaremba proved that a contingent equation defines a family of trajectories and, in that sense, defines a generalized dynamical system. The present paper treats systematically the problem on how a contingent equation defines a generalized dynamical system, giving for example existence and uniqueness theorems, which were not given before because this subject was not considered from this point of view.

The notation used in this paper is the following.

The "state variables" $x, y, \ldots$ will be points of the real n-dimensional euclidean space $X = R^n$, considered as vector space over the reals; $\|x\|$ will designate the usual norm.

The variable $t \in R$ will be called time. Curves $x(t)$ will usually be considered in R^{n+1} - space, its points $(x, t) \in X \times R$.

The distance between two points $x, y \in X$ is $\|x - y\|$. The distance between a point $x \in X$ and a set $A \subset X$ is defined by

$$\rho(x, A) = \rho(A, x) = \inf \{\|x - a\|; \ a \in A\}.$$

For two sets $A, B \subset X$, the "separation of A from B" is defined by

$$\rho^*(A, B) = \sup \{\rho(a, B); \ a \in A\}.$$

The "distance between A and B" is defined by

$$\rho(A, B) = \rho(B, A) = \max \{\rho^*(A, B), \rho^*(B, A)\}.$$

For compact sets, this distance is always finite and defines the Hausdorff metric.

For $r > 0$, the r-neighborhood of a set $A \subset X$ is $V_r(A) = \{x \in X; \rho(x, A) < r\}$.

The variable set $C(\alpha) \subset X$, where α belongs to some topological space, is said to be: a) continuous at α_o, if for every $\delta > 0$ there is some neighborhood of α_o, say V, such that for all $\alpha \in V$:

$$\rho(C(\alpha), C(\alpha_o)) < \delta;$$

b) upper semicontinuous at α_o if similarly, for all $\alpha \in V$:

$$\rho^*(C(\alpha), C(\alpha_o)) < \delta.$$

2. <u>Generalized dynamical systems</u>.

A usual dynamical system is given by a function $F(x_o, t_o, t) : X \times R^2 \to X$, which describes the movement of a point $x(t)$ from its initial position $x_o = x(t_o)$. This corresponds to physical systems whose evolution is uniquely determined by its initial conditions.

A generalized dynamical system (g.d.s.), is similarly given by a function $F(x_o, t_o, t)$ which to every $x_o \in X$, $t_o \in R$, $t \geq t_o$ makes correspond a set $F(x_o, t_o, t) \subset X$. This function F is called the "attainability function", because it represents states which are possible to reach from

(x_o, t_o) at time t. Therefore, generalized dynamical systems correspond to systems whose evolution is not uniquely determined by the initial condition only, for example control systems where some control action determines the evolution of the system.

The basic axioms assumed for the function $F(x_o, t_o, t)$ are the following.

I) $F(x_o, t_o, t)$ is a closed non-empty subset of X, defined for every $x_o \in X$; $t_o, t \in R$; $t \geq t_o$.

II) $F(x_o, t_o, t_o) = \{x_o\}$ for every $x_o \in X$, $t_o \in R$.

III) $t_o \leq t_1 \leq t_2$ implies

$$F(x_o, t_o, t_2) = \bigcup_{x_1 \in F(x_o, t_o, t_1)} F(x_1, t_1, t_2).$$

IV) Given $x_1 \in X$; $t_o, t_1 \in R$; $t_o \leq t_1$, there exists $x_o \in X$ such that

$$x_1 \in F(x_o, t_o, t_1).$$

V) $F(x_o, t_o, t)$ is continuous in t: given x_o, t_o, t_1; $t_o \leq t_1$, and $\epsilon > 0$, there is a $\delta > 0$ such that $\rho(F(x_o, t_o, t), F(x_o, t_o, t_1)) < \epsilon$ for all $|t - t_1| < \delta$ (being $t \geq t_o$).

VI) $F(x_o, t_o, t)$ is upper semicontinuous in (x_o, t_o), uniformly in any finite interval $t \in [t_1, t_2]$, $t_o \leq t_1 \leq t_2$: given x_o, t_o, t_1, t_2 and

$\epsilon > 0$, there is a $\delta > 0$ such that

$$\rho^*(F(x_0', t_0', t), F(x_0, t_0, t)) < \epsilon$$

for all x_0', t_0', t satisfying

$$\|x_0 - x_0'\| < \delta, \quad |t_0 - t_0'| < \delta, \quad t_1 \leq t \leq t_2.$$

From these axioms, many properties follow which are in agreement with what by intuition should be expected from a g.d.s.

In particular, a trajectory of a g.d.s. is defined to be a curve $x = \varphi(t)$ such that for every $t_1 \leq t_2$, $\varphi(t_2) \in F(\varphi(t_1), t_1, t_2)$. Intuitively this corresponds to the fact that $\varphi(t_2)$ is reachable from $\varphi(t_1)$ at the corresponding time.

The following fact can be proved for g.d. systems: if $x_2 \in F(x_1, t_1, t_2)$, there exists some trajectory $\varphi(t)$ of the g.d.s., such that $\varphi(t_1) = x_1$, $\varphi(t_2) = x_2$ (i.e. there is a trajectory going from x_1 to x_2 in the corresponding time interval).

Another important consequence of the axioms is the fact that the attainability function $F(x_0, t_0, t)$ can be extended backwards (for $t < t_0$), preserving almost all properties.

