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On the Antenna Radiation
Through a Plasma Sheath

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ON THE ANTENNA RADIATION THROUGH A PLASMA SHEATH

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ABSTRACT

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The influence of the plasma sheath on the slot antenna properties is considered.

It is shown that opening a slot in a thin overdense plasma sheath re-establishes

in many cases the radiation of an elementary cylinder antenna. The changes in

the impedance of a cavity backed slot antenna with the onset of a plasma sheath

are formulated on the basis of an energy theorem.   *author*

ACKNOWLEDGMENTS

I. INTRODUCTION

The difficulty in maintaining communications with the aerospace vehicle during those phases of the flight when a plasma sheath forms on the antenna surfaces is well known (Rotman and Meltz, 1960; Ellis, 1962).[1] The necessity of keeping radio contact with the vehicle at these times is equally clear (Huber and Nelson, 1962). It appears that a contribution to the solution of this problem will come from a quantitative understanding of the role the plasma sheath plays in modifying the absolute gain and the radiation pattern of the aerospace vehicle antennas. This requires developing analytical solutions and physical concepts to explain the effect on the far-zone antenna characteristics caused by the presence of reactive and lossy plasma sheath on the antenna surfaces. Once these interaction mechanisms are understood, a purposeful control of the plasma configurations by aerodynamic shaping of the vehicle, and perhaps by opening slots in the plasma sheath (Cuddihy, et al, 1963) or some other means will hopefully reduce the undesirable antenna pattern changes and signal attenuation to levels that are tolerable.

A simplified version of an aerospace vehicle is shown in Fig. 1-1a and 1-1b. It is a composite body consisting of a hemisphere, a section of a cone, and a section of a cylinder. Some sections of the body may be coated with a dielectric layer. A problem of current interest involves a radiating slot on the cone surface when the

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[1] Chapter references are found at the end of each chapter.

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Section A-A $0^{\circ} < \theta < 180^{\circ}$

FIG. 1-1b

θ

Slot Antenna

A

A

Plasma Sheath

Dielectric

Conductor
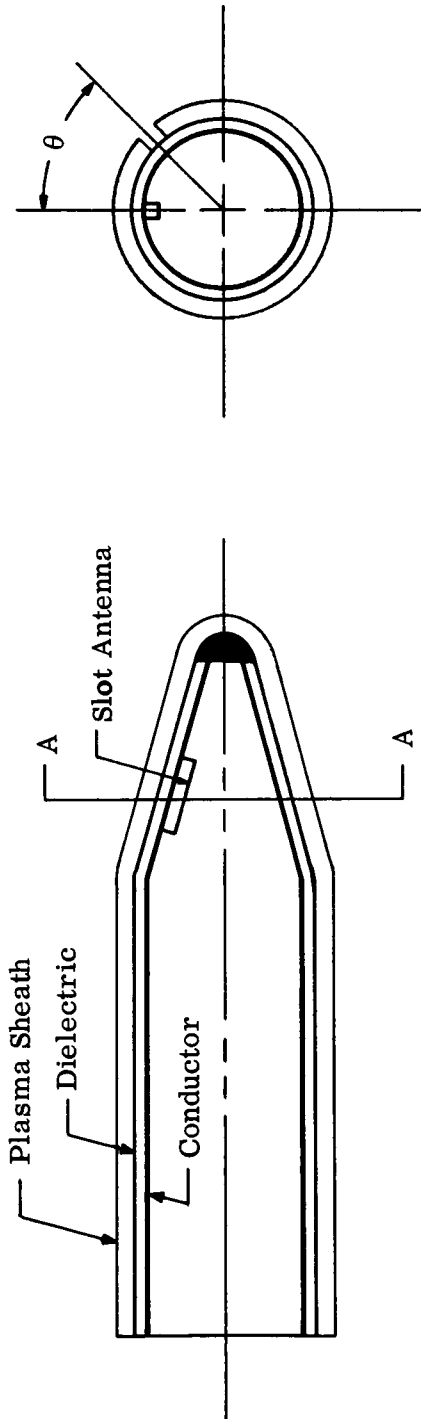
FIG. 1-1a

FIG. 1-1: AEROSPACE VEHICLE

vehicle is surrounded by a plasma sheath. We would like to find the changes in the antenna impedance, in the radiation pattern, and in the absolute gain with the onset of the plasma sheath. The plasma sheath may have an axial slot. The plasma slot may be displaced circumferentially from the cone slot. The arrangement of the various layers and the slots are shown in the cross section drawing, Fig. 1-1b.

It is clear that the configuration of this simple aerospace vehicle is such that the vector wave equation is not separable, and therefore the theoretical treatment of the antenna problem is extremely difficult. It appears that one approach to this problem should be that of first solving an appropriate canonical antenna problem and then extending the results of this analysis, using some physical insight and perhaps an experiment, to the vehicle shapes of interest. For the canonical problem it is advisable to choose the cylindrical geometry where the possibilities of a solution are favorable. The cross section view of the cylinder antenna would be the same as in Fig. 1-1b. A narrow axial slot of infinite length is cut in a perfectly conducting, infinitely long cylinder that is enclosed by a uniform dielectric layer. On the surface of the dielectric layer is a uniform plasma sheath. A narrow axial slot of infinite length is maintained in the plasma sheath. The slot of the plasma sheath may be above the cylinder slot, or displaced circumferentially from it. It is proposed that for a given tangential electric field in the cylinder slot the solution of the electromagnetic boundary value problem be obtained. In the first attack on this problem we consider a limiting case of some interest. When the plasma layer is then compared

to the wavelength and highly overdense, we can replace it by a vanishingly thin, perfectly conducting shell. This reduces the configuration to a two-region problem.

Furthermore, we take the dielectric under the shell to be the same as outside the shell. This problem has been formulated in Chapter II as a Fredholm integral equation of the first kind. It has been solved for the case of the narrow slot, and from this solution we derive the effect of the slotted shell on the radiation properties of the cylinder antenna. The cylinder diameters considered range from a small fraction of the wavelength to $1.8/\pi$ wavelength. The spacing between the cylinder and the shell is a small fraction of the wavelength. The principal conclusion is that opening a slot in the shell re-establishes the cylinder antenna radiation, except for certain angular separations of the cylinder slot from the shell slot. For details one should refer to the radiation curves in Chapter II.

In Chapter III we have an interesting new approach to the solution of the Fredholm integral equation that appeared in the previous chapter. The approach is to construct a solution to the Fredholm integral equation from a solution of a singular integral equation. The singular integral equation is derived by differentiating both sides of the Fredholm integral equation of the first kind. We hope that this solution will be of considerable value, especially for the discussion of the wide shell slot, in the future.

In Chapter IV we discuss a more general problem than those of II and III. We consider the impedance of a small slot excited by a co-axial transmission line

and backed by a cavity. Using the Energy Theorem we express the impedance in terms of certain integrals over the electromagnetic fields. From this analysis it appears that the antenna will be least unfavorably affected by the plasma sheath when it is tuned primarily by internal resonances, and the electric reactive energy is kept at a minimum on the antenna surfaces. It is doubtful if making the cavity lossy alone would be a desirable feature in devising plasma-proof antennas.

In Chapter V we discuss some suggestions for future work.

REFERENCES FOR CHAPTER I

Cuddihy, W. F., I. E. Beckwith and L. C. Schroeder (1963), "RAM B-2 Flight Tests of A Method for Reducing Radio Attenuation During Hypersonic Re-entry," NASA Report No. NASA-TM-X-902, October, 1963 (CONFIDENTIAL).

Ellis, C. E. Jr. (1962), Proceedings of the Second Symposium on the Plasma Sheath, Boston, Mass. (SECRET) 10-12 April.

Huber, P. W. and C. H. Nelson (1962), "Plasma Frequency and Radio Attenuation," NASA-University Conference on Science and Technology of Space Exploration, Chicago, November 1-3, Vol. 2, pp. 347-360.

Rotman, W. and G. Meltz (1961), Electromagnetic Effects of Re-entry, Pergamon Press, New York.

II. RADIATION OF AN ELEMENTARY CYLINDER ANTENNA THROUGH A SLOTTED ENCLOSURE. A. Olte

2.1 Introduction

In this chapter we treat the radiation problems of a cylinder antenna shrouded by a concentric axially sloted shell. The source of the cylinder antenna is an infinite axial slot uniformly excited, with the electric field in the circumferential direction. In the following five sections we present the analyses of the problem.

In the first section we reduce the boundary value problem of the antenna by employing a conventional series representation of the fields to a Fredholm integral equation of the first kind. The integral equation uniquely determines the tangential electric field of the slotted shell for the given source on the cylinder surface. The kernel of the integral equation is complex, non-Hermitian, and has a logarithmic singularity.

In the second section we briefly discuss the physical aspects of a singular integral equation which follows by differentiation of the Fredholm integral equation of the first kind. Hayashi was the first to derive the singular integral equation in a similar problem. This is discussed in detail in Chapter III of this report.

In the third section we report a solution of the Fredholm integral equation for the case of a narrow shell slot. Recognizing that in a narrow slot the field distribution is dominated by the edge singularity we are led to a slot field representation by a Fourier series of a kind where the first term is the dominant one. This idea has already been applied by Morse and Feshbach (1953) in discussing the scattering of

an electromagnetic wave normally incident on an axially slotted, perfectly conducting cylindrical shell.

In the fourth section of this chapter we report the numerical calculations which are based on the above solution of the Fredholm integral equation. The purpose of the calculations is to exhibit the influence of the slotted shell on the radiation of the cylinder antenna. The cylinder diameters considered are, in wavelength, from $0.2/\pi$ to $1.8/\pi$. The radial spacings between the cylinder and the shell are $0.1/\pi$, $0.05/\pi$, and $0.025/\pi$. It will be obvious that for the parameter values considered the slotted shell does not significantly modify the form of the cylinder antenna radiation pattern. However, the pattern is rotated by the angle between the source on the cylinder surface and the shell slot, although for the cylinder diameters considered the radiation is nearly omnidirectional anyway. Even for the largest diameter antenna considered the radiation field is omnidirectional to within ± 25 percent (Wait, 1959). Therefore, we have chosen to report the ratio of the radiated power with the slotted shell and without it, as a function of the source and the shell slot separation angle.

In the fifth section of this chapter we discuss the accuracy of the approximate solution of the integral equation, and the error reflected in the power radiated. We also discuss some of the physical implications of the solution.

In the last section we briefly discuss and summarize the main features of the results. The slotted shell prevents radiation only for certain parameter

combinations; for most others, the radiation remains the same and for some it is even enhanced.

2.2 Reduction of the Boundary Value Problem to a Fredholm Integral Equation of the First Kind.

We consider a wedge waveguide of width $2\theta_o$ feeding in the lowest order transverse electric mode a perfectly conducting circular cylinder of radius a, as shown in Fig. 2-1. The cylinder is concentrically shrouded by a vanishingly thin perfectly conducting shell of radius b. The shell has an axial slot of width $2\phi_o$. The center-to-center circumferential displacement of the shell slot and the cylinder slot is indicated by the angle θ. We employ a right-hand circular cylindrical coordinate system (r, ϕ, z) for which ϕ is measured counter-clockwise from the center of the shell slot and z is along the axis of the cylinder. The constitutive parameters ϵ and μ are assumed to be real. The rational MKS system of units is used and the time dependence of $e^{j\omega t}$ is implied for all field quantities.

We have a two-dimensional problem, since both the antenna structure and the source is independent of z. Furthermore, it is a three region problem: $r < a$; $a \leqslant r \leqslant b$; $r \geqslant b$. For this particular study we limit it to a two region problem by considering the tangential electric field of the cylinder slot as given. We set out to find the fields in the coaxial region and in the free space. The fields are given (Stratton, 1941) in a series form.

For $a \leqslant r \geqslant b$

$$H_z = k^2 \sum_{n=-\infty}^{\infty} \left\{ B_n J_n(kr) + C_n N_n(kr) \right\} e^{-jn\phi} \qquad (2.1)$$
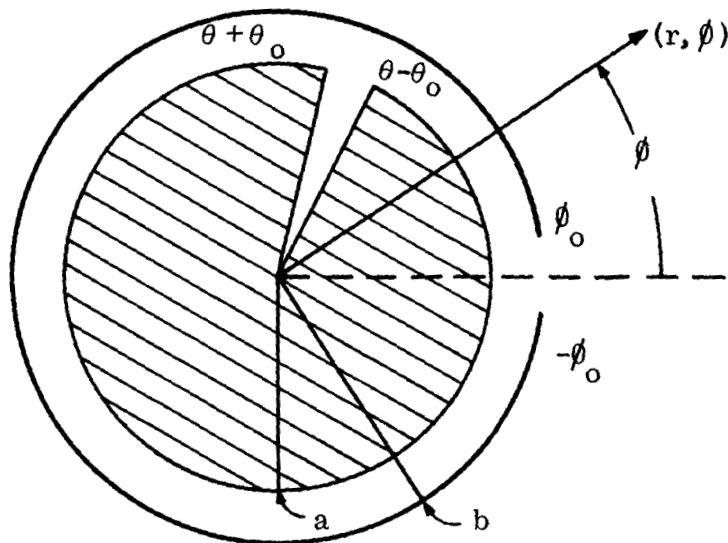
FIG. 2-1

CYLINDER ANTENNA SHROUDED BY A SLOTTED SHELL

$$E_\phi = j\omega\mu k \sum_{n=-\infty}^{\infty} \left\{ B_n J_n'(kr) + C_n N_n'(kr) \right\} e^{-jn\phi} \tag{2.2}$$

$$E_r = -\frac{\omega\mu}{r} \sum_{n=-\infty}^{\infty} n \left\{ B_n J_n(kr) + C_n N_n(kr) \right\} e^{-jn\phi}, \tag{2.3}$$

and for $r \geqslant b$

$$H_z = \sum_{n=-\infty}^{\infty} k^2 A_n H_n^{(2)}(kr) e^{-jn\phi} \tag{2.4}$$

$$E_\phi = \sum_{n=-\infty}^{\infty} k j\omega\mu A_n H_n^{(2)'}(kr) e^{-jn\phi} \tag{2.5}$$

$$E_r = - \sum_{n=-\infty}^{\infty} \frac{n\omega\mu}{r} A_n H_n^{(2)}(kr) e^{-jn\phi}, \tag{2.6}$$

where $k = \omega\sqrt{\mu\epsilon}$.

We regard

$$E_\phi(a, \phi) \equiv f(\phi), \qquad \theta - \theta_o \leqslant \phi \leqslant \theta + \theta_o$$

as given and we seek to find

$$E_\phi(b, \phi) \equiv E(\phi), \qquad -\phi_o \leqslant \phi \leqslant \phi_o.$$

We require that

$$j\omega\mu k \sum_{n=-\infty}^{\infty} \left\{ B_n J_n'(kb) + C_n N_n'(kb) \right\} e^{-jn\phi} = E(\phi), \quad -\phi_o \leqslant \phi \leqslant \phi_o$$

$$= 0, \quad \phi_o \leqslant \phi \leqslant 2\pi - \phi_o \tag{2.7}$$

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$$j\omega\mu k \sum_{n=-\infty}^{\infty} \left\{ B_n J'_n (ka) + C_n N'_n (ka) \right\} e^{-jn\phi}$$

$$= f(\phi), \quad \theta - \theta_o \leqslant \phi \leqslant \theta + \theta_o$$

$$= 0, \quad \theta + \theta_o \leqslant \phi \leqslant 2\pi + (\theta - \theta_o)$$

$$(2.8)$$

$$\sum_{n=-\infty}^{\infty} j\omega\mu k A_n H_n^{(2)'} (kb) e^{-jn\phi} = E(\phi), \quad -\phi_o \leqslant \phi \leqslant \phi_o$$

$$= 0, \quad \phi_o \leqslant \phi \leqslant 2\pi - \phi_o \; .$$

$$(2.9)$$

Because of the orthogonality of the circular functions we obtain from (2.7), (2.8)

and (2.9) respectively

$$j\omega\mu k \left\{ B_n J'_n (kb) + C_n N'_n (kb) \right\} 2\pi = \int_{-\phi_o}^{\phi_o} E(\phi') e^{jn\phi'} d\phi' \qquad (2.10)$$

$$j\omega\mu k \left\{ B_n J'_n (ka) + C_n N'_n (ka) \right\} 2\pi = \int_{\theta-\theta_o}^{\theta+\theta_o} f(\eta) e^{jn\eta} d\eta \qquad (2.11)$$

$$j\omega\mu k A_n H_n^{(2)'} (kb) 2\pi = \int_{-\phi_o}^{\phi_o} E(\phi') e^{jn\phi'} d\phi' \; , \qquad (2.12)$$

where ϕ' and η are dummy variables of integration. From (2.10) and (2.11)

we obtain

$$j2\pi \omega\mu k D_n(ka, kb) B_n = N_n'(kb) \int_{\theta-\theta_0}^{\theta+\theta_0} f(\eta)d^{jn\eta}d\eta$$

$$- N_n'(ka) \int_{-\phi_0}^{\phi_0} E(\phi')e^{jn\phi'}d\phi' , \qquad (2.13)$$

$$j2\pi \omega\mu k D_n(ka, kb) C_n = -J_n'(kb) \int_{\theta-\theta_0}^{\theta+\theta_0} f(\eta)e^{jn\eta}d\eta$$

$$+ J_n'(ka) \int_{-\phi_0}^{\phi_0} E(\phi')e^{jn\phi'}d\phi' , \qquad (2.14)$$

where

$$D_n(ka, kb) = J_n'(ka)N_n'(kb) - J_n'(kb)N_n'(ka). \qquad (2.15)$$

We observe that (2.12), (2.13) and (2.14) give us the Fourier coefficients of the fields for both regions once the electric slot fields are known. However, we do not know the shell slot field. We seek to find it by enforcing the remaining boundary condition: the continuity of the tangential magnetic field through the shell slot, i.e.

$$\sum_{n=-\infty}^{\infty} A_n H_n^{(2)}(kb)e^{-jn\phi} = \sum_{n=-\infty}^{\infty} \left[B_n J_n(kb) + C_n N_n(kb) \right] e^{-jn\phi}, \quad -\phi_0 \leqslant \phi \leqslant \phi_0 .$$

$$(2.16)$$

We eliminate the coefficients in (2.16), and after factoring and transposing of

terms obtain

$$\sum_{n=-\infty}^{\infty} \left[\frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{1}{D_n(ka, kb)} \int_{-\phi_0}^{\phi_0} E(\phi') e^{jn\phi'} d\phi' \right] e^{-jn\phi}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{D_n(ka, kb)} \int_{\theta-\theta_0}^{\theta+\theta_0} f(\eta) e^{jn\eta} d\eta \right] e^{-jn\phi} ,$$

$$-\phi_0 \leqslant \phi \leqslant \phi_0 . \tag{2.17}$$

Interchanging integration with summation on the left hand side we obtain an integral

equation of the form

$$\int_{-\phi_0}^{\phi_0} E(\phi') K(\phi', \phi) d\phi' = g(\theta, \phi); \quad -\phi_0 \leqslant \phi \leqslant \phi_0 , \tag{2.18}$$

where

$$K(\phi', \phi) \equiv \sum_{n=-\infty}^{\infty} \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{1}{D_n(ka, kb)} e^{jn(\phi'-\phi)}$$

$$g(\theta, \phi) \equiv \sum_{n=-\infty}^{\infty} \left[\frac{1}{D_n(ka, kb)} \int_{\theta-\theta_0}^{\theta+\theta_0} f(\eta) e^{jn\eta} d\eta \right] e^{-jn\phi} .$$

It is clear that (2.18) is a Fredholm integral equation of the first kind with a

complex non-Hermitian kernel, i.e.

$$K(\phi', \phi) \neq K^*(\phi, \phi'). \tag{2.19}$$

The kernel has also a logarithmic singularity. Interchanging integration with summation in the representation of the known function, we have

$$g(\theta,\phi)= \int_{\theta-\theta_0}^{\theta+\theta_0} f(\eta) K^{(a)}(\eta,\phi)\,d\eta,$$ (2.20)

where

$$K^{(a)}(\eta,\phi) \sum_{n=-\infty}^{\infty} \frac{1}{D_n(ka,\,kb)} e^{jn(\eta-\phi)}.$$

The kernel $K^{(a)}(\eta,\phi)$ is a continuous function of η and ϕ. We see that the known function of the integral equation (2.18) is obtained by transforming the electric field of the cylinder slot according to (2.20). We may further simplify the two kernels by observing that

$$D_{-n}(ka,\,kb) = D_n(ka,\,kb)$$

$$\frac{H^{(2)'}_{-n}(ka)}{H^{(2)'}_{-n}(kb)} = \frac{H^{(2)'}_n(ka)}{H^{(2)'}_n(kb)}$$

and introducing a factor ϵ_n such that

$$\epsilon_n = 1,\ n = 0$$

$$\epsilon_n = 2,\ n = 1,\,2,\,3,\,\ldots\,.$$

We have then

$$K(\phi',\phi) = \sum_{n=o}^{\infty} \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{\epsilon_n}{D_n(ka,kb)} \cos n(\phi'-\phi) \ , \tag{2.21}$$

and

$$K^{(a)}(\eta,\phi) = \sum_{n=0}^{\infty} \frac{\epsilon_n}{D_n(ka,kb)} \cos n\,(\eta-\phi). \tag{2.22}$$

We have reduced the boundary value problem to a Fredholm integral equation of the first kind with a non-Hermatian kernel. This is a unique statement of the original problem and no additional conditions need be imposed.

