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AVERAGING METHODS FOR DIFFERENTIAL EQUATIONS

WITH RETARDED ARGUMENTS AND A SMALL PARAMETER*

by

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1. Introduction.

In the last few years, there has been an increasing interest in the theory of differential equations with retarded arguments or systems in which the rate of change of a system may depend upon its past history. This is partially due to the fact that such equations arise in a natural manner in certain types of control problems. Much of the recent literature has been devoted to the extension of known results for ordinary differential equations to differential equations with retarded arguments. The present paper is another step in this direction.

More specifically, we shall indicate in what manner a particular form of the method of averaging of Krylov-Bogoliubov-Mitropolski-Diliberto can be extended to differential equations with hereditary dependence.

For ordinary differential equations, this method is well understood by most people who are concerned either with the computational aspects or the qualitative theory of nonlinear oscillations. In the development of this method for retarded systems, the basic difficulty lies in the fact that

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motions defined by the solutions of the equations cannot be described adequately in a finite dimensional space. The proper setting seems to be in an infinite dimensional space and, in the particular formulation given below, in a Banach space. To the author's knowledge, Krasovskii [1] was the first to exploit such equations in this setting in the extension of Lyapunov's second method.

The extension of the method of averaging to differential equations with retardation relies heavily upon the theory of linear equations with constant coefficients as developed by Shimanov [2, 3] and the author [4]. We will not give the details of the theory of linear systems, but merely apply the results to our problem, proceeding from its application to specific examples to the more general results.

2. Notation.

Let us digress for a moment and discuss the equation

$$(2.1) \quad \dot{x}(t) = f(t, x(t), x(t - r)), \quad r \geq 0,$$

["." represents the right hand derivative] from the point of view that is of interest to us in this paper. If x is in R^n (the n -dimensional Euclidean space), $f(t, x, y)$ is continuous in its arguments for all t, x, y , and φ is any continuous function mapping the interval $[-r, 0]$ into R^n , then for any t_0 one can show that there is a function $x(t_0, \varphi)$ which is defined on an interval $[t_0 - r, t_0 + A)$, $A > 0$, coincides with φ on $[t_0 - r, t_0]$ and satisfies (2.1) for $t \geq t_0$. We call such a function a solution of (2.1) with initial value φ at t_0 . Furthermore, if $f(t, x, y)$ is locally

Lipschitzian in x, y this solution is unique and depends continuously upon t_0 and φ . If we find another solution with initial value ψ at t_0 , then the corresponding solutions might behave as in Fig. 1.

The uniqueness property asserts that if two solutions coincide on any interval of length r , then they must coincide for all future time, but two distinct solutions may intersect many times on any interval of length r . This last remark suggests that the state of time t of a system described by (2.1) should be the collection of values of the solution on the interval $[t - r, t]$, or the restriction of x to the interval $[t - r, t]$. We designate this restriction by x_t (see Fig. 2). If we let $C = C([-r, 0], R^n)$ be the space of continuous functions mapping the interval $[-r, 0]$ into R^n with the uniform topology, then a solution $x(t_0, \varphi)$ of (2.1) yields for each fixed $t \geq t_0$ a mapping of C into C ; namely, the mapping $x_t(t_0, \varphi)$. Trajectories of (2.1) are then defined as the collection of points $(t, x_t(t_0, \varphi))$ in $R \times C$ $t_0 \leq t < t_0 + A$, as indicated in Fig. 3. Hereafter, we assume solutions defined for all $t \geq t_0 - r$; that is, $A = +\infty$.

The above definition of trajectories of (2.1) yields a situation which is analogous to ordinary differential equations. However, the reader should realize that the situation here is more complicated. First of all, trajectories in general are only defined to the right of t_0 and the mapping x_t is a smoothing operator if $r > 0$. In fact, for any t_0 , the mapping $x_t(t_0, \varphi)$ takes closed bounded subsets of C into compact subsets of C . This shows that $x_t(t_0, \varphi)$ cannot be a homeomorphism for $r > 0$ even if it is one-to-one. Secondly, the mapping $x_t(t_0, \varphi)$ need not be one-to-one even when

the uniqueness property holds. In fact, for the scalar equation

$$\dot{x}(t) = x(t - r) (1 - x(t)),$$

the solution $x(0, \varphi)$ corresponding to an initial function φ with $\varphi(0) = 1$ is such that $x_t(0, \varphi) = 1$ for $t \geq r$. Therefore, a subset of $C([-r, 0], R)$ which is the translate of a subspace of codimension 1 is such that the corresponding trajectories all coincide after r units of time.

If we let $F(t, \varphi)$ be a functional defined on $[0, \infty) \times C$ into R^n , then a rather general hereditary functional-differential equation can be defined as

$$(2.2) \quad \dot{x}(t) = F(t, x_t)$$

where x_t is the restriction of x to the interval $[t - r, t]$. The discussion below is concerned with these more general equations, but we devote much of our time to more specific types. Equations (2.2) are certainly more general than (2.1) and certainly include (2.1) with the functional F defined by

$$F(t, x_t) = f(t, x(t), x(t - r)).$$

3. A convenient coordinate system.

In this paper, we are interested in the oscillatory properties of perturbations of linear equations with constant coefficients. In ordinary differential equations experience has shown an understanding of oscillations

in perturbations of linear equations with constant coefficients is most easily accomplished by the introduction of a coordinate system which exhibits in an explicit manner the behavior of the unperturbed equation on the subspaces which correspond respectively to the eigenvalues which positive real parts, zero real parts and negative real parts. In this section, we indicate how this same end can be accomplished for hereditary functional-differential equations.

