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GUIDANCE APPLICATIONS OF LINEAR ANALYSIS

by LYLE R. DICKEY
Aero-Astrodynamics Laboratory

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Lyle R. Dickey

George C. Marshall Space Flight Center

Huntsville, Alabama

ABSTRACT

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The application of linear analysis in determining a guidance function is investigated. The differential equations of motion are linearized about a nominal calculus of variations solution. The result is an explicit expression for the cutoff radius error, Δr, and cutoff angle error, $\Delta\theta$, as a linear operation on deviations in initial conditions and several non-linear functions of thrust angle deviations and thrust acceleration deviations along the trajectory. With this expression available, a suitable form is selected for a function to determine thrust angle, χ. The coefficients of this function are mathematically determined from the explicit solution obtained for Δr and $\Delta\theta$ under the constraint that these values be as near zero as feasible for deviations in initial conditions and thrust acceleration whose values are arbitrary within their expected range of variation.

The results of employing this function to determine χ for a number of examples are shown. These results emphasize the advantage of mathematically imposing the mission criteria in determination of guidance coefficients as well as illustrate the value of linearization techniques in guidance analysis.

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Lyle R. Dickey

TECHNICAL AND SCIENTIFIC STAFF
AERO-ASTRODYNAMICS LABORATORY

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DEFINITION OF SYMBOLS

Symbol	Definition
A =	$\begin{bmatrix} 0 & 0 & 10^{-3} & 0 \\ 0 & 0 & 0 & 10^{-3} \\ h_1 & h_2 & 0 & 0 \\ k_1 & k_2 & 0 & 0 \end{bmatrix}$
B	intermediate matrix defined in equation (2.2.7)
B_2	intermediate matrix defined in equation (2.2.13)
B_3	intermediate matrix defined in equation (2.2.17)
$C(\delta r)$	intermediate matrix defined in equation (2.3.12)
$C(\delta\theta)$	intermediate matrix defined in equation (2.3.13)
$C_1(\delta r)$	intermediate matrix defined in equation (2.3.14)
$C_1(\delta\theta)$	intermediate matrix defined in equation (2.3.15)
$C_2(\delta r)$	intermediate matrix defined in equation (2.3.16)
$C_2(\delta\theta)$	intermediate matrix defined in equation (2.3.17)
$C_3(\delta r)$	intermediate matrix defined in equation (2.3.18)
$C_3(\delta\theta)$	intermediate matrix defined in equation (2.3.19)
$C_4(\delta r)$	intermediate matrix defined in equation (2.3.23)
$C_4(\delta\theta)$	intermediate matrix defined in equation (2.3.24)
$C_5(\delta r)$	intermediate matrix defined in equation (2.3.25)
$C_5(\delta\theta)$	intermediate matrix defined in equation (2.3.26)

DEFINITION OF SYMBOLS (Cont'd)

Symbol	Definition
$\left.\begin{matrix}\bar{C}_o \\ \bar{C}_1 \\ \bar{C}_2\end{matrix}\right\}$	coefficients of the polynomial defining $\bar{\chi} = \bar{C}_o + \bar{C}_1\tau + \bar{C}_2\tau^2$
D	intermediate matrix defined by equation (2.3.30)
D_1	intermediate matrix defined by equation (2.3.31)
D_2	intermediate matrix defined by equation (2.3.32)
D_3	intermediate matrix defined by equation (2.3.33)
D_4	intermediate matrix defined by equation (2.3.34)
D_5	intermediate matrix defined by equation (2.3.35)
F	the first row of the matrix $TU(t_n, t)\ H(t)$
G	the second row of the matrix $TU(t_n, t)\ H(t)$
$\bar{F}$ =	$F(t_j)\ \Delta t_j$
$\bar{G}$ =	$G(t_j)\ \Delta t_j$
H =	$(H_1\ \ H_2\ \ H_3\ \ H_4)$
H_1 =	$\frac{\pi}{180}\frac{f}{m}\begin{bmatrix} 0 \\ 0 \\ \cos\chi \\ -\sin\chi \end{bmatrix}$

Symbol	Definition
$H_2 =$	$\left(\frac{\pi}{180}\right)^2 \frac{f/m}{2} \begin{bmatrix} 0 \\ 0 \\ -\sin\chi \\ -\cos\chi \end{bmatrix}$
$H_3 =$	$\left(\frac{\pi}{180}\right)^3 \frac{f/m}{6} \begin{bmatrix} 0 \\ 0 \\ -\cos\chi \\ \sin\chi \end{bmatrix}$
$H_4 =$	$\begin{bmatrix} 0 \\ 0 \\ \sin\chi \\ \cos\chi \end{bmatrix}$
T	matrix defined by equation (1.5.9)
T_o	matrix defined by equation (1.3.8)
T_1	matrix defined by equation (1.3.5)
T_2	matrix defined by equation (1.4.4)
$U =$	$TU(t_n, t_o)$

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
U_1	the first row of the matrix U
U_2	the second row of the matrix U
$U(t_n, t)$	solution of the following matrix differential equation evaluated at $t_1 = t_n$ from initial conditions at t. $\frac{d}{dt_1} U(t_1, t) = A(t_1)\, U(t_1, t), \quad U(t, t) = I.$
$\bar{U}(t_j) =$	$U(t_n, t_j)\, \Delta t_j$
$V =$	$T\left(A_n - \frac{\ddot{X}_n\, T_1}{T_1\, \dot{X}_n}\right)$
V_1	the first row of the matrix V
V_2	the second row of the matrix V
$W_1 =$	$b_o + b_1\tau + b_2\tau^2$
$W_2 =$	$a_o + a_1\tau + a_2\tau^2$
$\dot{W}$	flow rate
$X =$	$\begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}$
$a =$	$\begin{bmatrix} a_o \\ a_1 \\ a_2 \end{bmatrix}$ Determined from equation (2.3.38).

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
a_o, a_1, a_2	coefficients for $W_2 = a_o + a_1\tau + a_2\tau^2$ determined from equation (2.3.38)
$b = \begin{bmatrix} b_o \\ b_1 \\ b_2 \end{bmatrix}$	
b_o, b_1, b_2	coefficients for $W_2 = b_o + b_1\tau + b_2\tau^2$ determined from equation (2.3.37)
$b' = \begin{bmatrix} b'_o \\ b'_1 \\ b'_2 \end{bmatrix}$	
b'_o, b'_1, b'_2	first approximation to b_o, b_1 and b_2 determined from equation (2.3.36)
$b^*(b)$	matrix defined by equation (2.3.20)
c	subscript referring to cutoff time

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
f	thrust force
f/m	thrust acceleration
f_1, f_2, f_3, f_4	elements of the row $F = (f_1 \;\; f_2 \;\; f_3 \;\; f_4)$
$\bar{f}_1$, $\bar{f}_2$, $\bar{f}_3$, $\bar{f}_4$	elements of the row $\bar{F} = (\bar{f}_1 \;\; \bar{f}_2 \;\; \bar{f}_3 \;\; \bar{f}_4)$
g	gravitational acceleration
g_o	gravitational acceleration at the earth's surface $g_o = 9.81$ m/sec^2
g_1, g_2, g_3, g_4	elements of the row $G = (g_1 \;\; g_2 \;\; g_3 \;\; g_4)$
g_5	defined by the following matrix equation

$$TH_1(t_n) = \begin{pmatrix} 0 \\ g_5 \end{pmatrix}$$

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
$\bar{g}_1$, $\bar{g}_2$, $\bar{g}_3$, $\bar{g}_4$	elements of the row $\bar{G} = (\bar{g}_1 \;\; \bar{g}_2 \;\; \bar{g}_3 \;\; \bar{g}_4)$
$h_1 =$	$\frac{\partial \ddot{x}_g}{\partial x}$
$h_2 =$	$\frac{\partial \ddot{x}_g}{\partial y}$
$k_1 =$	$\frac{\partial \ddot{y}_g}{\partial x}$
$k_2 =$	$\frac{\partial \ddot{y}_g}{\partial y}$
n	The number of subintervals into which the trajectory is subdivided for integration purposes. t_n is cutoff time on the standard trajectory. As a subscript, n denotes the value of the function for $t = t_n$.
r	radius distance from center of earth
s	subscript denoting that the function is evaluated on the standard trajectory
t	time measured on the standard trajectory time scale
t'	time measured on any other time scale
t_o	second stage ignition time on the standard trajectory
t_i	second stage ignition on any trajectory
t_n	cutoff time on the standard trajectory

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
t_c	cutoff time on any trajectory
t_k	time points defining the interval over which the A matrix is assumed to be constant $A = A(\xi_k) \quad t_{k-1} \leq t \leq t_k$
t_j	time points at which the integrand is evaluated for purposes of approximating the integral by summation
v	velocity
$\left.\begin{matrix} x \\ y \end{matrix}\right\}$	Cartesian coordinates with origin at the center of the earth; $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$, $\dddot{x}$, and $\dddot{y}$ represent their first, second and third time derivatives.
$\ddot{x}_g$	x-component of gravitational acceleration
$\ddot{y}_g$	y-component of gravitational acceleration
$\Delta C =$	$\begin{pmatrix} \Delta C_o \\ \Delta C_2 \end{pmatrix}$ evaluated by equation (2.2.18)
$\Delta C' =$	$\begin{pmatrix} \Delta C'_o \\ \Delta C'_2 \end{pmatrix}$ first approximation to ΔC evaluated by equation (3.1.3) or equation (2.2.10) whichever is appropriate
$\Delta C'^2 =$	$\begin{bmatrix} \Delta C_o'^2 \\ 2\Delta C'_o \ \Delta C'_2 \\ \Delta C_2'^2 \end{bmatrix}$

Symbol	Definition
$\Delta C'' =$	$\begin{pmatrix} \Delta C''_0 \\ \Delta C''_2 \end{pmatrix}$ second approximation to ΔC obtained from equation (2.2.14)
$\Delta C''^2 =$	$\begin{bmatrix} \Delta C''^{2}_0 \\ 2\Delta C''_0 \quad \Delta C''_2 \\ \Delta C''^{2}_2 \end{bmatrix}$
$\Delta C''^3 =$	$\begin{bmatrix} \Delta C''^{3}_0 \\ 3\Delta C''^{2}_0 \quad \Delta C''_2 \\ 3\Delta C''_0 \quad \Delta C''^{2}_2 \\ \Delta C''^{3}_2 \end{bmatrix}$
$\Delta F =$	$\begin{bmatrix} \left(1 + \frac{\Delta f/m}{f/m}\right) \Delta X \\ \left(1 + \frac{\Delta f/m}{f/m}\right) \Delta X^2 \\ \left(1 + \frac{\Delta f/m}{f/m}\right) \Delta X^3 \\ \Delta f/m \end{bmatrix}$
$\Delta R =$	$\begin{pmatrix} \Delta r \\ \Delta \theta \end{pmatrix}$

Symbol	Definition
$\Delta\bar{R}$ =	$\begin{bmatrix} \Delta\bar{r} \\ \Delta\bar{e} \end{bmatrix}$
ΔV_c =	$v(t_c) - v_s(t_n)$
ΔX =	$X(t') - X_s(t)$, $t' = t + \Delta t_o$
ΔX_c =	$X(t_c) - X_s(t_n)$
$\Delta f/m$ =	$\frac{f}{m}(t') - \frac{f_s}{m_s}(t)$, $t' = t + \Delta t_o$
Δr =	$\Delta r_c = r(t_c) - r_s(t_n)$
Δt =	$t_c - t_n - \Delta t_o$
Δt_o =	$t_i - t_o$
$\Delta\ddot{x}_g$ =	$\ddot{x}_g(t') - \ddot{x}_{gs}(t)$, $t' = t + \Delta t_o$
$\Delta\ddot{y}_g$ =	$\ddot{y}_g(t') - \ddot{y}_{gs}(t)$, $t' = t + \Delta t_o$
$\Delta\chi$ =	$\chi(t') - \chi_s(t)$, $t' = t + \Delta t_o$
$\Delta\chi_o$ =	$\Delta C_o + \Delta C_2\tau^2$, where ΔC_o and ΔC_2 are determined from equation (2.2.18)
$\Delta\theta$ =	$\Delta\theta_c = \theta(t_c) - \theta_s(t_n)$
χ	thrust angle measured from the y-axis
θ	angle of velocity vector measured from local vertical

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
$\xi_k =$	$\dfrac{t_k + t_{k-1}}{2}$, $k = 1, 2, \ldots, n$
$\tau =$	$\dfrac{t' - t_i}{100}$

TECHNICAL MEMORANDUM X-53166

GUIDANCE APPLICATIONS OF LINEAR ANALYSIS

SUMMARY

An explicit solution is obtained to the linearized differential equations of motion.[1] To illustrate the usefulness of such a solution it is used to impose the mission criteria in determining coefficients for a guidance function. The resulting function was employed in an available computer program designed to determine guidance function performance. The results were exceptionally good and serve to emphasize the advantages of this type of analysis.

INTRODUCTION

To determine a forcing function in such a way that it accomplishes a given result, it is useful to first determine the effect that this forcing function and other parameters and forcing functions have on this result. With this knowledge available, the task is considerably simplified. This report applies this principle to the problem of guidance. The equations of motion are linearized about a nominal trajectory. These equations are solved so that the important cutoff deviations, Δr and $\Delta\theta$, are obtained explicitly as a function of initial conditions, thrust acceleration and thrust angle. A form is selected for the function which determines the thrust angle, χ. The explicit solution for Δr and $\Delta\theta$ is employed to impose the mission criteria in determining the coefficients of this function which, as expected, accomplishes well the result for which it was designed. The method employed to obtain the explicit solution, the manner in which it was employed to determine the guidance function and the results obtained from using this guidance function in a number of examples are presented and discussed in detail.

The explicit solution to the linearized differential equations of motion was programmed on the IBM 1620 computer by Mr. Quintin Peasley, Technical and Scientific Staff, Aero-Astrodynamics Laboratory, George C. Marshall Space Flight Center.

[1] A similar analysis was effectively employed in Reference 3. The accuracy of the solution and the results it produced led directly to conclusions that would not likely have otherwise been suspected.