2.3 The Appearance of the Singular Integral Equation

It is well known that the normal component of the electric field must be continuous through an aperture. It is also well known that the continuity of the tangential magnetic field through the aperture automatically insures the continuity of the normal electrical field component as well. The reverse, however, is not true and therefore the continuity of the normal component of the electric field through the aperture is necessary, but not sufficient for a boundary condition. The continuity condition of the normal component of the electric field through the shell aperture in our case corresponds to differentiating the Fredholm integral equation of the first kind with respect to ϕ. We obtain

$$P\int_{-\phi_o}^{\phi_o} E(\phi')K'(\phi',\phi)d\phi' = \int_{\theta-\theta_o}^{\theta+\theta_o} f(\eta)K^{(a)'}(\eta,\phi)d\eta, \ -\phi_o < \phi < \phi_o, \tag{2.23}$$

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where the primes indicate that the kernels are to be differentiated with respect to ϕ.

The integral of this type in a similar problem was first obtained by Hayashi (see

Chapter III of this report). The singularity of the kernel $K(\phi', \phi)$ by differentiation

has been increased to a Cauchy type singularity and therefore we have to take the

integral in the principal value sense which we explicityly indicate by P in front of

the integration sign. Whereas the Fredholm integral equation exists as $\phi \rightarrow \pm \phi_o$,

(2.23) does not and we restrict ϕ to the open interval $-\phi_o < \phi < \phi_o$. The kernel

$K^{(a)'}(\eta, \phi)$ is continuous and the right hand side of (2.23) exists even as $0 \rightarrow \pm \phi_o$.

Evidently the Cauchy integral equation for this particular problem admits a set of

solutions. Somehow one has to choose a solution that satisfies the Fredholm

integral equation. One may possibly choose from the set a solution that satisfies the

well known condition of the edge singularity at $\phi = \pm \phi_o$. This solution is a sum of a

certain particular integral and a certain general solution of the homogeneous equation

of (2.23), both parts satisfying the edge singularity condition independent of each

other. The amplitude of the general solution of the homogeneous equation clearly re-

mains arbitrary. If finding the proper amplitude of this part of the solution gives us

the unique slot field, then it may be determined by substituting the total solution in

the Fredholm integral equation. These appear to be the physical aspects of the

singular integral equation method used by Hayashi in constructing a formal solution

to a similar boundary value problem in the subsequent chapter.

2.4 Solution of the Fredholm Integral Equation for the Case of a Narrow Slot.

The Fredholm integral equation of the first kind (2.18) was derived for the cylinder slot of arbitrary width $2\theta_o$. We simplify the problem, but retain its essential features by letting $\theta_o \to 0$, because then we can let the slot field assume a δ-function distribution, i.e.

$$f(\phi) = \frac{V_a}{a} \delta(\theta - \phi) \text{ as } \theta_o \to 0, \tag{2.24}$$

where V_a is the slot voltage.

In this case then

$$g(\theta, \phi) = V_a \int_{\theta - \theta_o}^{\theta + \theta_o} \frac{\delta(\theta - \eta)}{a} K^{(a)}(\eta, \phi) d\eta = \frac{1}{a} V_a K^{(a)}(\theta, \phi). \tag{2.25}$$

From (2.21) and (2.22) we observe that $K(\phi', \phi)$ and $K^{(a)}(\theta, \phi)$ may be divided into even and odd parts with respect to both variables, i.e.

$$K(\phi', \phi) = K_e(\phi', \phi) + K_o(\phi', \phi) \tag{2.26}$$

$$K^{(a)}(\theta, \phi) = K_e^{(a)}(\theta, \phi) + K_o^{(a)}(\theta, \phi), \tag{2.27}$$

where

$$K_e(\phi', \phi) = \sum_{n=0}^{\infty} \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{\epsilon_n}{D_n(ka, kb)} \cos(n\phi') \cos(n\phi) \tag{2.26a}$$

$$K_o(\phi'\phi) = \sum_{n=1}^{\infty} \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{2}{D_n(ka, kb)} \sin(n\phi')\sin(n\phi) \qquad (2.26\text{-}b)$$

$$K_e^{(a)}(\theta, \phi) = \sum_{n=0}^{\infty} \frac{\epsilon_n}{D_n(ka, kb)} \cos(n\theta)\cos(n\phi) \qquad (2.27\text{-}a)$$

$$K_o^{(a)}(\theta, \phi) = \sum_{n=1}^{\infty} \frac{2}{D_n(ka, kb)} \sin(n\theta)\sin(n\phi) \qquad (2.27\text{-}b)$$

Since the unknown slot field may also be represented by an even and an odd part, we have

$$E(\phi) = E_e(\phi) + E_o(\phi) \qquad (2.28)$$

and the Fredholm integral equation becomes

$$\int_{-\phi_o}^{\phi_o} E_e(\phi')K_e(\phi', \phi)d\phi' + \int_{-\phi_o}^{\phi_o} E_o(\phi')K_o(\phi', \phi)d\phi' = \frac{1}{a} V_a \left[K_e^{(a)}(\theta, \phi) + K_o^{(a)}(\theta, \phi)\right],$$
$$-\phi_o \leqslant \phi \leqslant \phi_o . \qquad (2.29)$$

We observe that the first integral is an even function in ϕ, while the second integral is an odd function, therefore we have that

$$\int_{-\phi_o}^{\phi_o} E_e(\phi')K_e(\phi', \phi)d\phi' = \frac{1}{a} V_a K_e^{(a)}(\theta, \phi), \quad -\phi_o < \phi < \phi_o \qquad (2.30)$$

$$\int_{-\phi_o}^{\phi_o} E_o(\phi')K_o(\phi', \phi)d\phi' = \frac{1}{a} V_a K_o^{(a)}(\theta, \phi), \quad -\phi_o < \phi < \phi_o \qquad (2.31)$$

We have succeeded in breaking the Fredholm integral equation into two integral equations of the same kind. The first one determines the even part of the shell slot field and the second one, the odd part. Stating the problem in this form means that we invert two small matrices instead of one large one in order to obtain the same accuracy in the solution.

Electromagnetic fields cannot have large spatial variations over distances that are small compared to the wavelength, except in the vicinity of the sources, at the discontinuities in the medium constitutive parameters, and at sharp conducting edges. The slot field of a narrow slot is therefore dominated by the edge singularity. We separate this out in the first term of a Fourier representation of the even and odd parts of the slot field, i.e.

$$E_e(\phi) = \frac{a_o}{\pi\sqrt{\phi_o^2 - \phi^2}} + \sum_{q=1}^{\infty} a_q \cos\frac{q\pi\phi}{\phi_o} \qquad (2.32)$$

and

$$E_o(\phi) = \frac{\phi b_1}{\pi\sqrt{\phi_o^2 - \phi^2}} + \sum_{q=2}^{\infty} b_q \sin\frac{q\pi\phi}{\phi_o} . \qquad (2.33)$$

We are guided in selecting these particular forms of the field singularities by the work of Sommerfeld (1964) and Millar (1960) on the diffraction by an infinite slit in a vanishingly thin perfectly conducting plane screen.

The left-hand side of the integral equation (2.30) takes on the form

$$a_o \int_{-\phi_o}^{\phi_o} \frac{K_e(\phi', \phi) d\phi'}{\pi \sqrt{\phi_o^2 - (\phi')^2}} + \int_{-\phi_o}^{\phi_o} d\phi' \sum_{q=1}^{\infty} a_q K_e(\phi', \phi) \cos \frac{q \pi \phi'}{\phi_o} \; .$$

For the first term we have after using the series representation for the kernel and by interchanging summation with integration

$$a_o \sum_{n=o}^{\infty} \epsilon_n \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{\cos(n\phi)}{D_n(ka, kb)} \int_{-\phi_o}^{\phi_o} \frac{\cos(n\phi') d\phi'}{\pi \sqrt{\phi_o^2 - (\phi')^2}} =$$

$$a_o \sum_{n=o}^{\infty} \epsilon_n \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(ka)} \frac{J_o(n\phi_o)}{D_n(ka, kb)} \cos(n\phi) \; .$$

Interchanging integration with summation in the second term we have to evaluate

$$\int_{-\phi_o}^{\phi_o} K_e(\phi', \phi) \cos \frac{q \pi \phi'}{\phi_o} \; d\phi'$$

which after using the series expansion for the kernel and again interchanging integration with summation becomes

$$\sum_{n=o}^{\infty} \epsilon_n \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{\cos(n\phi)}{D_n(ka, kb)} \int_{-\phi_o}^{\phi_o} \cos(n\phi') \cos \frac{q \pi \phi'}{\phi_o} \; d\phi' =$$

$$4\phi_o^2 \sum_{n=1}^{\infty} (-1)^{q+1} \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{n}{D_n(ka, kb)} \frac{\sin(n\phi_o)}{(q\pi)^2 - (n\phi_o)^2} \cos(n\phi) \; .$$

By this the integral equation (2.30) has been converted into the form

$$a_0 \sum_{n=0}^{\infty} \epsilon_n \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{J_0(n\phi_0)}{D_n(ka,kb)} \cos(n\phi) +$$

$$(2\phi_0)^2 \sum_{q=1}^{\infty} (-1)^{q+1} a_q \sum_{n=1}^{\infty} \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{1}{D_n(ka,kb)} \frac{n \sin(n\phi_0)}{(q\pi)^2 - (n\phi_0)^2} \cos(n\phi =$$

$$\frac{1}{a} V_a K_e^{(a)}(\theta, \phi) .$$ (2.34)

Integrating (2.34) term by term on the interval $-\phi_0 \leqslant \phi \leqslant \phi_0$ we have

$$a_0 \sum_{n=0}^{\infty} \epsilon_n \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{J_0(n\phi_0)}{D_n(ka,kb)} \frac{\sin(n\phi_0)}{n\phi_0} +$$

$$4\phi_0 \sum_{q=1}^{\infty} (-1)^{q+1} a_q \sum_{n=1}^{\infty} \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{1}{D_n(ka,kb)} \frac{\sin^2(n\phi_0)}{(q\pi)^2 - (n\phi_0)^2} =$$

$$\frac{1}{a} V_a \sum_{n=0}^{\infty} \frac{\epsilon_n}{D_n(ka,kb)} \frac{\sin(n\phi_0)}{n\phi_0} \cos(n\theta).$$ (2.35)

Multiplying (2.34) by $\cos(m\pi\phi/\phi_0)$ and integrating term by term on the same

interval we have

$$a_o \sum_{n=1}^{\infty} \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{J_o(n\phi_o)}{D_n(ka,kb)} \frac{n \sin(n\phi_o)}{(m\pi)^2 - (n\phi_o)^2} +$$

$$\sum_{q=1}^{\infty} (-1)^{q+1} a_q \sum_{n=1}^{\infty} \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{1}{D_n(ka,kb)} \frac{2(n\phi_o)^2 \sin^2(n\phi_o)}{\left[(q\pi)^2 - (n\phi_o)^2\right]\left[(m\pi)^2 - (n\phi_o)^2\right]} =$$

$$\frac{1}{a} V_a \sum_{n=1}^{\infty} \frac{n}{D_n(ka,kb)} \frac{\sin(n\phi_o)}{(m\pi)^2 - (n\phi_o)^2} \cos(n\theta). \tag{2.36}$$

We may summarize (2.35) and (2.36) in the matrix notation as

$$\left[h_{mq}^{(e)}\right] \left[a_q\right] = \frac{1}{a} V_a \left[\ell_m^{(e)}\right] \tag{2.37}$$

where

$$h_{oo}^{(e)} = \sum_{n=o}^{\infty} \epsilon_n \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{J_o(n\phi_o)}{D_n(ka,kb)} \frac{\sin(n\phi_o)}{n\phi_o}$$

$$h_{oq}^{(e)} = 4\phi_o (-1)^{q+1} \sum_{n=1}^{\infty} \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{1}{D_n(ka,kb)} \frac{\sin^2(n\phi_o)}{(q\pi)^2 - (n\phi_o)^2} , \quad q \leqslant 1$$

$$h_{mo}^{(e)} = \sum_{n=1}^{\infty} \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{J_o(n\phi_o)}{D_n(ka,kb)} \frac{n \sin(n\phi_o)}{(m\pi)^2 - (n\phi_o)^2} , \quad m \geqslant 1$$

$$h_{mq}^{(e)} = 2\phi_o^2 (-1)^{q+1} \sum_{n=1}^{\infty} \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{1}{D_n(ka,kb)} \frac{n^2 \sin^2(n\phi_o)}{\left[(q\pi)^2 - (n\phi_o)^2\right]\left[(m\pi)^2 - (n\phi_o)^2\right]}$$

$$m \geqslant 1, \ q \geqslant 1 ,$$

$$\ell_0^{(e)} = \sum_{n=0}^{\infty} \left[\frac{\epsilon_n}{D_n(ka,kb)} \frac{\sin(n\phi_o)}{n\phi_o} \right] \cos(n\theta)$$

$$\ell_m^{(e)} = \sum_{n=1}^{\infty} \left[\frac{n}{D_n(ka,kb)} \frac{\sin(n\phi_o)}{(m\pi)^2-(n\phi_o)^2} \right] \cos(n\theta), \ m \geq 1 .$$

Turning to the derivation of the odd part of the slot field, we observe that the left hand side of the integral equation (2.31), after the substitution of (2.33) becomes

$$b_1 \int_{-\phi_o}^{\phi_o} \frac{\phi' K_o(\phi',\phi)d\phi'}{\pi\sqrt{\phi_o^2-(\phi')^2}} + \int_{-\phi_o}^{\phi_o} d\phi' \sum_{q=2}^{\infty} b_q K_o(\phi',\phi) \sin(\frac{q\pi\phi'}{\phi_o}) .$$

Representing the kernel by the series (2.26-b), interchanging summation with integration and integrating once by parts the first term becomes

$$b_1 \sum_{n=1}^{\infty} \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{2\sin(n\phi)}{D_n(ka,kb)} \frac{2\phi_o}{\pi} \int_0^{n\phi_o} \sqrt{1-(\frac{x}{n\phi_o})^2} \cos x \, dx ,$$

and the second term after interchanging summation with integration twice becomes

$$4\phi_o \sum_{q=2}^{\infty} (-1)^{q+1} b_q \sum_{n=1}^{\infty} \left[\frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{1}{D_n(ka,kb)} \frac{q\pi\sin(n\phi_o)}{(q\pi)^2-(n\phi_o)^2} \right] \sin(n\phi) .$$

The integral equation (2.31) now takes on the form

$$b_1 \sum_{n=1}^{\infty} \left[\frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{2}{D_n(ka,kb)} \frac{2\phi_o}{\pi} \int_0^{n\phi_o} \sqrt{1-(\frac{x}{n\phi_o})^2} \cos x \, dx \right] \sin(n\phi) +$$

$$4\phi_o \sum_{q=2}^{\infty} (-1)^{q+1} b_q \sum_{n=1}^{\infty} \left[\frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{1}{D_n(ka,kb)} \frac{q\pi \sin(n\phi_o)}{(q\pi)^2 - (n\phi_o)^2} \right] \sin(n\phi) =$$

$$\frac{1}{a} V_a K_o^{(a)}(\theta,\phi) \quad . \qquad (2.38)$$

Multiplying (2.38) by $\sin(m\pi\phi / \phi_o)$ and integrating on the interval $-\phi_o \leqslant \phi \leqslant \phi_o$ we

obtain an infinite set of algebraic equations. We summarize them in the matrix

form as

$$\left[h_{mq}^{(o)} \right] \left[b_q \right] = \frac{1}{a} V_a \left[\ell_m^{(o)} \right] , \qquad (2.39)$$

where

$$h_{m1}^{(o)} = \frac{4\phi_o}{\pi} \sum_{n=1}^{\infty} \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{1}{D_n(ka,kb)} \frac{\sin(n\phi_o)}{(m\pi)^2 - (n\phi_o)^2} \int_0^{n\phi_o} \sqrt{1-(\frac{x}{n\phi_o})^2} \cos x \, dx,$$

$$h_{mq} = 4\phi_o (-1)^{q+1} \sum_{n=1}^{\infty} \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{1}{D_n(ka,kb)} \frac{q\pi \sin^2(n\phi_o)}{\left[(q\pi)^2 - (n\phi_o)^2\right]\left[(m\pi)^2 - (n\phi_o)^2\right]}$$

$$\ell_m^{(o)} = \sum_{n=1}^{\infty} \left[\frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{2}{D_n(ka,kb)} \frac{\sin(n\phi_o)}{(m\pi)^2 - (n\phi_o)^2} \right] \sin(n\theta) \quad , \quad m \geqslant 1 .$$

We have reduced the problem of finding the shell slot electric field to an inversion of two complex matrices. By the inclusion of the edge singularity in the first term for each part of the slot field we have enhanced the first column at the expense of the rest of the matrix. The enhancement is particularly large for a narrow slot. Thus the problem has been set up in a way so as to lead to the narrow slot approximation. However, before we delve into the numerical details, we should indicate the physical quantities we want to find.

We are interested in the narrow slot antennas. Further, we restrict the radial separation of the shell from the cylinder to a small fraction of the wavelength λ, i.e. kb-ka $<<$ 1. Under these conditions the form of the radiation pattern of the cylinder with the shell will be essentially the same as for the cylinder alone. However, the power radiated into the free space will depend most probably on the orientation of the shell. Therefore we want to show how the radiated power depends on the slot field in the next few paragraphs.

The power radiated per unit length by the shell slot is given by

$$P = \frac{1}{2} R_e \int_0^{2\pi} \overline{E} \times \overline{H}^* \cdot \overline{a}_r \, r d\phi$$

$$= \frac{1}{2} R_e \int_0^{2\pi} \sum_{n=-\infty}^{\infty} j\omega\mu k A_n H_n^{(2)}(kr) e^{-jn\phi} \sum_{n=-\infty}^{\infty} k^2 A_n^* H_n^{(2)*}(kr) e^{jn'\phi} r d\phi$$

$$= R_e \sum_{n=-\infty}^{\infty} j\omega\mu\pi k^3 A_n A_n^* r H_n^{(2)'}(kr) H_n^{(2)*}(kr) . \qquad (2.40)$$

When $kr >> 1$ and $n << kr$, we may represent the Hankel function and its derivative, respectively, by

$$H_n^{(2)}(kr) \sim \sqrt{\frac{2}{\pi kr}} \ e^{-j\left[kr - \frac{1+2n}{4}\pi\right]} \qquad (2.41)$$

$$H_n^{(2)'}(kr) \sim -j\sqrt{\frac{2}{\pi kr}} \ e^{-j\left[kr - \frac{1+2n}{4}\pi\right]} , \qquad (2.42)$$

and taking the limit of the general term of the series as $r \to \infty$, before we perform the summation on n, we obtain the exact result

$$P = \sum_{n=-\infty}^{\infty} 2\omega\mu k^2 A_n A_n^* . \qquad (2.43)$$

Using (2.12) we rewrite this as

$$P = \frac{1}{2\pi^2 \omega\mu} \sum_{n=-\infty}^{\infty} \frac{\left| \int_{-\phi_o}^{\phi_o} E(\phi)e^{jn\phi} d\phi \right|^2}{\left| H_n^{(2)'}(kb) \right|^2} =$$

$$\frac{1}{2\pi^2 \omega\mu} \sum_{n=o}^{\infty} \frac{\epsilon_n}{\left| H_n^{(2)'}(kb) \right|^2} \left\{ \left| \int_{-\phi_o}^{\phi_o} E_e(\phi)\cos n\phi \, d\phi \right|^2 + \right.$$

$$\left. \left| \int_{-\phi_o}^{\phi_o} E_o(\phi)\sin(n\phi)\, d\phi \right|^2 \right\}. \qquad (2.44)$$

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Substituting (2.32) and (2.33) in the preceding equation, we have, after the interchange of integration with summation

$$P = \frac{1}{2\pi^2 \omega \mu} \sum_{n=o}^{\infty} \frac{\epsilon_n}{\left| H_n^{(2)'}(kb) \right|^2} \left\{ \left| J_o(n\phi_o) a_o - 2\phi_o^2 n \sin(n\phi_o) \sum_{q=1}^{\infty} \frac{(-1)^q a_q}{(q\pi)^2 - (n\phi_o)^2} \right|^2 \right.$$

$$\left. + \left| 2\phi_o \sin(n\phi_o) \sum_{q=1}^{\infty} \frac{(-1)^q q\pi b_q}{(q\pi)^2 - (n\phi_o)^2} \right|^2 \right\}. \tag{2.45}$$

From the behavior of the denominator in the general term of the series we conclude that only $2kb$ terms need be considered independent of the slot width. A computation will show that for a narrow slot the second part of the numerator is negligible compared to a_o even for a comparable b_q. Only when b_q becomes substantially larger than a_o do we have to take, in some special cases, the second part into consideration. Also in the first part of the numerator the a_q term is negligible compared to the a_o of the narrow slot. Thus we have that for a narrow slot,

$$P(ka, kb, \theta) \simeq \frac{\left| a_o \right|^2}{2\pi^2 \omega \mu} \sum_{n=o}^{2kb} \frac{\epsilon_n J_o^2(n\phi_o)}{\left| H_n^{(2)'}(kb) \right|^2}. \tag{2.46}$$

The power radiated by the cylinder without the shell is given by

$$P(ka) \simeq \frac{1}{2\pi^2 \omega \mu} \left(\frac{V_a}{a} \right)^2 \sum_{n=o}^{2ka} \frac{\epsilon_n}{\left| H_n^{(2)'}(ka) \right|^2}. \tag{2.47}$$

We define the voltage of the slotted shell as

$$V_b \equiv b \int_{-\phi_o}^{\phi_o} E(\phi)d\phi \ ,$$

(2.48)

and it follows that

$$V_b = ba_o .$$

(2.49)

The ratio of equations (2.46) and (2.47) we may write as

$$\frac{P(ka, kb, \theta)}{P(ka)} \simeq \left| \frac{V_b}{V_a} \right|^2 \left(\frac{a}{b} \right)^2 F(ka, kb, \phi_o),$$

(2.50)

where

$$F(ka, kb, \phi_o) = \left[\sum_{n=0}^{2kb} \frac{\epsilon_n J_o^2(n\phi_o)}{\left| H_n^{(2)'}(kb) \right|^2} \right] \left[\sum_{n=0}^{2ka} \frac{\epsilon_n}{\left| H_n^{(2)'})(ka) \right|^2} \right]^{-1} \cdot$$

(2.51)

For further discussion we elect to keep V_a = a volts, whether or not the cylinder is enclosed by the shell. Then (2.50) takes the form

$$\frac{P(ka, kb, \theta)}{P(ka)} \simeq \left| a_o \right|^2 F(ka, kb, \phi_o),$$

(2.50-a)

and a_o we regard as a dimensionless quantity. The last formula gives the enhancement (or depression) of radiation when the cylinder is enclosed by the shell. Aside from the essentially geometric factor F, all depends on the amplitude of the a_o term in the shell slot field expression.