Generalized dynamical systems were introduced by Barbashin [1]. For proofs and more details the reader is referred to [5]. In this last reference the backwards extension of F is denoted by G, but in the present paper the same letter F is used.

3. Contingent equations.

Definition 3.1. Given a curve $x(t)$, the set of all $y \in X$ such that there exists a sequence t_i $(i = 1,2,3, \dots)$, $t_i \to t_o$, $t_i \neq t_o$, and $\lim_{i \to \infty} \frac{x(t_i) - x(t_o)}{t_i - t_o} = y$ is called the "contingent" (or "contingent derivative") of $x(t)$ at the point $x(t_o)$. It will be designated by $D^*x(t)\Big|_{t = t_o}$, or, in general, as $D^*x(t)$.

Note: Just as the analytical concept of derivative can be substituted by the geometrical one of the tangent line, one can also visualize the contingent as a set of straight lines through one point (a "cone") or two sets of half lines (a forward and a backward cone). This approach was used in the earlier papers ([3], [4], [10]).

An expression

$$D^*x \subset C(x, t), \tag{3.1}$$

where the set $C(x, t) \subset X$ depends on x and t, is called a contingent equation. To solve it means to find all curves $x(t)$ which satisfy (3.1); such curves are called trajectories or solutions of (3.1).

In 1936 Zaremba [10] proved the following theorem:

Theorem 3.1. If $C(x, t) \subset X$, defined in some closed neighborhood $\bar{V}$ of (x_o, t_o), is compact, convex and upper semicontinuous in (x, t) there, then, at least one solution $x = \varphi(t)$ of (3.1) exists, passing through

(x_o, t_o) (i.e. $\varphi(t_o) = x_o$), and this solution can be continued until reaching the boundary of $\bar{\bar{V}}$.

<u>Remark 3.1</u>. The hypothesis $D^*x(t) \subset C(x, t)$ compact insures that every trajectory $x(t)$ is locally lipschitzian and therefore, for example, rectifiable. Trajectories are, therefore, relatively smooth curves.

As Wazewski pointed out [9], under the assumed hypothesis, the condition

$$\begin{cases} \varphi(t) \text{ continuous} \\ D^*\varphi(t) \subset C(\varphi(t), t) \text{ everywhere} \end{cases}$$

is equivalent to

$$\begin{cases} \varphi(t) \text{ absolutely continuous,} \\ \varphi'(t) \in C(\varphi(t), t) \text{ almost everywhere.} \end{cases}$$

<u>Condition f.e.t.</u> If, for every $x \in X$ and every $t \in R$, the set $C(x, t)$ is contained in a fixed compact set C_o:

$$C(x, t) \subset C_o, \quad C_o \text{ compact,} \tag{3.2}$$

then for every trajectory $x = \varphi(t)$ and $t_1, t_2 \in R$,

$$\|\varphi(t_2) - \varphi(t_1)\| \leq k \cdot |t_2 - t_1|$$

4. Contingent equations for generalized dynamical systems.

Definition 4.1. Given the g.d.s. $F(x_o, t_o, t)$ and a point (x_o, t_o), the contingent of the g.d.s. at that point, designated by $D^*F(x_o, t_o, t)$, is defined to be the union of the contingents (at that point) of all the trajectories $\varphi(t)$ of the g.d.s., passing through (x_o, t_o):

$$D^*F(x_o, t_o, t) = \bigcup_{\varphi(t_o) = x_o} D^*\varphi(t)\Big|_{t = t_o}. \tag{4.1}$$

Given a g.d.s., the above definition determines $D^*F(x_o, t_o, t)$ at each point (x_o, t_o). Inversely, the contingent equation

$$D^*F(x_o, t_o, t) = C(x_o, t_o) \tag{4.2}$$

can be written, $C(x, t)$ being a given (variable) subset of X; the problem now is the determination of the g.d.s. $F(x_o, t_o, t)$ satisfying (4.2).

Of course, not for any arbitrary $C(x, t)$ it will be possible to find a solution of (4.2), so that the question arises to give sufficient conditions to insure the existence of a solution.

The contingent equation

$$D^*x \subset C(x, t) \tag{4.3}$$

for curves $x(t)$, will be called "associated" to equation (4.2).

If $C(x, t)$ satisfies the conditions of Theorem 3.2, that is if $C(x, t)$ is compact, convex, upper semicontinuous and satisfies condition f.e.t. (3.3), then the associated equation (4.3) defines a family of trajectories $\varphi(t)$. With the aid of these trajectories, a g.d.s. $F(x_o, t_o, t)$ can be defined as seen in Theorem 3.2. This g.d.s. satisfies therefore the relation

$$D^*F(x_o, t_o, t) \subset C(x_o, t_o), \tag{4.4}$$

because by definition of $F(x_o, t_o, t)$, every one of its trajectories satisfies (4.3).

<u>Example 4.1</u>. A very easy example shows that the above mentioned conditions are not sufficient to insure the existence of a solution. It suffices to take, in (4.2)

$$\begin{cases} C(x_o, t_o) = 0 & \text{for} \quad x \neq 0, \\ C(0, t_o) = \{\text{unit ball} & \|x\| \leq 1\}. \end{cases}$$

Thus $C(x, t)$ satisfies the above conditions. The corresponding g.d.s. is $F(x_o, t_o, t) \equiv x_o$, and $D^*F(x_o, t_o, t) = 0$, so that equation (4.2) is not satisfied at the origin $x = 0$.