CHAPTER I. EXPLICIT SOLUTION

Section 1. Linearized Equations of Motion

The explicit solution desired requires the solution to the linearized equations which describe the motion of the vehicle. For this analysis, the motion was assumed to be in a plane and described by the following differential equations:

$$\ddot{x} = \frac{f}{m} \sin \chi + \ddot{x}_g. \tag{1.1.1}$$

$$\ddot{y} = \frac{f}{m} \cos \chi + \ddot{y}_g. \tag{1.1.2}$$

The solution to the above set of differential equations is assumed to be known numerically for a standard set of conditions. The question concerning the solution under nonstandard conditions is now considered. Employing no approximations yet, the equations under nonstandard conditions can be written as a function of the standard solution and deviations from the standard as follows:

$$\ddot{x} + \Delta\ddot{x} = \left(\frac{f}{m} + \frac{\Delta f}{m}\right)(\sin \chi + \Delta \sin \chi) + \ddot{x}_g + \Delta\ddot{x}_g.$$

$$\ddot{y} + \Delta\ddot{y} = \left(\frac{f}{m} + \frac{\Delta f}{m}\right)(\cos \chi + \Delta \cos \chi) + \ddot{y}_g + \Delta\ddot{y}_g.$$

These can be rearranged to form

$$\ddot{x} + \Delta\ddot{x} = \frac{f}{m} \sin \chi + \ddot{x}_g + \left(1 + \frac{\Delta f/m}{f/m}\right)\frac{f}{m}\,\Delta \sin \chi + \sin \chi \frac{\Delta f}{m} + \Delta\ddot{x}_g.$$

$$\ddot{y} + \Delta\ddot{y} = \frac{f}{m} \cos \chi + \ddot{y}_g + \left(1 + \frac{\Delta f/m}{f/m}\right)\frac{f}{m}\,\Delta \cos \chi + \cos \chi \frac{\Delta f}{m} + \Delta\ddot{y}_g.$$

Substitution of equations (1.1.1) and (1.1.2) into the above expressions yields the following exact expressions:

$$\Delta\ddot{x} = \Delta\ddot{x}_g + \left(1 + \frac{\Delta f/m}{f/m}\right) \Delta \sin \chi + \sin \chi \frac{\Delta f}{m}.$$

$$\Delta\ddot{y} = \Delta\ddot{y}_g + \left(1 + \frac{\Delta f/m}{f/m}\right) \Delta \cos \chi + \cos \chi \frac{\Delta f}{m}.$$

At this point, the first approximation is employed to determine $\Delta\ddot{x}_g$ and $\Delta\ddot{y}_g$ as functions of position deviations Δx and Δy.

$$\Delta\ddot{x}_g = h_1 \Delta x + h_2 \Delta y. \tag{1.1.3}$$

$$\Delta\ddot{y}_g = k_1 \Delta x + k_2 \Delta y, \tag{1.1.4}$$

where

$$h_1 = \frac{\partial\ddot{x}_g}{\partial x}, \quad h_2 = \frac{\partial\ddot{x}_g}{\partial y}$$

and

$$k_1 = \frac{\partial\ddot{y}_g}{\partial x}, \quad k_2 = \frac{\partial\ddot{y}_g}{\partial y} .$$

The next approximations can be extended to include as many terms as necessary. The three terms included below were found sufficient for this report.

$$\Delta \sin \chi = \frac{\pi}{180} \cos \chi \, \Delta\chi - \left(\frac{\pi}{180}\right)^2 \sin \chi \frac{\Delta\chi^2}{2!} - \left(\frac{\pi}{180}\right)^3 \cos \chi \frac{\Delta\chi^3}{3!} + \ldots$$

$$\Delta \cos \chi = -\frac{\pi}{180} \sin \chi \, \Delta\chi - \left(\frac{\pi}{180}\right)^2 \cos \chi \frac{\Delta\chi^2}{2!} + \left(\frac{\pi}{180}\right)^3 \sin \chi \frac{\Delta\chi^3}{3!} + \ldots,$$

where $\Delta\chi$ is given in degrees.

The linearized differential equations to be solved are

$$\Delta\ddot{x} = h_1 \Delta x + h_2 \Delta y + \left(1 + \frac{\Delta f/m}{f/m}\right)\frac{f}{m}\left[\frac{\pi}{180} \cos \chi \, \Delta\chi - \left(\frac{\pi}{180}\right)^2 \sin \chi \frac{\Delta\chi^2}{2!} - \left(\frac{\pi}{180}\right)^3 \cos \chi \frac{\Delta\chi^3}{3!}\right] + \sin \chi \frac{\Delta f}{m}.$$

$$\Delta\ddot{y} = k_1 \Delta x + k_2 \Delta y + \left(1 + \frac{\Delta f/m}{f/m}\right)\frac{f}{m}\left[-\frac{\pi}{180} \sin \chi \, \Delta\chi - \left(\frac{\pi}{180}\right)^2 \cos \chi \frac{\Delta\chi^2}{2!} + \left(\frac{\pi}{180}\right)^3 \sin \chi \frac{\Delta\chi^3}{3!}\right] + \cos \chi \frac{\Delta f}{m}.$$

With the following definitions, the system can be expressed in more convenient matrix notation.

$$A = \begin{bmatrix} 0 & 0 & 10^{-3} & 0 \\ 0 & 0 & 0 & 10^{-3} \\ h_1 & h_2 & 0 & 0 \\ k_1 & k_2 & 0 & 0 \end{bmatrix}$$

$$H_1 = \frac{\pi}{180}\,\frac{f}{m}\begin{bmatrix} 0 \\ 0 \\ \cos\chi \\ -\sin\chi \end{bmatrix}, \qquad H_2 = \left(\frac{\pi}{180}\right)^2 \frac{f/m}{2}\begin{bmatrix} 0 \\ 0 \\ -\sin\chi \\ -\cos\chi \end{bmatrix},$$

$$H_3 = \left(\frac{\pi}{18}\right)^3 \frac{f/m}{6}\begin{bmatrix} 0 \\ 0 \\ -\cos\chi \\ \sin\chi \end{bmatrix}, \qquad H_4 = \begin{bmatrix} 0 \\ 0 \\ \sin\chi \\ \cos\chi \end{bmatrix}$$

$$H(t) = (H_1 \quad H_2 \quad H_3 \quad H_4)$$

$$\Delta F(t) = \begin{bmatrix} \left(1 + \frac{\Delta f/m}{f/m}\right)\Delta\chi \\ \left(1 + \frac{\Delta f/m}{f/m}\right)\Delta\chi^2 \\ \left(1 + \frac{\Delta f/m}{f/m}\right)\Delta\chi^3 \\ \Delta f/m \end{bmatrix}.$$

In this notation, the differential equations can be written

$$\Delta\dot{X} = A\Delta X + H\Delta F, \tag{1.1.5}$$

where

$$\Delta X = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta\dot{x} \\ \Delta\dot{y} \end{bmatrix}.$$

The terms Δx and Δy are expressed in km, $\Delta\dot{x}$ and $\Delta\dot{y}$ in m/sec, $\Delta\chi$ in degrees, and $\Delta f/m$ in m/sec^2.

The time variable deviations required in equation (1.1.5) are defined as

$$\Delta X = X(t + \Delta t_o) - X_s(t)$$

$$\Delta\chi = \chi(t + \Delta t_o) - \chi_s(t)$$

$$\Delta f/m = \frac{f}{m}(t + \Delta t_o) - \frac{f}{m}(t)$$

$$\Delta t_o = t_i - t_o,$$

where t_i is second stage ignition time on any trajectory, t_o is second stage ignition time on the standard trajectory, and the subscript s refers to values obtained from the standard trajectory.

Section 2. Explicit Solution to the Differential Equations

The solution to equation (1.1.5) is shown in Reference 1 to be of the following form:

$$\Delta X_n = U(t_n, t_o)\ \Delta X_o + \int_{t_o}^{t_n} U(t_n, t)\ H(t)\ \Delta F(t)\ dt, \qquad (1.2.1)$$

where $U(t_n, t)$ is the solution to the differential equation below evaluated at $t_1 = t_n$ from initial conditions at t.

$$\frac{d}{dt_1} U(t_1, t) = A(t_1)\ U(t_1, t),\ U(t, t) = I. \qquad (1.2.2)$$

The solution to equation (1.2.2) is obtained by assuming that the elements of the A matrix are constant over each of a number of small intervals. This assumption yields the solution at the end of this small interval which is then used to provide initial conditions from which the solution at the end of the next small interval is obtained. Continuing in this manner, the solution $U(t_n, t)$ can be obtained for $t = t_o$ and any of a number of values of t, $t_o \leq t \leq t_n$, where t_n is cutoff time on the standard trajectory.

Under the assumption just made, the differential equations which will be solved are

$$\Delta\dot{X}(t) = A(\xi_k)\ \Delta X(t) + H(t)\ \Delta F(t), \qquad t_{k-1} \leq t \leq t_k$$

where

$$\xi_k = \frac{t_k + t_{k-1}}{2}, \qquad k = 1, 2, \ldots, n.$$

For this system, the solution to equation (1.2.2) can be written as

$$U(t_k, t_{k-1}) = e^{A(\xi_k)\Delta t_k} = \sum_{i=0}^{\infty} \frac{\left[A(\xi_k)\Delta t_k\right]^i}{i!} . \qquad (1.2.3)$$

This solution appears in Reference 1 and is adapted to general application to large systems in Reference 2. Truncation error associated with the series in equation (1.2.3) is sometimes a problem. However, the A matrix considered in this application has such small elements and is changing so slowly with time that $\Delta t_k \leqq 5$ sec was found adequate and only the following terms were included in the series approximation:

$$e^{A(\xi_k)\Delta t_k} = I + A(\xi_k)\ \Delta t_k.$$

Then

$$U(t_n, t_{n-1}) = e^{A(\xi_n)\Delta t_n}$$

and

$$U(t_n, t_{k-1}) = U(t_n, t_k)\ e^{A(\xi_k)\Delta t_k}, \quad k = n - 1, \ldots, 1.$$

In this manner, $U(t_n, t_o)$ can be evaluated numerically and $U(t_n, t)$ for any value of t desired. Although the integration indicated in equation (1.2.1) might require more sophisticated techniques, the functions encountered in this application are sufficiently smooth that this integral can be evaluated quite well by a summation, where each element of the sum consists of the integrand evaluated at the midpoint of an interval $t_{k-1} \leqq t \leqq t_k$ multiplied by the length of the interval Δt_k for values of Δt_k as large as forty seconds. For extreme accuracy, however, Δt_k was chosen at values on the order of 5 seconds.

Section 3. Extrapolation to Cutoff Time

The procedure just described provides the solution ΔX at standard cutoff time t_n. The following equation will be used to determine ΔX_c, the deviations in state variables obtained by subtracting their standard value at t_n from their actual value at a different cutoff time, t_c:

$$\Delta X_c = \Delta X_n + \int_{t_n}^{t_c - \Delta t_o} \dot{X}(t + \Delta t_o)\, dt, \qquad (1.3.1)$$

where $\dot{X}(t + \Delta t_o)$ is evaluated on the nonstandard trajectory. This can be written

$$\dot{X}(t + \Delta t_o) = \dot{X}_s(t) + \dot{X}(t + \Delta t_o) - \dot{X}_s(t)$$

$$\dot{X}(t + \Delta t_o) = \dot{X}_s(t) + \Delta\dot{X}(t).$$

These terms can be further decomposed:

$$\dot{X}_s(t) = \dot{X}_n + \int_{t_n}^{t} \ddot{X}_s(t)\, dt.$$

$$\Delta\dot{X}(t) = A\Delta X(t) + H(t)\ \Delta F(t).$$

Subscript s refers to values obtained from the standard trajectory. The subscript n refers to values at cutoff time, t_n, on the standard trajectory.

The term ΔX_c can be written as

$$\Delta X_c = \Delta X_n + \dot{X}_n \Delta t + \int_{t_n}^{t_c - \Delta t_o} dt_1 \int_{t_n}^{t_1} \ddot{X}_s(t)\, dt + \int_{t_n}^{t_c - \Delta t_o} [A\Delta X + H\Delta F]\, dt,$$

where

$$\Delta t = t_c - t_n - \Delta t_o.$$

This can be further simplified, by notation, to the following expression:

$$\Delta X_c = \dot{X}_n \Delta t + E, \tag{1.3.2}$$

where

$$E = \Delta X_n + \int_{t_n}^{t_c - \Delta t_o} dt_1 \int_{t_n}^{t_1} \ddot{X}_s(t)\, dt + \int_{t_n}^{t_c - \Delta t_o} [A\Delta X + H\Delta F]\, dt. \tag{1.3.3}$$

The value of Δt in equation (1.3.2) depends on the cutoff criterion. Under the assumption of cutoff at a constant velocity, the following will be used to determine Δt.

$$\Delta V_c = \frac{\dot{x}_n}{v_n} \Delta \dot{x}_c + \frac{\dot{y}_n}{v_n} \Delta \dot{y}_c = 0,$$

or

$$\Delta V_c = T_1 \Delta X_c = 0, \tag{1.3.4}$$

where

$$T_1 = \frac{1}{v_n} (0 \quad 0 \quad \dot{x}_n \quad \dot{y}_n). \tag{1.3.5}$$

Equations (1.3.2) and (1.3.4) give the following relationship:

$$T_1 \, \Delta X_c = T_1 \, \dot{X}_n \, \Delta t + T_1 E = 0,$$

and

$$\Delta t = - \frac{T_1 E}{T_1 \, \dot{X}_n} . \tag{1.3.6}$$

Substitution of this expression for Δt into equation (1.3.2) yields

$$\Delta X_c = - \frac{\dot{X}_n \, T_1 E}{T_1 \, \dot{X}_n} + E,$$

or

$$\Delta X_c = T_o E, \tag{1.3.7}$$

where

$$T_o = I - \frac{\dot{X}_n \, T_1}{T_1 \, \dot{X}_n} , \tag{1.3.8}$$

and E is defined in equation (1.3.3). This expression for E requires some assumptions or approximations in order to actually be evaluated. The following approximations are used:

$$\ddot{X}_s(t) = \ddot{X}_n$$

and

$$A \Delta X(t) + H(t) = A_n \, \Delta X_n + H_n \, \Delta F_n$$

for

$$\left\{\begin{array}{ll} t_n \leqq t \leqq t_n + \Delta t, & \Delta t > 0 \\ t_n + \Delta t \leqq t \leqq t_n, & \Delta t < 0 \end{array}\right\}.$$

These approximations essentially keep second order terms in the expression for E, discarding only terms of third order or higher. The resulting expression for E with these approximations is

$$E = \Delta X_n + \ddot{X}_n \frac{\Delta t^2}{2} + A_n \Delta X_n \Delta t + H_n \Delta t \Delta F_n. \tag{1.3.9}$$

Equations (1.3.7), (1.3.8), and (1.3.9) give Δt and ΔX_c as functions of ΔX_n. Equation (1.2.1) yields the solution of ΔX_n as a function of initial conditions, ΔX_0, deviations in thrust acceleration, $\Delta f/m$, and deviations in thrust angle, ΔX, for $t_0 \leqq t \leqq t_n$. The other functions appearing in these equations are evaluated from data on the standard trajectory. Thus, with a given standard trajectory, an explicit expression for cutoff deviations for nonstandard trajectories has been obtained.

