ATTRACTORS IN DYNAMICAL SYSTEMS *

by

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Introduction

In the study of topological properties of ordinary differential equations, the stability theory of compact invariant sets, (which may be regarded as generalizations of critical points and limit cycles) plays a central role. While a multiplicity of stability conditions have been developed, the most prominent are Liapunov stability and asymptotic stability. By Liapunov stability (or just stability) of the compact invariant set M, we mean that every orbit starting sufficiently close to M will remain in a neighborhood of M. The set M is asymptotically stable if it is stable, and is also an "attractor" - that is, all orbits in a neighborhood $A(M)$ of M approach M.

By means of Liapunov functions and other techniques, asymptotic stability has been intensively studied in the literature of differential equations. It seems reasonable, therefore, to study the properties of attractors, without explicitly assuming stability. This is the object of the present work.

In sections 1 and 2 we review some of the basic notions of dynamical systems and stability theory and discuss several examples of attractors. In section 3 we use the prolongation of a point, and its close relative, the

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prolongational limit set of an orbit, to clarify the connection between attractors and stability. For example, theorem 1 tells us that the prolongation of an attractor is always asymptotically stable, with the same region of attraction, and theorems 4 and 5 give necessary and sufficient conditions for stability of an attractor. Theorem 5 casts some light on Zubov's (erroneous) stability condition.

The concluding section concerns the assumption that the prolongational limit set of $A(M) - M$ is contained in M. In certain important cases this is equivalent to asymptotic stability. Our final result is that this assumption (suitably localized) is "in general" valid for attractors.

As far as we know, the only previous systematic study of attractors is a paper of Pinchas Mendelson [5]. In his paper, he gives an example of an unstable attractor in the plane, which we discuss in section 2. A number of our results, (for example, theorems 1 and 6) are similar to Mendelson's. However for the sake of completeness, and since Mendelson's proofs depend, in part, on an unpublished manuscript, we prove all our results here.

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1. Notations and Elementary Concepts.

In what follows, X denotes a locally compact metric space with metric d. If $M \subset X$ and $x \in X$, we write

$$d(x, M) = \inf\{d(x, y) \mid y \in M\}, \quad S(M, r) = \{x \in X \mid d(x, M) < r\},$$

$$B(M, r) = \{x \in X \mid d(x, M) \leq r\},$$

and

$$H(M, r) = B(M, r) - S(M, r).$$

The closure of M will be denoted by $\bar{M}$, and its boundary, $\bar{M} \cap (\overline{X - M})$, by ∂M.

If, for each $x \in X$, $\varphi(x)$ is a subset of X, and $A \subset X$, then

$$\varphi(A) = \cup\{\varphi(a) \mid a \in A\} .$$

R denotes the real numbers, and R^+ and R^- the non-negative and non-positive reals, respectively.

A continuous map π of the product space $X \times R$ into X defines a <u>dynamical system</u> or <u>flow</u> on X if the two following conditions are satisfied:

(I) $\quad \pi(x, 0) = x$ for all $x \in X$,

(II) $\quad \pi(\pi(x, t_1), t_2) = \pi(x, t_1 + t_2)$ for all $t_1, t_2 \in R$, $x \in X$.

We remark that for every fixed t the map $\pi(x, t)$ is a topological map of X onto itself, so that π defines a group of homeomorphisms. For a given x, the set $\gamma(x) = \pi(x, R)$ is called the <u>trajectory</u> or <u>orbit</u> through x. The sets $\gamma^+(x) = \pi(x, R^+)$ and $\gamma^-(x) = \pi(x, R^-)$ are called, respectively, the positive and negative semi-orbits through x. The standard example of a dynamical system is given by the solutions of a differential system $dx/dt = f(x)$, where $x \in R^n$, $f \in R^n$ and f satisfies conditions to insure the existence, uniqueness, continuous dependence on the initial value, and unlimited extendability of solutions [6].

A subset M of X is said to be <u>invariant</u> if $\gamma(M) = M$, and positively (negatively) invariant if $\gamma^+(M) = M$ $(\gamma^-(M) = M)$.

The <u>positive</u> or <u>omega limit set</u> of an orbit $\gamma(x)$ is the set $\Lambda^+(x)$ consisting of all points y in X such that there is a sequence $\{t_n\}$ of reals with $t_n \to +\infty$ and $\pi(x, t_n) \to y$. It is readily verified that

$$\Lambda^+(x) = \cap\{\overline{\gamma^+(\pi(x, t))} \mid t \in R\} = \cap\{\overline{\gamma^+(\pi(x, t))} \mid t \geq t_o\},$$

for any real t_o, and, using the continuity of the map π, that

$$\overline{\gamma^+(x)} = \gamma^+(x) \cup \Lambda^+(x).$$

The negative or alpha limit set $\Lambda^-(x)$ of an orbit $\gamma(x)$ is defined similarly: $y \in \Lambda^-(x)$ if and only if there is a sequence $\{t_n\}$ with $t_n \to -\infty$ and $\pi(x, t_n) \to y$. The analogous expressions for $\Lambda^-(x)$ and $\overline{\gamma^-(x)}$ are, are, of course, valid.

2. Stability, Attractors, and Stable Attractors.

From now on, M will denote a non-empty compact invariant subset of X.

The set M is said to be

(a) (positively) stable, if, for each $\epsilon > 0$, there is a $\delta > 0$ such that $\gamma^+(y) \subset S(M, \epsilon)$ whenever $y \in S(M, \delta)$.

(b) a (positive) attractor, if, for some $\delta > 0$, $y \in S(M, \delta)$ implies $\Lambda^+(y)$ is a non-empty subset of M.

(c) (positively) asymptotically stable if it is a stable attractor - that is, if both (a) and (b) are satisfied.