5. <u>Existence theorem.</u>

<u>Lemma 5.1</u>. If C, C' are compact, convex subsets of R^n, $\rho(C, C') \leq \delta > 0$, ρ being the Hausdorff metric, if S_r designates the ball

of radius r and some fixed center x_o : $S_r = \{x; \|x - x_o\| \leq r\}$ and if $|r - r'| \leq \delta$, $S_r \cap C \neq \emptyset$, $S_{r'} \cap C' \neq \emptyset$, then there is $\epsilon = \epsilon(\delta, r) > 0$ such that $\rho(C \cap S_r, C' \cap S_{r'}) \leq \epsilon$ and $\epsilon(\delta, r) \to 0$ for r fixed and $\delta \to 0$.

Proof: The thesis means that:

i) $x \in C' \cap S_{r'} \Longrightarrow$ there is $y \in C \cap S_r$ such that $\|x - y\| \leq \epsilon$,

and

ii) $x \in C \cap S_r \Longrightarrow$ there is $y \in C' \cap S_{r'}$ such that $\|x - y\| \leq \epsilon$;

iii) $\epsilon(\delta) \to 0$ with $\delta \to 0$ for fixed r.

Conditions (i) and (ii) being symmetrical, it suffices to prove (i). By hypothesis,

$$x \in S_{r'} \cap C' \subset S_{r+\delta} \cap \bar{V}_\delta(C).$$

Therefore there exist $u \in S_r$, $v \in C$, such that $\|u - x\| \leq \delta$, $\|v - x\| \leq \delta$, therefore $\|u - v\| \leq 2\delta$ and $v \in S_{r+2\delta}$. As $C \cap S_r \neq 0$, there exists $z \in C \cap S_r$. On the segment $\overline{vz}$ $(v, z \in C)$ there is some y such that $y \in S_r$, $\|y - v\| \leq \sqrt{(r + 2\delta)^2 - r^2} = \epsilon'(\delta)$. (See Fig. 1.) Then

$$\|y - x\| \leq \|y - v\| + \|v - x\| \leq \epsilon'(\delta) + \delta = \epsilon(\delta).$$

(iii) follows from the expression of $\epsilon(\delta) = \delta + \sqrt{4r\delta + 4\delta^2}$, so this lemma is proved.

<u>Corollary 5.1.</u> If: i) $C = C(\alpha) \subset X$ is a compact convex set which is a continuous function of a variable α; ii) also $r = r(\alpha)$ is a continuous real positive function iii) $C_o = C(\alpha_o)$, $r_o = r(\alpha_o)$ and iv) $C(\alpha) \cap S_{r(\alpha)} \neq \emptyset$ for all α belonging to some neighborhood of α_o, then, given any $\epsilon > 0$, there is a neighborhood V of α_o such that for every $\alpha \in V$, $\rho(C(\alpha) \cap S_{r(\alpha)}, C_o \cap S_{r_o}) \leq \epsilon$.

In other words, $C(\alpha) \cap S_{r(\alpha)}$ is a continuous function of α at α_o.

Proof: According to the lemma, there is $\delta(\epsilon) > 0$ such that $\rho(C(\alpha), C(\alpha_o)) \leq \delta$, $|r(\alpha) - r(\alpha_o)| \leq \delta$, implies $\rho(C(\alpha) \cap S_{r(\alpha)}, C(\alpha_o) \cap S_{r(\alpha_o)}) \leq \epsilon$. The rest follows from the continuity of the functions $C(\alpha)$ and $r(\alpha)$.

<u>Theorem 5.1.</u> If the variable set $C(x, t) \subset X$ is compact, convex, continuous in (x, t) and satisfies condition f.e.t. (3.3), then there exists a g.d.s. $F(x_o, t_o, t)$ satisfying equation (4.2).

Proof: Defining $F(x_o, t_o, t)$ as in Theorem 3.2 by the trajectories of the associated equation (4.3), the relation (4.4) shows that only $D^*F(x_o, t_o, t) \supset C(x_o, t_o)$ remains to be proved. In other words, given any $y_o \in C(x_o, t_o)$, a trajectory $\varphi(t)$ satisfying (4.3) has to be shown, for which $\varphi(t_o) = x_o$ and $y_o \in D^*\varphi(t)\big|_{t = t_o}$.

Actually, $\varphi(t)$ satisfying (4.3) will be constructed so that $\varphi(t_o) = x_o$ and

$$\frac{d\varphi}{dt} = D^*\varphi(t)\Big|_{t = t_o} = y_o.$$

Having fixed x_o, t_o, y_o, define

$$\gamma(r) = \sup \{\rho(y_o, C(x, t)); \ \|x - x_o\| \leq r, \ |t - t_o| \leq r\}$$

for all $r \geq 0$. Then $\gamma(r)$ is continuous, non-negative and $\gamma(0) = 0$.

Define

$$h(x, t) = \max (\|x - x_o\|, \ |t - t_o|),$$

and

$$C'(x, t) = S_{2\gamma(h)}(y_o) = \{y; \|y - y_o\| \leq 2\gamma(h)\}.$$

It follows that

$$C^*(x, t) = C(x, t) \cap C'(x, t) \neq \emptyset.$$

Here Corollary 5.1 may be applied with $\alpha = (x, t)$, $r(\alpha) = 2\gamma(h(x, t))$, $C(\alpha) = C(x, t)$ as continuous functions of (x, t). Therefore $C^*(x, t)$ is continuous and satisfies all the conditions of Theorem 3.1; therefore for some trajectory $\varphi(t)$:

i) $D^*\varphi(t) \subset C^*(x, t) \subset C(x, t)$.

ii) $\varphi(t_o) = x_o$.

iii) $D^*\varphi(t)\Big|_{t = t_o} = C^*(x_o, t_o) = \{y_o\}$.