Section 4. End Conditions

To meet the mission, certain end conditions are required. The particular variables for which an explicit solution is to be obtained depend on the mission. Under the assumption of an orbital mission, Δr and $\Delta\theta$ are the variables of concern. The following approximations can be used to determine these variables as a function of the vector ΔX_c:

$$\Delta r = \frac{x_n}{r_n} \Delta x_c + \frac{y_n}{r_n} \Delta y_c \tag{1.4.1}$$

$$\Delta(r_c \, v_c \cos \theta_c) = \Delta(x_c \, \dot{x}_c + y_c \, \dot{y}_c).$$

$$\Delta(r_c \, v_c) \cos \theta_n + r_n v_n \, \Delta \cos \theta_c = \Delta x_c \, \dot{x}_n + x_n \, \Delta\dot{x}_c + \Delta y_c \, \dot{y}_n + y_n \, \Delta\dot{y}_c.$$

Although the following assumption is not necessary, it is convenient at this point to take advantage of the fact that the mission under consideration is a circular orbit at a fixed radius. Then r_c is constant and $\Delta(r_c V_c) = 0$. Then the following approximation is used:

$$\Delta \cos \theta_c = - \frac{\pi}{180} \sin \theta_n \Delta\theta_c.$$

Since $\sin \theta_n = 1$, we have

$$\Delta \cos \theta_c = - \frac{\pi}{180} \Delta\theta_c.$$

The following expression is then obtained for $\Delta\theta_c$.

$$\Delta\theta_c = - \frac{180}{\pi r_n v_n} (\dot{x}_n \Delta x_c + \dot{y}_n \Delta y_c + x_n \Delta\dot{x}_c + y_n \Delta\dot{y}_c). \qquad (1.4.2)$$

Equations (1.4.1) and (1.4.2) can then be combined into one matrix equation.

$$\Delta R = T_2 \Delta X_c, \qquad (1.4.3)$$

where

$$\Delta R = \begin{bmatrix} \Delta r_c \\ \Delta\theta_c \end{bmatrix} = \begin{bmatrix} \Delta r \\ \Delta\theta \end{bmatrix}$$

and

$$T_2 = \begin{bmatrix} \frac{x_n}{r_n} & \frac{y_n}{r_n} & 0 & 0 \\ \frac{-180\dot{x}_n}{\pi r_n v_n} & \frac{-180\dot{y}_n}{\pi r_n v_n} & \frac{-180x_n}{\pi r_n v_n} & \frac{-180y_n}{\pi r_n v_n} \end{bmatrix} \qquad (1.4.4)$$

Equations (1.2.1), (1.3.7), (1.3.8), (1.3.9), (1.4.3) and (1.4.4) give us the means of determining the variables Δr and $\Delta\theta$ as a function of initial condition deviations, ΔX_o, thrust acceleration deviations, $\frac{\Delta f}{m}(t)$, and thrust angle deviations, $\Delta\chi(t)$, $(t_o \leq t \leq t_n)$. This expression can be used to actually evaluate errors of individual trajectories in the neighborhood of the standard if all the deviations mentioned are known. It also provides considerable insight into the mechanics of solving differential equations of motion by showing, term by term, the effect on mission error of

$$\Delta X_o, \ \frac{\Delta f}{m}(t), \text{ and } \Delta\chi(t).$$

In addition, for this particular problem, the explicit representation provides a means of determining χ as a function of ΔX_o and $\frac{\Delta f}{m}(t)$ so that Δr and $\Delta\theta$ are as near zero as this analysis and the assumptions concerning the form of χ will allow. Although ΔX_o, $\frac{\Delta f}{m}(t)$, and $\Delta\chi(t)$ were the only parameters considered in this analysis, with little additional effort other forcing functions or parameters could have been included. For the present, the concern will remain with the forcing functions and parameters already considered and an actual application will be demonstrated.

Section 5. Numerical Example

Several calculus of variations solutions were available for an early SA-6 second stage vehicle. The standard trajectory had the following initial and end point conditions:

Initial Conditions: (1.5.1)

$$x_o = 153.98343 \text{ km} \qquad t_o = 146.815 \text{ sec}$$

$$y_o = 6435.8783 \text{ km} \qquad \frac{f}{m}(\tau) = \frac{8.78065}{1.3751 - .20888\tau},$$

$$\dot{x}_o = 2818.3294 \text{ m/sec}$$

$$\dot{y}_o = 988.35767 \text{ m/sec} \qquad \tau = \frac{t - t_o}{100}.$$

Final Conditions: (1.5.2)

$x_n = 2326.37$ km

$y_n = 6128.51$ km

$\dot{x}_n = 7285.34$ m/sec

$\dot{y}_n = -2765.50$ m/sec

$\ddot{x}_n = 19.047$ m/sec^2

$\ddot{y}_n = -13.596$ m/sec^2

$\dddot{x}_n = .1001$ m/sec^3

$\dddot{y}_n = -.0655$ m/sec^3

$v_n = 7792$ m/sec

$r_n = 6555.200$ km

$t_n = 620.679$ sec

$\frac{f}{m}(t_n) = 22.793$ m/sec^2

$\chi_n = 102.506°$

$\theta_n = 90°$.

The gravity components were defined by

$$\ddot{x}_g = \frac{x}{r} g$$

$$\ddot{y}_g = \frac{y}{r} g,$$

where

$$g = -\frac{g_o r_o^2}{r^2}, \quad g_o = 9.81 \text{ m/sec}^2, \quad r_o = 6370 \text{ km}.$$

From this, the following elements of the A matrix are determined.

$$\left.\begin{aligned}
h_1 &= \frac{\partial \ddot{x}_g}{\partial x} = -\frac{g_o\, r_o^2}{r^3}\,[1 - 3(x/r)^2] \\
h_2 &= \frac{\partial \ddot{x}_g}{\partial y} = \frac{g_o\, r_o^2}{r^3}\left[\frac{3xy}{r^2}\right] \\
k_1 &= \frac{\partial \ddot{y}_g}{\partial x} = \frac{g_o\, r_o^2}{r^3}\left[\frac{3xy}{r^2}\right] \\
k_2 &= \frac{\partial \ddot{y}_g}{\partial y} = -\frac{g_o\, r_o^2}{r^3}\,[1 - 3(y/r)^2]
\end{aligned}\right\} \qquad (1.5.3)$$

The values of x, y, and τ obtained from the standard trajectory are listed in Table 1.1. They are listed as a function of t where t designates time on the standard trajectory. In addition, the quantity Δt_k is listed which describes the length of the interval over which the differential equations were assumed to be a constant coefficient system.

TABLE 1.1

STANDARD TRAJECTORY DATA

t(sec)	$\tau(10^2$sec)	Δt_k(sec)	x(km)	y(km)
150	.0318	13.18	163.0	6435.6
180	.3318	40	250.4	6461.9
220	.7318	40	374.8	6488.2
260	1.1318	40	508.6	6504.4
300	1.5318	40	652.7	6510.5
340	1.9318	40	807.8	6506.2
380	2.3318	40	975.1	6491.0
420	2.7318	40	1155.7	6464.5
460	3.1318	40	1351.0	6426.0
500	3.5318	40	1562.7	6374.5
540	3.9318	40	1793.0	6308.8
580	4.3318	40	2044.7	6227.1
610	4.6318	20.68	2249.9	6153.9.

The data in Table 1.1 can be used to evaluate the elements of the A matrix defined in equations (1.5.3). The results are shown in Table (1.2) which for convenience have been multiplied by 1000.

TABLE 1.2

ELEMENTS OF A MATRIX

t	h_1	h_2	k_1	k_2
150	-1.489	.113	.113	2.327
180	-1.465	.171	.171	2.937
220	-1.436	.251	.251	2.886
260	-1.407	.334	.334	2.840
300	-1.378	.423	.423	2.799
340	-1.348	.518	.518	2.761
380	-1.315	.621	.621	2.722
420	-1.275	.731	.731	2.682
460	-1.228	.849	.849	2.633
500	-1.168	.977	.977	2.576
540	-1.113	1.079	1.079	2.505
580	-1.001	1.257	1.257	2.415
610	- .912	1.370	1.370	2.327

NOTE: All of the above elements have been multiplied by 1000. To use in the construction of the A matrix, they must first be multiplied by 10^{-3}.

The matrix $U(t_n, t)$ can be determined from the information in Tables 1.1 and 1.2 for the values of t listed as follows.

$$U(t_n, t_{n-1}) = I + A(\xi_n)\,\Delta t_n$$

where

$$A(\xi_n) = \begin{bmatrix} 0 & 0 & 10^{-3} & 0 \\ 0 & 0 & 0 & 10^{-3} \\ h_1(\xi_n) & h_2(\xi_n) & 0 & 0 \\ k_1(\xi_n) & k_2(\xi_n) & 0 & 0 \end{bmatrix}.$$

Choosing $\xi_n = 610$ and $t_{n-1} = 600$, $t_n = 620.68$, then the following matrix is determined.

$$U(t_n, 600) = \begin{bmatrix} 1 & 0 & .02068 & 0 \\ 0 & 1 & 0 & .02068 \\ -.01886 & .02833 & 1 & 0 \\ .02833 & .04812 & 0 & 1 \end{bmatrix}$$

Then $U(t_n, 560) = U(t_n, 600)\, U(600, 560)$ where $U(600, 560) = I + 40A(580)$.

Continuing in this manner, $U(t_n, t)$ can be determined for a number of values of t back to and including $t = t_o$. The following matrix $U(t_n, t_o)$ was obtained from the data in Tables 1.1 and 1.2.

$$U(t_n, t_o) = \begin{bmatrix} .86115 & .04792 & .45637 & .00905 \\ .04503 & 1.29901 & .00890 & .51145 \\ -.57740 & .35385 & .88097 & .09625 \\ .31006 & 1.38267 & .09284 & 1.27619 \end{bmatrix}. \qquad (1.5.4)$$

The intermediate matrices $U(t_n, t)$ are tabulated in Appendix I.

The expression for $U(t_k, t_{k-1})$ is the truncation of a series expansion. If the elements in the A matrix are large, more terms are required. If they are extremely large, $U(t_k, t_{k-1})$ must be evaluated by special methods described in Reference 2. In addition, the interval over which the system is assumed to have a constant A matrix depends on the particular problem at hand. The numerical example being used as an illustration assumes a constant coefficient system for intervals of forty seconds. A more accurate solution was employed to determine the guidance function in Chapter II. The system was assumed constant for intervals of 5 seconds or less, and the integration was carried out by summing over five second intervals. The solution differed generally in the third significant figure from that obtained from the present example where 40 seconds was used.

In addition to the matrix $U(t_n, t_o)$, the solution given by equation (1.2.1) requires that an integration be performed. The functions in this example are sufficiently smooth that a very good solution can be obtained by summing $U(t_n, t_k)\ H(t_k)\ \Delta F(t_k)\ \Delta t_k$. If the functions in the integrand are extremely variable, this technique may not give accurate values for the integral. Although techniques outlined in Reference 2 overcome this problem, they are not necessary for this example. With the following definition, ΔX_n can be written as a linear sum of deviations in initial conditions ΔX_o and the vector $\Delta F(t)$ evaluated for several values of t.

$$\bar{U}(t_j) = U(t_n, t_j)\ H(t_j)\ \Delta t_j. \tag{1.5.5}$$

Then,

$$\Delta X_n = U(t_n, t_o)\ \Delta X_o + \sum_{j=1}^{n} \bar{U}(t_j)\ \Delta F(t_j). \tag{1.5.6}$$

Table 1.3 lists the data taken from the standard trajectory necessary to evaluate the matrix $H(t_j)$ for the values of t_j indicated. In addition, the value of Δt_j is listed which was used to determine $\bar{U}(t_j)$ as defined in equation (1.5.5). Appendix I lists the corresponding matrices $U(t_n, t_j)$ which, with the data in Table 1.3, is sufficient to evaluate the elements of $\bar{U}(t_j)$ necessary to perform the summation indicated in equation (1.5.6).

TABLE 1.3

DATA REQUIRED FOR $H(t_j)\ \Delta t_j$

t_j (sec)	Δt_j (sec)	χ (°)	$\frac{f}{m}$ (m/sec^2)
160	33.18	59.255	6.516
200	40	62.460	6.946
240	40	65.745	7.438
280	40	69.119	8.005
320	40	72.590	8.665
360	40	76.168	9.444
400	40	79.861	10.376
440	40	83.676	11.513
480	40	87.620	12.930
520	40	91.693	14.743
560	40	95.896	17.149
600	40.68	100.223	20.493

The above values, together with $U(t_n, t_j)$ listed in Appendix I, were used to determine $\bar{U}(t_j)$ from equation (1.5.5). The elements of $\bar{U}(t_j)$ are listed in Appendix II. All of the information necessary to evaluate ΔX_n is available and the necessary coefficients evaluated; ΔX_o, $\Delta X(t_j)$ and $\frac{\Delta f}{m}(t_j)$ remain explicit.