We will first discuss the procedure for the simple equation

$$(3.1) \quad \dot{u}(t) = -\alpha u(t - r), \quad \alpha r = \pi/2, \quad r > 0,$$

and the perturbed equation

$$(3.2) \quad \dot{x}(t) = -\alpha x(t - r) + \epsilon f(t, x_t)$$

where x_t denotes the restriction of x to the interval $[t - r, t]$. Later we state a result for a more general equation.

If a function $e^{\lambda t}$ is a solution of (3.1), then λ must satisfy the characteristic equation

$$(3.3) \quad \lambda = -\alpha e^{-\lambda r}$$

Since $\alpha r = \pi/2$, it is not difficult to show that the roots of (3.3) all have negative real parts except for two which are equal to $\pm i\alpha$. Furthermore, every periodic solution of (3.1) must be of the form

$$(3.4) \quad a \sin \alpha t + b \cos \alpha t$$

for some constant a and b and every solution of (3.1) is exponentially asymptotic to a function of the form (3.4) as $t \rightarrow \infty$.

Now let us interpret these remarks in the space C . Let u be a periodic solution of (3.1). Then for $-r \leq \theta \leq 0$,

$$\begin{aligned} u_t(\theta) &= u(t + \theta) = a \sin \alpha(t + \theta) + b \cos \alpha(t + \theta) \\ &= (a \cos \alpha t - b \sin \alpha t) \sin \alpha \theta + (a \sin \alpha t + b \cos \alpha t) \cos \alpha \theta \end{aligned}$$

$$\stackrel{\text{def}}{=} y_1(t)\varphi_1(\theta) + y_2(t)\varphi_2(\theta)$$

where we have defined

$$(3.5) \quad \varphi_1(\theta) = \sin \alpha \theta, \quad \varphi_2(\theta) = \cos \alpha \theta, \quad -r \leq \theta \leq 0.$$

and y_1, y_2 are the corresponding coefficients of these functions.

Now φ_1, φ_2 are linear independent elements of C and thus generate a two-dimensional linear subspace P of C ; that is,

$$(3.6) \quad P = \{\varphi \in C: \varphi = a\varphi_1 + b\varphi_2, \quad a, b \text{ real}\}.$$

What the above computations have shown is that all of the periodic solutions of (3.1) must lie in P . Also, it can be shown that any solution of (3.1) approaches P exponentially as $t \rightarrow \infty$. The paths in P are closed curves and the motion in time is described by $y_1(t), y_2(t)$. Notice that $y_1(t), y_2(t)$ satisfy the ordinary differential equations

$$\dot{y}_1 = -\alpha y_2$$

$$\dot{y}_2 = \alpha y_1$$

If we could find another subspace Q of C (which necessarily must be infinite dimensional) which is positively invariant under the solutions of (3.1) and complementary to P in the sense that for every φ in C there exist unique elements φ_P in P , φ_Q in Q such that $\varphi = \varphi_P + \varphi_Q$, then pictorially the motions in C would be as shown in Fig. 4.

The existence of such a space Q follows from the general theory of linear operators since P is an eigenspace of the semigroup of bounded linear operators $U(t)$, $t \geq 0$, defined on C by $U(t)\varphi = u_t(\varphi)$, where $u(\varphi)$ is the solution of (3.1) with initial function φ at 0. On the other hand, if Q can be described analytically, then we will be in a position to introduce a coordinate system in C which will provide a natural means of extending perturbation theory.

This is accomplished by means of the equation

$$(3.7) \quad \dot{v}(s) = \alpha v(s + r)$$

"adjoint" to (3.1) with respect to the bilinear form

$$(3.8) \quad (\psi, \varphi) = \psi(0)\varphi(0) - \alpha \int_{-r}^0 \psi(\xi + r)\varphi(\xi)d\xi$$

defined for all ψ in $C([0, r], R)$ and φ in $C([-r, 0], R)$. This bilinear form has the property that if v is a solution of (3.7) defined for $s \geq 0$ and u is a solution of (3.1) defined for $t \geq 0$, then

$$(v_t, u_t) = \text{constant for } t \geq 0.$$

If A is a column vector of dimension k whose elements a_i belong to $C([0, r], R^n)$ and B is a row vector of dimension m whose elements b_i belong to $C([-r, 0], R^n)$, then we let (A, B) denote the $k \times m$ matrix whose $(i, j)^{th}$ element is given by (a_i, b_j) .

Equation (3.7) also has two linearly independent periodic solutions $\sin \alpha s, \cos \alpha s$ defined for s in $(-\infty, \infty)$. Define ψ_1, ψ_2 in $C([0, r], R)$ by

$$(3.9) \quad \psi_1(\theta) = \sin \alpha \theta, \quad \psi_2(\theta) = \cos \alpha \theta, \quad 0 \leq \theta \leq r$$

and
$$\Psi = (\psi_1, \psi_2), \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \Phi = (\varphi_1, \varphi_2).$$

Then a simple computation shows that the matrix

$$(\Psi, \Phi) \stackrel{\text{def}}{=} ((\psi_i, \varphi_j)) = \frac{1}{2} \begin{pmatrix} 1 & -\frac{\pi}{2} \\ \frac{\pi}{2} & 1 \end{pmatrix}$$

and it is nonsingular. It is convenient to define

$$\Psi^* = \Psi(\Psi, \Phi)^{-1} = \Psi \frac{2}{1 + \frac{\pi^2}{4}} \begin{pmatrix} 1 & -\frac{\pi}{2} \\ \frac{\pi}{2} & 1 \end{pmatrix}$$

since

$$(\Psi^*, \Phi) = I, \text{ the identity.}$$

We are now in a position to introduce a coordinate system in C and define the space Q complementary to P . In fact, for any φ in C we let

$$\varphi = \Phi c + \bar{\varphi}, \quad c = (\Psi^*, \varphi),$$

which gives a unique decomposition of every element φ in C . The subspace Q complementary to P is defined by

$$Q = \{\varphi \text{ in } C: (\Psi^*, \varphi) = 0\}.$$

For the integral representation below, it is necessary to extend the definition of Q to piecewise continuous functions. It is clear that this is possible and our decomposition is valid in this larger space. Hereafter, Q will denote this set.