Negative stability and negative attractors can also be defined, but we will not be concerned with them here. Therefore, in the future, we will omit the adjective "positive" when referring to stability and attractors.

By the region of attraction $A(M)$ of the set M (which need not be an attractor) we mean the union of all trajectories with the property that their positive limit sets are non-empty and contained in M. Then M is an attractor if and only if $A(M)$ is a neighborhood of M.

Lemma 1. If M is an attractor, then $A(M)$ is an open invariant set.

Proof. The invariance is obvious from the definition. In order to show that $A(M)$ is open, choose $\delta > 0$ such that $S(M, \delta) \subset A(M)$.

If $y \in A(M)$, there exists $\tau > 0$ such that $\pi(y, \tau) \in S(M, \delta) - M$, which is open. Thus there is a neighborhood N of $\pi(y, \tau)$ such that $N \subset S(M, \delta) - M$. Due to the continuity of π, the set $\pi(N, -\tau)$ is a neighborhood of y, and since $\Lambda^+(N) \subset M$, we have $\Lambda^+(\pi(N, -\tau)) = \Lambda^+(N) \subset M$. Thus $\pi(N, -\tau) \subset A(M)$, which shows that $A(M)$ is indeed open.

As we remarked in the introduction, stable attractors have been studied extensively in the literature. It is appropriate at this point, therefore, to mention several examples of unstable attractors.

Consider a dynamical system on a 1-sphere with a single critical point P, the complement of P being a single orbit both limit sets of which coincide with P. Obviously, P is an unstable attractor. Generally, we will call the union of an orbit γ and a critical point P, such that $\Lambda^+(\gamma) = \Lambda^-(\gamma) = \{P\}$, a <u>path monogon</u>. An analogous example exists on the torus: A closed orbit is approached spirally in both the positive and negative senses by all other orbits.

It is somewhat more difficult to find unstable attractors in non-compact spaces. An instructive example in the plane was provided by Mendelson [5], (Figure 1). There is a single critical point $\{P\}$ and if $x \in R^2$, $\Lambda^+(x) = \{P\}$. There is a path monogon consisting of $\{P\}$ and an orbit γ_1, which bounds a "nodal region" N; that is, an invariant set consisting of orbits tending to P in both senses. The orbits outside N have empty alpha limit sets. $\{P\}$ is an unstable attractor because of the nodal region N.

A simple analytical example of an unstable attractor is given by the following pair of differential equations (in polar coordinates):

$$\frac{dr}{dt} = r(1 - r), \qquad \frac{d\theta}{dt} = \sin^2 \frac{\theta}{2} .$$

There are two critical points, the origin and the point $(1, 0)$. Moreover, the unit circle constitutes a path monogon. An easy analysis shows that in both examples the solutions outside the path monogons are topologically equivalent. In the present case, the flow inside the unit circle is obtained, qualitatively, by a reflection at the unit circle. Consequently, the point $(1, 0)$ is an unstable attractor, the region of attraction being the plane without the origin (Figure 2).

3. <u>The Prolongation and the Prolongational Limit Set.</u>

If $x \in X$, the (<u>first</u>) (<u>positive</u>) <u>prolongation</u> of x, denoted by $D^+(x)$, is the set

$$D^+(x) = \bigcap_{\epsilon > 0} \overline{\gamma^+(S(x, \epsilon))} .$$

It is easy to see that $y \in D^+(x)$ if and only if there are sequences $\{x_n\}$ in X and $\{t_n\}$ of non-negative reals such that $x_n \to x$ and $\pi(x_n, t_n) \to y$.

Obviously, $D^+(x)$ contains the positive orbit closure $\overline{\gamma^+(x)}$. In general, $\overline{\gamma^+(x)}$ is a proper subset of $D^+(x)$. For example, let P be a saddle point of a plane dynamical system and let γ_1, γ_3 be the orbits tending to P as $t \to +\infty$, and γ_2, γ_4 those tending to P as $t \to -\infty$. Then the prolongation of any point x on γ_1 or γ_3 contains, besides its positive orbit closure, the two paths γ_2, γ_4.

The prolongation has been intensively studied in a series of papers, [1], [2], [7], [8]. Here we only require the following lemma:

Lemma 2. (i) If M is a compact subset of X then $D^+(M) = \cup \{D^+(x) | x \in M\}$ is closed and positively invariant.

(ii) The compact invariant set M is stable if and only if $D^+(M) = M$.

Proof. (i) follows easily from the definitions. For the proof of (ii), see [8], p.341.

It turns out to be useful, in the study of attractors, to single out a certain subset of the prolongation. If $x \in X$, the prolongational limit set of x, denoted by $\Lambda_D^+(x)$, is the set of all $y \in X$, such that there exist sequences $\{x_n\}$ in X, $\{t_n\}$ of reals, with $t_n \to +\infty$, such that $x_n \to x$, and $\pi(x_n, t_n) \to y$.

The prolongational limit set occupies a position with respect to the prolongation, analogous to that of the omega limit set relative to the positive orbit closure. Indeed we have

Lemma 3. Let $x \in X$. Then

(i) $D^+(x) = \gamma^+(x) \cup \Lambda_D^+(x)$

(ii) $\Lambda_D^+(x) = \cap\{D^+(\pi(x,t)) | t \in R\} = \cap\{D^+(\pi(x, t)) | t \geq t_o\}$

for any real t_o.

(iii) $\Lambda_D^+(x)$ is (positively and negatively) invariant.

(iv) If $t \in R$, $\Lambda_D^+(\pi(x, t)) = \Lambda_D^+(x)$.