This proves the theorem.

Remark 5.1. If the condition f.e.t. is not satisfied but all the other conditions of Theorem 5.1 hold, the result is still valid if the existence of the g.d.s. F is understood in a local sense. In order to give an exact meaning to this statement, one can, for example, change the given $C(x, t)$ outside a certain compact $G \subset X \times R$, in such a way that the new $C(x, t)$ satisfies condition f.e.t., and consider the so defined g.d.s. only in G.

Remark 5.2. From the symmetry of the assumptions with respect to a change of sign of t, it follows that $F(x_o, t_o, t)$ is also compact for all $t < t_o$, a fact which is not always true for the most general g.d.s. (see [5]).

Remark 5.3. From the proof of Theorem 5.1 follows that, under the assumptions made there, if $y_o \in C(x_o, t_o)$, there is a trajectory $x(t)$ of the associated equation (4.3), such that the derivative $x'(t)$ exists at $t = t_o$ and $x'(t_o) = y_o$.

6. Another definition of contingent of a g.d.s.

By similarity with definition 3.1, one can define a kind of generalized contingent of the g.d.s. $F(x_o, t_o, t)$ at the point (x_o, t_o). To avoid confusion with the contingent defined by definition 4.1, the notation $D'F(x_o, t_o, t)$ will be used here.

Definition 6.1. The set of all $y \in X$ such that there exist sequences t_i, x_i $(i = 1,2,3, \dots)$, $t_i \neq t_o$, $t_i \to t_o$, $x_i \in F(x_o, t_o, t_i)$ and $\lim_{i \to \infty} \frac{x_i - x_o}{t_i - t_o} = y$, will be called general contingent of the g.d.s. F at the point (x_o, t_o) and designated by $D'F(x_o, t_o, t)$.

Note: In this definition, also values $t < t_o$ are to be considered in the expression $F(x_o, t_o, t)$, which is then the backwards extension of the original F defined for $t \geq t_o$.

Remark 6.1. According to definitions 3.1 and 4.1, the contingent $D^*F(x_o, t_o, t)$ is obtained from $D'F(x_o, t_o, t)$ restricting, in definition 6.1, the points x_i to lie all on the same trajectory: $x_i = \varphi(t_i)$. Therefore

$$D^*F(x_o, t_o, t) \subset D'F(x_o, t_o, t).$$

Under the assumptions of Theorem 5.1, it will be shown that $D^*F = D'F$, but first some auxiliary lemmas will be stated.

Lemma 6.1. If $C \subset X$ is compact and convex, $a < b$, the vector function $x(t)$ is absolutely continuous in $[a, b]$ and $\frac{dx}{dt} \in C$ for almost every $t \in [a, b]$, then

$$\frac{x(t) - x(a)}{b - a} \in C.$$

The proof can be found in [8] or [6].

Lemma 6.2. If in a compact domain $G \subset X \times R$, and for the g.d.s. F, the set $D^*F(x_o, t_o, t) = C(x_o, t_o)$ is compact, convex and continuous, then there is a constant k such that for every trajectory $\varphi(t)$ in G

$$\|\varphi(t_2) - \varphi(t_1)\| \leq k \cdot |t_2 - t_1|.$$

The proof follows from the fact that $\varphi(t)$ is absolutely continuous (remark 3.1) and almost everywhere

$$\left\|\frac{d\varphi}{dt}\right\| \leq \sup \{\|y\|;\ y \in C(\varphi(t), t)\} \leq k;$$

k may be taken independent of $(\varphi(t), t)$ because of the continuity of $C(x, t)$ in G.

The inequality may also be written

$$\|(\varphi(t_2), t_2) - (\varphi(t_1), t_1)\| \leq k_1 \cdot |t_2 - t_1|$$

with $k_1 = k + 1$.

Lemma 6.3. Under the assumptions of Lemma 6.2, given $\epsilon > 0$ there is a $\delta > 0$ such that as long as all points considered belong to G, $|t_1 - t_o| < \delta$ implies $\dfrac{x_1 - x_o}{t_1 + t_o} \in \overline{V}_\epsilon(C(x_o, t_o))$ for all $x_1 \in F(x_o, t_o, t_1)$.

Here $\bar{V}_\epsilon(C)$ designates the closed ϵ-neighborhood of the set C.

Proof: By continuity of $C(x, t)$ in the compact set G, given $\epsilon > 0$ there is $\delta_1 > 0$ such that $\|(x, t) - (x_o, t_o)\| < \delta_1$ implies $\rho(C(x, t), C(x_o, t_o)) < \epsilon$ uniformly in G, or

$$(6.1) \qquad C(x, t) \subset V_\epsilon(C(x_o, t_o)).$$

By Lemma 6.2, there is $\delta > 0$ such that

$$(6.2) \qquad |t - t_o| < \delta \text{ implies } \|(\varphi(t), t) - (\varphi(t_o), t_o)\| < \delta_1$$

for every trajectory $\varphi(t)$ in G.