If x is a solution of (3.2) with initial value φ at σ , σ in $(-\infty, \infty)$, and

$$(3.10) \quad x_t = \Phi y(t) + \bar{x}_t, \quad \varphi = \Phi b + \bar{\varphi},$$

then $y(t)$, \bar{x}_t must satisfy

$$(3.11) \quad \begin{aligned} \dot{y}(t) &= B y(t) + \epsilon \Psi^{*T}(0) f(\Phi y(t) + \bar{x}_t), \quad y(\sigma) = b, \\ \bar{x}_t &= u_{t-\sigma}(\bar{\varphi}) + \epsilon \int_{\sigma}^t u_{t-\tau}(\bar{x}_0) f(\Phi y(\tau) + \bar{x}_\tau) d\tau \end{aligned}$$

where

$$B = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix},$$

$$\Psi^*(0) = \frac{2}{1+\frac{\pi^2}{4}} \begin{pmatrix} \pi/2 \\ 1 \end{pmatrix}$$

$\bar{\phi}, \bar{x}_0$ are in Q , $u(\Psi)$ is the solution of (3.1) with initial value Ψ at 0, $x_0(\theta) = 0$, $-r \leq \theta \leq 0$, $x_0(0) = I$, the identity. If $\bar{\phi}$ is in Q , then there are positive K, α such that

$$(3.12) \quad \|u_t(\bar{\phi})\| \leq Ke^{-\alpha t} \|\bar{\phi}\|, \quad t \geq 0.$$

This relationship expresses more precisely our stability property of the set P mentioned before.

Now let us see what this coordinate is like for the general linear equation,

$$(3.13) \quad \dot{u}(t) = \int_{-r}^0 [d\eta(\theta)] u(t + \theta),$$

where η is an $n \times n$ matrix whose elements are real functions of bounded variation, and the perturbed equation

$$(3.14) \quad \dot{x}(t) = \int_{-r}^0 [d\eta(\theta)] x(t + \theta) + \epsilon f(t, x_t).$$

We wish to indicate how a coordinate system can be introduced into C in such a way as to obtain a set of equations equivalent to (3.14) which is of the form (3.11) with $u_t(\bar{\varphi})$ satisfying (3.12) and the corresponding matrix B having eigenvalues which coincide with the characteristic values of (3.13) which have real parts ≥ 0 . The characteristic values of (3.13) are the roots of the equation

$$(3.15) \quad \det [\lambda I - \int_{-r}^0 [d\eta(\theta)] e^{\lambda\theta}] = 0,$$

and to any characteristic value, λ , there is a solution of (3.13) of the form $e^{\lambda t} b$ for some b and all t in $(-\infty, \infty)$.

As is to be suspected from the previous discussion, a basic role is played by the equation*

$$(3.16) \quad \mathcal{H}(s) = - \int_{-r}^0 [d\eta^T(\theta)] v(s - \theta)$$

"adjoint" to (3.13) with respect to the bilinear form,

$$(3.17) \quad (\psi, \varphi) = \psi^T(0)\varphi(0) - \int_{-r}^0 \int_0^\theta \psi^T(\xi - \theta) [d\eta(\theta)] \varphi(\xi) d\xi,$$

defined for all ψ in $C([0, r], R^n)$, φ in $C([-r, 0], R^n)$. The characteristic values of the adjoint equation are the roots of the equation

$$(3.18) \quad \det [\lambda I - \int_{-r}^0 [d\eta^T(\theta)] e^{\lambda\theta}] = 0,$$

* If A is a matrix A^T denotes the transpose of A .

and to each such root, λ , there is a solution of (3.16) of the form $e^{-\lambda s} b$ for some constant vector b and all s in $(-\infty, \infty)$. Notice that the solutions of equations (3.15) and (3.18) are the same.

Suppose $\lambda_1, \dots, \lambda_k$ are the characteristic values of (3.13) with real parts ≥ 0 . There are only a finite number, say m , of linearly independent solutions of (3.13) of the form $\sum_{j=1}^k p_j(t) e^{\lambda_j t}$ where the p_j are polynomials. Let $\Phi = (\phi_1, \dots, \phi_m)$ where ϕ_1, \dots, ϕ_m are the restrictions of these functions to $[-r, 0]$. Similarly, there are only m linearly independent solutions of (3.16) of the form $\sum_{j=1}^k q_j(s) e^{-\lambda_j s}$ where the q_j are polynomials. Let $\Psi = \text{col}(\psi_1, \dots, \psi_m)$ where ψ_1, \dots, ψ_m are the restriction of these functions to $[0, r]$.

It follows directly from the differential equations that there is a square matrix B with only the eigenvalues $\lambda_1, \dots, \lambda_k$ such that

$$(3.19) \quad \Phi(\theta) = \Phi(0) e^{B\theta}, \quad -r \leq \theta \leq 0.$$

Furthermore, one can show (see Hale [4]) that the matrix (Ψ, Φ) is nonsingular and, therefore, by a change of the basis Ψ , one can take

(Ψ, ϕ) to be the identity. Finally, the transformation (3.10) with ϕ, Ψ as above applied to (3.14) yields an equation of the form (3.11) with $u_t(\bar{\phi})$ satisfying (3.12) and the matrix B given by (3.19).