Since Maxwell's equations are linear the result (2.50) is independent of the amplitude of the source voltage V_a; thus the right hand sides of (2.50) and (2.50a) must be equal, and we obtain

$$\left| \frac{V_b}{V_a} \right| = \frac{kb}{ka} \left| a_o \right| . \tag{2.51-a}$$

The last formula expresses the transformer properties of the shell. We may regard $(kb/ka) \left| a_o \right|$ as the transformer turns ratio.

We are fortunate that for a narrow slot the a_o term is also dominant in the slot field representation. This assertion is expected to hold when

$$\phi_o << \pi \tag{2.52-a}$$

$$2b\phi_o << \lambda \tag{2.52-b}$$

$$b-a \geqslant 2b\phi_o \tag{2.52-c}$$

From (2.37) with the understanding that $V_a = a$, we have

$$a_o \simeq \frac{\ell_o^{(e)}}{h_{oo}^{(e)}} \tag{2.53}$$

When $D_n(ka, kb) \longrightarrow 0$, then (2.53) takes on the particularly simple form

$$a_o \simeq \frac{H_n^{(2)'}(kb)}{H_n^{(2)'}(ka)} \frac{\cos(n\theta)}{J_o(n\phi_o)} . \tag{2.53-a}$$

The lowest order root of $D_1(ka, kb)$ occurs (Truell, 1943) when $kb \simeq 2-ka$, $ka \leqslant 1$, that is, when the mean circumference is approximately equal to the wavelength. The

lowest order root of $D_2(ka, kb)$ occurs when $kb \simeq 4-ka$, $ka \leqslant 2$. Of particular interest are also the roots of $D_0(ka, kb)$, because then a_o does not depend on the angle θ. We list the lowest order root of $D_0(ka, kb)$ (Jahnke, Emde, Losch, 1960) for some parameter values of possible interest in Table II-1.

TABLE II-1; Parameters for Lowest Order Root
of $D_0(ka, kb)$

ka	kb/ka
31.427	1.1
15.7275	1.2
10.4993	1.3
7.8875	1.4
6.2702	1.5

2.5 The Results

We are left with the task of computing a_o. The series expansion of the numerator and denominator of a_o involve Bessel functions and Neumann functions. In view of the intended applications of the results of this study, we restrict the radius of the shell to $kb \leqslant 2$.

We use the exact series for computing the cylindrical functions of order 0, 1, 2, 3, 4 and 5. Recursion relations are used to compute orders 6 and 7. For orders greater than 7 we use the small argument (Jahnke-Emde-Losch, 1960) approximation to obtain

$$\frac{1}{D_n(ka, kb)} \simeq - \frac{\pi (kb)^2}{n \left[1 - (\frac{ka}{kb})^{2n} \right]} (\frac{ka}{kb})^{n+1}, \qquad (2.54\text{-a})$$

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$$\frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \simeq (\frac{kb}{ka})^{n+1} \quad . \tag{2.54-b}$$

We sum 90 terms in the series determining $h_{oo}^{(e)}$ and $\ell_o^{(e)}$. We retain only five

terms in the $F(ka, kb, \phi_o)$ series. The computations were performed on a digital

computer for the slot width of 0.06 radians, and ka in steps of 0.2 from 0.2 to 1.8,

for each step, kb−ka = 0.05, 0.10, 0.20. We elect to present the results in the

following form.

$$10 \log \frac{P(ka, kb, \theta)}{P(ka)}$$

we choose as the ordinate and θ as the abscissa in Figures 2-2 through 2-10.

A zero on the ordinate axis means that the source radiates the same amount

of power into the free space with and without the slotted shell. A negative number,

say −20, means that introduction of the slotted shell decreases the radiated power

to one hundredth of the previous value, while +20 means that the radiated power

is increased by a factor on one hundred. The origin on the ordinate axis ($\theta = 0^{o}$)

means that the cylinder slot is under the shell slot, and $\pm\theta = 180^{o}$ means that

the slots are on the opposite sides of the cylinder. The radiation curves are even

functions in θ.