(Compare (i) and (ii) with the representations of $\overline{\gamma^+(x)}$ and $\Lambda^+(x)$ given in the preceding section)

Proof. (i) and (iii) are immediate consequences of the definition. To prove (ii) observe first that $\Lambda_D^+(x)$ is certainly contained in both expressions on the right. If $y \in D^+(\pi(x,t))$, for all (sufficiently large) t, then for any t, and any $\epsilon > 0$ there is a $\tau > 0$ and an $x' \in X$, with $d(x, x') < \epsilon$ such that $d(y, \pi(x', t + \tau)) < \epsilon$. It follows immediately that $y \in \Lambda_D^+(x)$. Finally,

$$\Lambda_D^+(\pi(x,t)) = \cap\{D^+(\pi(\pi(x, t), s))|s \in R\}$$

$$= \cap\{D^+(\pi(x,t + s)|s \in R\} = \cap\{D^+(\pi(x, s))|s \in R\}$$

$= \Lambda_D^+(x)$. This proves (iv).

Note that (iv) tells us that it is meaningful to speak of the prolongational limit set of an orbit.

In the case of the Mendelson example, the prolongational limit set of the orbit marked γ_2 in figure 1 consists of the path monogon $\{P\} \cup \gamma_1$. We shall see later that this phenomenon is typical of unstable attractors.

Now we proceed to a deeper study of the prolongational limit set and attractors. The next two lemmas will be used constantly.

Lemma 4. If $x \in X$ and $\omega \in \Lambda^+(x)$, then $\Lambda_D^+(x) \subset D^+(\omega)$. (Consequently $\Lambda_D^+(x) \subset D^+(\Lambda^+(x))$).

Proof. Let $y \in \Lambda_D^+(x)$. Then there are sequences $\{x_n\}$ and $\{t_n\}$ with $x_n \to x$, $t_n \to +\infty$, and $\pi(x_n, t_n) \to y$. Since $\omega \in \Lambda^+(x)$, there is a sequence $\{\tau_n\}$ with $\tau_n \to \infty$ such that $\pi(x, \tau_n) \to \omega$. We may suppose without loss of generality that $t_n - \tau_n > 0$ for each n. Consider the sequences $\{\pi(x_n, \tau_1)\}$, $\{\pi(x_n, \tau_2)\}$, ... By continuity of π, we have $\pi(x_n, \tau_k) \to \pi(x, \tau_k)$ for each fixed k. We may choose subsequences $\{x_n'\}$ of $\{x_n\}$ with the property: $d(\pi(x_n', \tau_n), \pi(x, \tau_n)) \leq \frac{1}{n}$ and $d(\pi(x_m, \tau_n), \pi(x, \tau_n)) \leq \frac{1}{n}$ for $m \geq r$ where $x_n' = x_r$. The sequence $\{\pi(x_n', \tau_n)\}$ tends to ω. Indeed, $d(\pi(x_n', \tau_n), \omega) \leq d(\pi(x_n', \tau_n), \pi(x, \tau_n)) + d(\pi(x, \tau_n), \omega) \leq \frac{1}{n} + d(\pi(x, \tau_n), \omega) \to 0$. Note further, that if $\{t_n'\}$ is the subsequence of $\{t_n\}$ corresponding to the subsequence $\{x_n'\}$ of $\{x_n\}$, then $t_n' - \tau_n > 0$ for each n. Also, since $\{\pi(x_n', t_n')\}$ is a subsequence of $\{\pi(x_n, t_n)\}$, it follows that $\pi(x_n', t_n') \to y$. But $\pi(x_n', t_n') = \pi(\pi(x_n', \tau_n), t_n' - \tau_n)$, and since $\pi(x_n', \tau_n) \to \omega$ and $t_n' - \tau_n > 0$, we have $y \in D^+(\omega)$. This completes the proof.

<u>Lemma 5.</u> <u>Let</u> M <u>be an attractor and let</u> $\epsilon > 0$. <u>Then there exists</u> $T > 0$ <u>such that</u> $D^+(M) \subset \pi(B(M, \epsilon), [0, T])$.

<u>Proof.</u> Let $\epsilon > 0$. By decreasing ϵ if necessary, we may suppose that $B(M, \epsilon)$ is a compact subset of $A(M)$. For $x \in H(M, \epsilon)$, define $\tau(x) = \inf \{t > 0 | \pi(x, t) \subset B(M, \epsilon)\}$; since $x \in A(M)$, $\tau(x)$ is defined. Set $T = \{\sup \tau(x) | x \in H(M, \epsilon)\}$. We show $T < \infty$. If this is not the case there is a sequence $\{x_n\}$ in $H(M, \epsilon)$ for which $\tau(x_n) \to \infty$. We may suppose $x_n \to x \in H(M, \epsilon)$. Let $\tau > 0$ such that $\pi(x, \tau) \in S(M, \epsilon)$. Then, if n is sufficiently large $\pi(x_n, \tau) \in S(M, \epsilon)$, and it follows that $\tau(x_n) < \tau$, which contradicts $\tau(x_n) \to \infty$. Now let $y \in D^+(M) - B(M, \epsilon)$. Then there are sequences $\{x_n\}$ and $\{t_n\}$ with $x_n \to x \in M$, and $t_n \geq 0$ such that

$\pi(x_n, t_n) \to y$. Then, for all sufficiently large n, there is an s_n, $0 < s_n < t_n$ such that $\pi(x_n, s_n) \subset H(M, \epsilon)$, and $\pi(x, t) \notin B(M, \epsilon)$ for $s_n < t \leq t_n$. By the first part of this proof, $0 < t_n - s_n < T$. Then $\pi(x_n, t_n) = \pi(\pi(x_n, s_n), t_n - s_n) \in \pi(B(M, \epsilon), [0, T])$. Therefore, $y \in \pi(B(M, \epsilon), [0, T])$, since this set is closed.