4. Perturbation theory.

It was indicated in the previous section that there is a transformation $x_t = \phi y(t) + \bar{x}_t$, \bar{x}_t in Q , which takes the general system (3.14) into an equivalent system of the form

$$\begin{aligned} \dot{y}(t) &= By(t) + \epsilon \Psi(0)f(t, \phi y(t) + \bar{x}_t), \quad y(0) = b \\ (4.1) \quad \bar{x}_t &= u_{t-0}(\bar{\phi}) + \epsilon \int_0^t u_{t-\tau}(\bar{x}_0)f(\tau, \phi y(\tau) + \bar{x}_\tau) d\tau \end{aligned}$$

where $\phi = \phi b + \bar{\phi}$, $\bar{\phi}$ in Q , the eigenvalues of B have nonnegative real parts, $u(\phi)$ is the solution of (3.13) with initial values ϕ at 0 and

$$\|u_t(\bar{\phi})\| \leq Ke^{-\alpha t} \|\bar{\phi}\|, \quad t \geq 0, \quad K > 0, \quad \alpha > 0,$$

for any $\bar{\phi}$ in Q .

Equations (4.1) are now in a form which is very similar to that which is encountered in the theory of oscillations in ordinary differential equations. One can show that any solution of (4.1) which is bounded on $(-\infty, \infty)$ must be of such a nature that $\bar{x}_t = O(\epsilon)$ as $\epsilon \rightarrow 0$. Consequently, if our analysis is based upon an approximation procedure which can be justified to be correct by investigating only the terms of order ϵ , then the basic problem lies in the investigation of the ordinary differential equation

$$(4.2) \quad \dot{y}(t) = By(t) + \epsilon \bar{y}^T(0)f(t, \Phi y(t)).$$

The analysis of (4.2) is well understood and usually proceeds by the introduction of convenient combinations of polar coordinates and rectangular coordinates and the application of averaging procedures and successive approximations.

For simplicity, let us make the assumption that all the characteristic values of (3.13) have nonpositive real parts. Then B in (4.1) has all eigenvalues purely imaginary and a combination of polar and rectangular changes of coordinates in the components of y (see the examples in section 5 for the types of coordinates involved) leads to a set of equations of the form

$$(4.3) \quad \begin{aligned} \dot{\zeta} &= d + \epsilon \Theta(t, \zeta, \rho, \bar{x}_t) \\ \dot{\rho} &= \epsilon R(t, \zeta, \rho, \bar{x}_t) \\ \bar{x}_t &= u_{t-\sigma}(\bar{\varphi}) + \epsilon \int_{\sigma}^t u_{t-\tau}(\bar{x}_0) F(\tau, \zeta, \rho, \bar{x}_\tau) d\tau \end{aligned}$$

where ζ is a p -dimension vector, ρ is a q -dimensional vector, \bar{x}_t is an element of the Banach space C . The vector d is a constant vector with positive components and the functions Θ, R, F are multiply periodic in the vector ζ .

Assume that the functions Θ, R, F with arguments $t, \zeta, \rho, \bar{\varphi}$ have continuous second derivatives with respect to $\zeta, \rho, \bar{\varphi}$ and are almost

periodic in t uniformly with respect to $\zeta, \rho, \bar{\varphi}$ in some set. Let

$$\zeta + \tau = (\zeta_1 + \tau, \dots, \zeta_p + \tau)$$

and assume that

$$(4.4) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T R(t + \tau, \zeta + \tau, \rho, 0) d\tau \stackrel{\text{def}}{=} R_0(\rho)$$

is independent of t, ζ . We define the averaged equations associated with (4.3) to be the equations

$$(4.5) \quad \dot{\rho} = \epsilon R_0(\rho).$$

Notice that the averaged equations (4.5) are obtained from $R(t, \zeta, \rho, 0)$ and, therefore in a specific problem, they arise from an investigation of the ordinary differential equation (4.2).

Theorem 4.1. If system (4.3) satisfies the conditions enumerated above and if there exists a vector ρ_0 such that $R_0(\rho_0) = 0$ and the eigenvalues of the matrix $\partial R_0(\rho_0)/\partial \rho$ have nonzero real parts, then there exists an $\epsilon_0 > 0$ and functions $g(t, \zeta, \epsilon), h(t, \zeta, \epsilon), 0 \leq \epsilon \leq \epsilon_0, g$ in R^q, h in C , $g(t, \zeta, 0) = \rho_0, h(t, \zeta, 0) = 0$, multiply periodic in ζ and almost periodic in t such that the set $S_\epsilon, 0 \leq \epsilon \leq \epsilon_0$, defined by

$$S_\epsilon \{ (t, \zeta, \rho, \bar{\varphi}) : \rho = g(t, \zeta, \epsilon), \bar{\varphi} = h(t, \zeta, \epsilon), \\ -\infty < t < \infty, -\infty < \zeta_j < \infty, j=1, 2, \dots, p \}$$

is an integral manifold of system (4.3). If the functions Θ, R, F are independent of t (or periodic in t of period ω), then the functions g, h are independent of t or periodic in t of period ω . Furthermore, if all eigenvalues of $\partial R_0(\rho_0)/\partial \rho$ have negative real parts, S_ϵ is asymptotically stable for $0 < \epsilon \leq \epsilon_0$ and if one eigenvalue has a positive real part then S_ϵ is unstable for $0 < \epsilon \leq \epsilon_0$.

We merely give an indication of the proof of this theorem since it is so analogous to the proof for the case of ordinary differential equations given in Bogolubov and Mitropolski [5] and Hale [6].