The last step necessary to obtain an explicit expression for Δr and $\Delta\theta$ is to evaluate the matrices defined in equations (1.3.7), (1.3.9), and (1.4.3). From the end conditions given in (1.5.2), T_1, defined in equation (1.3.5), is evaluated.

$$T_1 = (0 \quad 0 \quad .9350 \quad -.3549). \tag{1.5.7}$$

Evaluation of the elements in T_2, defined by equation (1.4.4), gives the following result.

$$T_2 = \begin{bmatrix} .3549 & .9349 & 0 & 0 \\ -.008172 & .003102 & -.002609 & -.006875 \end{bmatrix},$$

$$\dot{X}_n = \begin{bmatrix} 7.2853 \\ -2.7655 \\ 19.047 \\ -13.596 \end{bmatrix} \quad \text{and} \quad \ddot{X}_n = \begin{bmatrix} .01905 \\ -.01360 \\ .1001 \\ -.0655 \end{bmatrix}.$$

Equation (1.3.7) provides the relationship,

$$\Delta X_c = T_o E,$$

where

$$T_o = I - \frac{\dot{X}_n \; T_1}{T_1 \; \dot{X}_n} .$$

Equation (1.4.3) gives the relationship

$$\Delta R = T_2 \, \Delta X_c .$$

Combining these expressions yields

$$\Delta R = T_2 \left(I - \frac{\dot{X}_n \; T_1}{T_1 \; \dot{X}_n} \right) E = TE, \qquad (1.5.8)$$

where

$$T = T_2 \left(I - \frac{\dot{X}_n \; T_1}{T_1 \; \dot{X}_n} \right). \tag{1.5.9}$$

Numerical evaluation of the elements gives the following matrix T.

$$T = \begin{bmatrix} .3549 & .9349 & 0 & 0 \\ -.008172 & .003102 & -.001603 & -.007262 \end{bmatrix} \tag{1.5.10}$$

The final expression for E is given by equation (1.3.9).

$$E = \Delta X_n + \ddot{X}_n \frac{\Delta t^2}{2} + A_n \; \Delta X_n \; \Delta t + H_n \; \Delta t \; \Delta F_n .$$

In addition to the quantities already evaluated, A_n and H_n must be evaluated in order to investigate the second order terms. From cutoff data, given by (1.5.2), the following can be determined.

$$A_n = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -.882 & 1.407 & 0 & 0 \\ 1.407 & 2.297 & 0 & 0 \end{bmatrix} \times 10^{-3}$$

$$H_n = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -.0853 & -.0068 & .0000 & -.2152 \\ -.3870 & .0015 & .0001 & .9766 \end{bmatrix}.$$

The effect of these matrices on ΔR is determined by equation (1.5.8) and (1.3.9).

$$\Delta R = TE = T\Delta X_n + \left[T\ddot{X}_n \frac{\Delta t}{2} + TA_n \Delta X_n + TH_n \Delta F_n\right] \Delta t. \qquad (1.5.11)$$

Equation (1.3.6) gives the following expression for Δt.

$$\Delta t = - \frac{T_1 E}{T_1 \dot{X}_n} .$$

Using the first order approximation for E, this expression becomes

$$\Delta t = - \frac{T_1 \Delta X_n}{T_1 \dot{X}_n} .$$

This can be substituted for Δt inside the brackets in equation (1.5.11), and the following expression results:

$$\Delta R = T\Delta X_n + \left[T\left(A_n - \frac{\ddot{X}_n T_1}{2T_1 \dot{X}_n}\right) \Delta X_n + TH_n \Delta F_n\right] \Delta t. \qquad (1.5.12)$$

A brief look at the values of the elements of the matrices inside the brackets will give an estimate of the contribution of these second order terms.

Cutoff data from the standard trajectory, equations (1.5.2), provide the information necessary to evaluate these matrices.

$$T\left(A_n - \frac{\ddot{X}_n T_1}{2T_1 \dot{X}_n}\right) = \begin{bmatrix} 0 & 0 & .4779 & .8882 \\ -.0088 & -.0189 & -.0106 & .0040 \end{bmatrix} \times 10^{-3}.$$

To get an idea of the effect this term might have some extreme values can be assumed for ΔX_n. First, it will be assumed that $\Delta x_n = \Delta y_n = 100$ km and $\Delta \dot{x}_n = -\Delta \dot{y}_n = 100$ m/sec. Then,

$$T\left(A_n - \frac{\ddot{X}_n T_1}{2T_1 \dot{X}_n}\right) \Delta X_n = \begin{pmatrix} -41.02 \\ -4.23 \end{pmatrix} \times 10^{-3}.$$

This means that the contribution to Δr is -.041 km or -41 m for each second of additional second stage burning time Δt. However, for $\Delta\theta$ the contribution is only -.004° for each second Δt. The signs associated with the deviations assumed for ΔX_n were chosen to have the greatest effect on $\Delta\theta$. With this in mind, it can be concluded that these second order terms are negligible with respect to $\Delta\theta$. An extreme example for Δr might be $\Delta x_n = \Delta y_n = 100$ km and $\Delta \dot{x}_n = \Delta \dot{y}_n = 100$ m/sec. This gives

$$T\left(A_n - \frac{\ddot{X}_n T_1}{2T_1 \dot{X}_n}\right) \Delta X_n = \begin{pmatrix} 136.61 \\ -3.43 \end{pmatrix} \times 10^{-3}.$$

This gives a smaller error in $\Delta\theta$, but in Δr it would produce about 137 m for each second of additional second stage burning time Δt. Whether this term is negligible or not depends on a more critical investigation of $\Delta \dot{x}_n$ and $\Delta \dot{y}_n$ which are not independent of each other. In most of the cases considered later, $\Delta \dot{x}_n$ and $\Delta \dot{y}_n$ were of opposite sign so that their effect on Δr tended to cancel rather than add. A more detailed examination of the relationship of $\Delta \dot{x}_n$ and $\Delta \dot{y}_n$ and their effect on the error in Δr will not be pursued in this report. However, in the event that any decisions depended on the assumption that this contribution be negligible, further investigations could be completed at that time.

The other term in the brackets in equation (1.5.12) is

$$TH_n = \begin{bmatrix} 0 & 0 & 0 & 0 \\ .00295 & .0000 & .0000 & -.0062 \end{bmatrix}.$$

Choosing $\Delta X_n = 3°$ and

$$\frac{\Delta f}{m}(t_n) = -.01\ \frac{f}{m}(t_n) = -.22\ \mathrm{m/sec}^2$$

gives

$$\Delta F_n = \begin{bmatrix} 3 \\ 9 \\ 27 \\ -.22 \end{bmatrix}.$$

Then

$$TH_n\ \Delta F_n = \begin{bmatrix} 0 \\ .00885 \quad + \quad .0014 \end{bmatrix}.$$

The contribution from ΔX_n is .00885° per second and from $\Delta f/m$, only .0014° per second. The contribution of $\Delta f/m$ will be neglected, but the contribution of ΔX_n will be kept since it may be several hundredths of a degree.

Having investigated the possible contribution of various second order terms and finding several of them to be insignificant, we will discard these and rewrite the expression for ΔR.

Defining two row vectors V_1 and V_2 by the following equation,

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = T\left(A_n - \frac{\ddot{X}_n\ T_1}{T_1\ \dot{X}_n}\right),$$

then,

$$\Delta R = T\Delta X_n + V\Delta X_n\ \Delta t + TH_n\ \Delta F_n. \tag{1.5.13}$$

Equation (1.2.1) gave the following expression for ΔX_n.

$$\Delta X_n = U(t_n, t_o) + \int_{t_o}^{t_n} U(t_n, t) H(t) \Delta F(t) dt.$$

Substitution of this expression for ΔX_n into the linear term $T\Delta X_n$ appearing in (1.5.13) gives

$$\Delta R = TU(t_n, t_o) + \int_{t_o}^{t_n} TU(t_n, t) H(t) \Delta F(t) dt + V\Delta X_n \Delta t + TH_n \Delta F_n.$$

For convenience, these definitions will be made:

$$TU(t_n, t_o) = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

$$TU(t_n, t) H(t) = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ g_1 & g_2 & g_3 & g_4 \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}.$$

Since all but one of the elements in the matrix TH_n were zero or considered negligible, the following simplification is used:

$$TH_n \Delta F_n = \begin{pmatrix} 0 \\ g_5 \end{pmatrix} \Delta X_n.$$

With these definitions and simplifications, the following expressions for Δr and $\Delta\theta$ are obtained:

$$\Delta r = U_1 \Delta X_o + \int_{t_o}^{t_n} F \Delta F \, dt + V_1 \Delta X_n \Delta t \qquad (1.5.14)$$

and

$$\Delta\theta = U_2 \Delta X_o + \int_{t_o}^{t_n} G \Delta F \, dt + g_5 \Delta X_n \Delta t. \qquad (1.5.15)$$

The numerical values obtained for these expressions are listed below.

$$U_1 = \begin{pmatrix} .34772 & 1.23145 & .17029 & .48137 \end{pmatrix} \qquad (1.5.16)$$

$$U_2 = \begin{pmatrix} -.008224 & -.006970 & -.005788 & -.007909 \end{pmatrix} \qquad (1.5.17)$$

$$V_1 = \begin{pmatrix} 0 & 0 & .4779 & .8882 \end{pmatrix} \times 10^{-3} \qquad (1.5.18)$$

$$g_5 = .00295.$$

The term ΔX_n is defined by equation (1.5.6). $U(t_n, t_o)$ is found in Appendix I and the elements of $\bar{U}(t_j)$ are listed in Appendix II.

The first order approximation for Δt is

$$\Delta t = - \frac{T_1 \Delta X_n}{T_1 \dot{X}_n} = (0 \quad 0 \quad -.04131 \quad .01568) \; \Delta X_n,$$

where ΔX_n can be evaluated from equation (1.5.6) as already mentioned.

The integrals appearing in equations (1.5.14) and (1.5.15) can be evaluated by summation.

$$\int_{t_o}^{t_n} G \, \Delta F \, dt = \sum_{j=1}^{n} G(t_j) \, \Delta F(t_j) \, \Delta t_j$$

and

$$\int_{t_o}^{t_n} F \, \Delta F \, dt = \sum_{j=1}^{n} F(t_j) \, \Delta F(t_j) \, \Delta t_j .$$

The multiplying factors for $\Delta F(t_j)$ will be defined as follows:

$$\bar{G}(t_j) = G(t_j) \, \Delta t_j = \begin{pmatrix} \bar{g}_1(t_j) & \bar{g}_2(t_j) & \bar{g}_3(t_j) & \bar{g}_4(t_j) \end{pmatrix}$$

$$\bar{F}(t_j) = F(t_j) \, \Delta t_j = \begin{pmatrix} \bar{f}_1(t_j) & \bar{f}_2(t_j) & \bar{f}_3(t_j) & \bar{f}_4(t_j) \end{pmatrix} .$$

These quantities are listed in Tables 1.4 and 1.5. The Δt_j employed to determine these values is listed in Table 1.3.

TABLE 1.4

$\bar{F}(t_j)$

t_j	$\bar{f}_1$	$\bar{f}_2(10^{-2})$	$\bar{f}_3(10^{-4})$	$\bar{f}_4$
160	-1.1877	-1.284	.6030	12.930
200	-1.4612	-1.388	.7419	13.118
240	-1.4790	-1.262	.7508	11.144
280	-1.4843	-1.133	.7535	9.297
320	-1.4741	-1.000	.7484	7.581
360	-1.4444	- .863	.7333	6.004
400	-1.3897	- .723	.7055	4.576
440	-1.3023	- .581	.6612	3.313
480	-1.1715	- .431	.5948	2.219
520	- .9802	- .297	.4977	1.319
560	- .7023	- .163	.3566	.624
600	- .2958	- .047	.1502	.151
600*	- .2363	- .037	.1199	.151

*The values in this row were obtained by using Δt_j = 32.5 sec instead of 40.68 sec. If $\Delta X = 0$ for $t > 612.5$, these values should be used instead of those listed in the regular table for t_j = 600 seconds.

TABLE 1.5

$\bar{G}(t_j)$

t_j	$\bar{g}_1(10^{-2})$	$\bar{g}_2(10^{-4})$	$\bar{g}_3(10^{-6})$	$\bar{g}_4$
160	1.438	2.972	-.730	-.2995
200	2.068	3.487	-1.049	-.3297
240	2.440	3.436	-1.239	-.3033
280	2.860	3.366	-1.452	-.2761
320	3.338	3.271	-1.695	-.2479
360	3.887	3.144	-1.973	-.2186
400	4.525	2.973	-2.297	-.1881
440	5.276	2.747	-2.679	-.1567
480	6.179	2.436	-3.137	-.1237
520	7.289	2.015	-3.701	-.0897
560	8.702	1.427	-4.418	-.0547
600	10.764	.582	-5.465	-.0186
600*	8.599	.465	-4.366	-.0186

*The values in this row were obtained by using Δt_j = 32.5 sec instead of 40.68 seconds. If $\Delta X = 0$ for $t > 612.5$, these values should be used instead of those listed in the regular table for t_j = 600 seconds.

The results just described give the influence coefficients for ΔX, $\Delta f/m$ and ΔX_0 as they affect Δr and $\Delta\theta$. In that respect they are of interest in themselves for the insight they provide. For computational convenience, large integration steps were taken. This was done on a desk calculator, and the procedure outlined can be checked against these results by a desk calculator to verify the steps involved. Although these results are accurate through two significant figures, this is not sufficient for later work where the assumption of a constant coefficient system and the integration steps were necessarily reduced from 40 second intervals to 5 second intervals. These more accurate results were obtained from an IBM 1620 digital computer program and used to derive the guidance function discussed in the following chapter. Nevertheless, the numerical example with stepwise evaluation included in this chapter should serve to illustrate the procedure described.