FIG. 2-2: RADIATION THROUGH THE SLOTTED SHELL FOR ka = 0.20

FIG. 2-3: RADIATION THROUGH THE SLOTTED SHELL FOR ka = 0.40

FIG. 2-4: RADIATION THROUGH THE SLOTTED SHELL FOR ka = 0.60

FIG. 2-5: RADIATION THROUGH THE SLOTTED SHELL FOR ka = 0.80

FIG. 2-6: RADIATION THROUGH THE SLOTTED SHELL FOR ka = 1.0

FIG. 2-7: RADIATION THROUGH THE SLOTTED SHELL FOR ka = 1.2
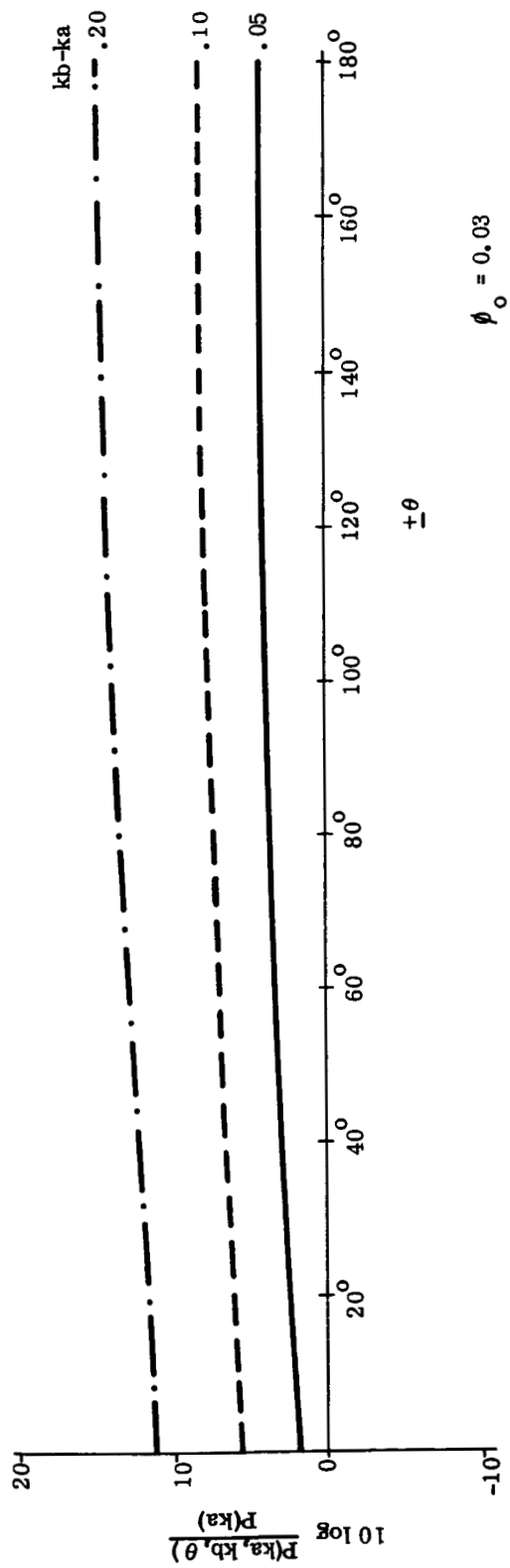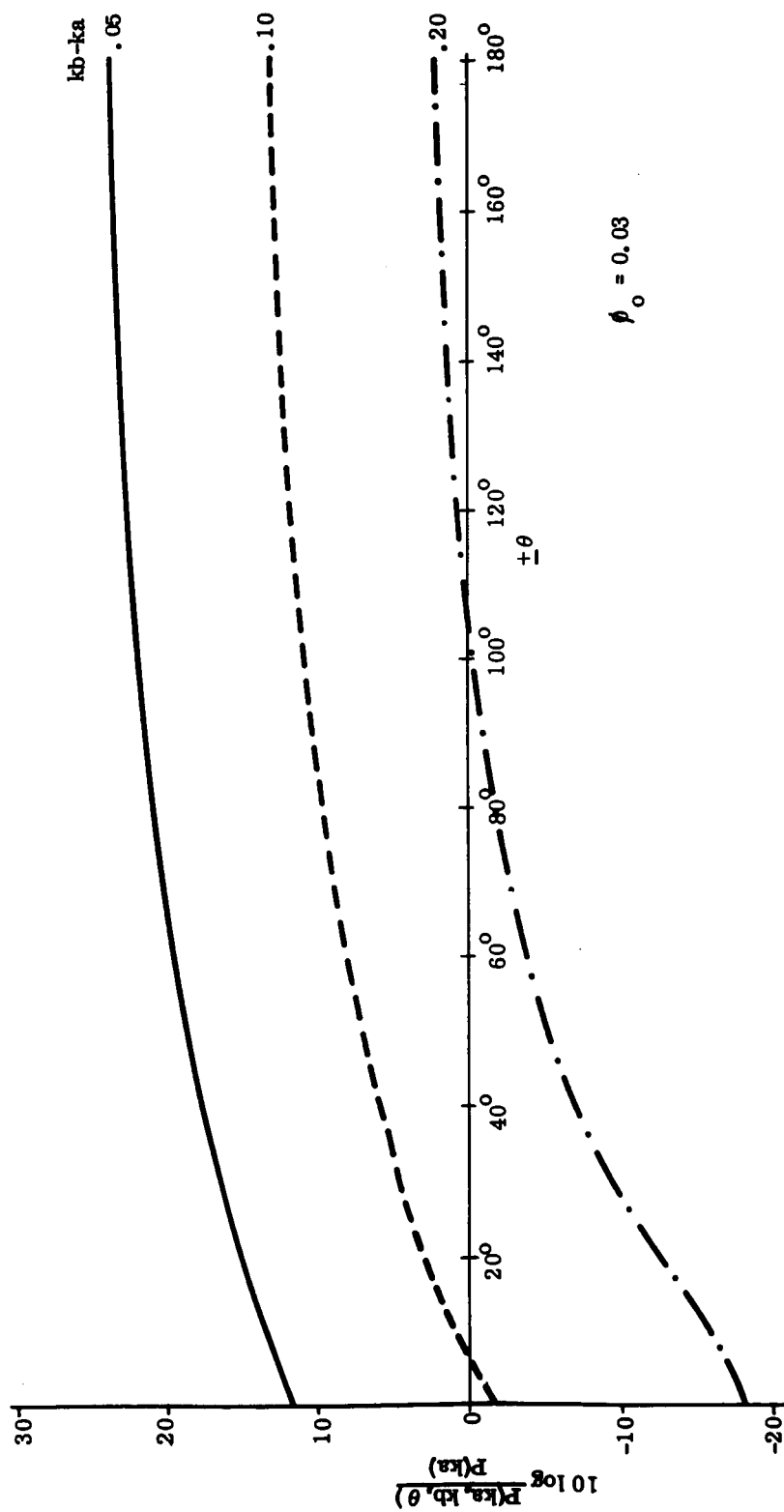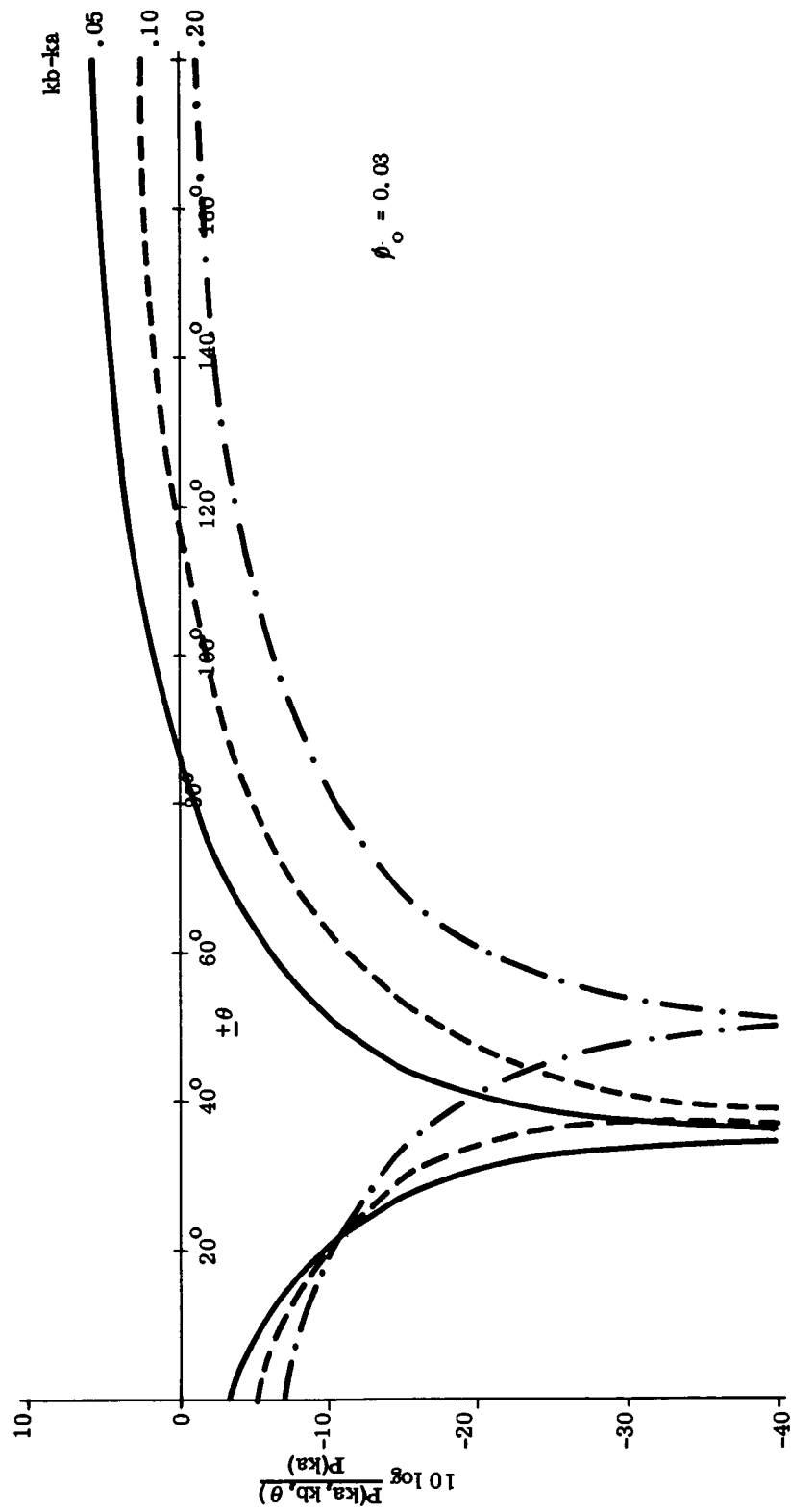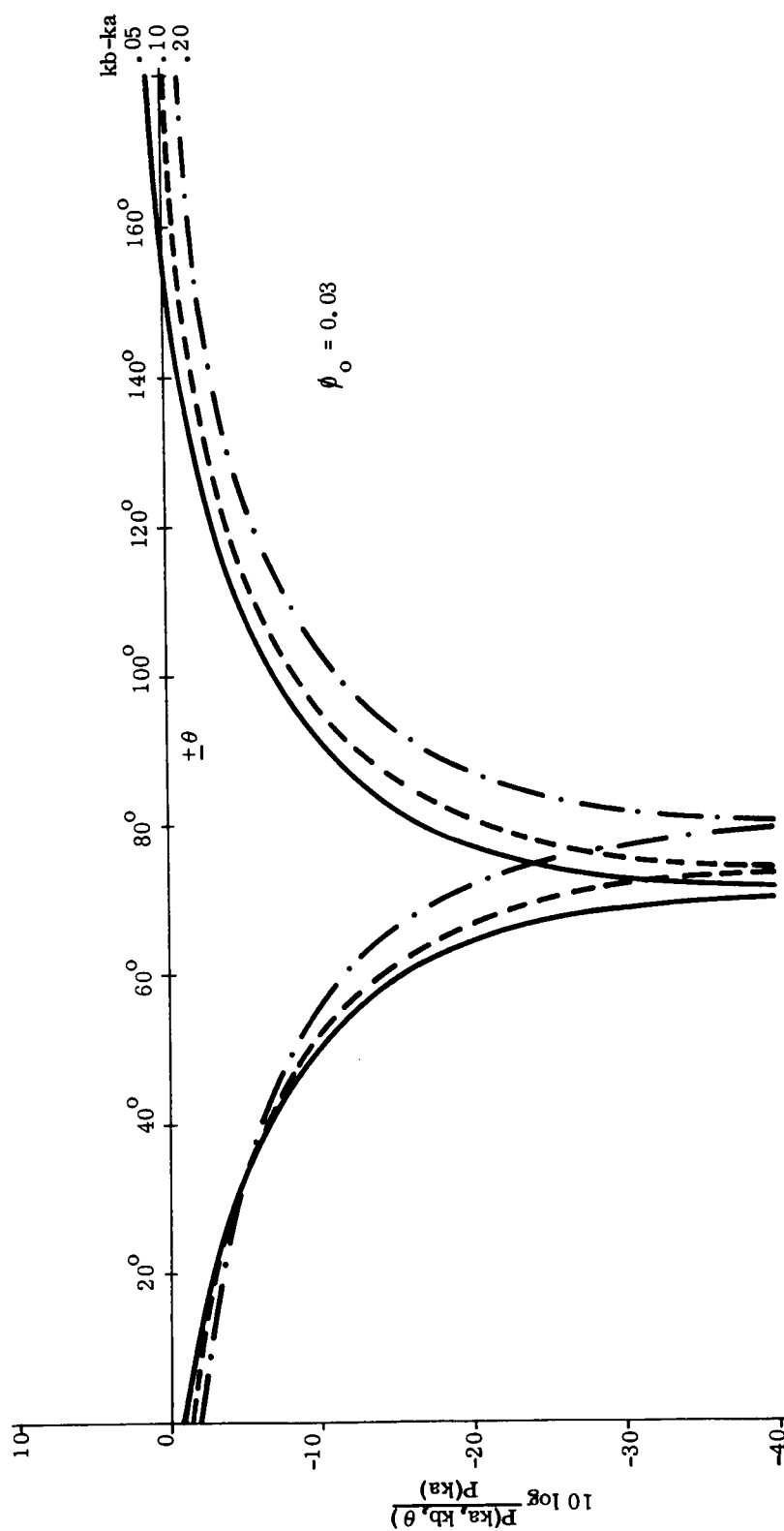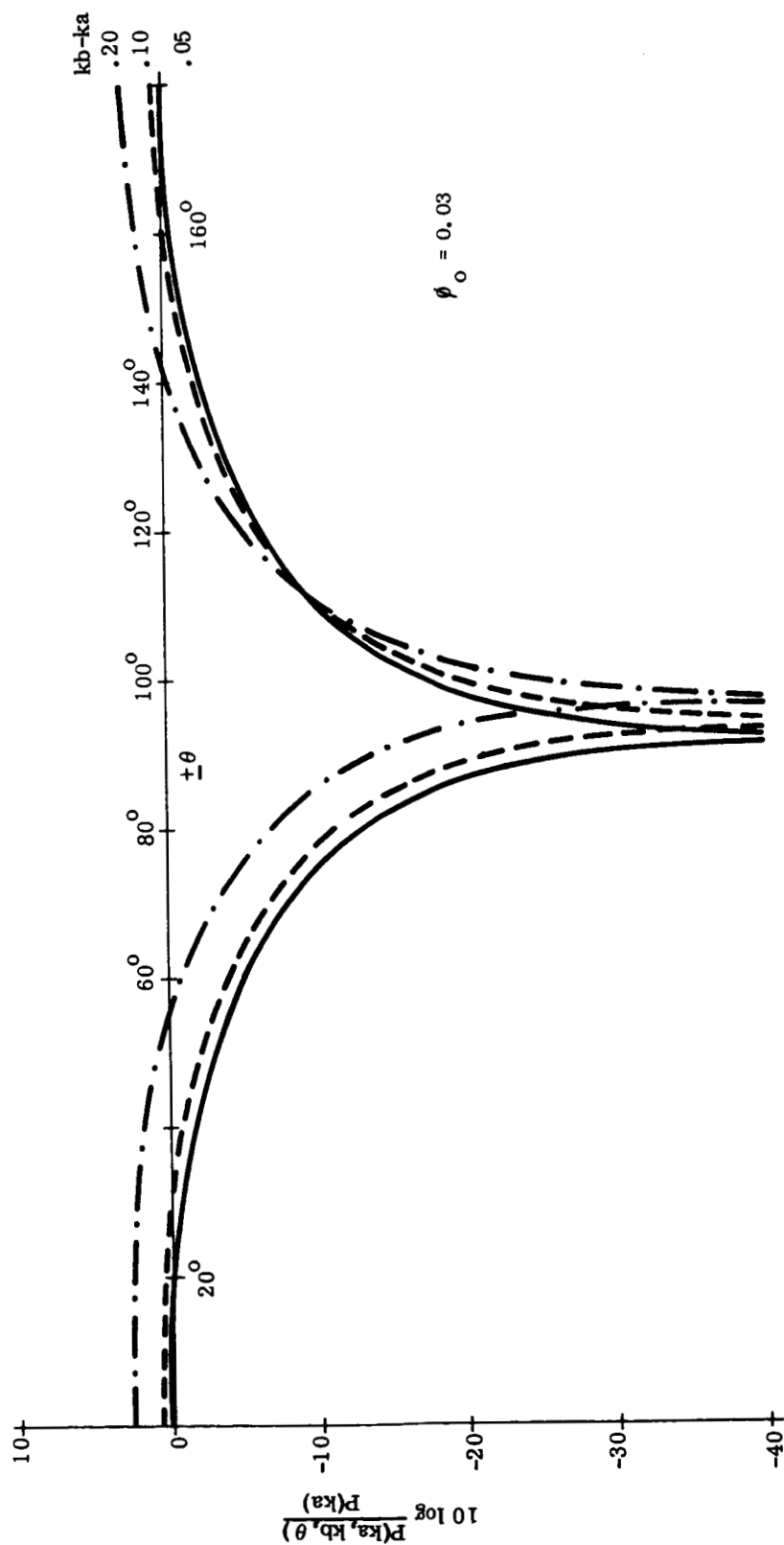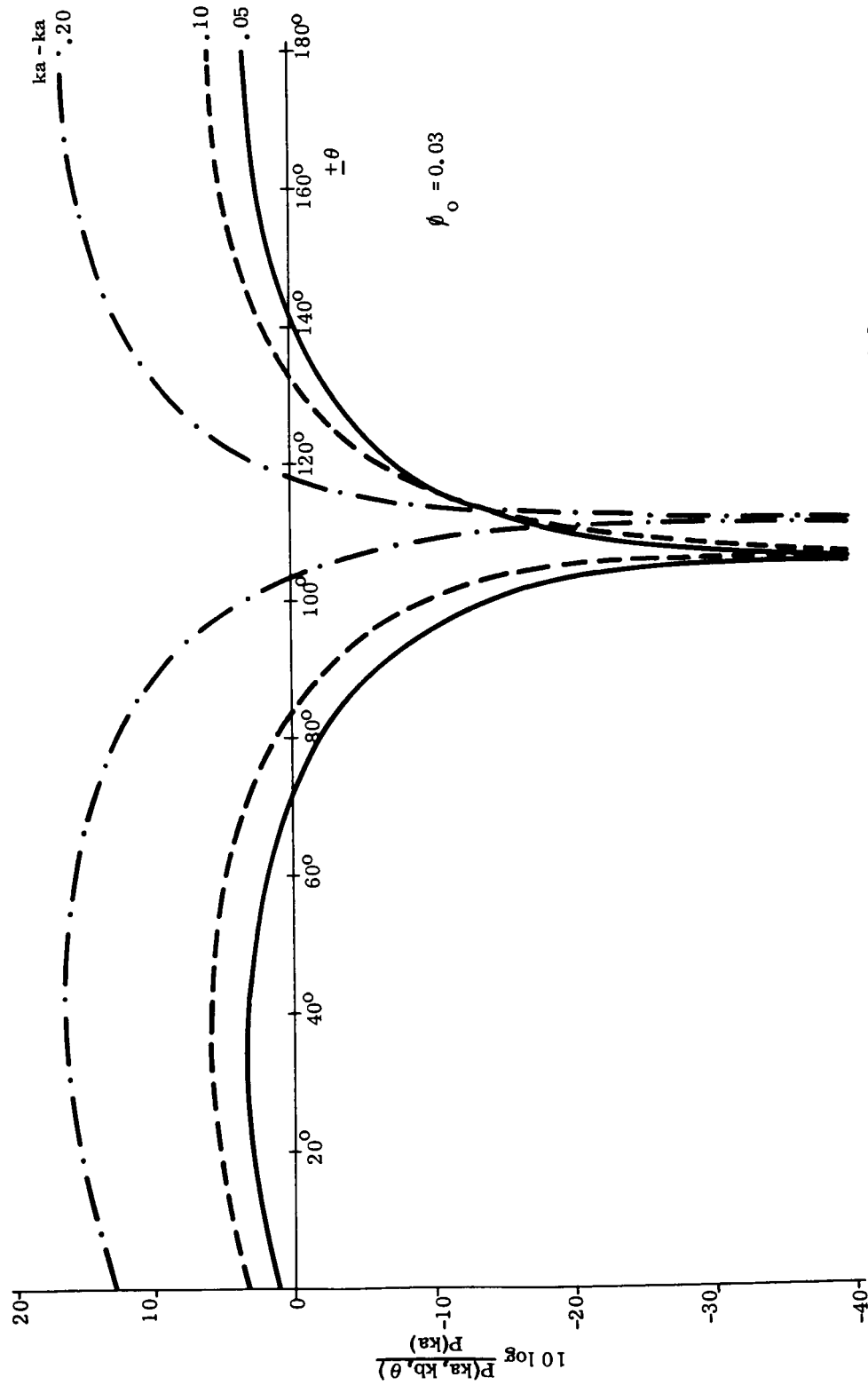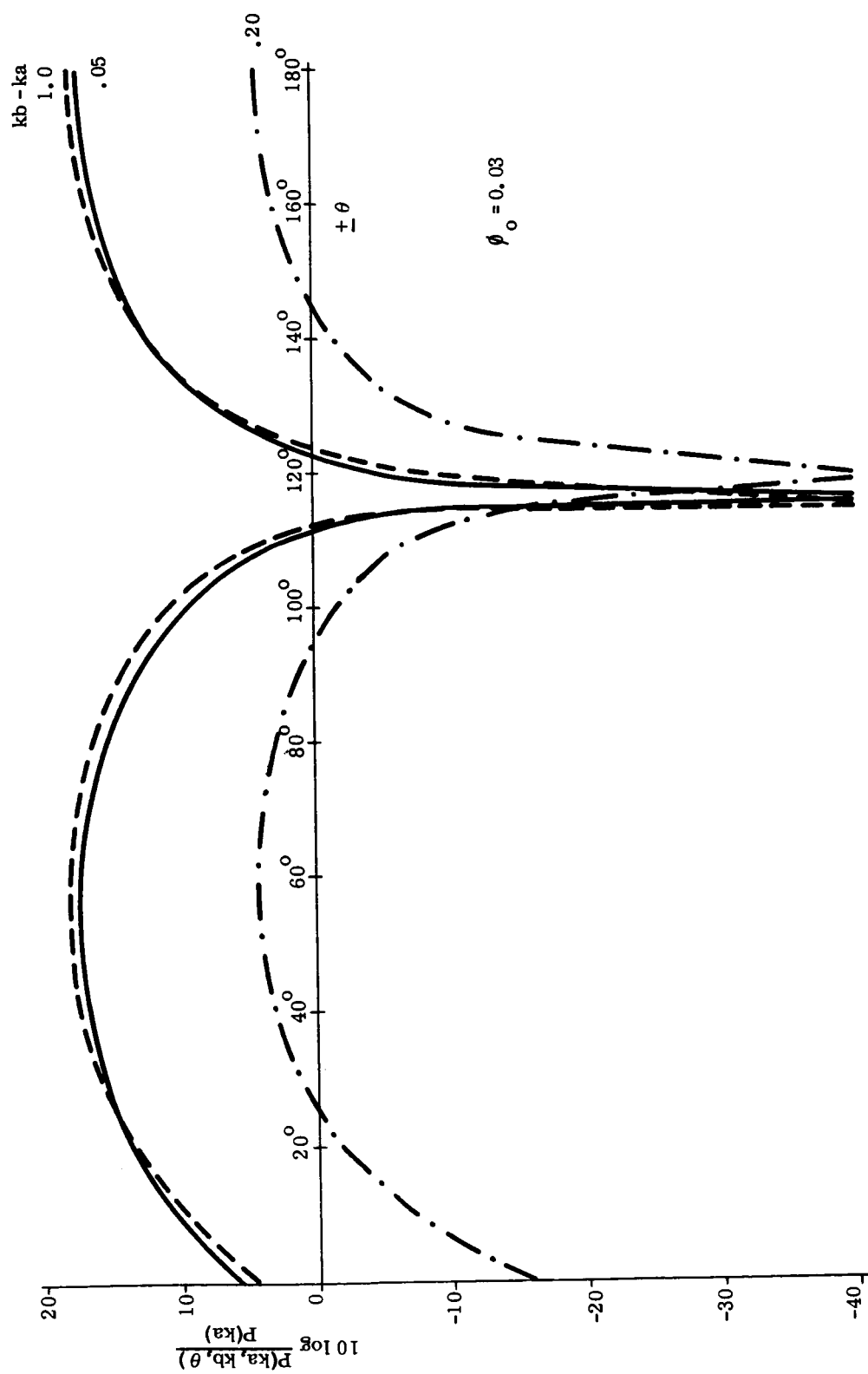
FIG. 2-8: RADIATION THROUGH THE SLOTTED SHELL FOR ka=1.4
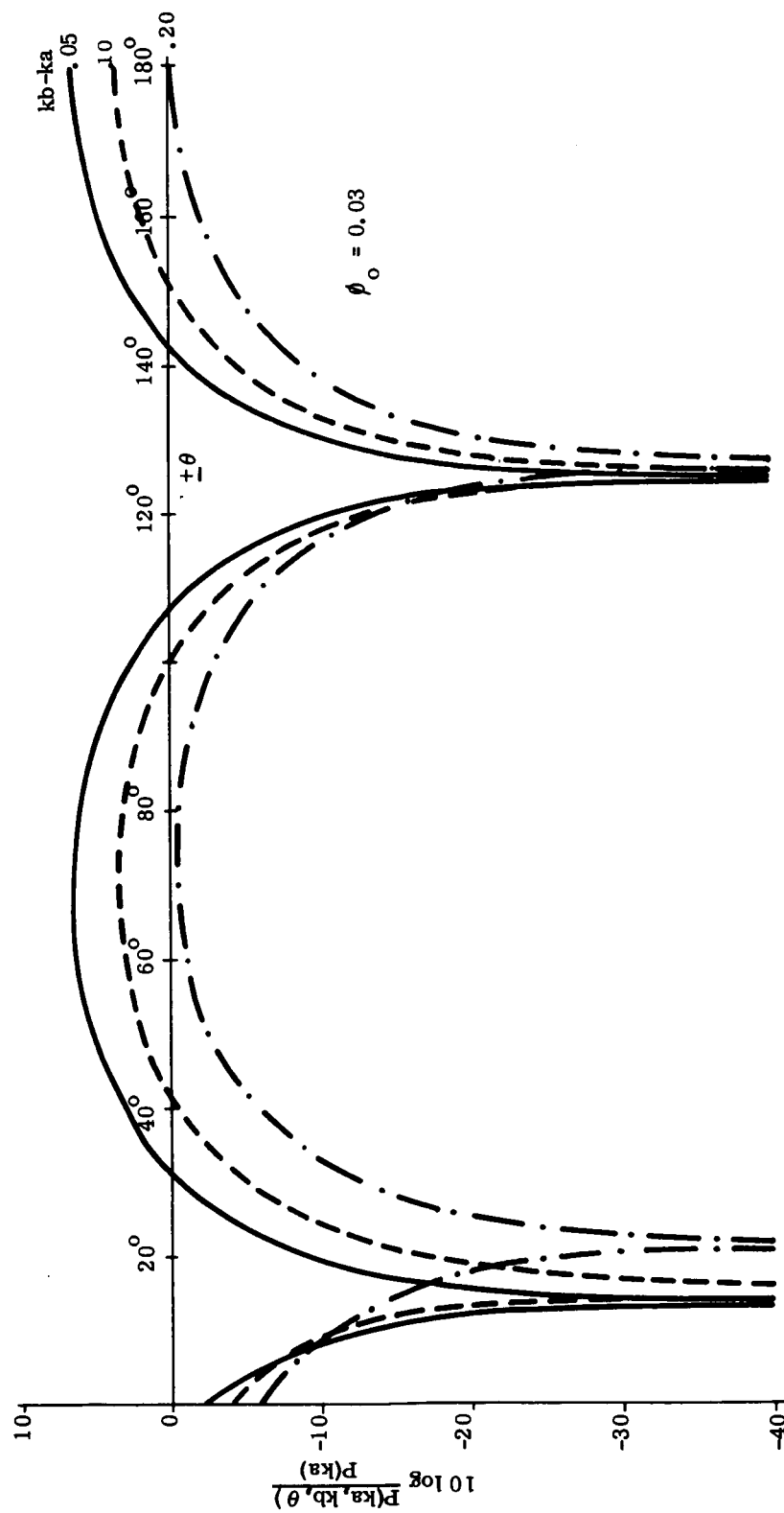
FIG. 2-9: RADIATION THROUGH THE SLOTTED SHELL FOR ka = 1.6
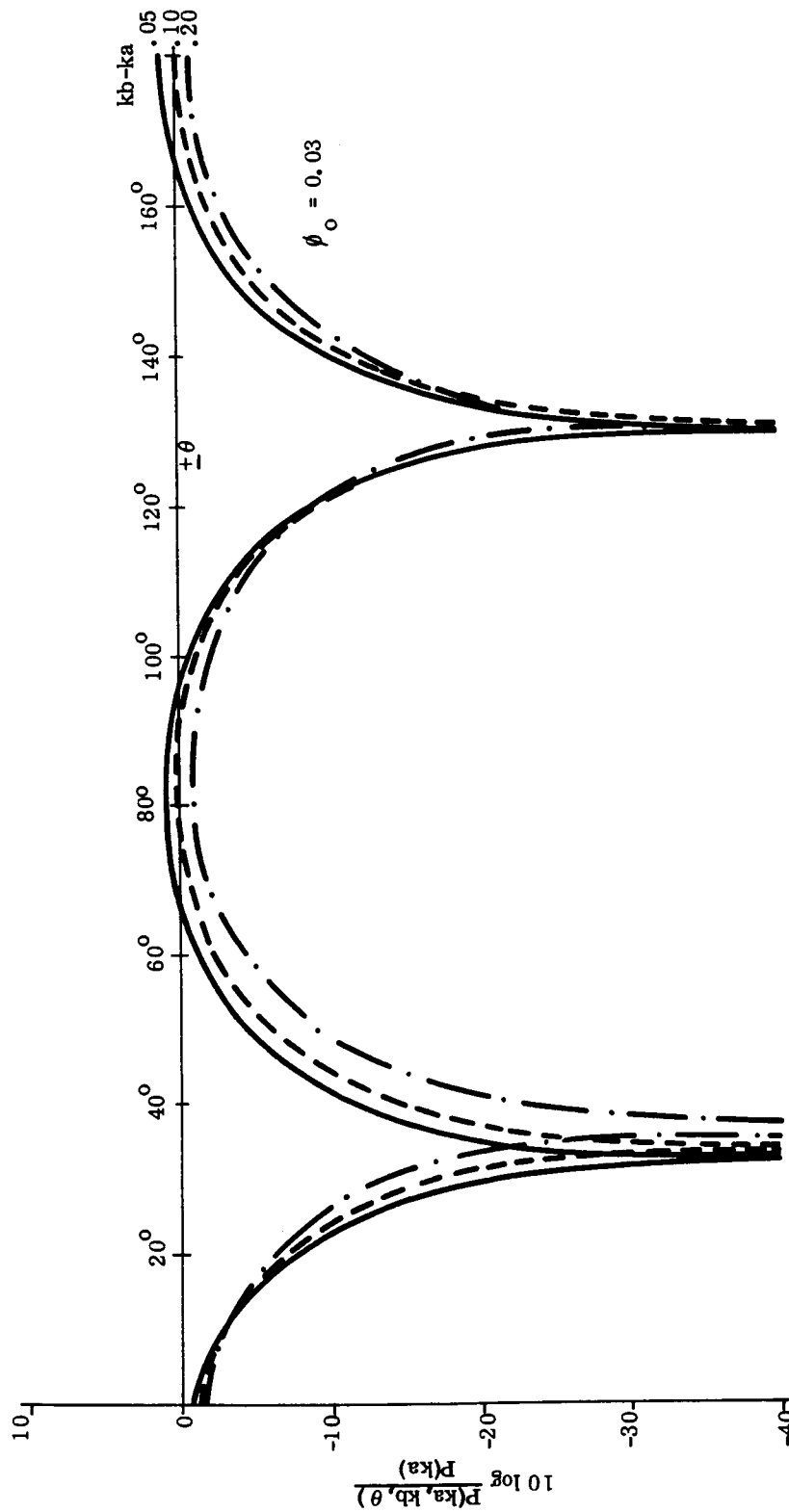
FIG 2-10: RADIATION THROUGH THE SLOTTED SHELL FOR ka = 1.8

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All along we assume that the source amplitude on the cylinder remains unchanged, i.e. the voltage of the cylinder slot remains at 'a' volts. Thus these calculations do not include the de-tuning of the antenna that must arise in most cases when it is surrounded by a perfectly conducting slotted shell. However, these calculations do show the effect the slotted shell has on the coupling between the line source of the cylindrical antenna and the radiation field in the free space.

In Fig. 2-2, the antenna diameter is $\frac{0.1}{\pi} \lambda$ (ka = 0.2), where λ is the free space wavelength. Enclosing this antenna by the slotted shell increases the radiated power, the bigger the separation between the antenna and the shell, the more radiation we get, which is only weakly dependent on the source and the slot separation angle θ. For kb-ka = 0.05, 0.10, 0.20 the radiated power is increased by 3, 7.5 and 13db, respectively. The maximum increase is when $\theta = 180^{\circ}$ and the minimum when $\theta = 0$. The difference between maximum and minimum for a given kb-ka is only about 2 db.

In Fig. 2-3, we have increased the antenna diameter to ka = 0.40 and the closer the shell is to the antenna the more radiation we get, which is just the opposite situation we had when ka = 0.2 in the preceding figure. When kb-ka = 0.05 the radiation is increased by 11.5db when $\theta = 0^{\circ}$, and increased to 23.5db when $\theta \rightarrow 180^{\circ}$. When we increase kb - ka to 0.10 the radiation very significantly decreases; when $\theta = 0^{\circ}$ we have only -1db, however the radiation increases to 13db as θ increases to 180°. Increasing kb-ka to 0.20 further reduces the radiation;

at $\theta = 0^0$ we have -18db and the radiation increases only to +2db as $\theta \rightarrow 180^0$.

In Fig. 2-4, we have ka = 0.6 and the farther the shell is from the antenna, the more it depresses the radiation. When kb-ka = 0.05 the radiation is -3db at θ =0, but decreases to a deep minimum (the approximation gives zero radiated power, but this is not expected to be true) at $\theta = 36^0$, and from then rapidly rises to +6db as $\theta \rightarrow 180^0$. Increasing kb-ka to 0.10 depresses radiation at $\theta = 0^0$ to -5db and the deep minima moves to $\theta = 41^0$; as $\theta \rightarrow 180^0$ the radiation recovers to +2.5db. As kb-ka $\rightarrow$ 0.2 the curve is depressed all along below zero, and the deep minima moves further to the right.

In Fig. 2-5 the increase of ka to 0.80 has brought, maintaining the same order, all three curves closer together, and minima have moved further to the right. When $\theta = 0^0$ the radiation is a few db below zero; at 180^0 either just above or below zero.

In Fig. 2-6 we have ka = 1.0. The minima have moved farther to the right and for the three curves occur between 90^0 and 100^0. The order of the curves has been reversed: the larger the kb-ka value, the more radiation we get into the free space. At $\theta = 0^0$, and 180^0, kb-ka = 0.05 gives 0 db, kb-ka = 0.10 gives + 0.5 db and kb-ka = 0.20 gives 2 db.

In Fig. 2-7 we have ka=1.2 and the minima are occurring between 105^0 and 115^0. The curve ordering remains the same as in the preceding figure, but the radiation is enhanced for most θ angles, especially so for kb-ka = 0.20. For

this curve the radiation at $\theta = 0^{\circ}$ and 180° is +13db and +16db, respectively.

In Fig. 2-8 we have increased the antenna radius to ka = 1.4, and the minimas now occur between 115° and 120°. The curves have started to reverse the order. The kb-ka=0.20 curve has dropped about 15db below the other curves and is approximately where the other two curves were in the preceding figure. For the high curves the maximum radiation is 18 db. All three curves show a second minimum starting to form at $\theta = 0^{\circ}$.