In [6, Ch. 12], it is shown that there is a function $w(t, \zeta, \rho, \epsilon)$, multiply periodic in ζ and almost periodic in t such that the transformation

$$\rho \rightarrow \rho + \epsilon w(t, \zeta, \rho, \epsilon)$$

applied to the equation $\dot{\rho} = \epsilon R(t, \zeta, \rho, 0)$ yields a new equation of the form $\dot{\rho} = \epsilon R_0(\rho) + \epsilon R_1(t, \zeta, \rho, \epsilon)$ where $R_1(t, \zeta, \rho, \epsilon)$ is zero for $\epsilon = 0$. Consequently, if this transformation is applied to (4.3), we obtain a system of the form

$$\dot{\zeta} = d + \epsilon \Theta_1(t, \zeta, \rho, \bar{x}_t, \epsilon)$$

$$\dot{\rho} = \epsilon R_0(\rho) + \epsilon R_1(t, \zeta, \rho, \epsilon) + \epsilon R_2(t, \zeta, \rho, \bar{x}_t, \epsilon)$$

$$\bar{x}_t = u_{t-\sigma}(\bar{\varphi}) + \epsilon \int_{\sigma}^t u_{t-\tau}(\bar{x}_0) F_1(\tau, \zeta, \rho, \bar{x}_\tau, \epsilon) d\tau$$

where Θ_1, F_1 are the same types of functions as Θ, F and

$$R_1(t, \zeta, \rho, 0) = 0, \quad R_2(t, \zeta, \rho, 0, \epsilon) = 0, \quad |R_2(t, \zeta, \rho, \bar{\varphi}, \epsilon) - R_2(t, \zeta, \rho, \bar{\psi}, \epsilon)| \leq K \|\bar{\varphi} - \bar{\psi}\|$$

for some constant K and $\rho, \bar{\varphi}$ in a bounded set.

One now proceeds in a manner completely analogous to that given in [5], [6] to show that the functions g, h mentioned in the theorem are the fixed points of an integral operator. The stability of the integral manifold must be investigated separately and is easily supplied using the ideas developed in [8] in connection with a saddle point for functional-differential equations.

As in [5], [6], one can also prove

Theorem 4.2. Suppose the averaged equations (4.5) have a nonconstant periodic solution $\rho = \rho^0(t)$ of period T such that $q-1$ of the characteristic exponents of the associated linear variational equations have nonzero real parts. Then there exists an $\epsilon_0 > 0$ and functions $g(t, \zeta, \psi, \epsilon)$, $h(t, \zeta, \psi, \epsilon)$, $0 \leq \epsilon \leq \epsilon_0$, g in R^q , h in C , $g(t, \zeta, \psi, 0) = \rho^0(\psi)$, $0 \leq \psi \leq T$, $h(t, \zeta, \psi, 0) = 0$, multiply periodic in ζ , periodic in ψ of period T and almost periodic in t such that the set S_ϵ , $0 \leq \epsilon \leq \epsilon_0$ defined by

$$S_\epsilon = \{(t, \zeta, \rho, \bar{\varphi}) : \rho = g(t, \zeta, \psi, \epsilon), \quad \bar{\varphi} = h(t, \zeta, \psi, \epsilon),$$

$$-\infty < t < \infty; \quad -\infty < \zeta_j < \infty, \quad j = 1, 2, \dots, p; \quad 0 \leq \psi \leq T\}$$

is an integral manifold of system (4.3). If Θ, R, F are independent of t (or periodic in t of period ω), then the functions g, h are

independent of t (or periodic in t of period ω). Furthermore, the stability properties of S_ϵ are the same as those of the periodic solution $\rho^0(t)$ of (4.5).

We now state some important corollaries of these theorems before turning to specific examples. Consider the equation

$$(4.6) \quad \dot{x}(t) = \epsilon f(t, x_t)$$

where $\epsilon > 0$ is a parameter, $f(t, \varphi)$ is almost periodic in t uniformly with respect to φ in some subset of $C([-r, 0], \mathbb{R}^n)$, and has a continuous second Frechet derivative with respect to φ . Let

$$(4.7) \quad f_0(\varphi) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\tau, \varphi) d\tau.$$

Halanay [7] has discussed, for small ϵ , some of the relationships between the solutions [on the interval $(0, \infty)$], of $\dot{x}(t) = \epsilon f_0(x_t)$ and the solutions of (4.6) for the case in which the retardation interval is of order ϵ . We now show that Theorems 4.1 and 4.2 imply that his results and even more are valid without any restriction on the retardation interval. In fact, we can prove the following two theorems. In the statement of these theorems, y sometimes denotes a vector in n -dimensional Euclidean space and sometimes a vector of constant functions in $C([-r, 0], \mathbb{R}^n)$, but it is clear from the context which meaning is implied. The averaged equations of (4.6) are then defined to be the ordinary differential equation

$$(4.8) \quad \dot{y} = \epsilon f_0(y).$$

Theorem 4.3. If the averaged equations (4.8) have an equilibrium point y_0 such that the matrix of coefficients of the linear variational equations has no eigenvalues on the imaginary axis, then, for ϵ sufficiently small, (4.6) has a unique almost periodic solution $x = g(t, \epsilon)$ in a neighborhood of $x = y_0$, $g(t, 0) = y_0$, and the stability properties of g are the same as the stability properties of y_0 .