Given any trajectory $\varphi(t)$, by remark 3.1 it is absolutely continuous and, by (6.2) and (6.1)

$$\frac{d\varphi}{dt} \in D^*\varphi(t) \subset D^*F(\varphi(t), t, \tau) = C(\varphi(t), t) \subset V_\epsilon(C(x_o, t_o))$$

almost everywhere in the interval $|t - t_o| < \delta$.

Therefore, if $|t_1 - t_o| < \delta$, Lemma 6.1 may be applied and

$$(6.3) \qquad \frac{\varphi(t_1) - \varphi(t_o)}{t_1 - t_o} \in \bar{V}_\epsilon(C(x_o, t_o)).$$

Now, if $x_1 \in F(x_o, t_o, t_1)$, there is a trajectory $\varphi(t)$ such that $\varphi(t_o) = x_o$, $\varphi(t_1) = x_1$. Applying (6.3), the lemma is proved.

An immediate result is the following.

Theorem 6.1. Given a g.d.s. F, if $D^*F(x_o, t_o, t) = C(x_o, t_o)$ is compact, convex and continuous, then $D'F = D^*F$.

Indeed, any possible limit of $\frac{x_1 - x_o}{t_1 - t_o}$ $(x_1 \in F(x_o, t_o, t_1))$ belongs to $\bar{V}_\epsilon(C(x_o, t_o))$ for every $\epsilon > 0$; therefore $D'F \subset C = D^*F$.

Lemma 6.4. If the g.d.s. F satisfies the contingent equation

$$D^*F = C(x, t), \tag{6.4}$$

where $C(x, t)$ is compact, convex and continuous, and if for the constants $(x_o, t_o) \in X \times R$, $y_o \in X$, $r_1 > 0$, $r_2 > 0$, $\epsilon > 0$, $\delta > 0$, $\|y_o\| + \epsilon + \delta \leq \frac{r_1}{r_2}$ and all (x, t) belonging to the set

$$S = \{(x, t);\ \|x - x_o\| \leq r_1,\ |t - t_o| \leq r_2\},$$

the relation

$$\rho(y_o, C(x, t)) < \epsilon$$

holds, then for every t such that $|t_o - t| \leq r_2$,

$$\rho(y_o, \{\frac{F(x_o, t_o, t) - x_o}{t - t_o}\}) < \epsilon.$$

Proof: Assume $t_o < t^* \leq t_o + r_2$; it will be proved that for any $\eta > 0$, there is a $x^* \in F(x_o, t_o, t^*)$ such that

$$\left\| \frac{x^* - x_o}{t^* - t_o} - y_o \right\| < \epsilon + \eta. \tag{6.5}$$

This will prove the lemma for $t^* > t$; for $t^* < t_o$ the same proof applies, making some obvious changes.

By hypothesis, $\rho(y_o, C(x_o, t_o)) < \epsilon$, so that there is some $y_1 \in C(x_o, t_o)$ such that $\|y_1 - y_o\| < \epsilon$. As $y_1 \in D^*F(x_o, t_o, t)$, there is (x_1, t_1) such that $t_o < t_1 \leq t^*$, $x_1 \in F(x_o, t_o, t_1)$ and

$$\left\| \frac{x_1 - x_o}{t_1 - t_o} - y_1 \right\| < \eta.$$

Therefore

$$\left\| \frac{x_1 - x_o}{t_1 - t_o} - y_o \right\| < \eta + \epsilon,$$

or

$$\|x_1 - x_o - y_o(t_1 - t_o)\| < (\eta + \epsilon)(t_1 - t_o).$$

It may be assumed that $\eta < \delta$ and therefore

$$\|x_1 - x_o\| < (\|y_o\| + \eta + \epsilon)(t_1 - t_o) < r_1,$$

so that $(x_1, t_1) \in S$.