In Fig. 2-9 we have increased ka to 1.6, and the reversal of the curve order has been completed, i.e. increased kb-ka decreases the radiation. Also two sets of minima have formed: the new set is between 10° to 20° and the old set has moved to 125° to 130^{0}. The radiation is increased only by 6.5 db at the maximum.

In Fig. 2-10 we have ka= 1.8, and all three curves have moved very closely together. The first minima occur at about 33° and the second at about 130°. At 0° the radiation is -1db; at 80° and 180°, it is ± 1 db.

2.6 Discussion

In the figures presented, we have increased the radius of the cylinder, a, in nine equal steps from $0.1/\pi \lambda$ to $0.9/\pi \lambda$. In all nine cases we have shown the effect of the slotted shell on the radiation when the radial separation between the cylinder and the shell is $0.025/\pi \lambda$, $0.05/\pi \lambda$, and $0.10/\pi \lambda$. The shell increases radiation independently of θ when a $< 0.1/\pi \lambda$. A very substantial increase in radiation is maintained as the radius of the cylinder is increased to $0.2/\pi \lambda$. As a is increased to $0.3/\pi \lambda$ and beyond, a deep minimum appears in the curves which

indicates that for those angles the slotted shell decouples the cylindrical antenna from the free space. When a is $0.4/\pi\,\lambda$, or $0.5/\pi\,\lambda$ the slotted shell leaves the antenna radiation largely unaffected for extensive ranges of θ, except when the slot is in the 90° range from the source where then deep minima occur. As a is increased to $0.6/\pi\,\lambda$ the antenna radiation is enhanced, and also becomes sensitive to the cylinder and the slotted shell separation distance. The same thing remains true as a is further increased to $0.7/\pi\,\lambda$, except that a new minimum appears to form at $\theta = 0$. As a is further increased to $0.8/\pi\,\lambda$, both the radiation enhancement and the sensitivity of the radiation enhancement on the cylindrical antenna and the slotted shell separation markedly decrease for all θ values. Two deep minima have formed as well. Increasing a further to $0.9/\pi\,\lambda$ reduces the radiation enhancement practically to zero, and the radiation becomes independent of the antenna and the slotted shell radial separation. This also occurred at a = $0.5/\pi\,\lambda$. This phenomenon appears to be associated with the 'resonances' in the coaxial cavity formed by the cylindrical antenna and the coaxial shell. The 'resonances' occur when $D_n(ka, kb) = 0$. For our range of variables the first 'resonance' appears when ka$\sim$1, the second when ka$\sim$2, i.e. a$\sim 0.5/\pi\,\lambda$, and $1/\pi\,\lambda$, respectively.

In equation (2.50-a) essentially a geometrical factor relates the radiated power per unit length to the square of the amplitude of the a_o coefficient. This factor we denoted by F, and in particular cases considered here, the infinite series was approximated by five terms, i.e.

$$F(ka, kb, \phi_o) = \left[\sum_{n=o}^{5} \frac{\epsilon_n J_o^2 (n\phi_o)}{\left| H_n^{(2)'}(kb) \right|^2} \right] \left[\sum_{n=o}^{5} \frac{\epsilon_n}{\left| H_n^{(2)'}(ka) \right|^2} \right]^{-1} . \qquad (2.55)$$

We plot $F(ka, kb, \phi_o)$ in Fig. 2-11 for the same ranges of variables as appeared in the preceding figures. Since the shell is close to the cylinder in the three cases considered we have that the factor F is close to unity when $ka > 1$. Only for $ka < 1$ do we have a substantial increase of the factor above unity. Using this factor we can very simply obtain the ratio of the slot voltages from the preceding figures. From (2.50) we have

$$10 \log \left| \frac{V_b}{V_a} \right|^2 = 10 \log \frac{P(ka, kb, \theta)}{P(ka)} - 10 \log \left[(\frac{ka}{kb})^2 F(ka, kb, \phi_o) \right] . \qquad (2.56)$$

We notice that in the second term on the right hand side the argument of the log is close to unity. Thus for most parameter configurations the Figures 2-2 through 2-10 give also directly the ratio of the slot voltages as a function of θ. We may add that in this approximation the phase of V_a is constant between the minima, and it suffers a 180° change as one goes through a minimum. The slot voltage phase is given by the phase of $1/h_{oo}^{(e)}$ and the 180° phase change comes from the sign reversal of $\ell_o^{(e)}$ on going through the minimum.

We have presented the approximate solution of the integral equation (2.30), and discussed to some extent the physical consequences of this solution. The method of approximation suggested itself from the results of the narrow slot in

$$F = \frac{\displaystyle\sum_{n=0}^{5} \frac{\epsilon_n J_n^2 (n\phi_o)}{\left| H_n^{(2)'} (kb) \right|^2}}{\displaystyle\sum_{n=0}^{5} \frac{\epsilon_n}{\left| H_n^{(2)'} (ka) \right|^2}}$$

kb – ka	
○	0.05
□	0.10
△	0.20
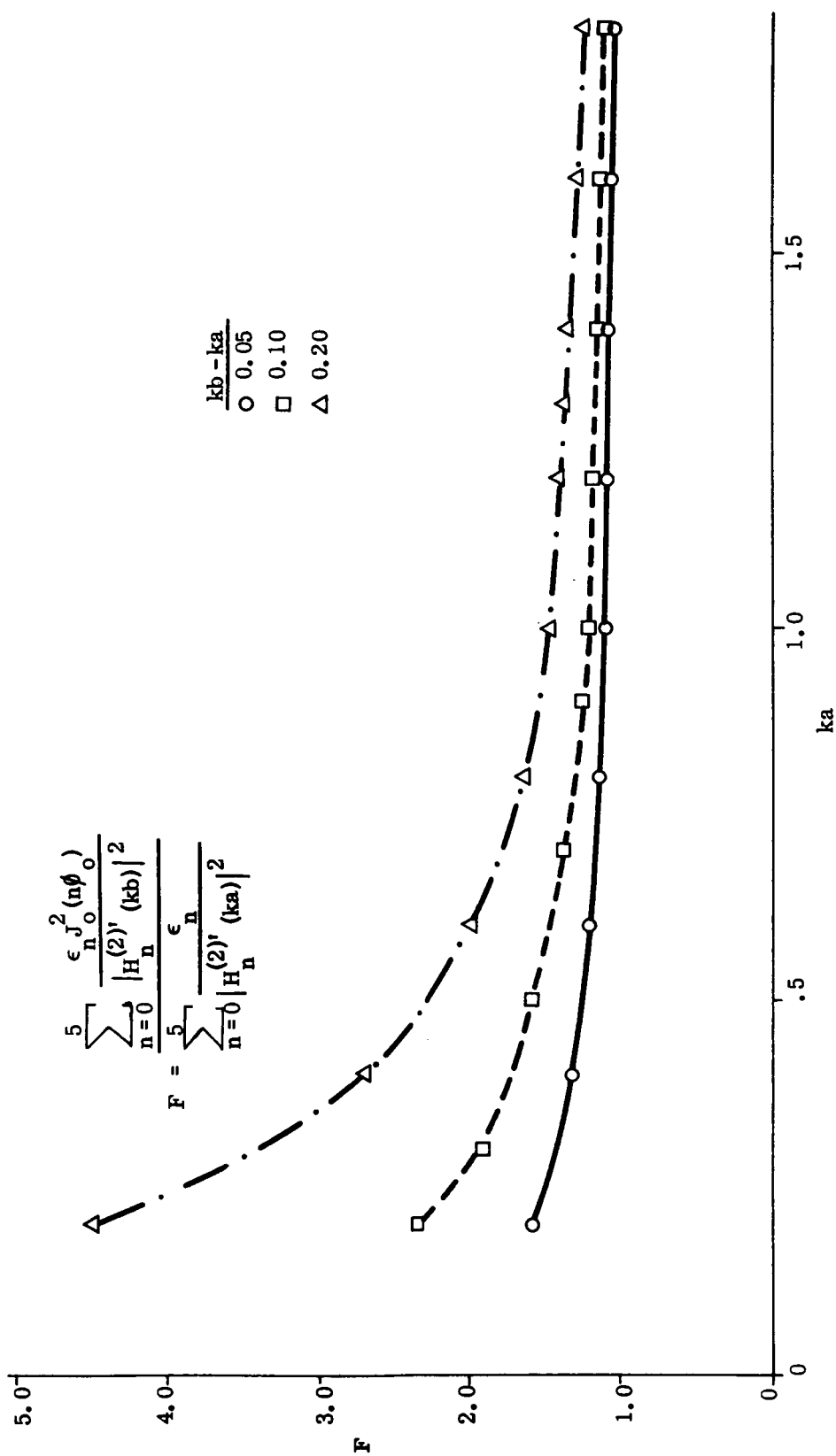
FIG. 2-11: COMPARATIVE RADIATION EFFICIENCY OF CYLINDERS

a plane screen. The use of the same leading terms in the slot field representation is justified largely on the physical grounds and it leads to the geometrical restrictions (2.52). Although we feel the approximation is a good one for the range of parameters discussed, a quantitative statement of the slot voltage approximation would be very desirable. We may truncate the matrix in (2.37) and invert it. This procedure is laborious, the results may not be conclusive as to the error in any case, and particularly so when the original approximation is a good one. We choose to go back to the integral equation itself. We rewrite it in the form

$$\left[K_e^{(a)}(\theta,\phi)\right]^{-1} \int_{-\phi_o}^{\phi_o} E_e(\phi')K_e(\phi',\phi)d\phi' - 1 = 0. \tag{2.57}$$

When we use the approximate solution, the left hand side of the above equation will not be quite zero. This difference we denote by $\Delta^{(e)}(\phi)$, i.e.

$$\Delta^{(e)}(\phi) = \left[K_e^{(a)}(\theta,\phi)\right]^{-1} \int_{-\phi_o}^{\phi_o} E_e'(\phi')K_e(\phi',\phi)d\phi' - 1, \quad -\phi_o \leqslant \phi \leqslant \phi_o \tag{2.58}$$

where the prime on the field indicates that it is the approximate solution we have obtained, i.e.

$$E_e'(\phi') = \frac{\ell_o^{(e)}(\theta)}{h_{oo}^{(e)}} \frac{1}{\pi\sqrt{\phi_o^2-(\phi')^2}}. \tag{2.59}$$

We find that

$$\Delta^{(e)}(\phi) = \frac{\ell_o^{(e)}(\theta)}{h_{oo}^{(e)}} \left[K_e^{(a)}(\theta,\phi) \right]^{-1} \sum_{n=o}^{\infty} \frac{H_n^{(2)'}(ka)}{H_n^{(2)'}(kb)} \frac{\epsilon_n J_o(n\phi_o)}{D_n(ka,kb)} \cos(n\phi)-1 \ .$$

(2.60)

In Table II-2 we present some of the calculations from (2.60) for the parameters of Figs. 2-2, 2-5, 2-6 and 2-9. The $\ell_o^{(e)}(\theta)$ and the $K_e^{(a)}(\theta,\phi)$ series have been summed to 90 terms, the other two series to 200 terms. The calculations showed that the approximate solution (2.59) 'satisfies' the integral equation (2.57) essentially independent of the angle θ. In the table we have shown how $\Delta^{(e)}(\phi)$ depends on the field point coordinate ϕ. We have carried out the computations for $\pm \phi$ = 0, 0.015, and 0.030, i.e. at the slot center, half-way to either end, and at either slot edge.

TABLE II-2: VALUES OF $\Delta^{(e)}(\phi)$

$\Delta^{(e)}$ for ϕ_o = 0.030

ka	kb−ka	ϕ =0	ϕ =± 0.015	ϕ =± 0.030
0.40	0.05	0.030−j0.013	−0.063+j0.032	0.20+j0.10
0.40	0.10	−0.015−j0.0045	0.037+j0.0097	−0.12−j0.030
0.40	0.20	−0.010−j0.0017	0.022+j0.0035	−0.072−j0.011
1.0	0.05	0.0004−j0.0000002	−0.0004+j0.000001	0.0014−j0.000004
1.0	0.10	0.0010−j0.000008	−0.0016+j0.00002	0.005−j0.00006
1.0	0.20	0.0040−j0.00023	−0.0077+j0.0005	0.024−j0.0016
1.2	0.05	0.0052−j0.00001	−0.004+j0.0002	0.017−j0.00067
1.2	0.10	0.0082−j0.00055	−0.013+j0.0015	0.043−j0.0048
1.2	0.20	0.024−j0.041	−0.047+j0.090	0.14−j0.27
1.8	0.05	−0.0060+j0.000027	0.0033+j0.000085	0.013−j0.00033
1.8	0.10	−0.0050−j0.000037	0.0046+j0.00018	−0.015−j0.00062
1.8	0.20	−0.0036−j0.0001	0.0053+j0.00026	0.017−j0.00083

The approximate solution 'satisfies' the integral equation better at the slot center and the worst at the slot edges. Also it appears that the integral equation is 'satisfied' better when the radiation is insensitive to the cylinder and the shell radial separation, i.e. when we are closer to some particular co-axial 'resonance' than when in between them.

In order to be able to make a quantitative estimate of the error we rewrite (2.57) in the form

$$\int_{-\phi_o}^{\phi_o} \left\{ E_e(\phi') - E_e'(\phi') \right\} K_e(\phi', \phi) d\phi' = K_e^{(a)}(\theta, \phi) \left[-\Delta^{(e)}(\phi) \right], \qquad (2.61)$$

Taking the absolute values we have

$$\left| \int_{-\phi_o}^{\phi_o} \left\{ E_e(\phi') - E_e'(\phi^1) \right\} K_e(\phi', \phi) d\phi' \right| \leqslant \left| K_e^{(a)}(\theta, \phi) \right| \left| \Delta^{(e)}(\phi_o) \right|. \qquad (2.62)$$

Since $K_e^{(a)}(\theta, \phi)$ is a real function we may argue that at the maximum

$$E_e(\phi) - E_e'(\phi) = \left| \Delta^{(e)}(\phi_o) \right| E_e(\phi), \qquad (2.63)$$

and hence

$$E_e'(\phi) = \left[1 - \left| \Delta^{(e)}(\phi_o) \right| \right] E_e(\phi). \qquad (2.64)$$

From Table II-2 and the last formula we compute that the maximum possible error in the slot field is 22 percent for ka = 0.40, 2.4 percent for ka = 1.0, 32 percent for ka = 1.2 and 1.7 percent for ka = 1.8. For the radiation this corresponds to maximum errors of 1.7, 0.2, 2.4 and 0.16 db, respectively. Thus we feel that the

maximum possible error in the data plotted with the exception of the radiation

minima, should be a few tenths of a db when the curves are close together, and a

few db when they are separated, i.e. when we are far from a particular co-axial

resonance. The nature of (2.45) indicates the minima in the radiation coupling curves

for the slot width considered should be at least 25 db deep. Some further work is

necessary to establish the depth of the minima.

We have computed the radiation through the shell for three different shell

spacings from the cylinder. For the parameters we have selected it would appear

that bringing the shell closer to the cylinder in most cases gives stronger radiation

than letting the shell be farther away from the cylinder. However, the type of the

problems we have does not allow us to extrapolate these results in either direction.

Further calculations are necessary to establish this behavior. However, we may

offer some comments based on simple physical arguments. It would appear that the

radiation goes to zero as the slotted shell coalesces with the cylinder for

$\phi_0 < \theta < 2\pi - \phi_0$, i.e. the source is not under the shell slot, because then the

source is enclosed by a perfectly conducting medium which precludes radiation.

When the source is under the shell slot $(-\phi_0 < \theta < \phi_0)$ as $b \to a$, then of course we

have the familiar situation of a source on the perfectly conducting cylinder. These

comments are also supported by some additional calculations for the case of

Fig. 2-3 (ka = 0.40). These were done for kb-ka=0.025, 0.010, 0.005, and the

radiation was successfully reduced as kb-ka was decreased from 0.050. The other

limit, that of increasing kb-ka to infinity for a given ka has no practical significance. Increasing kb-ka beyond 0.20 we may expect the radiation alternatively to increase and decrease.

We may also ask the physical question: How does the radiation get out when the source is not directly visible through the slot? In an attempt to read some physics into the mathematics we may compute the Fourier coefficient of the E_ϕ component of the electric field (2.3) in the co-axial space. Taking the cylinder slot voltage at 'a' volts, we have that

$$j\omega\mu \, k\left[B_n J_n'(kr) + C_n N_n'(kr)\right] \simeq \frac{1}{2\pi \, D_n(ka, kb)}\left\{\left[J_n'(kr)N_n'(kb) - N_n'(kr)J_n'(kb)\right] e^{jn\theta}\right.$$

$$\left. - \left[J_n'(kr) \, N_n'(ka) - N_n'(kr) \, J_n'(ka)\right] a_o J_o(n\phi_o)\right\} , \qquad (2.65)$$

where the approximate sign enters because we use the narrow slot approximation for the shell slot field. The first few coefficients we cannot discuss without some numerical computations. However, when ka < 2 and n >> kb, we may use the small argument approximation of the cylindrical functions, and obtain

$$j\omega\mu \, k\left[B_n J_n'(kr) + C_n N_n'(kr)\right] \simeq \frac{1}{2\pi}\left\{\left[(\frac{ka}{kr})^{n-1} - (\frac{ka}{kb})^{n+1}(\frac{kr}{kb})^{n-1}\right] e^{jn\theta}\right.$$

$$\left. + \left[(\frac{kr}{kb})^{n-1} - (\frac{ka}{kr})^{n+1}(\frac{ka}{kb})^{n-1}\right] a_o J_o(n\phi_o)\right\}, \quad kb < 2, \, n >> kb. \quad (2.65\text{-a})$$

From (2.65-a) we have that on the cylinder surface

$$j\omega\mu \, k\left[B_n J_n'(ka) + C_n N_n'(ka)\right] \simeq \frac{1}{2\pi} e^{jn\theta} \qquad (2.65\text{-b})$$

and on the shell surface

$$j\omega\mu k \left[B_n J'_n (kb) + C_n N'_n (kb) \right] \simeq \frac{1}{2\pi} a_o J_o(n\phi_o). \qquad (2.65\text{-}c)$$

It is clear that many coefficients are necessary to approximate reasonably well the E_ϕ component in the co-axial space. The same statement applies to the other two field components. No Fourier coefficient alone under any conditions dominates the field components in the co-axial region, even in the case when for some particular coefficient $D_n(ka, kb) \rightarrow 0$. From this behavior of the Fourier coefficients we may conclude that no simple model may be devised to explain the transfer of power through the co-axial region. The power transfer results from the interference of very many Fourier coefficients.

2.7 Conclusions

We briefly summarize the salient features of the numerical results of this study. With a fixed magnetic line source on the cylinder the addition of a relatively close fitting slotted shell:

1) enhances radiation for all source and shell slot separation angles θ when the cylinder diameter $< 0.5/\pi\lambda$,

2) leaves radiation roughly unchanged when $0.5/\pi \lambda <$ the cylinder diameter $< 1.1/ \pi \lambda$, except for a deep minima in the vicinity of $\theta \sim \pm 90^o$,

3) enhances radiation when $1.1/\pi \lambda <$ the cylinder diameter $< 1.5/\pi \lambda$ except for a deep minima in the vicinity of $\theta \sim \pm 110^o$,

4) leaves radiation roughly unchanged when $1.5/\pi\,\lambda <$ the cylinder diameter $<2/\pi\,\lambda$, except for a deep minima in the vicinity of $\theta\sim\pm\,45^{\circ}$, $\pm\,135^{\circ}$.

The deep minima in case 2) is associated with the lowest root of $D_1(ka, kb)=0$, and in case 4) with the lowest root of $D_2(ka, kb)=0$. It appears that radiation will have no deep minima when $D_0(ka, kb)=0$, or very close to zero. Some of the antenna parameters for which this will occur are shown in Table II-1.

We also note that the theoretical work on the problem has been carried to the point where the numerical calculations can be carried out for wider slots, if we so desire. In this particular theoretical work the solution of the Fredholm integral equation of the first kind is based on the truncation of two infinite matrices. It would appear that this particular method of solving the fundamental integral equation of the problem has been put in the most favorable form for carrying out further computations.

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III. THEORY OF ELECTROMAGNETIC FIELDS FOR COAXIAL CIRCULAR
CYLINDERS WITH A SLOT AND MAGNETIC LINE SOURCE. Y. Hayashi

3.1 Introduction

In this chapter it is shown how to solve the Maxwell's equations for coaxial

circular cylinders with an axial slot of arbitrary width and an axial magnetic line

source. This is a simplified version of the solution for coaxial circular cylinders

with arbitrary number of slots and arbitrary number of axial electric and magnetic

line sources (Hayashi, 1964a).

The result obtained here is exact and rigorous, because it satisfies the

Maxwell's equations in the domain, the boundary condition on the walls of the

cylinders, the radiation condition at infinity, the edge condition at the edges of the

slot and the continuity condition at the slot. The result is also true for any values

of wave number and radii of the cylinders.

First of all, the field is represented by Fourier series with unknown

coefficients. Then a singular integral equation is derived for the unknown coefficients.

The singular integral equation is solved exactly by the theory developed by the author.

As an example, the interesting case of a narrow slot is presented in a form ready

for a numerical calculation.