Theorem 1. Let M be an attractor. Then $D^+(M)$ is a compact invariant set which is a stable attractor. Its region of attraction $A(D^+(M))$ coincides with $A(M)$. Moreover $D^+(M)$ is the smallest stable attractor containing M.

Proof. By lemma 5, $D^+(M)$ is a closed subset of the compact set $\pi(B(M, \epsilon), [0, T])$, so $D^+(M)$ is compact. Let $x \in M$, $y \in D^+(x)$, and $t \in R$. Then $\pi(y, t) \in \pi(D^+(x), t) = \pi(\gamma^+(x) \cup \Lambda_D^+(x), t) \subset M \cup \Lambda_D^+(x) \subset D^+(M)$ (lemma 3). This shows $D^+(M)$ is invariant.

Now, $D^+(M) \subset \pi(B(M, \epsilon), [0, T]) \subset \pi(A(M), R) = A(M)$. That is, $A(M)$ is an open neighborhood of $D^+(M)$, and, since $A(M)$ is invariant, every trajectory tending to $D^+(M)$ is contained in $A(M)$. This proves that $A(D^+(M)) = A(M)$.

To show $D^+(M)$ is stable, observe first that $\Lambda^+(D^+(M)) \subset \Lambda^+(A(M)) \subset M$. Now, if $z \in D^+(M)$, $D^+(z) = \gamma^+(z) \cup \Lambda_D^+(z) \subset D^+(M) \cup D^+(\Lambda^+(z)) \subset D^+(M) \cup D^+(M) = D^+(M)$, by lemmas 3 and 4. That is $D^+(D^+(M)) \subset D^+(M)$, and, by lemma 2, $D^+(M)$ is stable.

Finally, let M_1 be any set such that $M \subset M_1 \subset D^+(M)$. Then

$D^+(M) \subset D^+(M_1) \subset D^+(D^+(M)) = D^+(M)$, and $D^+(M_1) = D^+(M)$.

If M_1 is stable, then $M_1 = D^+(M_1) = D^+(M)$. The proof is completed.

<u>Theorem 2.</u> <u>If</u> M <u>is an attractor and</u> $y \in D^+(M)$, <u>then</u> $\Lambda^-(y) \cap M \neq \Phi$.

<u>Proof.</u> If $y \in M$, $\Lambda^-(y) \subset M$, since M is compact invariant. Suppose $y \notin M$. If $\epsilon > 0$, there is, by lemma 5, a $t < 0$ such that $\pi(y, t) \in B(M, \epsilon)$. Let $\{t_n\}$ be a sequence of negative reals such that $\pi(y, t_n) \to x \in M$. If this sequence is bounded below, it follows that $\gamma(y) \cap M \neq \Phi$, which contradicts the invariance of M. Therefore $t_n \to -\infty$, and $x \in \Lambda^-(y) \cap M$.

It is not in general true that the negative limit set of a point of $D^+(M)$ is contained entirely in M, as the following example of an attractor on a torus shows (This example is due to Carlos Perello):

Consider a flow on a torus containing a path monogon which consists of a critical point P and a path γ_0, where γ_0 is not contractible into P. Suppose that all other orbits approach this path monogon spirally in the negative sense and tend to P in the positive sense (Figure 3). These conditions determine the flow topologically. The orbit γ_0 and another one, which we denote by γ_1, together split the neighborhood of P into three regions, two of which are hyperbolic (i.e., contain no complete semi-orbits), the third parabolic (consisting of positive semi-orbits) (Figure 4). Evidently P is an unstable attractor, its prolongation being the whole torus, because P is a negative limit point of every other orbit. On the other hand, the negative limit sets of the orbits outside the path monogon contain the orbit γ_0 as well as P.

However, we do have the following.

Theorem 3. Consider a dynamical system defined in a planar region by the system of differential equations

$$\dot{x} = f(x, y)$$

$$\dot{y} = g(x, y)$$

Suppose M is a compact connected attractor, and let $x \in D^+(M)$. Then $\Lambda^-(x) \subset M$.

Proof. Let $x \in D^+(M)$, $x \notin M$. Let $y \in \Lambda^-(x)$ and suppose, if possible, $y \notin M$. Certainly y is not a critical point, nor is $y \in \gamma(x)$. For if y were a critical point, we would have $\Lambda^+(y) = \{y\} \subset M$, since $D^+(M)$ is in the region of attraction of M (theorem 1). Again, $y \in \gamma(x)$ would imply that $\gamma(x)$ is recurrent, hence periodic, and again $y \in M$. Therefore y is a regular point and $y \notin \gamma(x)$. In this case we can draw a transversal ℓ through y, such that $\gamma^-(x)$ will intersect ℓ in a monotone sequence of points $\{P_n\}$, $\{P_n\} \to y$ as $n \to \infty$ [6]. The portion of the semi-orbit $\gamma^-(x)$ between any two successive points, say P_1 and P_2, of this sequence and the part of the transversal between them form a Jordan curve J. This curve divides the plane into two connected sets A and B which are disjoint. The two sets $\Lambda^-(x)$ and $\Lambda^+(x)$ are connected, ([6] chapter V), so one must be contained in A, the other in B.

$\Lambda^+(x)$ and $\Lambda^-(x)$ are therefore disjoint. Since M is compact, we can assume that the segment of ℓ joining P_1 and P_2 does not meet M. Then M and J are disjoint. Since M is also connected, and

$A^+(x) \subset M$, M is contained either in A or B, whichever contains $A^+(x)$. Thus $A^-(x)$ and M are disjoint. This however contradicts theorem 2. The proof is completed.

The authors strongly suspect that it is possible to dispense with the differentiability hypothesis in this theorem.

Now, we address ourselves to the problem of finding conditions under which an attractor is stable.

Theorem 4. Let M be a compact invariant set. Then M is a stable attractor if and only if there is a neighborhood U of M such that $\Lambda_D^+(U) \subset M$. In this case, the set of $x \in X$ for which $\Lambda_D^+(x) \subset M$ coincides with $A(M)$.