CHAPTER II

DETERMINATION OF A GUIDANCE FUNCTION

Section 1. Fitting the Nominal Trajectory

Equations (1.5.14) and (1.5.15) of the preceding chapter provide an explicit solution for Δr and $\Delta\theta$ as a function of ΔX_o, $\frac{\Delta f}{m}(t)$ and $\Delta X(t)$. These equations can be employed to ensure that $\bar{X}$, a function derived to approximate the function X on the standard trajectory, will meet the required end conditions, $\Delta r = \Delta\theta = 0$. Under standard conditions $\Delta X_o = 0$ and $\frac{\Delta f}{m}(t) = 0$. In addition to this, if $\bar{X}$ sufficiently well approximates the nominal value of X, the contribution of higher order terms becomes negligible. It will be assumed that this can be done sufficiently well by describing $\bar{X}$ as a quadratic in t. For convenience, the following transformation will be employed.

$$\tau = \frac{t' - t_i}{100} , \tag{2.1.1}$$

where t' denotes time on any trajectory and t_i denotes ignition time on that same trajectory. The term t' on an arbitrary trajectory will be related to t on the standard trajectory by the following equation,

$$t' = t + \Delta t_o, \tag{2.1.2}$$

where

$$\Delta t_o = t_i - t_o. \tag{2.1.3}$$

Substituting these expressions into equation (2.1.1) gives

$$\tau = \frac{t - t_o}{100} .$$

This definition was already used in equation (1.5.1). Thus, equation (2.1.1) offers no contradiction.

The term $\bar{\chi}$ will now be defined as

$$\bar{\chi} = \bar{C}_o + \bar{C}_1\tau + \bar{C}_2\tau^2. \tag{2.1.4}$$

With this definition and otherwise standard conditions, equations (1.5.14) and (1.5.15) become, respectively,

$$\Delta r = \int_{t_o}^{t_n} f_1 \left[\bar{C}_o + \bar{C}_1\tau + \bar{C}_2\tau^2 - \chi_s(t)\right] dt \tag{2.1.5}$$

$$\Delta\theta = \int_{t_o}^{t_n} g_1 \left[\bar{C}_o + \bar{C}_1\tau + \bar{C}_2\tau^2 - \chi_s(t)\right] dt. \tag{2.1.6}$$

The higher order terms in these expressions have been considered negligible. The term $\chi_s(t)$ is the nominal calculus of variations solution for χ. To ensure that the end conditions are met, the following constraints are imposed on the coefficients $\bar{C}_o$, $\bar{C}_1$ and $\bar{C}_2$.

$$\bar{C}_o \int_{t_o}^{t_n} f_1\, dt + \bar{C}_1 \int_{t_o}^{t_n} f_1\tau\, dt = - \bar{C}_2 \int_{t_o}^{t_n} f_1\tau^2\, dt + \int_{t_o}^{t_n} f_1\chi_s(t)\, dt. \tag{2.1.7}$$

$$\bar{C}_o \int_{t_o}^{t_n} g_1\, dt + \bar{C}_1 \int_{t_o}^{t_n} g_1\tau\, dt = - \bar{C}_2 \int_{t_o}^{t_n} g_1\tau^2\, dt + \int_{t_o}^{t_n} g_1\chi_s(t)\, dt. \tag{2.1.8}$$

The integrals in equations (2.1.7) and (2.1.8) can be evaluated quite well from the coefficients in Tables 1.4 and 1.5 together with the values of τ and χ_s tabulated in Table 2.1. Although these results may differ in the third or fourth significant figure from the values shown below, the values presented were those actually used. Since the resulting function $\bar{\chi}$ fits sufficiently well, the results of the following constraints were not corrected.

$$14.284\ \bar{C}_o + 28.8545\ \bar{C}_1 = -\ 79.49615\ \bar{C}_2 + 1080.9536$$

$$.5872\ \bar{C}_o + 1.80745\ \bar{C}_1 = -\ 6.491105\ \bar{C}_2 + 50.32710.$$

Solving for $\bar{C}_o$ and $\bar{C}_1$ yields

$$\bar{C}_o = 4.914469\ \bar{C}_2 + 56.523779 \tag{2.1.9}$$

$$\bar{C}_1 = -\ 5.187907\ \bar{C}_2 + 9.480948. \tag{2.1.10}$$

Substituting these values into the expression for $\bar{\chi}$ shown in equation (2.1.4) yields

$$\bar{\chi} = a(\tau)\bar{C}_2 + b(\tau), \tag{2.1.11}$$

where

$$a(\tau) = 4.914469 - 5.187907\tau + \tau^2$$

and

$$b(\tau) = 56.523779 + 9.480948\tau.$$

Any choice of $\bar{C}_2$ in equation (2.1.11) will produce a Δr and $\Delta\theta$ of zero providing the resultant value of $\Delta\chi$ is sufficiently small that the higher order terms are negligible. A value of $\bar{C}_2$ must be chosen with this in mind. The sum of the squared residuals are given in the following expression:

$$S = \sum_{j=1}^{n} \left[\bar{X}(\tau_j) - X(t_j) \right]^2 .$$

Minimizing the value with respect to the choice of $\bar{C}_2$ gives the following least squares constraint.

$$\frac{\partial S}{\partial \bar{C}_2} = 2 \sum_{j=1}^{n} \left[\bar{X}(t_j) - X_s(t_j) \right] \frac{\partial \bar{X}(\tau_j)}{\partial \bar{C}_2} = 0.$$

Since

$$\frac{\partial \bar{X}(\tau_j)}{\partial \bar{C}_2} = a(\tau_j),$$

this becomes

$$\bar{C}_2 \sum_{j=1}^{n} \left[a(\tau_j) \right]^2 = \sum_{j=1}^{n} a(\tau_j) \left[b(\tau_j) - X_s(t_j) \right]. \qquad (2.1.12)$$

Using the expressions for $a(\tau)$ and $b(\tau)$ indicated in equation (2.1.11) and the values of τ_j and $X_s(t_j)$ indicated in Table 2.1 gives the following numerical expression for equation (2.1.12):

$$40.78235 \, \bar{C}_2 = 14.57552.$$

This result, together with equations (2.1.9) and (2.1.10), yields the following coefficients for $\bar{X}$.

$$\left. \begin{aligned} \bar{C}_0 &= 58.2802 \\ \bar{C}_1 &= 7.6268 \\ \bar{C}_2 &= .3574 \end{aligned} \right\} , \qquad (2.1.13)$$

and

$$\bar{\chi} = 58.2802 + 7.6268\tau + .3574\tau^2. \tag{2.1.14}$$

$\bar{\chi}$ in the above expression is given in degrees and τ is defined by equation (2.1.1).

Table 2.1 shows a comparison of $\bar{\chi}$ with χ_s, the function to which it was fitted. The resulting residuals are sufficiently small that their effect on second order terms is negligible.

TABLE 2.1

COMPARISON OF χ AND χ_s

t_j (sec)	τ_j (10^2sec)	$\bar{\chi}(\tau_j)$ (°)	$\chi_s(t_j)$ (°)	$(\bar{\chi} - \chi_s)$ (°)
160	.1319	59.292	59.225*	.067
200	.5319	62.438	62.460	-.022
240	.9319	65.698	65.745	-.047
280	1.3319	69.072	69.119	-.047
320	1.7319	72.561	72.590	-.029
360	2.1319	76.164	76.168	-.004
400	2.5319	79.882	79.861	.021
440	2.9319	83.713	83.676	.037
480	3.3319	87.660	87.620	.040
520	3.7319	91.720	91.693	.027
560	4.1319	95.895	95.896	-.001
600	4.5319	100.184	100.223	-.039

* The value 59.225 was erroneously used in the fit shown here. The actual value, as shown in Table 1.3, was 59.255. This would make the actual residual at this point .040 instead of .067. Since the coefficients in equation (2.1.13) were used in later work and since the error was of minor significance, for consistency and economy of effort, this correction was not made and the coefficients in $\bar{\chi}$ have been employed as they appear in equation (2.1.14).

Section 2. Initial Conditions

If we take another look at equations (1.5.14) and (1.5.15), they appear as follows:

$$\Delta r = U_1 \, \Delta X_o + \int_{t_o}^{t_n} F \, \Delta F \, dt + V_1 \, \Delta X_n \, \Delta t,$$

and

$$\Delta\theta = U_2 \, \Delta X_o + \int_{t_o}^{t_n} G \, \Delta F \, dt + g_5 \, \Delta X_n \, \Delta t,$$

where

$$\Delta F = \begin{bmatrix} \left(1 + \frac{\Delta f/m}{f/m}\right) \Delta X \\ \\ \left(1 + \frac{\Delta f/m}{f/m}\right) \Delta X^2 \\ \\ \left(1 + \frac{\Delta f/m}{f/m}\right) \Delta X^3 \\ \\ \Delta f/m \end{bmatrix}.$$

To keep the guidance function as simple as possible, the second order terms must be negligible or made so. The term $V_1 \, \Delta X_n \, \Delta t$ is extremely difficult to evaluate at second stage ignition. Both Δt and ΔX_n are functions of $\Delta f/m$ and ΔX along the entire trajectory. Because of the difficulties involved, this term will first be ignored, and if after making this assumption, the resulting function is not sufficiently accurate, the problem can again be taken up at that time.

The second order term $g_5 \Delta X_n \Delta t$, however, poses a different problem. The forcing function which we wish to determine, ΔX, appears as a multiplier. This term was the only one considered significant in the following term from the expression for E, equation (1.3.3), after it was substituted into equation (1.5.8).

$$\int_{t_n}^{t_c - \Delta t_o} TH \, \Delta F \, dt \sim \int_{t_n}^{t_c - \Delta t_o} TH_1 \left(1 + \frac{\Delta f/m}{f/m}\right) \Delta X \, dt. \qquad (2.2.1)$$

If ΔX is chosen to be zero throughout the interval defined by the limits of integration, the entire integral would be zero. This can be assured if a δ is determined such that $\Delta t \geqq \delta$ for all likely nonstandard trajectories ($\Delta t = t_c - t_n - \Delta t_o$). By defining ΔX such that $\Delta X = 0$, $t > t_n + \delta$, the integral appearing on the right in equation (2.2.1) will always be zero and the second order term $g_5 \Delta X_n \Delta t$ has been eliminated from the expression for $\Delta\theta$.

At second stage ignition time, the only information available is ΔX_o; $\frac{\Delta f}{m}(t)$ will not be known for $t > t_o$ until a later time, and not completely known until cutoff. A value of X must be determined, however. With this in mind, ΔX will be defined as the sum of two functions. One function, ΔX_o, will be determined at second stage ignition to meet the required end conditions if $\frac{\Delta f}{m}(t)$ is zero throughout the second stage. The other function δX will be determined as a function of $\Delta f/m$ as this function becomes known. Thus, ΔX is defined as

$$\Delta X = \Delta X_o + \delta X. \qquad (2.2.2)$$

Under the assumptions just described, ΔX_o must satisfy the following equations.

$$\Delta r = U_1 \, \Delta X_o + \int_{t_o}^{t_n+\delta} f_1 \, \Delta X_o \, dt + \int_{t_o}^{t_n+\delta} f_2 \, \Delta X_o^2 dt + \int_{t_o}^{t_n+\delta} f_3 \, \Delta X_o^3 \, dt = 0, \quad (2.2.3)$$

and

$$\Delta\theta = U_2 \, \Delta X_o + \int_{t_o}^{t_n+\delta} g_1 \, \Delta X_o \, dt + \int_{t_o}^{t_n+\delta} g_2 \, \Delta X_o^2 \, dt + \int_{t_o}^{t_n+\delta} g_3 \, \Delta X_o^3 \, dt = 0. \qquad (2.2.4)$$

The solution in the example given in Chapter I was obtained by assuming a constant coefficient system of differential equations for intervals of 40 seconds and the integral evaluated by summing over 40-second intervals. This was done to simplify the problem to the point that the numerical results could be obtained by simple desk calculator operations. The numerical values employed in the following sections assumed a constant coefficient system over five-second intervals and evaluated the integral by summing over five-second intervals. This was done on an IBM 1620 digital computer and although differing only in the third significant figure in most cases, was considered necessary for the accuracy desired.

From observation of the results of several different calculus of variations solutions, it was expected the following form would be adequate to represent ΔX_o.

$$\Delta X_o = \begin{cases} \Delta C_o + \Delta C_2 \tau^2, & \tau \leq \tau_n + \dfrac{\delta}{100} \\ 0 \quad , & \tau > \tau_n + \dfrac{\delta}{100} \end{cases} \qquad (2.2.5)$$

Since the standard trajectory is expected to always be included among the likely trajectories, δ will always be less than or equal to zero. The value assigned to δ in this report is -8.18 seconds.