3.2 Reduction of the Problem to a Singular Integral Equation

Suppose that the structure of the perfectly conducting cylindrical shells are

described, in the cylindrical coordinates as follows (Fig. 3-1)

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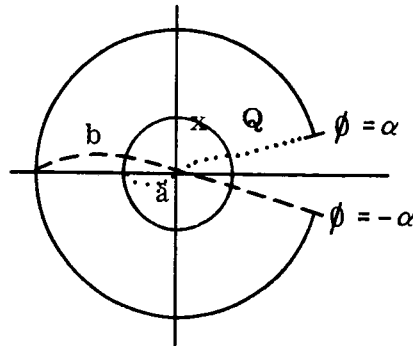
FIG. 3-1

Wall: $r = a$, $0 \leq \phi \leq 2\pi$ and $r = b$, $\alpha \leq \phi \leq 2\pi - \alpha$

Slot: $r = b$, $-\alpha < \phi < \alpha$

Assume that the axial line source is located at

$$Q : \quad r = a, \qquad \phi = \phi_0$$

In this case, as is well known, Maxwell's equations are reduced to the two-dimensional wave equation, i.e.

$$\Delta u + k^2 u = 0 \tag{3.1}$$

where $u = H_z$ = axial magnetic field and $k^2 = \omega^2 \epsilon \mu$. The rational MKS units are employed and $e^{i\omega t}$ time dependence is suppressed throughout.

The requirements on $u = H_z$ are as follows,

The radiation condition

$$\lim_{r \to \infty} r^{1/2} \left\{ \frac{\partial u}{\partial r} + i k u \right\} = 0. \tag{3.2}$$

The boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on the wall.} \tag{3.3}$$

The continuity condition, i.e.

$$u \text{ and } \frac{\partial u}{\partial n} \text{ are continuous through the slot.} \qquad (3.4)$$

The edge condition

$$u = 0(\rho^{-1/2}) \qquad (3.5)$$

where ρ is the distance from an edge of the slot.

As a solution of (3.1) which satisfies (3.2), u is represented as

$$u = u_e = \sum_{n=-\infty}^{\infty} A_n H_n(kr) e^{in\phi}, \qquad (b < r)$$

$$u = u_i = \sum_{n=-\infty}^{\infty} \left\{ B_n J_n(kr) + C_n H_n(kr) \right\} e^{in\phi} + f H_0(kR),$$

$$(a \leqslant r < b) \qquad (3.6)$$

where J_n and H_n are the Bessel function and the Hankel function of the second kind, f is an arbitrary source constant, and R is the distance from the source to the point of observation.

It is easy to see that (3.3) and (3.4) are equivalent to

$$\frac{\partial u_i}{\partial r} = 0 \qquad (r = a, \ 0 \leqslant \phi \leqslant 2\pi) \qquad (3.7)$$

$$\frac{\partial u_e}{\partial r} = \frac{\partial u_i}{\partial r} \qquad (r = b, \ 0 \leqslant \phi \leqslant 2\pi) \qquad (3.8)$$

$$\frac{\partial u_e}{\partial r} = 0 \qquad (r = b, \ \alpha < \phi < 2\pi - \alpha) \qquad (3.9)$$

$$u_e = u_i \qquad (r = b, \quad -\alpha < \phi < \alpha) \tag{3.10}$$

On substituting (3.6) into (3.7) and (3.8) and making use of the orthogonality of $\left\{ e^{in\phi} \right\}$, simultaneous linear equations for A_n, B_n and C_n are derived with which B_n and C_n are determined in terms of A_n as follows,

$$B_n = \frac{-H'_n(kb)}{T_n(a,b)} \left\{ H'_n(ka) A_n + \frac{i}{\pi ka} f_n \right\}$$

$$C_n = \frac{J'_n(ka)}{T_n(a,b)} \left[H'_n(kb)A_n - f_n \left\{ J_n(ka)H'_n(kb) - J'_n(kb) H_n(ka) \right\} \right]$$

$$- \frac{i}{\pi ka} \frac{J'_n(kb)}{T_n(a,b)} f_n \tag{3.11}$$

where

$$f_n = f e^{-in\phi_o}$$

$$T_n(a,b) = J'_n(ka)H'_n(kb) - J'_n(kb)H'_n(ka),$$

and (') means the derivative with respect to the arguments.

Eqs. (3.9) turns out, by the substitution of (3.6), to be

$$\sum_{n=-\infty}^{\infty} A_n H'_n(kb) e^{in\phi} = \begin{cases} 0 & ; \alpha < \phi < 2\pi - \alpha \\ 2\pi \tau(\phi); & -\alpha < \phi < \alpha. \end{cases} \tag{3.12}$$

The upper line of the right hand side of (3.12) is the direct consequence of (3.9), while $\tau(\phi)$ in the lower line is an unknown quantity which is proportional to E_ϕ in the slot.

Similarly, from (3.10), we have

$$\sum_{n=-\infty}^{\infty} \frac{e^{in\phi}}{T_n(a,b)} \left\{ H'_n(ka) A_n + \frac{i}{\pi ka} f_n \right\} = 0 \qquad -\alpha < \phi < \alpha, \qquad (3.13)$$

(3.12) and (3.13) are the "Dual series equations" with respect to A_n.

It is easy to see that (3.12) is equivalent to

$$A_n = \frac{1}{H'_n(kb)} \int_L \tau(\theta) e^{-in\theta} d\theta \qquad (3.14)$$

where L stands for $-\alpha < \phi < \alpha$. Then the substitution of (3.14) into (3.13) gives

$$\sum_{n=-\infty}^{\infty} S_n \int_L \tau(\theta) e^{in\Theta} d\theta = -i\pi(kb)^2 f(\phi) \qquad (3.15)$$

where

$$S_n = \frac{H'_n(ka)}{T_n(a,b) H'_n(kb)}, \quad \Theta = \phi - \theta,$$

$$f(\phi) = \frac{f}{\pi^2 (ka)(kb)^2} \sum_{n=-\infty}^{\infty} \frac{e^{in(\phi-\phi_o)}}{T_n(a,b)}$$

Interchanging integration with summation in (3.15) we obtain an integral equation with respect to the unknown function τ. If we solve the integral equation, then A_n's are defined by (3.14) in terms of τ, B_n's and C_n's are determined by (3.11), in terms of A_n's and finally, u_e and u_i are obtained by (3.6). It is not difficult to prove that u, obtained by this way, satisfies all the requirements (3.1), (3.2), (3.3), (3.4), and (3.5) if the solution τ has a singularity at the edge points, i.e.,

$$\tau = 0(\rho^{-1/2}) \qquad (3.16)$$

Therefore, the original problem is equivalent to solving (3.15) with respect to τ which satisfies edge condition (3.16).

According to the formulas due to the author (Hayashi, 1964b) the Bessel and Hankel functions are represented as follows:

$$J_n(\rho) = \frac{1}{n!} \left(\frac{\rho}{2}\right)^n \left\{1 + j_n\right\}. \qquad J'_n(\rho) = \frac{1}{2(n-1)!} \left(\frac{\rho}{2}\right)^{n-1} \left\{1 + j'_n\right\}$$

$$H_n(\rho) = \frac{i(n-1)!}{\pi} \left(\frac{2}{\rho}\right)^n \left\{1 + h_n\right\}. \qquad H'_n(\rho) = \frac{(-i)n!}{2\pi} \left(\frac{\rho}{2}\right)^{n+1} \left\{1 + h'_n\right\} \qquad (3.17)$$

where j_n, j'_n, h_n and h'_n are proved to be quantities of order $\frac{\rho^2}{4n}$. Hence, for a positive number N which is sufficiently large, S_n is estimated, when $n > N$, as

$$S_n = \frac{-i\pi(kb)^2}{n} \left\{1 + s_n\right\} \qquad (3.18)$$

where s_n is of order $\rho^2/4n$. For negative n (3.18) is also true, because we can prove that $S_{-n} = S_n$ with the help of well-known formulas $J_n = (-1)^n J_{-n}, H_n = (-1)^n H_{-n}, J'_n = (-1)^n J'_n$ and $H'_n = (-1)^n H'_{-n}$.

By virtue of (3.18) and the formula

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} = \frac{1}{2} \ln \frac{1}{2 - 2\cos\theta}, \quad \theta \neq 0$$

and interchanging integration with summation, (3.15) is converted into

$$\int_L \tau(\theta) N(\phi, \theta) d\theta = f(\phi) \qquad (3.19)$$

where

$$N(\phi, \theta) = \ln\frac{1}{2-2\cos\theta} + \sum_{n=1}^{N} \frac{2 s_n}{n} \cos n\theta + \frac{iS_o}{\pi(kb)^2} \tag{3.20}$$

Differentiating (3.19) with respect to ϕ, we have

$$\int_L \tau(\theta) \left\{ \frac{i}{2} - \frac{\sin\theta}{2-2\cos\theta} - k(\phi,\theta) \right\} d\theta = \frac{1}{2} f'(\phi) \tag{3.21}$$

where

$$k(\phi, \theta) = \frac{i}{2} \left[1 + \sum_{n=1}^{N} \left\{ 1 - \frac{inS_n}{\pi(kb)^2} \right\} \left(e^{in\theta} - e^{-n\theta} \right) \right] \tag{3.22}$$

Equation (3.21) is a singular integral equation with a Cauchy kernel. We will solve it in the following section by the application of the general theory on the singular integral equations developed by the author.

3.3 Solution of the Singular Integral Equation

Suppose that $z = r e^{i\theta}$ is a point in the complex z-plane, then

$$t = be^{i\theta}, \qquad t_o = be^{i\phi}, \qquad -\alpha < \theta, \quad \phi < \alpha \tag{3.23}$$

correspond to points on L, i.e., in the slot, and

$$c_1 = be^{\alpha}, \quad c_{-1} = be^{-\alpha}$$

correspond to the edge points of the slot.

Then it is not difficult to see that (3.21) is equivalent to

$$\hat{k}\tau = \frac{1}{\pi i} \int_L \left\{ \frac{1}{t-t_o} - k(t_o, t) \right\} \tau(t)\, dt = h(t_o) \tag{3.24}$$

where $\tau(t)$ and $h(t_o)$ are functions which correspond to $i\pi\tau(\theta)$ and $\frac{1}{2} f'(\phi)$

respectively under the transformation (3.23). $k(t_o, t)$ is a kernel corresponding to $k(\phi, \theta)$ and is

$$k(t_o, t) = \sum_{n=-N}^{N} k_n \frac{t_o^n}{t^{n+1}} \qquad (3.25)$$

where

$$k_n = \begin{cases} \frac{1}{2}\left[1 - \frac{in}{\pi(kb)^2}S_n\right], & n > 0 \\ \frac{1}{2} & n = 0 \\ \frac{-1}{2}\left[1 + \frac{in}{\pi(kb)^2}S_n\right], & n < 0 \end{cases}$$

The solution $\tau(t)$ of (3.24) must have, because of (3.16), a singularity at $c_{\pm 1}$, that is

$$\lim_{t \to c_{\pm 1}} \tau(t) = 0(1/\sqrt{t - c_{\pm}}) \qquad (3.26)$$

Suppose that M is a set of functions, the element $\psi(z)$ of which satisfies the following conditions (i)$\sim$(v),

(i) $\psi(z)$ is holomorphic in z–plane, except on L, at 0 and ∞.

(ii) $\psi(z) = 0(z^N)$, $|z| \to \infty$

(iii) $\psi(z) = 0(z^{-N})$, $|z| \to 0$

(iv) $\lim_{z \to c_{\pm 1}} \psi(z) = 0(1/\sqrt{z - c_{\pm 1}})$

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$$(v) \quad \psi^{+}(t_o) + \psi^{-}(t_o) = h(t_o), \quad t_o \in \lambda L$$

where $\psi^{+}(t_o) \left(\psi^{-}(t_o) \right)$ means the limiting value of $\psi(z)$ when z tends to a point $t_o \in L$ from the left (right) hand side of L. Then we can see (Muskhelishvili, 1953a) that a function defined by

$$\phi(z) = \frac{1}{2\pi i} \int_L \left\{ \frac{1}{t-z} - k(z,t) \right\} \tau(t) dt$$

satisfies (i)$\sim$(v), if τ is a solution of (3.24) which satisfies the requirement (3.26) Furthermore, we have,

$$\tau(t_o) = \phi^{+}(t_o) - \phi^{-}(t_o). \tag{3.27}$$

On the other hand, it is not difficult to see that the general expression of a function $\psi(z)$ which satisfies (i)$\sim$(v) is given by

$$\psi(z) = \frac{x(z)}{2\pi i} \int_L \left\{ \frac{1}{t-z} - \lambda(z,t) \right\} \frac{h(t)}{x(t)} dt + x(z) \sum_{n=-N}^{N+1} p_n z^n$$

where $p_n (n = -N, \ldots, N+1)$ are constants, $\lambda(z,t)$ is a function similar to $k(z,t)$ which will be determined later, and

$$x(z) = \frac{1}{\sqrt{(z-c_{+1})(z-c_{-1})}}$$

Thus we have found that a solution τ of (3.24) exists among these functions defined by

$$\tau(t_o) = \psi^{+}(t_o) - \psi^{-}(t_o)$$

or

$$\tau(t_o) = \hat{\lambda} h(t_o) + \tau_1(t_o) \tag{3.28}$$

where

$$\hat{\lambda} \, h(t_0) = \frac{x(t_0)}{\pi i} \int_L \left\{ \frac{1}{t - t_0} - \lambda(t_0, t) \right\} \frac{h(t)}{x(t)} \, dt$$

$$\tau_1(t_0) = x(t_0) \sum_{n=-N}^{N+1} p_n t_0^n \, .$$

Conversely, on substituting (3.28) into (3.24), we have (Muskhelishvili, 1953b)

$$\hat{k} \, \tau_1 = Kh \qquad\qquad (3.29)$$

where

$$Kh = \int_L K(t_0, t) h(t) \, dt$$

$$K(t_0, t) = \frac{1}{\pi^2 x(t)} \int_L x(\zeta) \left\{ \frac{1}{\zeta - t_0} - k(t_0 \zeta) \right\} \left\{ \frac{1}{t - \zeta} - \lambda(\zeta, t) \right\} d\zeta \qquad (3.30)$$

Because (3.29) is of the same form as (3.24), we can see that τ_1 is given by

$\tau_1 = \hat{\lambda} K h + \tau_2$ where τ_2 is again a solution of $\hat{k} \tau_2 = K^2 h$. Repeating this process,

we have

$$\tau_m = \hat{\lambda} K^m h + \tau_{m+1} \qquad\qquad (3.31)$$

where K^m is an operator with k-th iterated kernel $K_m(t_0, t)$ of $K(t_0, t)$. By the

addition of (3.31) from $m = 0$ to $m = m$, we have

$$\tau = \hat{\lambda} \sum_{n=0}^{m} K^n h + \tau_{m+1}$$

If λ is chosen so that $\|K\| < 1$, then $\displaystyle\sum_{n=0}^{m} K^n h$ converges, and, because we have

$\lim_{n \to \infty} K^n h = 0$ as the necessary condition for the convergence, then

$$\tau^{(0)} = \lim_{n \to \infty} \tau_n \text{ must be a solution of}$$

$$\hat{k}\tau^{(0)} = 0 \tag{3.32}$$

Thus we have obtained the solution of (3.24), that is

$$\tau = \hat{\lambda} \sum_{n=0}^{\infty} K^n h + \tau^{(0)} \tag{3.33}$$

Especially, if $K = 0$, then (3.33) is reduced to

$$\tau = \hat{\lambda} h + \tau^{(o)}. \tag{3.34}$$

$\tau^{(0)}$ has the same form as (3.28), i.e.

$$\tau^{(0)}(t_o) = x(t_o) \sum_{n=-N}^{N+1} p_n t_o^n \tag{3.35}$$

must be a solution of (3.32). In other words, $p_n (n = -N, \ldots, N+1)$ must be determined so that (3.35) satisfies (3.32). On substituting (3.35) into (3.32), we have

$$\sum_{n=-N}^{N+1} p_n \int_L \frac{x(t)t^n}{t-t_o} dt = \sum_{n=-N}^{N+1} p_n \int_L k(t_o, t) x(t) t^n dt \tag{3.36}$$

which is the necessary and sufficient condition for p_n with which (3.35) satisfies

(3.32). After manipulations, we can reduce (3.36) to

$$\sum_{n=-N}^{N+1} k_m \alpha_{n-m} p_n = - \sum_{n=m}^{-N} \gamma_{m-n} p_n, \quad -N \leq m \leq -1$$

$$\sum_{n=-N}^{N+1} k_m \alpha_{n-m} p_n = - \sum_{n=m+1}^{N+1} \beta_{m-n} p_n, \quad 0 \leq m \leq N \tag{3.37}$$

where

$$\alpha_{n-m} = \frac{1}{\pi i} \int_L x(t) t^{n-m-1} dt \tag{3.38}$$

and β_{m-n} and γ_{m-n} are constants defined by

$$x(z) z^n = \begin{cases} \displaystyle\sum_{m=-\infty}^{n-1} \beta_{m-n} z^m, & |z| > b \\[2em] \displaystyle\sum_{m=n}^{\infty} \gamma_{m-n} z^m & |z| < b. \end{cases} \tag{3.39}$$

(3.37) are $2N + 1$ conditions for $2N + 2$ constants p_n ($n = -N, \ldots, N+1$) which are equivalent to (3.36). Thus, p_n ($n = -N, \ldots N+1$) are determined up to a constant factor.

3.4 A Note On Solution of Non-Singular Integral Equations

Although we have solved (3.24), or (3.21) a solution of (3.21) does not necessarily satisfy the original equation (3.19) because (3.31) has been derived by the differentiation of (3.19) with respect to ϕ. However, we can pick the unique solution of (3.19) from the general solution of (3.21), by taking appropriate values of

$p_n (n = -N_1, \ldots, N+1)$. This will be discussed in the following.

Suppose that a set of $\{p_n\}$ satisfies (3.37), then

$$\tau^{(o)} = x(\theta) \sum_{n=-N}^{N+1} c_n e^{in\theta}$$

where

$$c_n = b^n p_n; \qquad x(\theta) = \frac{1}{b\sqrt{(e^{i\theta} - e^{i\alpha})(e^{i\theta} - e^{-i\alpha})}} \tag{3.40}$$

satisfies

$$\hat{k} \, \tau^{(o)} = 0,$$

or,

$$\int_L \tau^{(o)}(\theta) \, N'(\phi, \theta) \, d\theta = 0 \tag{3.41}$$

where $N'(\phi, \theta)$ is the kernel in the left hand side of (3.21). Furthermore,

$$\tau(\theta) = \tau_p(\theta) + \tau^{(o)}(\theta)$$

where $\tau_p(\theta)$ is the particular solution $\hat{\lambda}$ f, satisfies (3.21), or

$$\int_L \left\{ \tau_p(\theta) + \tau^{(o)}(\theta) \right\} N'(\phi, \theta) d\theta = f'(\phi) \tag{3.42}$$

From (3.41) and (3.42), we have

$$\int_L \tau_p(\theta) N'(\phi, \theta) d\theta = f'(\phi)$$

or, on integrating the last equation with respect to ϕ, we have

$$\int_L \tau_p(\theta) N(\phi, \theta) d\theta = f(\phi) - g \tag{3.43}$$

where g is a known constant. On the other hand, (3.41) turns out to be

$$\int_L \tau^{(0)}(\theta) N(\phi, \theta) d\theta = \beta$$

where β is a known constant corresponding to particular values of $\left\{ p_n \right\}$.

If we change $\left\{ p_n \right\}$ by $\left\{ \frac{g}{\beta} p_n \right\}$, or what is the same thing,

if we change $\tau^{(o)}$ by $\frac{g}{\beta} \tau^{(o)}$, then

$\tau \equiv \tau_p + \frac{g}{\beta} \tau^{(o)}$ satisfies (3.19). In fact,

$$\int_L \left\{ \tau_p(\theta) + \frac{g}{\beta} \tau^{(o)}(\theta) \right\} N(\phi, \theta) d\theta$$

$$= \int_L \tau_p(\theta) N(\phi, \theta) d\theta + \frac{g}{\beta} \int_L \tau^{(o)}(\theta) N(\phi, \theta) d\theta = f(\phi) - g + \frac{g}{\beta} \beta = f(\phi).$$

3.5 Electromagnetic Field for a Narrow Slot

As an example of the application of the general theory obtained in the previous sections, we will solve the field of a narrow slot. We assume that the slot is narrow in the sense that

$$\alpha^2 << 1 \tag{3.44}$$

and also that

$$\left| ka \right|, \left| kb \right| < 1. \tag{3.45}$$

Then, s_n, which is of the order $(kb)^2 / 4n$, is negligible for $n \geqslant 2$. Hence, we can assume that $N = 1$, and we have

$$\frac{1}{t-t_o} - k(t_o, t) \doteq \frac{+ie^{-i\theta}}{b} \left[\frac{\sin \Theta}{2-2\cos\Theta} + s_1 \sin\Theta \right],$$

where $\Theta = \phi - \theta$. Because $|\Theta| = |\phi = \theta| \lessgtr |2\alpha| < 1$,

$\sin\Theta$ and $\cos\Theta$ can be expanded in power series with respect to Θ, and we have

$$1 - k(t_o, t)(t-t_o) \doteq 1 + \frac{i}{2}(\phi - \theta) + 0(\alpha^2), \qquad (3.46)$$

$\left[\text{If } |s_1| \text{ is so large that} |s_1| \alpha^2 \sim 1, \text{ then we have to take one more term in}\right.$

(3.46). Here, we have assumed that $\left. |s_1| \alpha^2 < 1.\right]$

On taking $\lambda(t_o, t)$ to be equal to $k(t_o, t)$, one has

$$K(t_o, t) = \frac{1}{\pi^2 x(t)(t-t_o)} \int_L x(\zeta) \left\{ \frac{1}{\zeta-t_o} - \frac{1}{\zeta-t} \right\} g(t_o, t, \zeta) d\zeta,$$

where

$$g(t_o, t, \zeta) \equiv \left\{ 1 - k(t_o, \zeta)(\zeta - t_o) \right\} \left\{ 1 - k(\zeta, t)(t-\zeta) \right\}.$$

By virtue of (3.46), we have

$$g(t_o, t, \zeta) = 1 + \frac{i}{2}(\phi - \theta) + 0(\alpha^2)$$

which is independent of ζ if we neglect terms of $0(\alpha^2)$. Hence, $K(t_o, t)$ will be

$$K(t_o, t) = \frac{g(t_o, t)}{\pi^2 x(t)(t-t_o)} \int_L x(\zeta) \left\{ \frac{1}{\zeta-t_o} - \frac{1}{\zeta-t} \right\} d\zeta = 0$$

where $g(t_o, t) \equiv g(t_o, t, \zeta)$.