Proof. If M is a stable attractor, and $x \in A(M)$, then, by lemma 4, $\Lambda_D^+(x) \subset D^+(\Lambda^+(x)) \subset D^+(M) = M$.

Conversely, if U is a neighborhood of M such that $\Lambda_D^+(U) \subset M$, and $x \in U$, then $\Lambda^+(x) \subset \Lambda_D^+(x) \subset M$, so M is an attractor. If $x \in M$, then $D^+(x) = \gamma^+(x) \cup \Lambda_D^+(x) \subset M \cup \Lambda_D^+(U) \subset M$. This shows stability.

We have already shown that $\Lambda_D^+(A(M)) \subset M$. If $\Lambda_D^+(x) \subset M$, then $\Lambda^+(x) \subset \Lambda_D^+(x) \subset M$, and $x \in A(M)$. The proof is completed.

In [9], Zubov stated that a necessary and sufficient condition for stability of the compact invariant set M is that M contain no alpha limit points of orbits outside M. This condition is obviously necessary for stability of M. However, Mendelson and Bass [3] observed that it is not in general sufficient. (In order to give a correct condition, Bass introduced the notion of a <u>strongly negatively linked sequence of saddle sets</u>, which in our terminology means a sequence of closed invariant sets, each of which contains in its prolongation all its successors). Our next theorem shows that in the case of an attractor, Zubov's condition is indeed necessary and sufficient.

<u>Theorem 5.</u> Let M be an attractor. Then the following are equivalent

(1) M <u>is stable.</u>

(2) $A(M)$ <u>contains no alpha limit points of orbits in</u> $A(M) - M$.

(3) M <u>contains no alpha limit points of orbits in</u> $A(M) - M$.

<u>Proof.</u>

(1) $\Rightarrow$ (2): Let $z \in A(M) - M$, and suppose $y \in \Lambda^-(z) \cap A(M)$. Then, since M is stable, $z \in \Lambda_D^+(y) \subset \Lambda_D^+(A(M)) \subset M$, by theorem 4. This is a contradiction.

Obviously, (2) implies (3). (3) $\Rightarrow$ (1): If M is not stable, there is a $y \in D^+(M) - M$. Now (3) tells us that $\Lambda^-(y) \cap M = \Phi$. But this contradicts theorem 2.

4. <u>The Hypothesis (#)</u>.

By definition, a compact invariant set M is an attractor if and only if $\Lambda^+(U - M) \subset M$ for some neighborhood U of M. Now, the analogous condition $\Lambda_D^+(U - M) \subset M$ is certainly necessary for asymptotic stability of

M, as theorem 4 shows. On the other hand, consideration of our first example of an unstable attractor (the path monogon on the 1 sphere) shows that this condition does not in general imply that an attractor is stable. Nevertheless, this condition has some interesting consequences, and, as we shall see (theorem 7 and corollary 1) does imply asymptotic stability under reasonable hypotheses.

Now, suppose our condition is satisfied; then there is a neighborhood U of M for which $\Lambda_D^+(U - M) \subset M$. Then, if we write $A^*(M) = A(M) - M$, and $z \in A^*(M)$, there is a $t > 0$ such that $\pi(z, t) \in U$. Now $\Lambda_D^+(z) = \Lambda_D^+(\pi(z, t)) \subset \Lambda_D^+(U - M) \subset M$. That is, $\Lambda_D^+(A^*(M)) \subset M$; we call this <u>hypothesis</u> ($\mathcal{J}$).

In order to state our first result concerning hypothesis ($\mathcal{J}$) we require a few definitions. A dynamical system in a space Y is said to be <u>parallelizable</u> if there is a set $S \subset Y$ which intersects every orbit of the dynamical system, and a homeomorphism h of Y onto $S \times R$ such that $h(\pi(x, t)) = (x, t)$, for $x \in S$. In [4], it is shown that a dynamical system is parallelizable if and only if it is <u>dispersive</u> - that is, if $y_1, y_2 \in Y$, there are neighborhoods U_1 and U_2 of y_1 and y_2 respectively and a positive number T such that $\pi(U_1, t) \cap U_2 = \Phi$ for $t \geq T$. It is easy to see that this is equivalent to the requirement that $\Lambda_D^+(y) = \Phi$ for all $y \in Y$. Using this characterization of a parallelizable dynamical system, we may obtain a condition equivalent to ($\mathcal{J}$).

<u>Theorem 6.</u> <u>Let</u> M <u>be an attractor.</u> <u>Then</u> (#) <u>holds if and only if</u> $A^*(M)$ <u>is parallelizable.</u>

<u>Proof.</u> If (#) holds, $\Lambda_D^+(A^*(M)) \cap A^*(M) \subset M \cap A^*(M) = \Phi$, and $A^*(M)$ is parallelizable. Suppose $A^*(M)$ is parallelizable, and let $x \in A^*(M)$. Then $\Lambda_D^+(x) \subset M \cup \partial A(M)$. Now $\gamma^+(x) \cup M$ is compact, and thus possesses a compact neighborhood K with $K \cap \partial A(M) = \Phi$. Now, by a fundamental property of the prolongation ([1], p. 456), $D^+(x) \subset K$, or $D^+(x) \cap \partial K \neq \Phi$. Using the representation $D^+(x) = \gamma^+(x) \cup \Lambda_D^+(x)$, we see that the latter possibility is excluded. Hence $D^+(x) \subset K$, and $\Lambda_D^+(x) \subset K \cap (M \cup \partial A(M)) \subset M$.