Theorem 4.4. If (4.8) has a nonconstant periodic solution $y = y^{(0)}(t)$ of period T , such that the linear variation equation has $n-1$ of its characteristic exponents not on the imaginary axis, then, for ϵ sufficiently small, there exists a function $g(t, \zeta, \epsilon)$ in C , almost periodic in t uniformly with respect to ζ , periodic in ζ of period T , $g(t, \zeta, 0) = y_{\zeta}^{(0)}$, $y_{\zeta}^{(0)}(\theta) = y^{(0)}(\zeta + \theta)$, $-r \leq \theta \leq 0$, such that the surface S_{ϵ} in $C \times (-\infty, \infty)$ defined by

$$S_{\epsilon} = \{(\varphi, t): \varphi = g(t, \zeta, \epsilon), \quad 0 \leq \zeta \leq T, \quad -\infty < t < \infty\}$$

is an integral manifold of (4.6). Furthermore, S_{ϵ} is unique in a neighborhood of $S_0 = \{(\varphi, t): \varphi = y_{\zeta}^{(0)}, \quad 0 \leq \zeta \leq T, \quad -\infty < t < \infty\}$ and has the same stability properties as S_0 . If f in (4.6) is independent of t , then g is independent of t , and if f is periodic in t , then g is periodic in t with the same period.

To show that these results are consequences of Theorems 4.1 and 4.2, we proceed as follows. For any φ in $C([-r, 0], R^n)$, the decomposition

$$\varphi = b + \bar{\varphi},$$

is unique if b is the constant function whose value is $\varphi(0)$. If, in (4.6),

$$x_t = y(t) + \bar{x}_t$$

then

$$\dot{y}(t) = \epsilon f(t, y(t) + \bar{x}_t)$$

$$\bar{x}_t = u_{t-\sigma}(\bar{\varphi}) + \epsilon \int_{\sigma}^t u_{t-\tau}(\bar{x}_0) f(\tau, y(\tau) + \bar{x}_\tau) d\tau$$

and

$$\|u_t(\bar{\varphi})\| \leq K e^{-\alpha t} \|\bar{\varphi}\|, \quad t \geq 0$$

for some positive K, α . This last relation is obviously true in this case since $u_t(\bar{\varphi}) = 0$ for all $\bar{\varphi}$ and $t \geq r$. System (4.9) is a special case of (4.3) and one obtains Theorems 4.3 and 4.4 from Theorems 4.1 and 4.2.

5. Some specific examples.

Let us first discuss the oscillatory properties of equation (3.2), namely, the equation

$$\dot{x}(t) = -\alpha x(t - r) + \epsilon f(t, x_t) \quad (5.1)$$

$$\alpha r = \pi/2, \quad \epsilon > 0.$$

We have seen in section 3 that this equation is equivalent to the system (3.11), that is, the equation

$$\begin{aligned} \dot{y}_1 &= -\alpha y_2 + \epsilon \pi \mu^2 f(t, \phi y + \bar{x}_t) \\ \dot{y}_2 &= \alpha y_1 + 2\epsilon \mu^2 f(t, \phi y + \bar{x}_t) \\ \bar{x}_t &= u_{t-\sigma}(\bar{\phi}) + \epsilon \int_{\sigma}^t u_{t-\tau}(\bar{X}_0) f(\tau, \phi y(\tau) + \bar{x}_\tau) d\tau \\ \phi(\theta) &= (\phi_1(\theta), \phi_2(\theta)) \stackrel{\text{def}}{=} (\sin \alpha \theta, \cos \alpha \theta), \quad -r \leq \theta \leq 0 \end{aligned} \quad (5.2)$$

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \mu^2 = \frac{1}{1 + \frac{\pi^2}{4}}.$$

If we let

$$\begin{aligned} y_1 &= \rho \sin \alpha \zeta \\ y_2 &= -\rho \cos \alpha \zeta \end{aligned} \quad (5.3)$$

then system (5.2) becomes

$$\begin{aligned} \dot{\zeta} &= 1 + \frac{\epsilon \mu^2}{\rho} (\pi f \cos \alpha \zeta + 2f \sin \alpha \zeta) \stackrel{\text{def}}{=} 1 + \dots, \zeta, \rho, \bar{x}_t) \\ \dot{\rho} &= \epsilon \mu^2 (\pi f \sin \alpha \zeta - 2f \cos \alpha \zeta) \stackrel{\text{def}}{=} \in R(t, \zeta, \rho, \bar{x}_t) \\ \bar{x}_t &= u_{t-\sigma}(\bar{\phi}) + \epsilon \int_{\sigma}^t u_{t-\tau}(\bar{X}_0) f(\tau, \phi y(\tau) + \bar{x}_\tau) d\tau \end{aligned} \quad (5.4)$$

where $f = f(t, \phi y + \bar{x}_t)$ and y is given in (5.3).

If the function f is almost periodic in t uniformly with respect to the other arguments, then the averaged equations for this particular case are

$$\begin{aligned} \dot{\rho} &= \epsilon R_0(\rho) \\ R_0(\rho) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mu^2 [\pi f \sin \alpha(\zeta + \tau) - 2f \cos \alpha(\zeta + \tau)] d\tau \\ (5.5) \quad f &= f(t + \tau, \rho[\phi_1 \sin \alpha(\zeta + \tau) - \phi_2 \cos \alpha(\zeta + \tau)]) \\ \phi_1(\theta) &= \sin \alpha\theta, \quad \phi_2(\theta) = \cos \alpha\theta, \quad -\pi \leq \theta \leq 0, \end{aligned}$$

where we always assume that this limit is independent of t, ζ .

Notice that equation (5.5) are the same equations that are obtained by introducing the polar coordinate transformation (5.3) to the ordinary differential equation

$$\begin{aligned} \dot{y}_1 &= -\alpha y_2 + \pi \epsilon \mu^2 f(t, \phi y) \\ (5.6) \quad \dot{y}_2 &= \alpha y_1 + 2\epsilon \mu^2 f(t, \phi y) \end{aligned}$$

and then taking the average.

Let us now take some specific functions f in (5.1) to show that important information is obtained by this method.