Thus we have proved, under the assumptions (3.44), (3.45) and $0(\alpha^2)=0$, that

$K(t_o, t) = 0$. Therefore, by virtue of (3.34), the solution τ of (3.21), or the tangen-

tial component of the electric field E in the slot is obtained as

$$\tau(\theta) = \frac{-i\omega\epsilon}{2\pi k} E_\theta(\theta)$$

$$= \frac{-f x(\theta)}{\pi^4 (ka)(kb)^2} \int_{-\alpha}^{\alpha} \frac{1}{x(\psi)(\theta-\psi)} \sum_{n=1}^{\infty} \frac{n \sin n(\psi-\phi_o)}{T_n(a,b)} d\psi$$

$$+ x(\theta) \sum_{n=-1}^{2} p_n b^n e^{in\theta} . \tag{3.47}$$

(3.47) is rewritten, by virtue of

$$\frac{1}{T_n(a,b)} = \frac{-i\pi (kb)^2}{n} \left(\frac{a}{b}\right)^{n+1} \left\{ 1 + 0\left(\frac{(kb)^2}{4n}\right) \right\} \tag{3.48}$$

as

$$\tau(\theta) = \frac{-i\omega\epsilon}{2\pi k} E_\theta(\theta)$$

$$= \frac{ifx(\theta)}{\pi^3 ka} \int_{-\alpha}^{\alpha} \frac{1}{x(\psi)(\theta-\psi)} \sum_{n=1}^{\infty} \left(\frac{a}{b}\right)^{n+1} \sin n(\psi-\phi_o) d\psi$$

$$+ x(\theta) \sum_{n=-1}^{2} p_n b^n e^{in\theta} , \tag{3.49}$$

where $x(\theta) = \dfrac{1}{b e^{i\theta} \sqrt{\alpha^2 - \theta^2}}$.

[Note that if (3.48) is not true for $n=1$, then the first term of (3.49) should be modified.]

In (3.47), or (3.49), $\{p_n\}$ is determined by the following conditions

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$$(k_{-1}\alpha_0 + \gamma_0)\, p_{-1} + k_{-1}\, \alpha_1 \quad p_0 + k_{-1}\, \alpha_2\, p_1 + k_{-1}\, \alpha_3\, p_2 = 0$$

$$k_0\, \alpha_{-1}\, p_{-1} + k_0\, \alpha_0\, p_0 + (k_0 \alpha_1 + \beta_{-1})\, p_1 + (k_0\, \alpha_2 + \beta_{-2})\, p_2 = 0$$

$$k_1\, \alpha_{-2}\, p_{-1} + k_1\, \alpha_{-1}\, p_0 + k_1\, \alpha_0\, p_1 + (k_1\, \alpha_1 + \beta_{-1})\, p_2 = 0 \tag{3.50}$$

where

$$\alpha_n = \frac{b^n}{\pi} \int_{-\alpha}^{\alpha} x(\theta)\, e^{in\theta}\, d\theta .$$

With the knowledge of the distribution of E_θ in the slot obtained by (3.49), the fields are given by (3.14), (3.11) and (3.6).

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IV. CONSIDERATION OF ENERGY STORAGE AND DISSIPATION IN A CAVITY
BACKED SLOT ANTENNA AS IT AFFECTS INPUT IMPEDANCE AND
RADIATION POWER . A. Olte

4.1 Introduction

In this chapter we consider the interaction of a cavity backed slot antenna with

the plasma sheath. The slot is small with respect to the wavelength and hence the

antenna is very narrow band. With the onset of the over-dense plasma sheath, the

antenna becomes detuned and this in addition to the usual overdense plasma inter-

ference with the radio waves strongly limits transmission. One could broad-band

the antenna at the expense of the antenna's efficiency by introducing losses in the

cavity. Would this help in maintaining transmissions with the onset of the plasma

sheath? Can we in any other way limit the undesirable influence of the plasma

sheath? We undertook this study in an attempt to search out and define some

favorable modes of interaction. This work is reported in four sections. In the

first section we derive energy theorem for the cavity backed slot antenna. In the

second section we use the energy theorem to express the antenna impedance in terms

of certain integrals. In the third section we derive expressions for the antenna

impedance with and without the plasma sheath both for the loss-less and the lossy

cavity. In the fourth section we discuss and compare these expressions. The

cavity losses do not appear to play any useful role. It appears that the plasma sheath

interference will be kept at a minimum if the antenna resonance does not depend on

the outside reactive fields, and if the reactive fields on the antenna surfaces are of the

magnetic type. To the extent these conditions can be brought about by inside tuning is of continuing interest.

4.2 The Energy Theorem

We consider antenna structure as shown in Figure 4-1. A cavity is cut in a perfectly conducting body of a finite size. The cavity is connected to the free space by a slot which is center-fed by a co-axial line. We select the input terminals of the antenna at a reference plane A in the co-axial line. The volume of the co-axial line from the reference plane A to the opening at the slot we denote by V_1, the cavity volume by V_2, and the plasma volume by V_3. The free space volume bounded by the plasma surface and the geometrical surface S_4 is denoted by V_4. The following electromagnetic parameters are assumed for the respective volumes:

$$V_1: \quad \epsilon_1 \text{ (real)}, \quad \mu_o$$

$$V_2: \quad \epsilon_2 = \epsilon_2' - j \epsilon_2'', \quad \mu_o$$

$$V_3: \quad \epsilon_3 = \epsilon_3' - j \epsilon_3'', \quad \mu_o$$

$$V_4: \quad \epsilon_4 = \epsilon_o, \quad \mu_o$$

We have that $\quad \epsilon_o = \dfrac{1}{c^2 \mu_o}$, $\quad$ where $\mu_o = 4\pi \times 10^{-7}$, and c is the velocity of light in vacuum.

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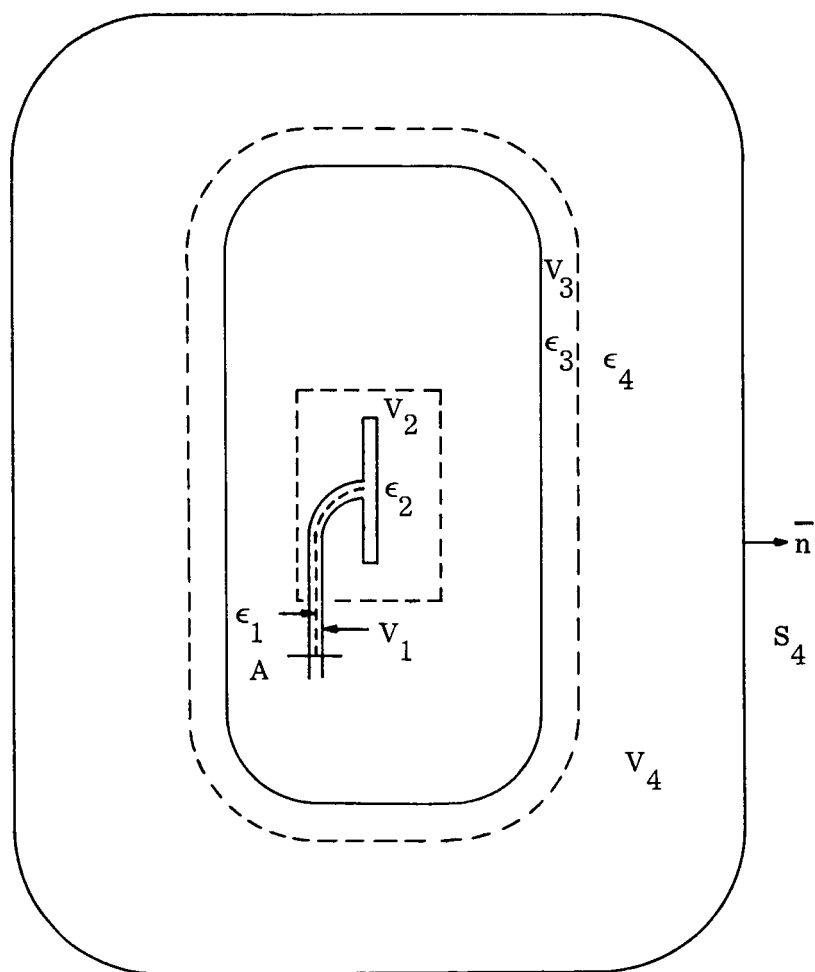
The Antenna Structure

Figure 4-1

We start our analysis with the vector identity

$$\nabla \cdot (\bar{E} \times \bar{H}^*) = \bar{H}^* \cdot \nabla \times \bar{E} - \bar{E} \cdot \nabla \times \bar{H}^*, \qquad (4.1)$$

where $\bar{E}$ and $\bar{H}$ are the complex amplitudes of the electric and magnetic

fields, respectively. The time dependence is taken as $e^{j\omega t}$. Assuming that no

sources are present in the regions enumerated, or on their boundaries, we obtain

from (4.1) after substituting the appropriate Maxwell's equations,

$$\nabla \cdot (\bar{E} \times \bar{H}^*) = j\omega \left[\epsilon^* \bar{E} \cdot \bar{E}^* - \mu_0 \bar{H} \cdot \bar{H}^* \right], \qquad (4.2)$$

We multiply Eq. (4.2) by the differential volume dv and integrate in each region. We

add the results together for the four regions. After transforming the left hand side by

the Divergence Theorem, only two surface integrals remain. The others either van-

ish or cancel each other because of the usual electro-magnetic boundary conditions.

The end result of this operation is

$$\int_{S_A} \bar{E}^{(1)} \times \bar{H}^{(1)*} \cdot \bar{n} ds + \int_{S_4} \bar{E}^{(4)} \times \bar{H}^{(4)*} \cdot n ds =$$

$$j\omega \sum_{i=1}^{4} \int_{V_i} \left[\bar{E}^{(i)} \cdot (\epsilon_{(i)} \bar{E}^{(i)})^* - \mu_0 \bar{H}^{(i)} \cdot \bar{H}^{(i)*} \right] dv, \qquad (4.3)$$

where $\bar{n}$ is the outward normal and the superscripts identify the fields of the

respective regions as shown in Figure 4-1.

The time average power radiated by the antenna, P_r, is

$$\frac{1}{2} R_e \int_{S_4} \overline{E}^{(4)} \times \overline{H}^{(4)*} \cdot \overline{n} \, ds \tag{4.4}$$

When the surface S_4 is sufficiently far removed from the antenna, then the power flow is radially outward and the electromagnetic fields locally assume a plane wave character, and

$$P_r \rightarrow \frac{1}{2} \int_{S_4} \overline{E}^{(4)} \times \overline{H}^{(4)*} \cdot \overline{n} ds \tag{4.5}$$

In order not to have the question always come up as to how far we should take S_4 from the antenna, we let $S_4 \rightarrow S_\infty$, the surface at infinity.

We assume that the radial dimenions of the co-axial line are such that only the TEM mode is propagating. We select the reference plane A sufficiently far removed from the co-ax opening (a fraction of the wavelength is sufficient) to have the higher order mode fields essentially zero. At the reference plane the fields are then the sum of the incident and the reflected TEM modes. The amplitudes of the electric and the magnetic fields may be associated with equivalent electric voltage V and current I, respectively. One of the defining expressions is taken as

$$VI^* = - \int_{S_A} \overline{E}^{(1)} \times \overline{H}^{(1)^*} \cdot \overline{n} \, ds \tag{4.6}$$

The minus sign is selected to conform to the convention that the positive direction of power flow is into the antenna. In case of the TEM mode the equivalent

transmission line voltages and currents may be made unique by taking the

characteristic impedance of the line, Z_o, as

$$Z_o = \frac{1}{2\pi} \sqrt{\frac{\mu_o}{\epsilon_1}} \ln\left(\frac{r_o}{r_i}\right), \tag{4.7}$$

where r_i and r_o are inside and outside cylinder diameters, respectively.

Substituting (4.5) and (4.6) in (4.3) we have

$$VI^* = 2P_r + j\omega \sum_{i=1}^{4} \int_{V_i} \left[\mu_o \overline{H}^{(i)} \cdot \overline{H}^{(1)*} - \overline{E}^{(i)} \cdot \left(\epsilon_{(i)} \overline{E}^{(1)*}\right) \right] dv, \tag{4.8}$$

Because we have let S_4 recede to infinity, the volume V_4 is infinite. However, this volume integral is finite because the fields assume the plane wave character in the far-zone and as a result the integrand vanishes there.

Substituting the respective values of permittivity, in Eq. (4.8), and

separating the right hand side of (4.8) into real and imaginary parts,

$$VI^* = 2P_r + \omega \sum_{i=2}^{3} \left[\int_{V_i} \epsilon_i'' \, \overline{E}^{(i)} \cdot \overline{E}^{(i)*} \, dv \right] + $$

$$+ j\omega \sum_{i=1}^{4} \int_{V_i} \left[\mu_o \overline{H}^{(i)} \cdot \overline{H}^{(i)*} - \epsilon_i' \, \overline{E}^{(i)} \cdot \overline{E}^{(i)*} \right] dv \tag{4.9}$$

We define

$$\mu_o \overline{H}^{(i)} \cdot \overline{H}^{(i)*} = 4 w_H^{(i)}$$

$$\epsilon_{(i)}' \, \overline{E}^{(i)} \cdot \overline{E}^{(i)*} = 4 w_E^{(i)} \tag{4.10}$$

We recognize that $w_H^{(i)}$ and $w_E^{(i)}$ is the time average energy density of the magnetic and electric fields, respectively, in the i-th region.

With these definitions equation (4.9) may be rewritten as

$$\frac{1}{2} VI^* = P_r + \frac{\omega}{2} \sum_{i=2}^{3} \int_{V_i} \frac{\epsilon_i''}{\epsilon_i'} w_E^{(i)} \, dv +$$

$$2j\omega \sum_{i=1}^{4} \int_{V_i} \left[w_H^{(i)} - w_E^{(i)} \right] \, dv \tag{4.11}$$

The right-hand side of (4.11) consists of a real part and an imaginary part. The real part consists of the first two expressions. The first expression is the average radiated power of the antenna. We identify the physical meaning of the terms in the second expression by observing that the divergence of the instantaneous Poynting's vector is equal to the negative instantaneous dissipated power density in the medium. From this, one may show (Landau and Lifshitz, 1960) that the time average density of the power dissipated in the permittivity losses is given by

$$\frac{1}{2} \omega \frac{\epsilon_i''}{\epsilon_i'} w_E^{(i)} \tag{4.12}$$

and consequently the average power losses in the cavity and the plasma sheath P_2 and P_3, respectively, are given by the terms of the second expression on the right hand side of Eq. (4-11). The last set of terms on the right hand side we interpret as proportional to the difference in the magnetic and the electric energies stored by the antenna in each region, the proportionality constant being 2ω.

4.3 The Circuit Representation

We define the antenna impedance at the reference plane A by

$$Z = R + jX = \frac{V}{I}.$$ (4.13)

Substituting this in Eq. (4-11) and separating the real and imaginary parts, we obtain

$$R = R_r + R_2 + R_3$$ (4.14)

where

$$R_r = \frac{1}{II^*} \int_{S_\infty} \overline{E}^{(4)} \times \overline{H}^{(4)*} \cdot \bar{n}ds$$

$$R_2 = \frac{\omega}{II^*} \int_{V_2} \frac{\epsilon_2''}{\epsilon_2'} w_E^{(2)} dv$$

$$R_3 = \frac{\omega}{II^*} \int_{V_3} \frac{\epsilon_3''}{\epsilon_3'} w_E^{(3)} dv;$$

and

$$X = \sum_{i=1}^{4} X_i,$$ (4.15)

where

$$X_i = \frac{4\omega}{II^*} \int_{V_i} \left[w_H^{(i)} - w_E^{(i)} \right] dv$$

From equation (4.14) we see that the resistance of the antenna consists of three resistances in series, we denote them by

the "radiation resistance," the "cavity resistance," and the plasma "sheath resistance."

The "radiation resistance" is given by twice the average power radiated divided by the antenna input current amplitude squared. The "cavity resistance" and the "plasma sheath resistance" is given by twice the average power dissipated in the cavity and in the plasma sheath, respectively, divided by the square of the antenna input current.

From Eq. (4-15) we see that the reactance of the antenna consists of four reactances in series: the "co-axial termination reactance," the "cavity reactance," the "plasma sheath reactance," and the "free space reactance," X_1, X_2, X_3, and X_4, respectively. Each reactance is given by the difference in the magnetic and the electric energies stored in the respective region multiplied by $\frac{4\omega}{II^*}$.

In this formulation of the equivalent circuit problem the current I is the independent variable. The equivalent antenna circuit may be represented by an ideal current generator that is connected to the impedance $Z = R_r + R_2 + R_3 + j\left[X_1 + X_2 + X_3 + X_4\right]$ as shown in Figure 4-2.
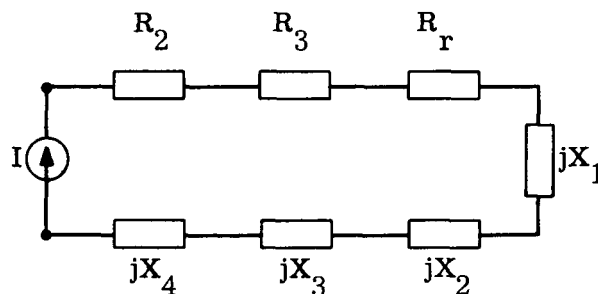

Figure 4-2.

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Specifying the input current in the one-port network corresponds to specifying the tangential magnetic field at the input terminals of the antenna. In the four regions we must then solve for the electromagnetic fields that satisfy the vector wave equation and the boundary conditions; a formidable problem for the antenna structure under discussion here. If we had specified the voltage V at the input terminals, then this would correspond to specifying the tangential electric field (in case of the TEM mode the total electric field) at the reference plane A and the appropriate circuit representation would involve an ideal voltage generator connected to an admittance which represents the antenna. The admittance Y is given by Z^{-1}.

The above energy balance frame-work does not give any outright answers to the antenna problem. It does provide a form which suggests possible mechanisms to account for and explain some of the observed antenna behaviour. Before we take up the discussion of antenna configurations of interest we should make some observation on the realizability of the equivalent circuit representation as well as discuss some of the circuit aspects of the problem. The formulation of the antenna impedance representation pre-supposes the availability of an ideal current generator, i.e. a device which produces a specified current that is independent of the load. The voltage across the load then is a variable and depends for a given current generator on the load connected to it. For quasi-static situations we have devices that approximate reasonably well the ideal current generator. For high frequencies such devices are not available, and the internal impedance of the generator enters the problem. However, for high frequencies instead of a constant current generator we can

physically approximate the ideal power generator: a device that sends a fixed

amount of power on the load independent of the load and absorbs any reflected power.

When the antenna and its surroundings are composed of a linear medium, and such a

case is assumed for this discussion, then the ratio of the reflected power to the

incident power is independent of the incident power amplitude, and this ratio then is

of some use in describing the antenna properties. The power of a particular incident

waveguide mode determines the electromagnetic field amplitude. The other main

feature of the mode is its phase. Therefore, the boundary value problem of an ideal

power generator source may be visualized as a waveguide mode incident on the

reference plane A (in our case the TEM mode of the co-ax) for which we seek to find

the amplitude and the phase of the reflected mode. Again we recall the assumption

that the TEM mode only may propagate in the co-ax, and thus the incident and the

reflected TEM modes constitute in our case the total field at the reference plane A,

for all practical purposes. We see that in the ideal power generator boundary value

problem we specify neither the tangential electric field nor the tangential magnetic

field at A, but a partial field of each, namely, the electromagnetic field of the

incident mode. The boundary value problem consists of finding the electromagnetic

field in each region subject to the local boundary conditions and the provision that in

the coax in addition to the incident TEM mode there exists an outward travelling TEM

mode, i.e., the reflected TEM mode. In this case the total tangential electric and

magnetic field at the antenna input terminals, depends on

the load, i.e. on any changes in the antenna structure. As before, we may associate equivalent current and voltage at the input terminals, but none of these are now independent of the load. However, formulas (4.14) and (4.15) are valid for computing the antenna resistance and reactance.