<u>Theorem 7.</u> <u>Let</u> M <u>be an attractor for which</u> (#) <u>holds.</u> <u>Suppose also that</u> $D^+(M) \neq X$, <u>and that</u> $X^* = X - M$ <u>is connected.</u> <u>Then</u> M <u>is stable.</u>

<u>Proof.</u> If M is not stable, $D^* = D^+(M) - M$ is non empty. Now D^* is closed in X^*, and, since X^* is connected $X^* - D^*$ is not closed in X^*. Since $D^+(M) \neq X$, $X^* - D^* \neq \Phi$. Hence, there is a $y \in D^*$ and a sequence $\{y_n\}$ in $X^* - D^*$ such that $y_n \to y$. Since $A(D^+(M)) = A(M)$ is open (lemma 1) we may suppose $y_n \in A(M)$. Choose $\epsilon > 0$ so that (i) $B(M, \epsilon)$ and $B(D^+(M), \epsilon)$ are compact, (ii) $y \notin S(M, \epsilon)$, and (iii) $B(D^+(M), \epsilon) \subset A(M)$. Now $y_n \in A(D^+(M)) - D^+(M)$, and since $D^+(M)$ is asymptotically stable, $\Lambda^-(y_n) \cap A(M) = \phi$ (theorems 1 and 5). Then there is a sequence $\{t_n\}$ of reals with $t_n \to -\infty$ such that $\pi(y_n, t_n) \in H(D^+(M), \epsilon)$. Since $H(D^+(M), \epsilon)$ is compact, we may suppose $\pi(y_n, t_n) \to x^* \in H(D^+(M), \epsilon)$. Now, $x^* \in A^*(M)$. But $y \in \Lambda_D^+(x^*)$, and $y \notin M$. This contradicts (#).

From theorems 1 and 7 we immediately obtain:

Corollary 1. Let M be an attractor for which (#) holds. If X is not compact and $X^* = X - M$ is connected, then M is stable.

In the Mendelson example hypothesis (#) is of course not satisfied; indeed as we observed earlier if $x \in \gamma_2$, $A_D^+(x) = \gamma_1 \cup \{P\}$. If we consider the dynamical system obtained by deleting the orbit γ_2, then (#) is satisfied, although $\{P\}$ is still not stable. This is not a counterexample to corollary 1, however, since the phase space $R^2 - \gamma_2$ is not locally compact.

Finally, we show that the condition (#) is, in a sense, "generic" for attractors. That is, the set of $x \in A^*(M)$ for which $A_D^+(x) \not\subset M$ is sparse in the category sense. To show this, we require several lemmas.

Lemma 6. The set of pairs (x, y) such that $y \in A_D^+(x)$ is closed in $X \times X$. That is, if $\{x_n\}$ and $\{y_n\}$ are sequences in X with $y_n \in A_D^+(x_n)$, $x_n \to x$, $y_n \to y$, then $y \in A_D^+(x)$.

The proof follows easily from the definition of prolongational limit set and is therefore omitted.

Lemma 7. Let M be an attractor, and let U be open in X. Then $A^{\#}(U) = [x \in A^*(M) \mid A_D^+(x) \subset U]$ is open.

Proof. If $A^{\#}(U)$ is not open, there is an $x \in A^{\#}(U)$, and a sequence $\{x_n\}$ such that $x_n \notin A^{\#}(U)$, and $x_n \to x$. Let

$y_n \in \Lambda_D^+(x_n) \cap (X - U)$. Now $y_n \in \Lambda_D^+(x_n) \subset \Lambda_D^+(A^*(M)) \subset D^+(M)$, since $D^+(M)$ is asymptotically stable. Since $D^+(M)$ is compact, we may suppose $y_n \to y \in D^+(M)$. Since $y_n \notin U$, $y \notin U$. But the previous lemma tells us that $y \in \Lambda_D^+(x) \subset U$. This is a contradiction.

Lemma 8. Let M be an attractor, and let N be an open neighborhood of M with $N \subset A(M)$. Then $A^{\#}(N)$ is open and dense in $A^*(M)$.

Proof. That $A^{\#}(N)$ is open has just been shown. Let K be a closed neighborhood of M such that $K \subset N$, and let U be a non-empty open set in $A(M)$. Choose a sequence of reals $\{t_n\}$ such that $t_n \to \infty$, and let $U_n = [x \in U | \pi(x, t) \in K,$ for $t \geq t_n]$. Since M is an attractor, $U = \cup U_n$, and by the Baire category theorem there is an n such that $W_n = \operatorname{int} \bar{U}_n$ is non-empty. Then $\pi(W_n, t) \subset K$, for $t \geq t_n$, and it follows easily that $\Lambda_D^+(x) \subset K \subset N$, for $x \in W_n$. But $x \in W_n \subset \bar{U}_n \subset \bar{U}$. This proves $A^{\#}(N)$ is dense in $A^*(M)$.

Now we can easily obtain the result we promised earlier.

Theorem. Let M be an attractor. Then $[x \in A^*(M) | \Lambda_D^+(x) \not\subset M]$ is a first category set in $A^*(M)$.

Proof. Let $C_n = [x \in A^*(M) | \Lambda_D^+(x) \not\subset S(M, \frac{1}{n})]$, $n = 1,2,\ldots$. Each C_n is nowhere dense (since, by lemma 8, its complement is open and dense), and $\bigcup_{n=1,2,\ldots} C_n = [x \in A(M) | \Lambda_D^+(x) \not\subset M]$. The proof is completed.