The requirements stated in equations (2.2.3) and (2.2.4) seem to imply that the simultaneous solution of two third-degree polynomials in two variables is needed. It should be remembered that if the linear terms are extremely large with respect to the higher order terms, the linear expression itself provides a good approximation to the solution. Such is the case in the problem at hand. The solution to the linear system can be used to approximate the second order term to effect an iteration on the solution. The linear system can be expressed as follows:

$$B \Delta C' = - U \, \Delta X_o, \qquad (2.2.6)$$

where

$$B = \int_{t_o}^{t_n+\delta} \begin{bmatrix} f_1 & f_1\tau^2 \\ g_1 & g_1\tau^2 \end{bmatrix} dt, \tag{2.2.7}$$

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = TU(t_n, t_o), \tag{2.2.8}$$

and

$$\Delta C' = \begin{bmatrix} \Delta C'_o \\ \Delta C'_2 \end{bmatrix}. \tag{2.2.9}$$

$\Delta C'_o$ and $\Delta C'_2$, the solution to the linear system, are determined as follows:

$$\Delta C' = - B^{-1} U \Delta X_o. \tag{2.2.10}$$

The second iteration gives

$$\Delta C'' = \begin{bmatrix} \Delta C''_o \\ \Delta C''_2 \end{bmatrix}$$

from the following equation:

$$B\Delta C'' + U \Delta X_o + B_2 \Delta C'^2 = 0 \tag{2.2.11}$$

where

$$\Delta C'^{2} = \begin{bmatrix} \Delta C_o'^{2} \\ 2\Delta C_o' \, \Delta C_2' \\ \Delta C_2'^{2} \end{bmatrix}, \tag{2.2.12}$$

and

$$B_2 = \int_{t_o}^{t_n+\delta} \begin{bmatrix} f_2 & f_2\tau^2 & f_2\tau^4 \\ g_2 & g_2\tau^2 & g_2\tau^4 \end{bmatrix} dt. \tag{2.2.13}$$

Then,

$$\Delta C'' = - B^{-1} \, U \, \Delta X_o - B^{-1} \, B_2 \, \Delta C'^{2}. \tag{2.2.14}$$

The third iteration including the third ordered term gives

$$\Delta C = \begin{bmatrix} \Delta C_o \\ \Delta C_2 \end{bmatrix},$$

the solution to the following equation:

$$B\Delta C + U \, \Delta X_o + B_2 \, \Delta C''^{2} + B_3 \, \Delta C''^{3} = 0, \tag{2.2.15}$$

where $\Delta C''^{2}$ is obtained from equation (2.2.12) by replacing $\Delta C_o'$ and $\Delta C_2'$ by $\Delta C_o''$ and $\Delta C_2''$, respectively,

$$\Delta C''^3 = \begin{bmatrix} \Delta C_o''^3 \\ 3\Delta C_o''^2 \, \Delta C_2'' \\ 3\Delta C_o'' \, \Delta C_2''^2 \\ \Delta C_2''^3 \end{bmatrix} \tag{2.2.16}$$

and

$$B_3 = \int_{t_o}^{t_n+\delta} \begin{bmatrix} f_3 & f_3\tau^2 & f_3\tau^4 & f_3\tau^6 \\ g_3 & g_3\tau^2 & g_3\tau^4 & g_3\tau^6 \end{bmatrix} dt. \tag{2.2.17}$$

The solution for ΔC is

$$\Delta C = \begin{bmatrix} \Delta C_o \\ \Delta C_2 \end{bmatrix} = - B^{-1} U \Delta X_o - B^{-1} B_2 \Delta C''^2 - B^{-1} B_3 \Delta C''^3. \tag{2.2.18}$$

The integrals and other numerical elements necessary to determine the matrices required of the above equations are listed in Appendix III. They have been obtained numerically under the assumption of a constant coefficient system over five-second intervals, and the integrals evaluated by summing over five-second intervals. These results yield the following matrices:

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} .349481 & 1.259049 & .170980 & .491843 \\ -.008175 & -.007129 & -.005744 & -.008092 \end{bmatrix}. \tag{2.2.19}$$

$$B = \begin{bmatrix} -14.523714 & -79.737064 \\ .570968 & 6.044026 \end{bmatrix}. \tag{2.2.20}$$

$$B_2 = \begin{bmatrix} -.092285 & -.328751 & -2.813252 \\ .003168 & .016477 & .171907 \end{bmatrix}. \tag{2.2.21}$$

$$B_3 = \begin{bmatrix} .000737 & .004048 & .043109 & .572059 \\ -.000029 & -.000307 & -.004554 & -.075637 \end{bmatrix}. \tag{2.2.22}$$

Inverting the matrix in equation (2.2.20) gives

$$- B^{-1} = \begin{bmatrix} .143038 & 1.887071 \\ -.013512 & -.343720 \end{bmatrix}. \tag{2.2.23}$$

The result, together with equations (2.2.19), (2.2.21) and (2.2.23), gives

$$- B^{-1}U = \begin{bmatrix} .034562 & .166639 & .013617 & .055082 \\ -.001912 & -.014562 & -.000336 & -.003864 \end{bmatrix}, \tag{2.2.24}$$

$$- B^{-1} B_2 = \begin{bmatrix} -.007222 & -.015931 & -.078001 \\ .000158 & -.001221 & -.021075 \end{bmatrix}, \tag{2.2.25}$$

and

$$- B^{-1} B_3 = \begin{bmatrix} .000051 & 0 & -.002427 & -.060906 \\ 0 & .000051 & .000983 & .018268 \end{bmatrix}. \tag{2.2.26}$$

Choosing the following initial condition deviations, an example of the determination of ΔC will be followed.

$$\Delta X_o = \begin{bmatrix} 17.2860 \\ 5.5463 \\ -4.6494 \\ -58.34471 \end{bmatrix} \qquad (2.2.27)$$

This, together with equations (2.2.10) and (2.2.24), gives

$$\Delta C' = \begin{bmatrix} -1.7554 \\ .11319 \end{bmatrix}. \qquad (2.2.28)$$

The term $\Delta C'^2$, defined in equation (2.2.12), is

$$\Delta C'^2 = \begin{bmatrix} 3.0814 \\ -.39739 \\ .012812 \end{bmatrix}.$$

This, with equations (2.2.14) and (2.2.25), gives

$$\Delta C'' = \begin{bmatrix} -1.7554 \\ .11319 \end{bmatrix} + \begin{bmatrix} -.01692 \\ .000702 \end{bmatrix} = \begin{bmatrix} -1.7723 \\ .11389 \end{bmatrix}.$$

This result can be used to determine $\Delta C''^2$ and $\Delta C''^3$ as defined by equations (2.2.12) and (2.2.16), respectively, to give

$$\Delta C''^2 = \begin{bmatrix} 3.1410 \\ -.40369 \\ .012971 \end{bmatrix} \quad \text{and} \quad \Delta C''^3 = \begin{bmatrix} -5.5668 \\ 1.07320 \\ -.06897 \\ .001477 \end{bmatrix}.$$

These results are used in equation (2.2.18), together with the matrices defined by equations (2.2.25) and (2.2.26), to give the following solution ΔC.

$$\Delta C = \begin{bmatrix} -1.7554 \\ .11319 \end{bmatrix} + \begin{bmatrix} -.0173 \\ .00007 \end{bmatrix} + \begin{bmatrix} -.0002 \\ .00001 \end{bmatrix}$$

$$\Delta C = \begin{bmatrix} \Delta C_o \\ \Delta C_2 \end{bmatrix} = \begin{bmatrix} -1.7727 \\ .11327 \end{bmatrix}. \tag{2.2.29}$$

Section 3. Second Stage Perturbations

Having obtained a function, ΔX_o, which should meet the required end conditions for a standard second stage, it remains only to determine δX such that the effect of $\Delta f/m$ on the end conditions is small. This requires further investigation of the expressions for Δr and $\Delta\theta$ which are obtained from equations (1.5.14) and (1.5.15). A restatement of these equations with the omission of the second order terms which have already been neglected or accounted for yields the following expressions.

$$\Delta r = U_1 \, \Delta X_o + \int_{t_o}^{t_n} F \, \Delta F \, dt, \tag{2.3.1}$$

and

$$\Delta\theta = U_2 \, \Delta X_o + \int_{t_o}^{t_n} G \, \Delta F \, dt. \tag{2.3.2}$$

Substitution of U_2 for U_1 and G for F transforms the expression for Δr into the expression for $\Delta\theta$. With this in mind, only the expression for Δr will be considered, and the corresponding result for $\Delta\theta$ can be obtained by the substitution just described. From equation (2.2.2), $\Delta X = \Delta X_o + \delta X$. Substituting this expression into ΔF in equation (2.3.1), and recalling that ΔF is defined as

$$\Delta F = \begin{bmatrix} \left(1 + \frac{\Delta f/m}{f/m}\right) \Delta X \\ \left(1 + \frac{\Delta f/m}{f/m}\right) \Delta X^2 \\ \left(1 + \frac{\Delta f/m}{f/m}\right) \Delta X^3 \\ \Delta f/m \end{bmatrix}, \tag{2.3.3}$$

gives, after expanding and neglecting all previously unaccounted terms higher than second order, the following expression.

$$\Delta r = U_1 \Delta X_o + \int_{t_o}^{t_n} f_1 \Delta X_o \, dt + \int_{t_o}^{t_n} f_2 \Delta X_o^2 \, dt + \int_{t_o}^{t_n} f_3 \Delta X_o^3 \, dt$$

$$+ \int_{t_o}^{t_n} \left[(f_1 + 2f_2 \Delta X_o) \, \delta X + \left(f_4 + \frac{f_1}{f/m} \Delta X_o\right) \frac{\Delta f}{m} + f_2 \, \delta X^2 \right.$$

$$\left. + \frac{f_1}{f/m} \frac{\Delta f}{m} \delta X \right] dt.$$

It will be recalled that the coefficients for ΔX_o were determined to satisfy equation (2.2.3). This constraint required the following relationship to exist.

$$U_1 \Delta X_o + \int_{t_o}^{t_n} f_1 \Delta X_o \, dt + \int_{t_o}^{t_n} f_2 \Delta X_o^2 \, dt + \int_{t_o}^{t_n} f_3 \Delta X_o^3 \, dt = 0.$$

Thus, the entire contribution to Δr stems from the last integral of the expression, namely,

$$\Delta r = \int_{t_o}^{t_n} \left[(f_1 + 2f_2 \Delta X_o) \, \delta X + \left(f_4 + \frac{f_1}{f/m} \Delta X_o \right) \frac{\Delta f}{m} + f_2 \, \delta X^2 \right.$$

$$\left. + \frac{f_1}{f/m} \frac{\Delta f}{m} \, \delta X \right] dt. \tag{2.3.4}$$

The evaluation of the above integral requires the knowledge of $\Delta f/m$ along the entire trajectory. This information is not available until after the cutoff conditions have been reached. However, in order that the integral be zero, it is sufficient that the integrand be zero everywhere. In determining an adequate guidance function, this restriction can be relaxed. It is sufficient that the integrand be sufficiently near zero that the integral is negligible. In order to accomplish this, δX must be a function of $\Delta f/m$. The following form is chosen for δX.

$$\delta X = W_1 \frac{\Delta f}{m} + W_2 \left(\frac{\Delta f}{m} \right)^2. \tag{2.3.5}$$

Substituting this expression into equation (2.3.4) and neglecting terms higher than second order gives

$$\Delta r = \int_{t_o}^{t_n} \left[\delta r_1(t) \frac{\Delta f}{m} + \delta r_2(t) \left(\frac{\Delta f}{m}\right)^2 \right] dt, \tag{2.3.6}$$

where

$$\delta r_1(t) = f_1 W_1 + f_4 + \frac{f_1}{f/m} \Delta X_o + 2 f_2 \Delta X_o W_1 \tag{2.3.7}$$

and

$$\delta r_2(t) = f_1 W_2 + f_2 W_1^2 + \frac{f_1}{f/m} W_1. \tag{2.3.8}$$

Similarly, for $\Delta\theta$, the following expression is obtained:

$$\Delta\theta = \int_{t_o}^{t_n} \left[\delta\theta_1(t) \frac{\Delta f}{m} + \delta\theta_2(t) \left(\frac{\Delta f}{m}\right)^2 \right] dt, \tag{2.3.9}$$

where

$$\delta\theta_1(t) = g_1 W_1 + g_4 + \frac{g_1}{f/m} \Delta X_o + 2 g_2 \Delta X_o W_1 \tag{2.3.10}$$

and

$$\delta\theta_2(t) = g_1 W_2 + g_2 W_1^2 + \frac{g_1}{f/m} W_1. \tag{2.3.11}$$

The term W_1 will be considered a quadratic function of τ and the coefficients determined to minimize the sum of squares of equations (2.3.7) and (2.3.10) simultaneously for a number of time points along the trajectory. W_2 will also be considered a quadratic function of τ and the coefficients determined similarly using equations (2.3.8) and (2.3.11). For convenience, the following definitions will be used where the indicated summation is intended to include a convenient number of time points evenly distributed over the entire trajectory.

$$C(\delta r) = \sum f_1{}^2 \begin{bmatrix} 1 & \tau & \tau^2 \\ \tau & \tau^2 & \tau^3 \\ \tau^2 & \tau^3 & \tau^4 \end{bmatrix}. \tag{2.3.12}$$

$$C(\delta\theta) = \sum g_1{}^2 \begin{bmatrix} 1 & \tau & \tau^2 \\ \tau & \tau^2 & \tau^3 \\ \tau^2 & \tau^3 & \tau^4 \end{bmatrix}. \tag{2.3.13}$$

$$C_1(\delta r) = - \sum f_1 f_4 \begin{bmatrix} 1 \\ \tau \\ \tau^2 \end{bmatrix}. \tag{2.3.14}$$

$$C_1(\delta\theta) = - \sum g_1 g_4 \begin{bmatrix} 1 \\ \tau \\ \tau^2 \end{bmatrix}. \tag{2.3.15}$$

$$C_2(\delta r) = - \sum \frac{f_1{}^2}{f/m} \begin{bmatrix} 1 & \tau^2 \\ \tau & \tau^3 \\ \tau^2 & \tau^4 \end{bmatrix}. \tag{2.3.16}$$

$$C_2(\delta\theta) = -\sum \frac{g_1^2}{f/m} \begin{bmatrix} 1 & \tau^2 \\ \tau & \tau^3 \\ \tau^2 & \tau^4 \end{bmatrix}. \tag{2.3.17}$$

$$C_3(\delta r) = -\sum f_1 f_2 \begin{bmatrix} 1 & \tau^2 \\ \tau & \tau^3 \\ \tau^2 & \tau^4 \\ \tau^3 & \tau^5 \\ \tau^4 & \tau^6 \end{bmatrix}. \tag{2.3.18}$$

$$C_3(\delta\theta) = -\sum g_1 g_2 \begin{bmatrix} 1 & \tau^2 \\ \tau & \tau^3 \\ \tau^2 & \tau^4 \\ \tau^3 & \tau^5 \\ \tau^4 & \tau^6 \end{bmatrix}. \tag{2.3.19}$$

$$b^*(b) = \begin{bmatrix} b_o & b_1 & b_2 & 0 & 0 \\ 0 & b_o & b_1 & b_2 & 0 \\ 0 & 0 & b_o & b_1 & b_2 \end{bmatrix}. \tag{2.3.20}$$

$$b = \begin{bmatrix} b_o \\ b_1 \\ b_2 \end{bmatrix}. \qquad (2.3.21)$$

$$b' = \begin{bmatrix} b_o' \\ b_1' \\ b_2' \end{bmatrix}. \qquad (2.3.22)$$

$$C_4(\delta r) = - \sum \frac{f_1{}^2}{f/m} \begin{bmatrix} 1 & \tau & \tau^2 \\ \tau & \tau^2 & \tau^3 \\ \tau^2 & \tau^3 & \tau^4 \end{bmatrix}. \qquad (2.3.23)$$

$$C_4(\delta\theta) = - \sum \frac{g_1{}^2}{f/m} \begin{bmatrix} 1 & \tau & \tau^2 \\ \tau & \tau^2 & \tau^3 \\ \tau^2 & \tau^3 & \tau^4 \end{bmatrix}. \qquad (2.3.24)$$

$$C_5(\delta r) = - \sum f_1 f_2 \begin{bmatrix} 1 & \tau & \tau^2 \\ \tau & \tau^2 & \tau^3 \\ \tau^2 & \tau^3 & \tau^4 \\ \tau^3 & \tau^4 & \tau^5 \\ \tau^4 & \tau^5 & \tau^6 \end{bmatrix}. \qquad (2.3.25)$$

$$C_5(\delta\theta) = - \sum g_1 g_2 \begin{bmatrix} 1 & \tau & \tau^2 \\ \tau & \tau^2 & \tau^3 \\ \tau^2 & \tau^3 & \tau^4 \\ \tau^3 & \tau^4 & \tau^5 \\ \tau^4 & \tau^5 & \tau^6 \end{bmatrix}. \tag{2.3.26}$$

$$a = \begin{bmatrix} a_o \\ a_1 \\ a_2 \end{bmatrix}. \tag{2.3.27}$$

With these definitions established, W_1 and W_2 will be defined as follows and a least squares fit obtained.