The characterization of the antenna that naturally fits the ideal power generator representation is the reflection coefficient, Γ. It is defined for any mode, as a ratio of the transverse electric field amplitudes of the reflected and incident modes, taken at the reference plane A. The same amount of physical information about the antenna is contained in the reflection coefficient representation as in the impedance representation. A well known simple transformation connects both representations:

$$\Gamma = \frac{Z - Z_o}{Z + Z_o} \tag{4.16}$$

The amplitude of the reflection coefficient squared is the fraction of the incident power reflected from the antenna. One often represents the antenna in terms of the voltage standing wave ratio (VSWR) given by

$$VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|} \tag{4.16a}$$

which is the ratio of the electric field maximum to minimum amplitudes in the waveguide feeding the antenna.

Whenever the circuit dimensions are small compared to the wavelength simple circuits are common in which the resistance and reactance may be varied independently from each other. In some cases the antenna reactance may possibly be varied without significantly affecting the radiation resistances. But this is not to be expected as a matter of course. Let us assume that in Eq. (4.16) the real part of Z is fixed and let us vary the imaginary part so as to obtain the minimum power reflections from the load for a given Z_o and R. The fraction of the reflected power from (4.16) is

$$|\Gamma|^2 = \frac{(R-Z_o)^2 + X^2}{(R+Z_o)^2 + X^2} \tag{4.17}$$

The extremum of $|\Gamma|^2$ under the conditions discussed above is

$$\frac{\partial}{\partial X}\left[|\Gamma|^2\right] = 0 \tag{4.18}$$

It leads to the condition that

$$XRZ_o = 0 \tag{4.19}$$

and this can only be satisfied when

$$X = 0 \tag{4.20}$$

Clearly this extremum point gives the minimum condition for the reflected power. And from Eq. (4.16a), we see that whenever $|\Gamma|$ is minimum then also VSWR is minimum. The absolute minimum ($\Gamma = 0$, and VSWR = 1) occurs when $Z = R = Z_o$. We then say that the antenna is matched to the feeding transmission line. In circuit theory we say that the circuit is in resonance when the reactive part of Z vanishes. This of course is the series resonance of a simple RLC circuit.

Similarly we say that the antenna is at resonance when $X = 0$. This condition implies

from Eq. (4.15) that the magnetic energy storage is equal to the electric energy stor-

age in the antenna. The same condition as in the RLC circuit resonance.

4.4 The Broad-Band and the Narrow-Band Antenna Problem

We want to discuss specific antenna configurations and possibly infer some

mechanism to explain the reported antenna behavior. We will use the current gener-

ator representation, since it contains the same amount of information as any other

one.

Case a. The cavity dielectric is loss-free: $P_2 = 0$, and no plasma sheath on

antenna surfaces, i.e. $\epsilon_3 = \epsilon_o$. We have then from (4.14) and (4.15) that

$$R_a = \frac{1}{\Pi^*} \int_{S_\infty} \bar{E}_a^{(4)} \times \bar{H}_a^{(4)*} \cdot \bar{n} \, ds \qquad (4.21)$$

and

$$X_a = \frac{4\omega}{\Pi^*} \sum_{i=1}^{\infty} \int_{V_i} \left[w_{Ha}^{(i)} - w_{Ea}^{(i)} \right] dv \qquad (4.22)$$

We have used the subscript a in Eq. (4.21) and (4.22) to specifically indicate

those quantities which will depend on the antenna structure changes initiated in the

later discussions, and the antenna configuration will be identified by them as well.

We imagine that we tune the antenna by changing the cavity volume. The changes are

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made on the wall opposite the cavity slot, so as to keep the spatial character of the slot field approximately the same. The antenna is tuned when $X_a = 0$ in Eq. (4.22). We then have a specific value of the antenna resistance which we denote by R_a. We desire to make the antenna resistances equal to the characteristic impedance of the feeder line as the optimum condition of the operation. The next question of interest is to determine what causes X_a to equal zero. Does each integral in Eq. (4.22) vanish separately, or do they cancel each other in some manner from which we may infer some physical meaning ? The first integral is over the volume of the co-ax feeder section. This is normally a small volume and the difference in the average energies stored is expected to be small compared with the other energies to be discussed. For the time being we leave it out of the discussion. Since the cavity volume is considerable, a significant amount of average energy may be stored there: electric energy will predominate when the cavity is below resonance while magnetic energies predominates above resonance. In the present case of no plasma sheath the common boundary between volumes V_3 and V_4 is only a geometrical boundary and the two volumes should be discussed together. Although V_4 is an infinite volume, the integral will exist because the far-zone fields assume a plane wave character which implies that the instantaneous energy storage in the magnetic and electric fields are equal, and hence the difference in the time average energy densities is zero as well i.e.

$$w_{Ha}^{(4)} - w_{Ea}^{(4)} = 0, \; r > R_o, \qquad (4.23)$$

where R_o is a sufficiently large number. Throughout volume V_3 we will get a significant contribution to the integral, this volume being close to the slot and also in the immediate vicinity of the cylinder. To the extent of neglecting the energy storage in the co-axial feeder section, one possibility of obtaining antenna resonance, physically a very appealing one, is to consider that the cavity difference energy storage is compensated by the difference energy storage outside the antenna, i.e.

$$\int_{V_2} \left[w_{Ha}^{(2)} - w_{Ea}^{(2)} \right] dv \simeq \sum_{i=3}^{4} \int_{V_i} \left[w_{Ha}^{(i)} - w_{Ea}^{(i)} \right] dv, \qquad (4.24)$$

In this picture the slot couples the two predominant energy storage systems: the cavity and the near-zone of the antenna. The excess energy surges back and forth through the slot alternating directions each half cycle. This would be expected to enhance the slot field amplitude, and thus to increase the radiated power of the antenna. Unfortunately, we have no analytic means at our disposal to make this picture of the antenna resonance unique for a configuration as in Fig. 4-1. We may also see from a slightly different viewpoint that the picture of resonance presented is a physically reasonable one. Let us assume that the cavity slot is closed, and the cavity is excited by a small probe on the opposite side. We select the cavity dimensions such that it is in the lowest mode of resonance at the exciting frequency ω. Now let us open a small slot. Energy from the cavity will leak through the slot into the outside space. The nature of the cavity fields close to the slot will also change radically, because of the changed boundary conditions. The cavity is no longer in "resonance" as it has been de-tuned by the slot. Opening the slot creates in fact a

new system which involves not only the cavity, but also the free space outside the
body of the antenna. We want to find again the lowest mode of resonance of this new
system. The simplest one and therefore possibly the lowest mode of resonance
appears when we change the cavity dimensions just sufficient to make the sum of the
difference energy storages in the cavity and the free space equal to zero. Only two
possibilities are open, either the difference energy storage in the cavity and the free
space vanish simultaneously, or they cancel each other. The possibility of the first
situation cannot be excluded for all cases. However, the latter possibility appears
to be physically much more probable than the first one. In our discussion, for
most part, we will consider only the most probable case.

When the coupling slot is small compared to the wavelength in both dimension,
only a small fraction of the cavity energy will leak out, and consequently we will
claim that the "Q" of the new system is just slightly lower than for the no slot
cavity. In such a situation in order to obtain a significant amount of radiation from
the antenna, the system must be brought to a resonance. That implies large field
amplitudes in the cavity, large surface currents over the conducting walls, and also
large electric fields in the slot, even when it is small compared to wavelengths.
With the proper transmission line coupling into the cavity and the system at
resonance, we should be able to radiate all the power from the feed line, no matter
how small the slot is. Such an antenna has a very narrow band-width and any changes
in configuration will de-tune it markedly. The fact that in our case the feeder excites

the slot directly does not change the substance of the discussion so far presented, it

only makes the slot field more complex and thus less amenable to a theoretical

treatment.

From this discussion we see that it would be of considerable value to establish

for some antenna shape the magnitude of the coupling provided by the slot for inside

and outside reactive energies. The antenna configuration should be such that the

exact solution of the boundary value problem can be carried out.

At this point we should make a digression to observe that the impedance of a

narrow slot in a perfectly conducting ground plane in free-space is related by

Babinets' principle to the impedance of the thin electric antenna. (A thin wire cut in

half and excited at the center by a voltage generator, also known as the electric

dipole). The electric dipole is the most common and most studied antenna and its

impedance is readily available. If Z_e is the electric dipole impedance and Z_m

the thin slot impedance then from Babinets' principle we have that

$$Z_m = \frac{\mu}{\epsilon} \; \frac{1}{Z_e} \tag{4.25}$$

A thin slot in a perfectly conducting ground plane is an example of the physical

realization of the magnetic dipole antenna. The magnetic dipole impedance may be

applicable directly to antennas where the radii of curvature are large with respect to

wavelength.

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Case b. We leave everything as in case a), except we add the plasma sheath. The permittivity of plasma we take as

$$\epsilon_3 = \epsilon_o \left[1 - \frac{\omega_p^2}{\omega^2 + \nu_m^2} - j \frac{\left(\frac{\nu_m}{\omega}\right)\omega_p^2}{\omega^2 + \nu_m^2} \right], \tag{4.26}$$

where ω_p is the plasma frequency, and ν_m the momentum transfer collision frequency. In case a) we had the resistance of the antenna given by (4-21) and the cavity was tuned such that the total reactance was zero. Maintaining the same current I at the input terminals we introduced the plasma sheath in V_3. Both the resistance and the reactance will change. We denote the new quantities by subscript b.

$$R_b = \frac{1}{II^*} \left\{ \int_{S_\infty} \bar{E}_b^{(4)} \times \bar{H}_b^{(4)*} \cdot \bar{n} \, ds + \epsilon_o \omega \int_{V_3} \frac{\nu_m}{\omega} \frac{\omega_p^2}{\omega^2 + \nu_m^2} \bar{E}_b^{(3)} \cdot \bar{E}_b^{(3)*} \, dv \right\} \tag{4.27}$$

$$X_b = \frac{4\omega}{II^*} \sum_{i=1}^{4} \int_{V_i} \left[w_{Hb}^{(i)} - w_{Eb}^{(i)} \right] dv \tag{4.28}$$

The antenna resistance now consists of two terms. The first term comes, as before, from the radiated power. The second term is new and it represents the power absorbed in the plasma sheath. The reactance, as before, consists of four

integrals. We would like very much to make a statement about how much R_b differs

from R_a, and X_b from zero. Unfortunately for this we have to calculate the fields

first. In that case we would also find the tangential electric field at the reference

plane A, and hence the voltage V for the specified I. There would be no need for

this integral representation. The main value of the above expressions are in that

they provide a qualitative picture of the components of the resistance and reactance,

and furthermore give a frame of reference for making comparisons if reasonable

approximations can be made in the fields. The best possible thing would be to solve

some canonical problems whose results could then be extended to the antenna shape

of interest by physical argument. We have not done that as yet and thus we have

to rely only on our physical feelings for the problem in order to make some further

comments.

Conditions have not changed in the cavity, thus the fields would tend to main-

tain themselves there at the previous values. Let us assume therefore that the

difference average energy storage in cavity remains approximately the same. What

happens to the fields in the plasma sheath? Let us consider a case where

$\omega^2 + \nu_m^2 < \omega_p^2$, then the real part of ϵ_3 is negative and hence $w_{Eb}^{(3)}$ is negative, and

the difference energy storage in the plasma sheath is the sum of the average electric

energy and average magnetic energy; always a positive quantity. The implication of

this is that plasma beyond cut-off appears always inductive, no matter in what con-

figuration it is placed. It is rather difficult to make some statements on the

difference average energy stored outside the plasma sheath. However, one over-all effect of the plasma sheath is obvious: the antenna will be de-tuned; probably $X_b > 0$, i.e. the antenna will appear to be inductive. This is the effect that the plasma sheath will play on the imaginary part of the antenna impedance. The real part of the antenna impedance consists of two parts, as pointed out before, the radiation resistance and the plasma sheath resistance. The radiation resistance with the plasma sheath beyond cut-off is definitely smaller than R_a because the sheath will impede radiation. The magnitude of the plasma sheath resistance will depend in large measure on the magnitude of the electric fields in the plasma sheath and the ratio $\dfrac{\nu_m}{\omega}$. It may be comparable to the radiation resistance.

Case c. The antenna has the same conditions as in case a), except the permittivity in the cavity is complex. We tune up the antenna, i.e. make $X_c = 0$, and we have a corresponding resistance R_c. In terms of the fields for this case we have

$$R_c = \frac{1}{\Pi^*} \left\{ \int_{S_\infty} \overline{E}_c^{(4)} \times H_c^{(4)*} \cdot \overline{n}\, ds + \omega \int_{V_2} \frac{\epsilon_2''}{\epsilon_2'}\, w_{Ec}^{(2)}\, dv \right\} \qquad (4.29)$$

and

$$X_c = \frac{4\omega}{\Pi^*} \sum_{i=1}^{4} \int_{V_i} \left[w_{Hc}^{(i)} - w_{Ec}^{(i)} \right] dv = 0. \qquad (4.30)$$

This antenna will have a wider band-width than in case a) because the losses are greater and hence the antenna Q is lower. The antenna Q is defined in the same manner as for circuits.

Case d) The antenna has the same conditions as in case c), except we add the plasma sheath and do not re-tune the antenna. The elements of the antenna impedance we now express as

$$R_d = \frac{1}{II^*} \left\{ \int_{S_\infty} \overline{E}_d^{(4)} \times \overline{H}_d^{(4)*} \cdot \overline{n} \, ds + \omega \int_{V_2} \frac{\epsilon_2''}{\epsilon_2'} \, w_{Ed}^{(2)} \, dv + \right.$$

$$\left. \epsilon_o \omega \int_{V_3} \frac{\nu_m}{\omega} \frac{\omega_p^2}{\omega^2 + \nu_m^2} \, \overline{E}_d^{(3)} \cdot \overline{E}_d^{(3)*} \, dv \right\} \tag{4.31}$$

$$X_d = \frac{4\omega}{II^*} \sum_{i=1}^{4} \int_{V_i} \left[w_{Hd}^{(i)} - w_{Ed}^{(i)} \right] \, dv \tag{4.32}$$

The same comments can be made about this case as in case b). Things are changed only to the extent that we have energy dissipation in the cavity and this entails the change of fields in every other region to some extent.

4.5 Discussion

If the slot is narrow and less than half a wavelength long, then the coupling between the cavity and the free space is not a maximum as would be expected when the

slot is approximately one half-wavelength long. In order to obtain a significant amount of radiation the antenna must be brought to a resonance and the shorter the slot the more narrow the antenna bandwidth will be. Thus when the slot is short as in Case a) we definitely have a narrow band antenna. In our further discussion we will refer to this case as the "narrow band" antenna, and the Case b) as the "narrow band" antenna with the plasma sheath. If we make the cavity dielectric lossy, we have Case c). The lossy cavity dielectric decreases the antenna Q and hence broadbands the antenna to some extent, causing a loss, of course, in the radiation efficiency. The Case c) we will denote in our discussion as the "broad band" antenna. And Case d) as the "broad band" antenna with the plasma sheath. We have represented the equivalent antenna impedance in terms of the fields in the appropriate regions. Each region is characterized by a resistance and a reactance. For the total effect they add in series. Now we want to discuss the power radiated in each case and make some comparisons. It is significant to discuss the antenna performance taking into account the internal impedance of the source. We select the voltage generator as the source and we consider a circuit as shown in Figure 4-3.
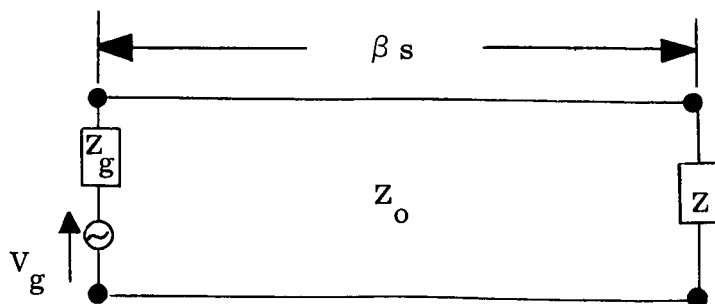

Figure 4 - 3

An ideal transmission line of length s, phase constant β and a characreristic impedance Z_o connects the antenna to the equivalent voltage generator which represents the transmitter. The length of the transmission line is from the reference plane A to the transmitter terminals. The transmitter is specified by voltage V_g and internal impedance Z_g. The current I through the impedance Z is given by

$$I = \frac{V_g}{Z_o (Z_g + Z_s)} \left[Z_o \cos(\beta s) - j Z_s \sin(\beta s) \right] , \qquad (4.33)$$

where

$$Z_s = Z_o \frac{Z + j Z_o \tan(\beta s)}{Z_o + j Z \tan(\beta s)} \qquad (4.34)$$

The above formulas result from elementary transmission line theory. The radiated power of the antenna is given by

$$P_r = \frac{1}{2} \left| I \right|^2 R_r \qquad (4.35)$$

The maximum power will be extracted from the generator and transferred by the transmission line to the load when the load impedance as seen at the generator terminals is equal to the conjugate of the generator impedance, a well known condition, i.e.

$$Z_s = Z_g^* \qquad (4.36)$$

Furthermore, when the generator is not matched to the line, the reflection coefficient in the line does not provide a means of relating the power absorbed by the

antenna with the antenna impedance changes. This is because the reflection

coefficient is simply the reflected wave amplitude divided by the incident wave

amplitude, and the incident wave amplitude changes as the load changes when the

generator is not matched to the line, i.e., $Z_g \neq Z_o$. A good way to show that the

generator is matched to the line is to monitor with a directional coupler the incident

power. If it remains constant as the load is changed, then the generator is indeed

matched to the line, and the reflection coefficient changes permit one to obtain a

measure of changes in the power absorbed by the antenna. In the subsequent dis-

cussion we will assume that the generator is matched to the transmission line.

The main question before us is the experimental observation that the plasma

sheath impeded the radiation field considerably more in the case of the "narrow-

band" antenna than in the "broad-band" antenna case. Even in spite of the fact that

the narrow band antenna is a more efficient radiator when in resonance than the

"broad band" antenna, the former gave a weaker far-zone field with the plasma

present. This observation is very difficult to explain. On the basis of the impedance

representation we can readily explain the antenna de-tuning when the plasma sheath

appears in the "narrow band" antenna case. The sheath destroys the resonance

condition and this introduces an appreciable reactance in the antenna impedance

which decreases the antenna current and hence the radiation power. Part of the

power taken by the antenna is dissipated in the plasma resistance and the remainder

is radiated.

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In case of the "broad band" antenna similar comments apply, with the exception that the cavity is now lossy, and in addition to the plasma resistance and radiation resistance we add cavity resistance. Also we are told the resonance experimentally occured under different conditions. It was necessary to introduce a stub tuner in the transmission line between the reference plane A and the co-ax opening in order to obtain a better match between the antenna and the co-ax. In fact this creates an energy storage volume that no longer can be neglected in the over-all balance. One must consider the fact that this may change the resonance pattern, and either just reduce the reactive fields outside in the vicinity of the slot without unduly decreasing the slot field itself, or it may cause outside reactive energy to shift more into the magnetic field. The appearance of the plasma sheath under these conditions is expected to decrease the radiation field to a lesser degree than in the "narrow band" antenna case.

The experimental geometry of the problem makes it impossible to consider an exact theoretical solution. We are left with considering a resonant structure where a solution is possible and to study the outside reactive fields and their control by changes in the inside energy storages with a view of making the above arguments more definite and unique than they are now. One of the main questions to be answered is: to what extent the energy storage outside is coupled to the inside energy storages, and to what extent it can be controlled by inside tuning? Can the antenna be tuned by some "inside resonances" which as a result does not require

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large reactive fields outside? In these areas there is very little theoretical information available. It must be added that for a resonant antenna of the type we are discussing the "plane wave" approximation to predict the loss through the plasma sheath does appear to be very poor indeed. This type of approximation is expected to hold much better for apertures that are comparable with wavelength in both dimensions, such as an open waveguide radiator. One must also be aware of the fact that the plasma sheath not only impedes the transfer of power into the radiation field, but also changes the amplitude of the slot field (usually decreasing it) and thus compounding the influence of the plasma sheath. The real antenna problem is not equivalent to some prescribed source on the antenna surface, because in reality the true source is all the way back in the electronic package. The slot field can be regarded as an "equivalent" source for the field outside the antenna. However, it lacks the requirements of a true source: the energy in the "equivalent source" is finite, while in a true source it must be infinite.

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V. FUTURE WORK

Some obvious extensions of the theoretical work accomplished so far include consideration of a dielectric under the shell that is different from that on the outside and the application of the method of Chapter II to the case of the narrow slot. On the basis of this study one would then be in a position to choose a dielectric constant ratio such that for a narrow frequency band of interest the cylinder antenna would radiate through the slotted shell for all angular positions of the shell slot.

Another extension is to consider a finite thickness, partially transparent plasma sheath with an infinite axial slot. The radiation then would depend not only on the energy that leaks out through the slot, but also on the leakage through the plasma sheath itself. This is a new boundary value problem and it is expected to be somewhat more difficult than the one discussed in Chapters II and III.

Another problem of some potential interest is to consider that the cylinder slot is excited by a wedge waveguide which has a magnetic line source at the origin. This in fact is a complete antenna problem. In order to find the slot fields of the cylinder slot and the sheath slot we have to solve two simultaneous integral equations. The solution of this problem would also show the extent to which the shell depresses the cylinder slot field.

The further problem of relating these solutions to practical antenna configurations is also of continuing importance. For this, physical intuition and understanding of the canonical problem may not be sufficient, and some judicious experiments may have to be undertaken.