Example 5.1 (Pinney [9]) $f = -rx(t-1) + \epsilon x^3(t-1), \alpha = \pi/2, r = 1.$

Since $x(t-1) = \phi(-1)y(t) = -y_1(t) = -\sin(\pi/2)$, it is easy to check that the average $R_0(\rho)$ in (5.5) is

$$R_0(\rho) = K\rho(1 - \frac{\gamma\rho^2}{4\gamma}),$$

where K is a positive constant, and the averaged equations (5.5) are

$$\dot{\rho} = \epsilon K\rho(1 - \frac{\gamma\rho^2}{4\gamma}).$$

If $\gamma\beta > 0$ the averaged equation has an equilibrium point $\rho_0 = \sqrt{4\gamma/3\beta}$ which is asymptotically stable if $\gamma > 0$ and unstable if $\gamma < 0$. Consequently, for $\epsilon > 0$ and sufficiently small, Theorem 4.1 asserts for $\gamma\beta > 0$ the existence of functions $g(\zeta, \epsilon)$, $h(\zeta, \epsilon)$ (these functions are independent of t since f is), $g(\zeta, 0) = \sqrt{4\gamma/3\beta}$, $h(\zeta, 0) = 0$, periodic in ζ of period $2\pi/\alpha$ such that $S_\epsilon = \{(t, \zeta, \rho, \bar{\varphi})$:

$\rho = g(\zeta, \epsilon)$, $\bar{\varphi} = h(\zeta, \epsilon)$, $-\infty < t < \infty$, $0 \leq \zeta \leq 2\pi/\alpha\}$ is an integral manifold of (5.4) which is stable for $\gamma < 0$. From (3.10) and (5.3), this implies that $T_\epsilon = \{(t, \varphi) : \varphi = (\varphi_1 \sin \alpha\zeta - \varphi_2 \cos \alpha\zeta)g(\zeta, \epsilon) + h(\zeta, \epsilon)$, $0 \leq \zeta \leq 2\pi/\alpha\}$ is an integral manifold of our original system. Such a cylinder T_ϵ in $R \times C$ obviously corresponds to a periodic solution of our system which is stable if $\gamma > 0$ and unstable if $\gamma < 0$ and has an amplitude approximately equal to $\sqrt{4\gamma/3\beta}$. The approximate period ω is obtained by solving the equation

$$\dot{\zeta} = 1 + \epsilon\mu^2(\gamma \sin \pi\zeta/2 - \beta\rho_0^3 \sin^3 \pi\zeta/2), \quad \rho_0 = \sqrt{4\gamma/3\beta},$$

and determining ω so that $\zeta(t + \omega) = \zeta(t) + 2\pi/\alpha$.

It is interesting to note that the second order system (5.6) for this example is actually equivalent to a second order scalar differential equation. The method of averaging should then allow one to obtain an "equivalent" linear second order equation in the sense of Krylov-Bogoliubov. This should in turn lead to methods which will yield important information about equations with retardation when ϵ is not small -- describing functions, etc. So far, this has not been exploited.

Example 5.2. Consider the equation

$$(5.7) \quad \ddot{x}(t) = - \left[\frac{\pi}{2} + \epsilon \eta(t) \right] x(t-1)(1 - \epsilon x^2(t))$$

where η is almost periodic in t .

This is a special case of (5.1) with $\alpha = \pi/2$, $r = 1$, and

$$f = \frac{\pi}{2} x^2(t)x(t-1) - \eta(t)x(t-1) + \epsilon \eta(t)x^2(t)x(t-1).$$

Using the fact that

$$x(t) = \Phi(0)y(t) = y_2(t) = -\rho \cos(\pi t/2)$$

$$x(t-1) = \Phi(-1)y(t) = -y_1(t) = -\rho \sin(\pi t/2)$$

the averaged equation (5.5) becomes

$$\rho = \epsilon \mu^2 \rho(\eta_0 - \frac{\pi}{8} \rho^2), \quad \eta_0 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \eta(t) dt$$

provided

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \eta(t) \begin{pmatrix} \cos \pi(\xi + t) \\ \cos \pi(\xi + t) \end{pmatrix} dt = 0.$$

Consequently, if $\eta_0 > 0$ and $\epsilon > 0$ is sufficiently small there exists a stable integral manifold of solutions of (5.7) whose parametric representation in C is almost periodic in t and periodic in ξ and for $\epsilon = 0$ is given by

$$\phi \begin{pmatrix} \rho_0 \sin \pi \xi / 2 \\ -\rho_0 \cos \pi \xi / 2 \end{pmatrix}, \quad \rho_0 = \sqrt{8\eta_0/\pi}.$$

If $\eta(t)$ is independent of t , then the parametric representation of the integral manifold is independent of t and one obtains a nonconstant periodic solution of (5.7) with amplitude approximately $\sqrt{8\eta/\pi}$. Jones [10] has discussed periodic solutions of (5.5) with η independent of t and even more general equations.

Example 5.3. Consider the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ (5.8) \quad \dot{x}_2(t) &= -\alpha^2 x_1(t) + \epsilon [1 - x_1^2(t - \tau)] x_2(t) \end{aligned}$$

where $\alpha > 0$, $r \geq 0$, $\epsilon > 0$ are parameters. For $r = 0$, this is van der Pol's equation. By using the preceding theory, we will investigate the existence and stability of limit cycles of (5.8) for ϵ small.