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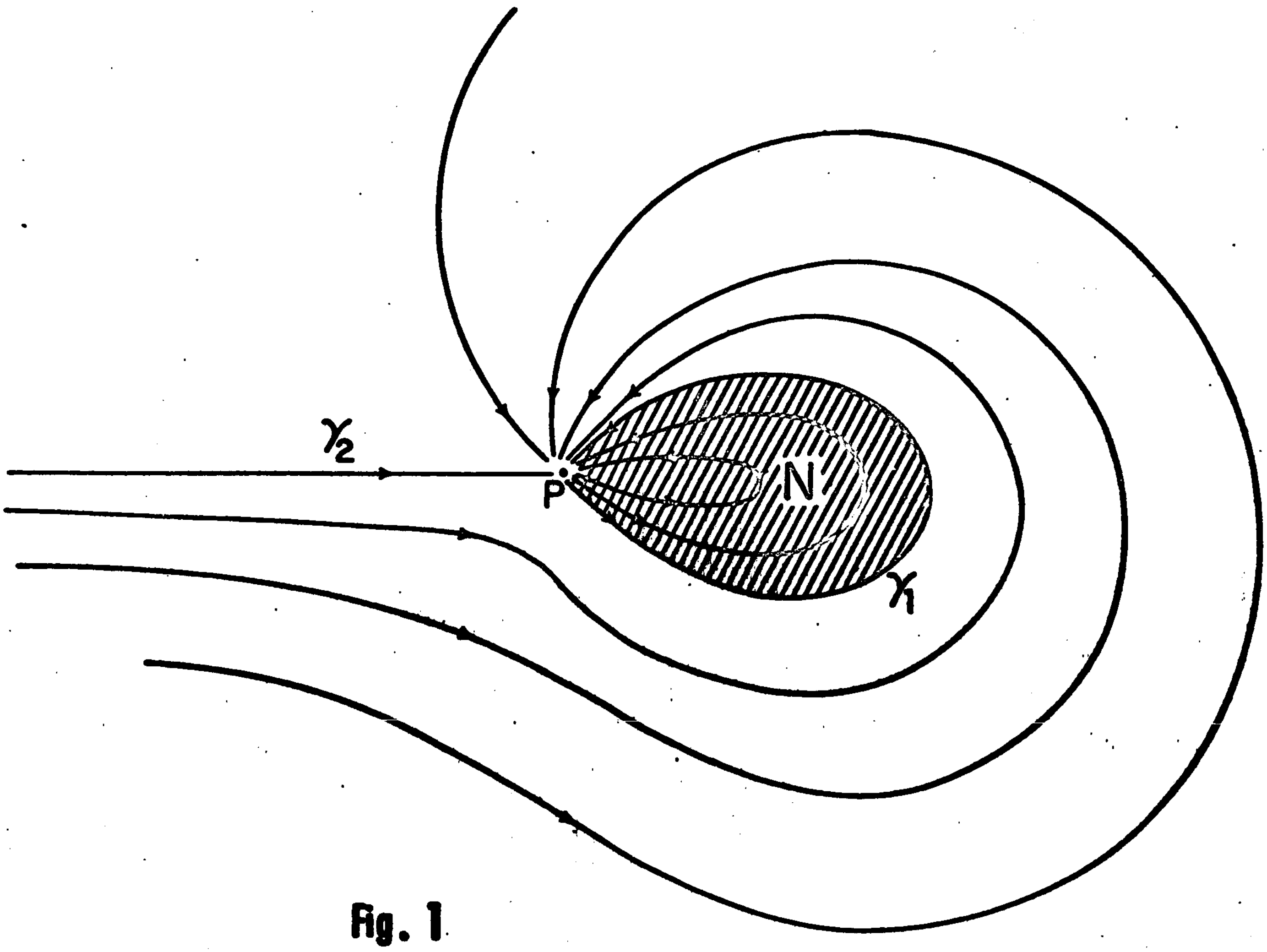