If $t_1 < t^*$, there is similarly (x_2, t_2) such that $t_1 < t_2 \leq t^*$ and

$$\|x_2 - x_1 - y_o(t_2 - t_1)\| < (\eta + \epsilon)(t_2 - t_1).$$

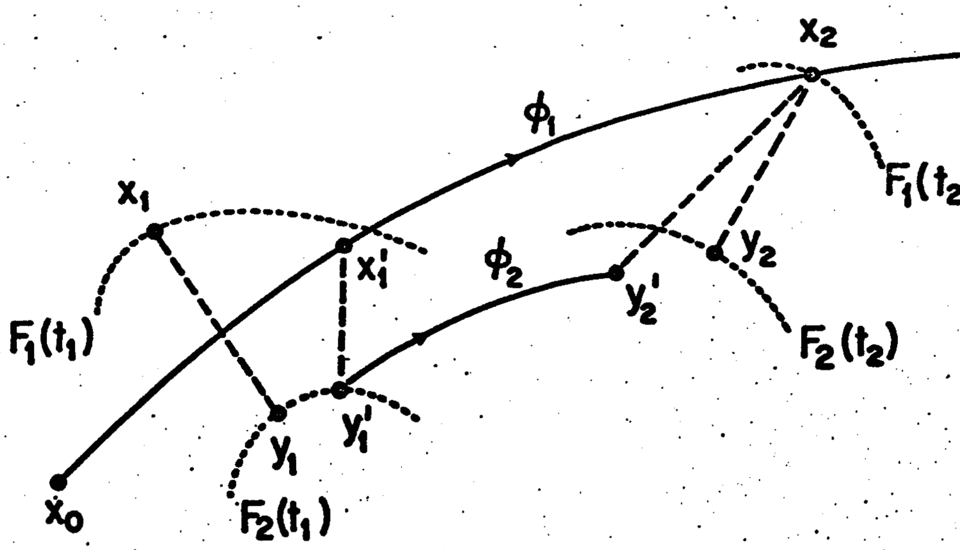

Fig. 4

A sequence $t_o < t_1 < t_2 < \dots < t_n$ is constructed in this way, and

$$\|x_n - x_o - y_o(t_n - t_o)\| =$$
$$\|x_n - x_{n-1} + x_{n-1} - x_{n-2} + \dots + x_1 - x_o - y_o(t_n - t_{n-1} + \dots + t_o)\| \leq$$
$$\|x_n - x_{n-1} - y_o(t_n - t_{n-1})\| + \dots + \|x_1 - x_o - y_o(t_1 - t_o)\| <$$
$$(\eta + \epsilon)\,(t_n - t_o).$$

Dividing by $(t_n - t_o) > 0$,

$$\left\| \frac{x_n - x_o}{t_n - t_o} - y_o \right\| < \eta + \epsilon$$

is obtained.

If, at any step, $t_n = t^*$, relation (6.5) is proved. If $t_n < t^*$ the sequence can be continued, and if $t_n \to t^*$ for $n \to \infty$, then by continuity of $F(x_o, t_o, t)$ in t, the existence of x^* satisfying (6.5) is insured. If there is an upper limit t_ω such that the sequence t_n cannot be extended farther than t_ω, but $t_n \to t_\omega$ for $n \to \infty$, then by continuity again there is a x_ω satisfying (6.5) and from (x_ω, t_ω) the sequence can be extended far more, contrary to the assumption; therefore t^* can be indefinitely approached. This proves the lemma.

Remark 6.2. If the g.d.s. F satisfies equation (6.4) with $C(x, t)$ compact, convex and continuous, and if $y_o \in C(x_o, t_o)$, then for any $\epsilon > 0$ there is $\delta > 0$ such that for $|t - t_o| < \delta$,

$$\rho(y_o, \{\frac{F(x_o, t_o, t) - x_o}{t - t_o}\}) < \epsilon.$$

Indeed, $C(x, t)$ being continuous, there is some neighborhood of (x_o, t_o) of the type of the set S in Lemma 6.4, where the assumptions of that lemma are satisfied.

Remark 6.3. The result of the preceding remark can be formulated: for $|t - t_o| < \delta$,

(6.6) $$\rho^*(C(x_o, t_o), \{\frac{F(x_o, t_o, t) - x_o}{t - t_o}\}) < \epsilon.$$

This is obvious, except perhaps for the fact that being $C(x_o, t_o)$ bounded, the set S can be chosen so that the relation $\|y_o\| < \frac{r_1}{r_2}$ of Lemma 6.4 is satisfied for all $y_o \in C(x_o, t_o)$.

Remark 6.4. Under the assumptions of Lemma 6.4, $C(x, t)$ is uniformly continuous and uniformly bounded in every compact set $G \subset X \times R$, therefore (6.6) holds uniformly in G.

Combining Lemmas 6.3 and 6.4, the following result is obtained.

Theorem 6.2. If $C(x, t)$ is compact, convex and continuous in the compact domain $G \subset X \times R$, then

$$\rho(C(x_o, t_o), \{\frac{F(x_o, t_o, t) - x_o}{t - t_o}\}) \to 0$$

for $t \to t_o$ uniformly in G.

7. Uniqueness.

Given a contingent equation (4.2) satisfying the conditions of Theorem 5.1, it is still possible that the solution is not unique. In other words, given $C(x, t)$ it is possible that there exist two essentially different g.d.s., F_1 and F_2, such that at every point (x_o, t_o):

$$D^*F_1(x_o, t_o, t) = D^*F_2(x_o, t_o, t) = C(x_o, t_o).$$

Example 7.1. Let $X = R$. The curves $x = \varphi(t) = (t + \text{const})^3$ define the g.d.s.

$$F_1(x_o, t_o, t) = \{(t + \sqrt[3]{x_o} - t_o)^3\} \quad \text{(the set of one point)}$$

(see fig. 2). Obviously it satisfies all axioms of a g.d.s.

All solution curves of the differential equation

$$\frac{dx}{dt} = 3x^{2/3} \tag{7.1}$$

also defines a g.d.s. F_2, which is different from F_1 because, for example, for $t > 0$,

$$F_2(0, 0, t) = [0, t^3]$$

is the set of all x, $0 \le x \le t^3$ (fig. 3).

On the other hand, F_1 satisfies the same differential equation (7.1) which is a special case of a contingent equation.

Definition 7.1. Given the contingent equation $D^*F = C(x, t)$ satisfying the conditions of Theorem 5.1, the g.d.s. F defined by the trajectories $\varphi(t)$ of the associated equation $D^*\varphi \subset C(\varphi, t)$, as:

$$\begin{cases} x \in F(x_o, t_o, t) \text{ if and only if there is a trajectory} \\ \varphi(t) \text{ such that } \varphi(t_o) = x_o, \quad \varphi(t_1) = x_1; \end{cases}$$

will be called maximal solution of the contingent equation F_{max}.