$$W_1 = b_o + b_1\tau + b_2\tau^2. \tag{2.3.28}$$

$$W_2 = a_o + a_1\tau + a_2\tau^2. \tag{2.3.29}$$

Minimizing the sum of squares of equation (2.3.7) over the specified number of time points yields the following constraint on the coefficients b_o, b_1 and b_2:

$$C(\delta r)b = C_1(\delta r) + \left[C_2(\delta r) + 2b^*(b)\ C_3(\delta r) \right] \Delta C.$$

The criterion for minimizing the sum of squares of $\delta\theta(t)$ defined in equation (2.3.10), however, is

$$C(\delta\theta)b = C_1(\delta\theta) + \left[C_2(\delta\theta) + 2b^*(b)\ C_3(\delta\theta) \right] \Delta C.$$

In general, the least squares criterion for $\delta r_1(t)$ differs from that required for $\delta\theta_1(t)$. A single solution requires one criterion which is a function of the two separate criteria just derived. The single criterion used in this report will be a weighted sum of the two separate criteria just derived. To this end, the following matrices will be defined.

$$D = C(\delta r) + \alpha^2 C(\delta\theta). \tag{2.3.30}$$

$$D_1 = C_1(\delta r) + \alpha^2\ C_1(\delta\theta). \tag{2.3.31}$$

$$D_2 = C_2(\delta r) + \alpha^2\ C_2(\delta\theta). \tag{2.3.32}$$

$$D_3 = C_3(\delta r) + \alpha^2\ C_3(\delta\theta). \tag{2.3.33}$$

$$D_4 = C_4(\delta r) + \alpha^2\ C_4(\delta\theta). \tag{2.3.34}$$

$$D_5 = C_5(\delta r) + \alpha^2\ C_5(\delta\theta). \tag{2.3.35}$$

The symbol α is a weight used to represent the relative importance of angular error $\Delta\theta$ to radial error, Δr. Combining the two separate criteria into one criterion in the manner described, yields the following constraint on b.

$$Db = D_1 + \left[D_2 + 2b^*(b)\ D_3\right] \Delta C.$$

The term in brackets represents a second order term and will be handled by obtaining an approximate solution b' by ignoring higher order terms and then determining b by using b' to approximate b in the expression $b^*(b)$. With this in mind, the solution to the following equations defines b'.

$$Db' = D_1.$$

Then,

$$b' = D^{-1} D_1. \tag{2.3.36}$$

With this first order approximation available, an iteration will be used to determine b from the equations

$$Db = D_1 + \left[D_2 + 2b^*(b') D_3 \right] \Delta C,$$

and

$$b = D^{-1} D_1 + D^{-1} \left[D_2 + 2b^*(b') D_3 \right] \Delta C,$$

or

$$b = b' + D^{-1} \left[D_2 + 2b^*(b') D_3 \right] \Delta C. \tag{2.3.37}$$

Having determined the coefficients for W_1, it remains only to determine the coefficients for W_2. Minimizing the sum of squares of $\delta r_2(t)$ gives

$$C(\delta r)a = b^*(b) C_5(\delta r)b + C_4(\delta r)b.$$

Similarly for $\delta\theta_2(t)$, the following result is obtained:

$$C(\delta\theta)a = b^*(b) D_5 b + D_4 b.$$

Combining these two expressions as before gives

$$Da = b^*(b) D_5 b + D_4 b.$$

The contribution of $W_2(\Delta f/m)^2$ is small compared to that of $W_1 \frac{\Delta f}{m}$. In addition, b' is a good first order approximation to b. In view of these facts and to avoid unnecessarily complicated expressions introduced by relatively insignificant terms, b' will be used in the above system thus making "a" independent of ΔC. Under these conditions, the following equation will be used to determine "a".

$$a = D^{-1} \left[D_4 b' + b^*(b') \, D_5 b' \right]. \tag{2.3.38}$$

Equations (2.3.36), (2.3.37) and (2.3.38) provide a means of determining the coefficients for W_1 and W_2 so as to minimize, as near as possible within the restrictions imposed, the sum of squares of $\delta r_1(t)$, $\delta r_2(t)$, $\delta\theta_1(t)$, and $\delta\theta_2(t)$ and hence their effect on Δr and $\Delta\theta$. An example will determine whether or not this has been accomplished sufficiently well for the result to be used as a guidance function.

The values for the f's and the g's were not available. The computer program employed yielded the f's and g's evaluated at five-second intervals and multiplied by $\Delta t = 5$ sec. Since every value used was multiplied by the same constant, the least squares solution is mathematically identical to that which would have been obtained if none of the values had been multiplied by Δt. Since the following summations were done on the desk calculator, only every fourth point was used and each summation involved 24 time points spaced 20 seconds apart beginning with t = 150 seconds. The following numerical values were obtained.

$$\left.\begin{aligned}
25 \sum f_1^2 &= .662385 & 25 \sum g_1^2 &= .118972 \times 10^{-2} \\
25 \sum f_1^2\tau &= 1.112071 & 25 \sum g_1^2\tau &= .427513 \times 10^{-2} \\
25 \sum f_1^2\tau^2 &= 2.782090 & 25 \sum g_1^2\tau^2 &= 1.664566 \times 10^{-2} \\
25 \sum f_1^2\tau^3 &= 8.040666 & 25 \sum g_1^2\tau^3 &= 6.737941 \times 10^{-2} \\
25 \sum f_1^2\tau^4 &= 25.186436 & 25 \sum g_1^2\tau^4 &= 27.921969 \times 10^{-2}
\end{aligned}\right\} \tag{2.3.39}$$

$$
\left.\begin{aligned}
25 \sum f_1 f_4 &= -3.417960 & \qquad 25 \sum g_1 g_4 &= -\ .26176 \times 10^{-2} \\
25 \sum f_1 f_4 \tau &= -4.104875 & \qquad 25 \sum g_1 g_4 \tau &= -\ .55670 \times 10^{-2} \\
25 \sum f_1 f_4 \tau^2 &= -7.961945 & \qquad 25 \sum g_1 g_4 \tau^2 &= -1.573233 \times 10^{-2}
\end{aligned}\right\} \quad (2.3.40)
$$

$$
\left.\begin{aligned}
25 \sum \frac{f_1^{\,2}}{f/m} &= .071016 & \qquad 25 \sum \frac{g_1^{\,2}}{f/m} &= .008462 \times 10^{-2} \\
25 \sum \frac{f_1^{\,2}\tau}{f/m} &= .107982 & \qquad 25 \sum \frac{g_1^{\,2}\tau}{f/m} &= .027354 \times 10^{-2} \\
25 \sum \frac{f_1^{\,2}\tau^2}{f/m} &= .244417 & \qquad 25 \sum \frac{g_1^{\,2}\tau^2}{f/m} &= .100396 \times 10^{-2} \\
25 \sum \frac{f_1^{\,2}\tau^3}{f/m} &= .656697 & \qquad 25 \sum \frac{g_1^{\,2}\tau^3}{f/m} &= .390356 \times 10^{-2} \\
25 \sum \frac{f_1^{\,2}\tau^4}{f/m} &= 1.942588 & \qquad 25 \sum \frac{g_1^{\,2}\tau^4}{f/m} &= 1.569404 \times 10^{-2}
\end{aligned}\right\} \quad (2.3.41)
$$

$$
\begin{aligned}
25\sum f_1f_2 &= .42248 \times 10^{-2} & 25\sum g_1g_2 &= .040996 \times 10^{-4}\\
25\sum f_1f_2\tau &= .59287 \times 10^{-2} & 25\sum g_1g_2\tau &= .102289 \times 10^{-4}\\
25\sum f_1f_2\tau^2 &= 1.27461 \times 10^{-2} & 25\sum g_1g_2\tau^2 &= .316953 \times 10^{-4}\\
25\sum f_1f_2\tau^3 &= 3.29295 \times 10^{-2} & 25\sum g_1g_2\tau^3 &= 1.079346 \times 10^{-4}\\
25\sum f_1f_2\tau^4 &= 9.43480 \times 10^{-2} & 25\sum g_1g_2\tau^4 &= 3.876986 \times 10^{-4}\\
25\sum f_1f_2\tau^5 &= 29.30378 \times 10^{-2} & 25\sum g_1g_2\tau^5 &= 14.529144 \times 10^{-4}\\
25\sum f_1f_2\tau^6 &= 95.57426 \times 10^{-2} & 25\sum g_1g_2\tau^6 &= 55.835969 \times 10^{-4}
\end{aligned}
\tag{2.3.42}
$$

To obtain one set of equations, instead of one for Δr and another for $\Delta\theta$, it is necessary to choose a suitable value for α. The value chosen for this report is $\alpha = 5$. This is equivalent to saying that an error in angle $\Delta\theta$ of 0.1 degree is five times as bad as an error in r, Δr, of 100 m. Or, equivalently, a value of Δr of 100m is weighted the same as $\Delta\theta = .02$ degrees. With $\alpha = 5$, $\alpha^2 = 25$. This, together with the sums evaluated in equations (2.3.39) through (2.3.42) gives the following matrices:

$$
25D = \begin{bmatrix} .652128 & 1.218949 & 3.198232 \\ 1.218949 & 3.198232 & 9.725151 \\ 3.198232 & 9.725151 & 8.355253 \end{bmatrix} \tag{2.3.43}
$$

$$25D_1 = \begin{bmatrix} 3.483400 \\ 4.244050 \\ 8.355253 \end{bmatrix} \tag{2.3.44}$$

$$25D_2 = \begin{bmatrix} -.073132 & -.269516 \\ -.114820 & -.754286 \\ -.269516 & -2.334939 \end{bmatrix} \tag{2.3.45}$$

$$25D_3 = \begin{bmatrix} -\ .432729 & -\ 1.353848 \\ -\ .618442 & -\ 3.562787 \\ -\ 1.353848 & -\ 10.404047 \\ -\ 3.562787 & -\ 32.936066 \\ -10.404047 & -109.533252 \end{bmatrix} \times 10^{-2} \tag{2.3.46}$$

$$25D_4 = \begin{bmatrix} -.073132 & -.114820 & -.269516 \\ -.114820 & -.269516 & -.754286 \\ -.269516 & -.754286 & -2.334939 \end{bmatrix} \tag{2.3.47}$$

$$25D_5 = \begin{bmatrix} -.432729 & -.618442 & -1.353848 \\ -.618442 & -1.353848 & -3.562787 \\ -1.353848 & -3.562787 & -10.404047 \\ -3.562787 & -10.404047 & -32.936066 \\ -10.404047 & -32.936066 & -109.533252 \end{bmatrix} \times 10^{-2}. \tag{2.3.48}$$

From equation (2.3.43), the following inverse is obtained.

$$\begin{bmatrix} 25D_1 \end{bmatrix}^{-1} = \frac{1}{25} D^{-1} = \begin{bmatrix} 11.368704 & -11.105430 & 2.227201 \\ -11.105430 & 14.724394 & -3.347514 \\ 2.227201 & -3.347514 & .821713 \end{bmatrix}. \quad (2.3.49)$$

Equations (2.3.36), (2.3.44), and (2.3.49) give the following solution for b'.

$$b' = \begin{bmatrix} 11.0786 \\ -4.1629 \\ .4168 \end{bmatrix}. \quad (2.3.50)$$

Equations (2.3.45) and (2.3.49) yield

$$D^{-1} D_2 = \begin{bmatrix} -.15656 & .11224 \\ .02372 & -.29707 \\ .00002 & .00607 \end{bmatrix}. \quad (2.3.51)$$

Referring to equations (2.3.20), (2.3.46), (2.3.49) and (2.3.50), the following result is obtained.

$$D^{-1} b^*(b') D_3 = \begin{bmatrix} -.11688 & .05583 \\ .06228 & -.15880 \\ -.00861 & .03506 \end{bmatrix}. \quad (2.3.52)$$

The results obtained from equations (2.3.50), (2.3.51), and (2.3.52) can be combined in equation (2.3.37) to give the following values for the coefficients of W_1.

$$b = \begin{bmatrix} b_o \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 11.0786 \\ -4.1629 \\ .4168 \end{bmatrix} + \begin{bmatrix} -.39032 & .22390 \\ .14828 & -.61467 \\ -.01720 & .07619 \end{bmatrix} \triangle C. \qquad (2.3.53)$$