If φ_1, φ_2 are the vectors in $C([-r, 0], R^2)$ defined by

$$\varphi_1(\theta) = \begin{pmatrix} \cos \alpha\theta \\ -\alpha \sin \alpha\theta \end{pmatrix}, \quad \varphi_2(\theta) = \begin{pmatrix} \frac{1}{\alpha} \sin \alpha\theta \\ \cos \alpha\theta \end{pmatrix}, \quad -r \leq \theta \leq 0$$

and $\Phi = (\varphi_1, \varphi_2)$ then the transformation

$$x_t = \Phi y(t) + \bar{x}_t, \quad y(t) = x(t), \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

applied to (5.8) yields the equivalent system

$$\begin{aligned} \dot{y}_1 &= y_2 \\ (5.9) \quad \dot{y}_2 &= -\alpha^2 y_1 + \epsilon [1 - (y_1 \cos \alpha r - \frac{1}{\alpha} y_2 \sin \alpha r + \bar{x}_t(-r))^2] y_2(t) \\ \bar{x}_t &= u_{t-\sigma}(\bar{\varphi}) + \epsilon \int_{\sigma}^t u_{t-\tau}(\bar{X}_0) F(y(\tau), \bar{x}_\tau) d\tau \end{aligned}$$

where $u_t(\bar{\varphi})$ has the same meaning as in the previous sections and F is a two-vector whose specific form is of no particular interest here.

If we introduce the polar coordinates

$$y_1 = \rho \sin \alpha \zeta$$

$$y_2 = \rho \alpha \cos \alpha \zeta$$

into (5.9) and set $\bar{x}_t = 0$, we obtain

$$\begin{aligned}\dot{\xi} &= 1 - \frac{\epsilon}{2\alpha} \sin 2\alpha\xi [1 - \rho^2 \{\sin \alpha(\xi - r)\}^2] \\ \dot{\rho} &= \epsilon \rho \cos^2 \alpha\xi [1 - \rho^2 \{\sin \alpha(\xi - r)\}^2]\end{aligned}$$

The averaged equation is

$$\dot{\rho} = \frac{\epsilon \rho}{2} \left[1 - \frac{1}{2} \left(1 - \frac{1}{2} \cos 2\alpha r \right) \rho^2 \right].$$

This equation has an equilibrium point $\rho_0 = \sqrt{2/r}$, $r = 1 - (\cos 2\alpha r)/2 > 0$ for every value of r and the linear variational equation relative to ρ_0 is $\dot{\rho} = -2\epsilon\rho$. Consequently, Theorem 4.1 implies as in Example 5.1 the existence of a stable periodic solution of (5.8) with amplitude approximately $\sqrt{2/r}$.

For physical examples of retarded equations and the importance of oscillatory phenomena, see Chapter 21 of N. Minorsky [11].

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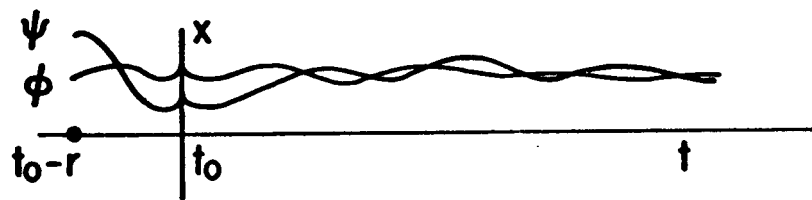


FIG. 1

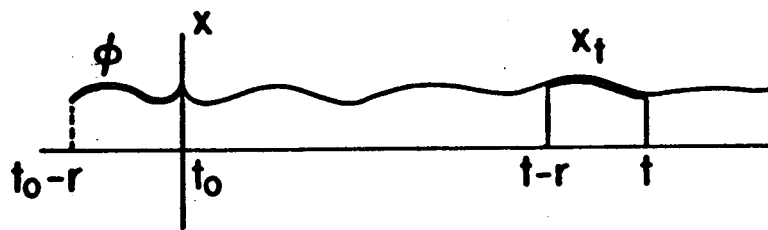


FIG. 2

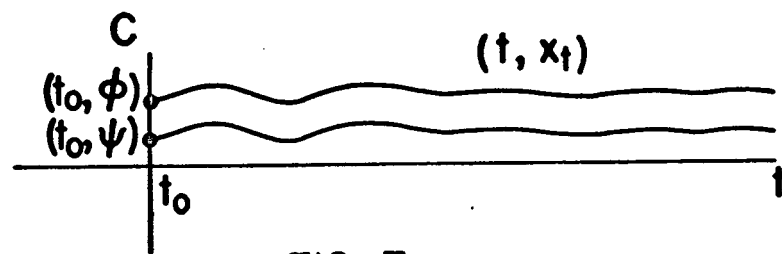


FIG. 3

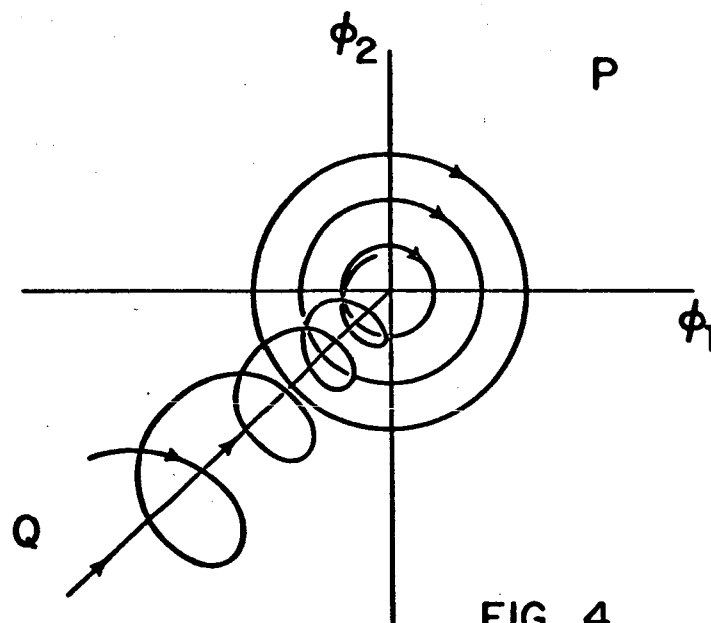


FIG. 4