This denomination is justified, because if $F_1(x_o, t_o, t)$ satisfies $D^*F_1 = C(x, t)$, and $x_1 \in F_1(x_o, t_o, t_1)$, there is a trajectory of this g.d.s. $\varphi(t)$, going from $\varphi(t_o) = x_o$ to $\varphi(t_1) = x_1$. But then $D^*\varphi \subset D^*F_1 = C(x, t)$, so that φ satisfies the associated equation and $x_1 \in F_{max}(x_o, t_o, t)$. Therefore $F_1 \subset F_{max}$.

Theorem 7.1. If the variable set $C(x, t)$, defined in $X \times R$ is compact, convex, continuous, satisfies condition f.e.t. (3.3) and satisfies also the following condition: (*)

$$(7.3) \qquad \rho(C(x_2, t), C(x_1, t)) \leq w(\|x_2 - x_1\|, t)$$

where $w(z, t)$ is defined for $z \geq 0$ and is non-negative, continuous, bounded and increasing with z and such that

----

(*) A similar conditioh was used by Turowicz in [7] and [8].

$$\frac{dz}{dt} = w(z, t); \quad z(0) = 0 \tag{7.4}$$

has the unique solution $z(t) \equiv 0$, then the solution of the contingent equation

$$D^*F = C(x_o, t_o) \tag{7.5}$$

exists and is unique.

Proof: The existence has already been proved in Theorem 5.1. To prove uniqueness assume that two different g.d.s. $F_1(x_o, t_o, t)$ and $F_2(x_o, t_o, t)$ satisfy (7.4). It will be shown that for every $t > t_o$,

$$\rho^*(F_1(x_o, t_o, t), \; F_2(x_o, t_o, t)) = 0. \tag{7.5}$$

Interchanging F_1 and F_2, $\rho(F_1, F_2) = 0$ follows, and being closed sets, $F_1 = F_2$. Changing the sign of t, the proof is still valid and the general result follows.

In order to prove (7.5), it may be assumed that (x_o, t_o) is fixed and all points (x, t) considered lie in some compact region $G \subset X \times R$; the arbitrariness of (x_o, t_o) and G makes the result general.

Designating

$$z(t) = \rho^*(F_1(x_o, t_o, t), \; F_2(x_o, t_o, t)),$$

$z(t)$ is lipschitzian by Lemmas 6.2 or 6.3, and therefore absolutely continuous. Also $z(0) = 0$.

To obtain a bound for the derivative $z' = \frac{dz}{dt}$ at some instant t_1, the following construction will be made (see fig. 4).

Assuming

$$z(t_1) = z_1 > 0,$$

there is $x_1 \in F_1(x_o, t_o, t_1)$, $y_1 = F_2(x_o, t_o, t_1)$ such that

$$z_1 = \rho^*(F_1(x_o, t_o, t_1), F_2(x_o, t_o, t_1)) =$$

$$= \rho(x_1, F_2(x_o, t_o, t_1)) = \|x_1 - y_1\|.$$

Now, assume $\epsilon > 0$ given arbitrarily and take $\delta > 0$ such that for $|t_2 - t_1| < \delta$,

$$\rho(C(x_1, t_1), \{\frac{F(x_1, t_1, t_2) - x_1}{t_2 - t_1}\}) < \epsilon \tag{7.6}$$

uniformly in G (Theorem 6.2), for both g.d.s. F_1 and F_2. Then, for any t_2 such that $|t_2 - t_1| < \delta$, $z_2 = z(t_2) > 0$ may also be assumed by continuity, and there are points

$$x_2 \in F_1(x_o, t_o, t_2), \quad y_2 \in F_2(x_o, t_o, t_2)$$

such that

$$z_2 = \rho^*(F_1(x_o, t_o, t_2), F_2(x_o, t_o, t_2))$$

$$= \rho(x_2, F_2(x_o, t_o, t_2)) = \|x_2 - y_2\|.$$

Take any trajectory $\varphi_1(t)$ of F_1 through $\varphi_1(t_o) = x_o$ and $\varphi_1(t_2) = x_2$. Calling $\varphi_1(t_1) = x_1'$, obviously $x_1' \in F_1(x_o, t_o, t_1)$.

There is a point $y_1' \in F_2(x_o, t_o, t_1)$ such that

$$\rho(x_1', F_2(x_o, t_o, t_1)) = \|x_1' - y_1'\|.$$

Choose a trajectory $\varphi_2(t)$ of F_2 through $\varphi_2(t_1) = y_1'$ such that, calling $\varphi_2(t_2) = y_2'$, $y_2' \in F_2(x_o, t_o, t_2)$ and

$$\left\| \frac{y_2' - y_1'}{t_2 - t_1} - \frac{x_2 - x_1'}{t_2 - t_1} \right\| \leq w(\|x_1' - y_1'\|, t_1) + 2\varepsilon.$$

This can be done because, by assumption (7.3)

$$\rho(C(x_1', t_1), C(y_1', t_1)) \leq w(\|x_1' - y_1'\|, t_1)$$

and by Lemma 6.3 and condition (7.6)

$$\frac{x_2 - x_1'}{t_2 - t_1} \in \overline{V}_\varepsilon(C(x_1', t_1)),$$

and

$$\frac{y_2' - y_1'}{t_2 - t_1} \in \overline{V}_\varepsilon(C(y_1', t_1)).$$

Now,

$$z_1 = \|x_1 - y_1\| \geq \|x_1' - y_1'\|,$$

$$z_2 = \|x_2 - y_2\| \leq \|x_2 - y_2'\|.$$

Therefore

$$\begin{aligned} z_2 - z_1 &\leq \|x_2 - y_2'\| - \|x_1' - y_1'\| \\ &\leq \|x_2 - y_2' - x_1' + y_1'\| \\ &= \|(x_2 - x_1') - (y_2' - y_1')\|. \end{aligned}$$