Fig. 1

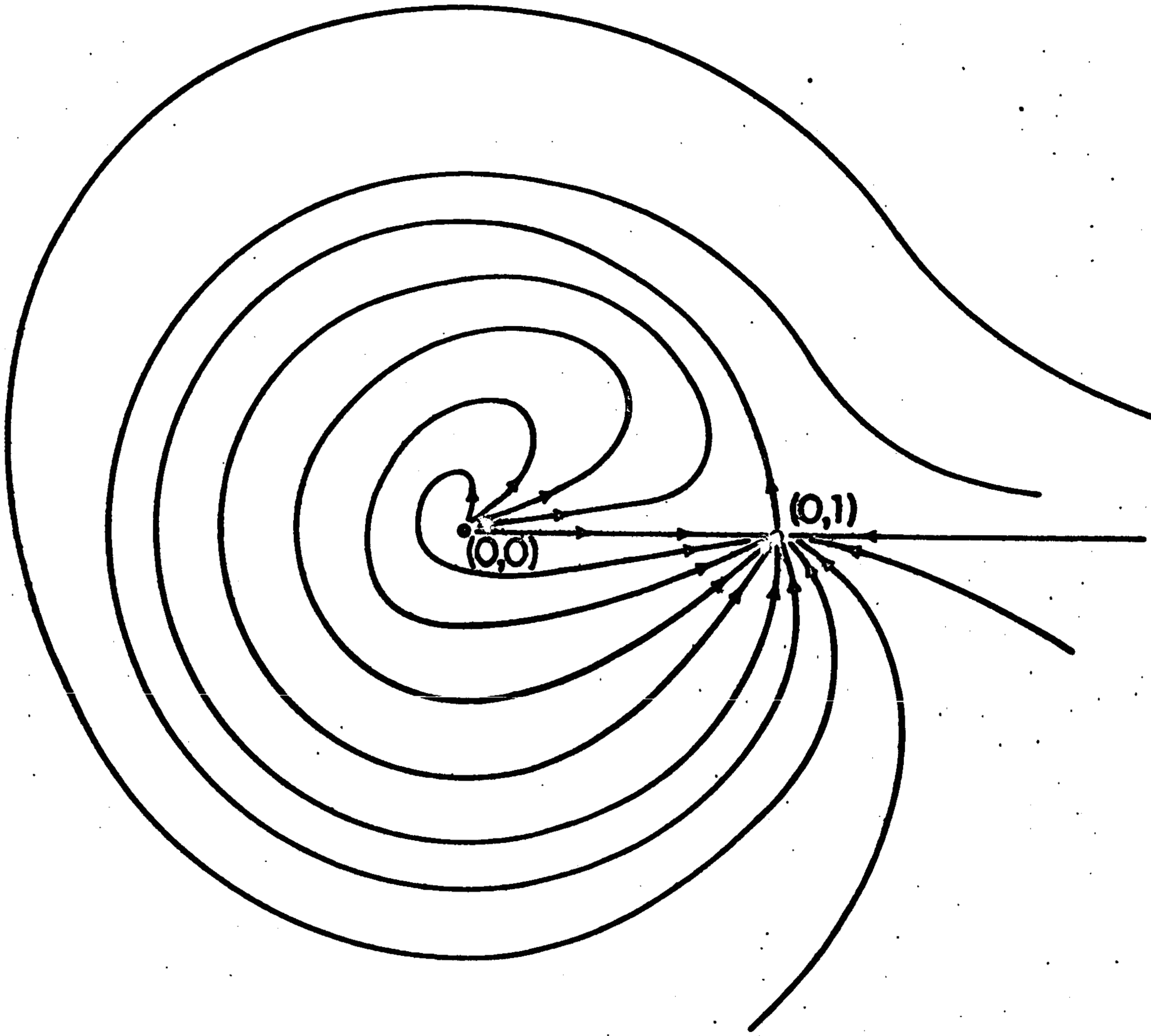

Fig. 2

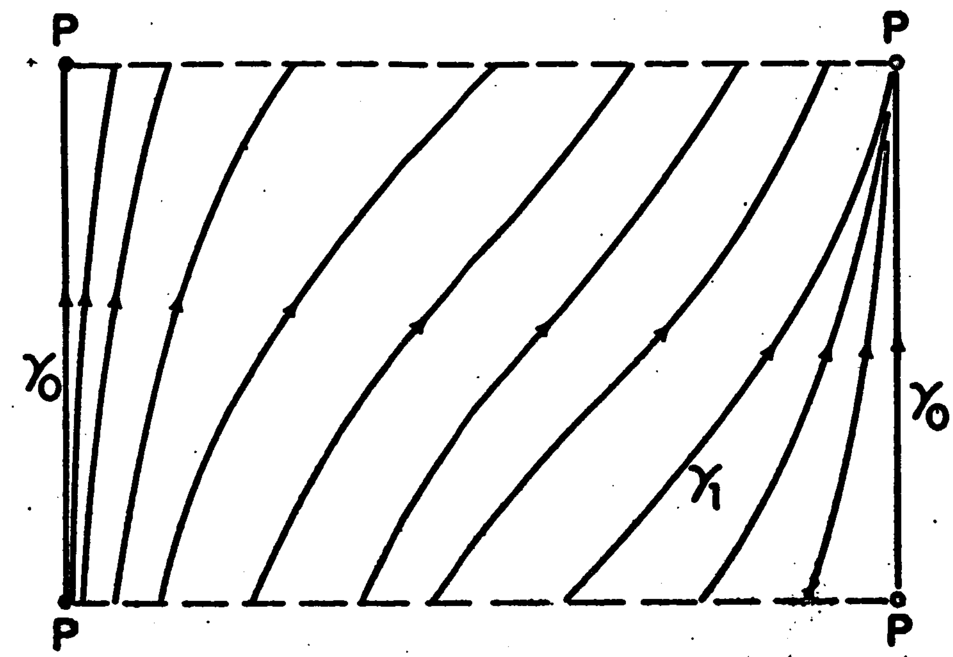

Fig. 3

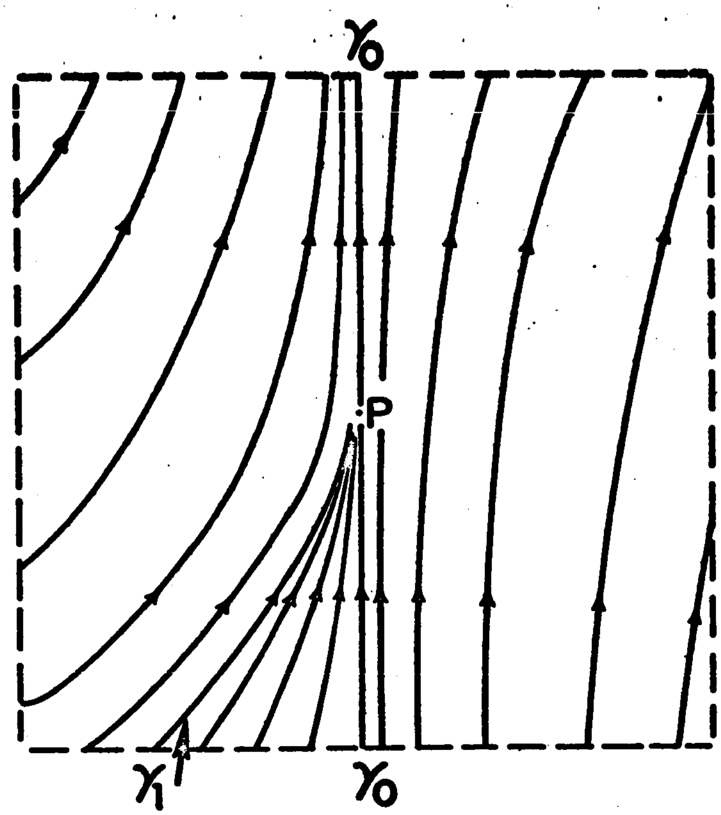

Fig. 4