Dividing by $t_2 - t_1$,

$$\begin{aligned} \frac{z_2 - z_1}{t_2 - t_1} &\leq \left\| \frac{x_2 - x_1'}{t_2 - t_1} - \frac{y_2' - y_1'}{t_2 - t_1} \right\| \\ &\leq w(\|x_1' - y_1'\|, t_1) + 2\varepsilon \leq w(\|x_1 - y_1\|, t_1) + 2\varepsilon \\ &= w(z_1, t_1) + 2\varepsilon \end{aligned}$$

is obtained. Making now $t_2 \to t_1$, ϵ being arbitrary

$$\left.\frac{dz}{dt}\right|_{t = t_1} \leq w(z(t_1), t_1).$$

Therefore $z(t)$ is majorized by the solution of (7.4), so that $z(t) \equiv 0$ and the theorem is proved.

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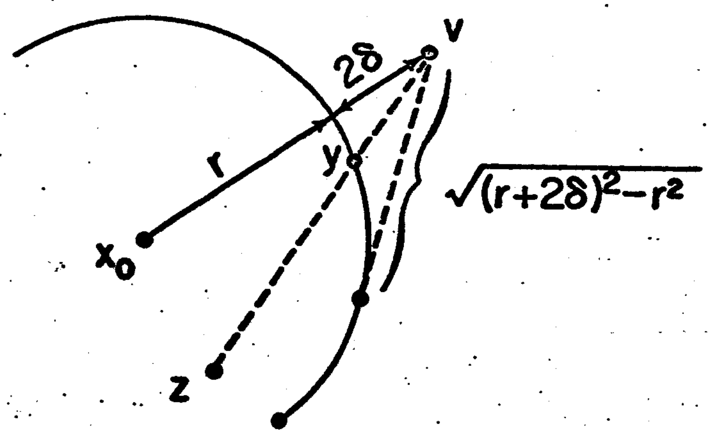

Fig. 1

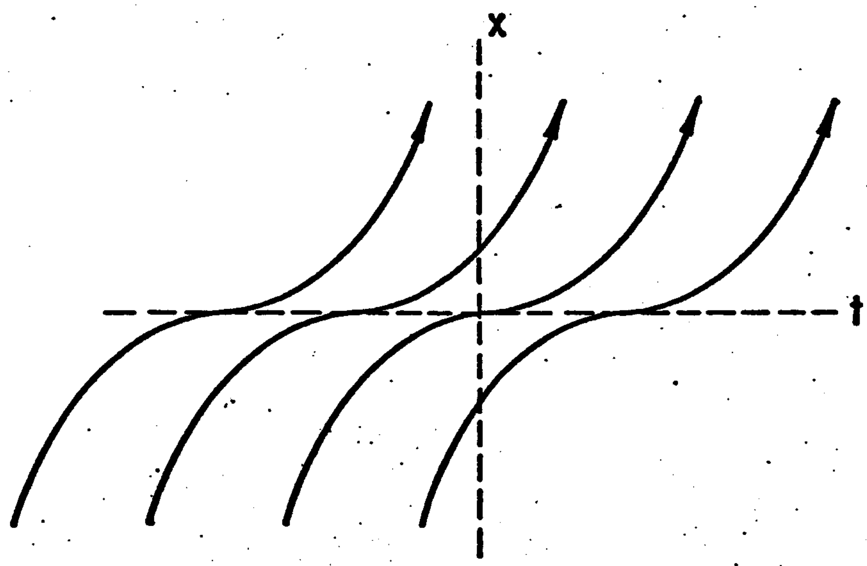

Fig. 2

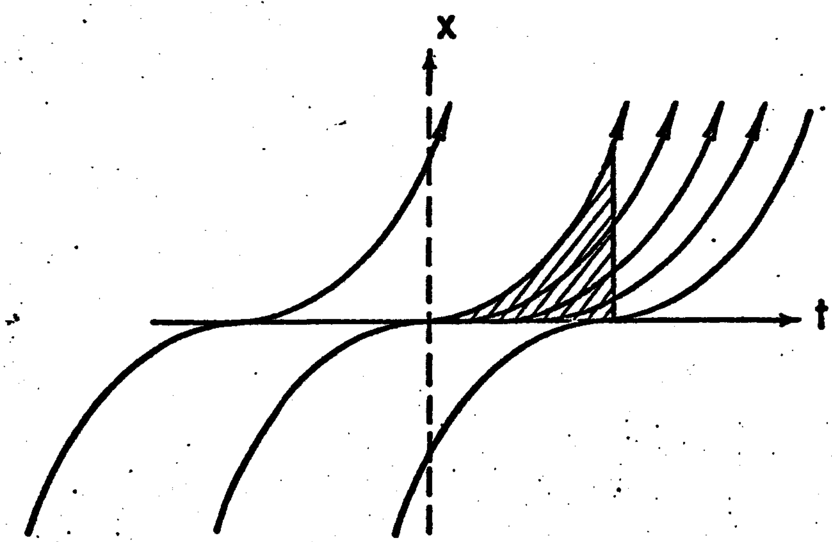

Fig. 3