From equations (2.3.47), (2.3.49) and (2.3.50), we get

$$D^{-1} D_4 b = \begin{bmatrix} -1.7201 \\ .8397 \\ -.1083 \end{bmatrix}. \qquad (2.3.54)$$

Equations (2.3.20), (2.3.48), (2.3.49) and (2.3.50) give

$$D^{-1} b^*(b') D_4 b' = \begin{bmatrix} -1.1589 \\ .7580 \\ -.11970 \end{bmatrix}. \qquad (2.3.55)$$

Using the expression obtained in equations (2.3.54) and (2.3.55) to evaluate equation (2.3.38) gives the following coefficients for W_2.

$$a = \begin{bmatrix} a_o \\ a_1 \\ a_2 \end{bmatrix} = D^{-1} \left[D_4 b' + b^*(b') D_5 b' \right] = \begin{bmatrix} -2.8790 \\ 1.5977 \\ -.2280 \end{bmatrix}. \qquad (2.3.56)$$

The complete expression for χ can now be stated. Equations (2.1.13), (2.2.2), (2.2.5), (2.3.5), (2.3.28), (2.3.29), (2.3.53), and (2.3.56) yield the following function.

$$\left.\begin{aligned}
\chi &= (58.2802 + \Delta C_o) + 7.6268\tau + (.3574 + \Delta C_2)\tau^2 + (b_o + b_1\tau + b_2\tau^2)\,\frac{\Delta f}{m} \\
&\quad + (-2.8790 + 1.5977\tau - .2280\tau^2)\;(\Delta f/m)^2, \\
&\qquad 0 \leqq \tau \leqq 4.65685 \\
\chi &= 58.2802 + 7.6268\tau + .3574\tau^2, \qquad \tau > 4.65685
\end{aligned}\right\} \tag{2.3.57}$$

The terms ΔC_o and ΔC_2 are determined at second stage ignition from equations (2.2.10), (2.2.14), and (2.2.18) using the matrices evaluated in equations (2.2.24), (2.2.25) and (2.2.26) with the definitions of equations (2.2.9), (2.2.12) and (2.2.16). Also b_o, b_1 and b_2 are determined at second stage ignition and are evaluated after ΔC_o and ΔC_2 by equation (2.3.53). The symbol τ is defined in equation (2.1.1), and $\Delta f/m$ is defined as follows.

$$\frac{\Delta f}{m}(t) = \frac{f}{m}(t + \Delta t_o) - \frac{8.78065}{1.3751 - .20888\tau},$$

where $\Delta f/m$ is in m/sec^2 and χ is in degrees.

This guidance function has been derived for trajectories in the neighborhood of the standard trajectory employed. The example already used to illustrate the method of determining ΔX_o employed the following initial condition deviations stated in equation (2.2.27) where Δx_o, Δy_o are in km and $\Delta \dot{x}_o$ and $\Delta \dot{y}_o$ are in m/sec.

$$\Delta X_o = \begin{bmatrix} \Delta x_o \\ \Delta y_o \\ \Delta \dot{x}_o \\ \Delta \dot{y}_o \end{bmatrix} = \begin{bmatrix} 17.2860 \\ 5.5463 \\ -4.6494 \\ -58.34471 \end{bmatrix}.$$

These values yielded the following values for ΔC_o and ΔC_2 (equation 2.2.29).

$$\Delta C = \begin{bmatrix} \Delta C_o \\ \Delta C_2 \end{bmatrix} = \begin{bmatrix} -1.7727 \\ .11327 \end{bmatrix}.$$

These values of ΔC_o and ΔC_2 can be used in equation (2.3.53) to give the following values for b_o, b_1 and b_2.

$$b = \begin{bmatrix} b_o \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 11.7959 \\ -4.4954 \\ .4559 \end{bmatrix}.$$

The function which would be used to determine χ for the deviations in initial conditions used in this example would be

$$\left.\begin{aligned} \chi &= 56.5075 + 7.6268\tau + .47067\tau^2 + (11.7959 - 4.4954\tau + .4559\tau^2)\frac{\Delta f}{m} \\ &\quad + (-2.8790 + 1.5977\tau - .2280\tau^2)\,(\Delta f/m)^2 \\ &\qquad 0 \leqq \tau \leqq 4.65685 \\ \chi &= 58.2802 + 7.6268\tau + .3574\tau^2, \qquad \tau > 4.65685 \end{aligned}\right\} \tag{2.3.58}$$

where

$$\frac{\Delta f}{m} = \frac{f}{m}\,(t + \Delta t_o) - \frac{8.78065}{1.3751 - .20888\tau}.$$

Section 4. Implementation

The employment of the guidance function just described would involve the precalculation of the following quantities which have been numerically evaluated for the mission under consideration.

$$\left.\begin{aligned} \bar{C}_o &= 58.2802 \\ \bar{C}_1 &= 7.6268 \\ \bar{C}_2 &= .3574 \end{aligned}\right\} \tag{2.4.1}$$

$$-B^{-1}U = \begin{bmatrix} .034562 & .166639 & .013617 & .055082 \\ -.001912 & -.014562 & -.000336 & -.003864 \end{bmatrix} \tag{2.4.2}$$

$$-B^{-1}B_2 = \begin{bmatrix} -.007222 & -.015931 & -.078001 \\ .000158 & -.001221 & -.021075 \end{bmatrix} \tag{2.4.3}$$

$$-B^{-1}B_3 = \begin{bmatrix} .000051 & 0 & -.002427 & -.060906 \\ 0 & .000051 & .000983 & .018268 \end{bmatrix} \tag{2.4.4}$$

$$b' = D^{-1}D_1 = \begin{bmatrix} 11.0786 \\ -4.1629 \\ .4168 \end{bmatrix} \tag{2.4.5}$$

$$D^{-1}\left[D_4 b' + b^*(b')\, D_5 b'\right] = \begin{bmatrix} -2.8790 \\ 1.5977 \\ -.2280 \end{bmatrix}. \tag{2.4.6}$$

With these quantities precomputed, they can be used to determine the following quantities as soon as ΔX_o is determined. This is determined by measuring x, y, $\dot{x}$ and $\dot{y}$ at ignition time of the second stage. From these measurements the elements of ΔX_o are determined.

$$\Delta X_o = \begin{bmatrix} x_o \\ y_o \\ \dot{x}_o \\ \dot{y}_o \end{bmatrix} - \begin{bmatrix} 153.983 \\ 6435.878 \\ 2818.329 \\ 988.358 \end{bmatrix}, \tag{2.4.7}$$

where x_o and y_o are measured in kilometers and $\dot{x}_o$ and $\dot{y}_o$ are in m/sec. Then ΔC is computed in the following sequence of operations.

$$\Delta C' = \begin{bmatrix} .034562 & .166639 & .013617 & .055082 \\ -.001912 & -.014562 & -.000336 & -.003864 \end{bmatrix} \Delta X_o = \begin{bmatrix} \Delta C'_o \\ \Delta C'_2 \end{bmatrix} \tag{2.4.8}$$

$$\Delta C'' = \Delta C' + \begin{bmatrix} -.007222 & -.015931 & -.078001 \\ .000158 & -.001221 & -.021075 \end{bmatrix} \begin{bmatrix} \Delta C_o'^{2} \\ 2\Delta C'_o \, \Delta C'^{2} \\ \Delta C_2'^{2} \end{bmatrix} = \begin{bmatrix} \Delta C''_o \\ \Delta C''_2 \end{bmatrix} \tag{2.4.9}$$

and

$$\Delta C = \Delta C' + \begin{bmatrix} -.007222 & -.015931 & -.078001 \\ .000158 & -.001221 & -.021075 \end{bmatrix} \begin{bmatrix} \Delta C''^{2} \\ 2\Delta C''_o \; \Delta C''_2 \\ \Delta C''^{2}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} .000051 & 0 & -.002427 & -.060906 \\ 0 & .000051 & .000983 & .018268 \end{bmatrix} \begin{bmatrix} \Delta C''^{3}_o \\ 3\Delta C''^{2}_o \; \Delta C''_2 \\ 3\Delta C''_o \; \Delta C''^{2}_2 \\ \Delta C''^{3}_2 \end{bmatrix} = \begin{bmatrix} \Delta C_o \\ \Delta C_2 \end{bmatrix}. \tag{2.4.10}$$

Having obtained the values of ΔC_o and ΔC_2, the values of b_o, b_1 and b_2 can be determined.

$$b = \begin{bmatrix} b_o \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 11.0786 \\ -4.1629 \\ .4168 \end{bmatrix} + \begin{bmatrix} -.39032 & .22390 \\ .14828 & -.61467 \\ -.01720 & .07619 \end{bmatrix} \Delta C. \tag{2.4.11}$$

With b_o, b_1, b_2, ΔC_o and ΔC_2 determined from the deviations at second stage ignition, ΔX_o, the thrust angle X is determined hereafter by the following function.

$$\chi = (58.2802 + \Delta C_o) + 7.6268\tau + (.3574 + \Delta C_2)\ \tau^2 + b_o \frac{\Delta f}{m} + b_1\tau \frac{\Delta f}{m}$$

$$+ b_2\tau^2 \frac{\Delta f}{m} - 2.8790\ (\Delta f/m)^2 + 1.5977\tau\ (\Delta f/m)^2 - .2280\tau^2\ (\Delta f/m)^2,$$

$$0 \leqq \tau \leqq 4.65685$$

$$\chi = 58.2802 + 7.6268\tau + .3574\tau^2, \qquad \tau > 4.65685$$

$$(2.4.12)$$

The term $\Delta f/m$ in the above expression is determined by the relationship

$$\frac{\Delta f}{m}(\tau) = \frac{f}{m}(\tau) - \frac{8.78065}{1.3751 - .20888\tau}, \tag{2.4.13}$$

where

$$\tau = \frac{t' - t_i}{100}. \tag{2.4.14}$$

The term t_i is second stage ignition on the nonstandard trajectory, t' is time measured on the nonstandard trajectory; t' and t_i are in seconds, f/m is in m/sec^2, and χ is in degrees.

CHAPTER III

RESULTS AND CONCLUSIONS

Section 1. Adjustment to a Different Standard Trajectory

In the event that performance of trajectories in the neighborhood of a different standard trajectory is to be investigated, it might be best to duplicate the computations which have been described in Chapters I and II for the new standard trajectory if it differs greatly from the standard which was originally assumed. However, if it differs only slightly, there will be small deviations in the desired end conditions if the function derived for $\bar{X}$ is used to define X on this trajectory. These deviations will be represented by the following vector.

$$\Delta\bar{R} = \begin{bmatrix} \Delta\bar{r} \\ \Delta\bar{\theta} \end{bmatrix}. \tag{3.1.1}$$

To meet the desired end conditions, equation (2.2.15) will be modified to the following:

$$\Delta R = B\Delta C + U\,\Delta X_o + B_2\,\Delta C''^2 + B_3\,\Delta C''^3 + \Delta\bar{R} = 0. \tag{3.1.2}$$

If the definition of $\Delta C'$ given by equation (2.2.6) is modified to the definition

$$\Delta C' = -\,B^{-1}\,U\,\Delta X_o - B^{-1}\,\Delta\bar{R}, \tag{3.1.3}$$

then all the subsequent computations can be performed exactly as before.

To demonstrate the accuracy of the function which has been derived, an available computer program was employed which was designed for this purpose. The differential equations of this program described the motion in three dimensions and used a gravity field which differed from that assumed in this report. The program was extremely intricate and difficult to alter so that it would duplicate the simple two-dimensional differential equations of motion which were assumed in deriving the guidance function presented in Chapter II. It was considered simpler to assume that this program represented a slightly different standard trajectory and alter the coefficients for X accordingly. The following values for Δr and $\Delta\theta$ were obtained when $\bar{X}$, defined by the coefficients in equation (2.1.13), was used.

$$\begin{aligned} \Delta r &= -1.8084 \text{ km} \\ \Delta\theta &= .05955^\circ \end{aligned} \tag{3.1.4}$$

and

$$\Delta\bar{R} = \begin{bmatrix} -1.8084 \\ .05955 \end{bmatrix}. \tag{3.1.5}$$

Using $-B^{-1}$ from equation (2.2.23), the following values are obtained.

$$-B^{-1}\,\Delta\bar{R} = \begin{bmatrix} -.1464 \\ .0040 \end{bmatrix}.$$

Employing this result in equation (3.1.3), together with $-B^{-1}U$ defined in equation (2.2.24), gives the following relationship for $\Delta C'$.

$$\Delta C' = \begin{bmatrix} .034562 & .166639 & .013617 & .055082 \\ -.001912 & -.014562 & -.000336 & -.003864 \end{bmatrix} \Delta X_o + \begin{bmatrix} -.1464 \\ .0040 \end{bmatrix}. \tag{3.1.6}$$

Section 2. Coefficient Computations

Having determined the guidance function and adjusted it to the standard trajectory of an available computer program, it was then verified by using this guidance function for a number of different sets of initial conditions. These initial conditions resulted from various nonstandard first stages and were combined with several different second stage perturbations. Table 3.1 lists these deviations in initial conditions along with the first stage deviation which caused them. Equation (3.1.6) was used to determine $\Delta C'$ instead of equation (2.4.8); ΔC was then determined from equations (2.4.9) and (2.4.10). These results along with b_0, b_1 and b_2 computed from equation (2.4.11) are listed in Table 3.2.

TABLE 3.1

DEVIATIONS IN INITIAL CONDITIONS

Example No.	First Stage Deviations	Δx (km)	Δy (km)	$\Delta \dot{x}$ (m/sec)	$\Delta \dot{y}$ (m/sec)
1	None	0	0	0	0
2	+5000 lb	-.93152	-1.1623	-30.2692	-22.78749
3	-5000 lb	.94444	1.1824	30.8335	23.36393
4	Engine #2 out at 100 sec.	17.28599	5.5463	-4.6494	-58.34471
5	Tail Wind	.10183	1.8180	2.7670	25.69483
6	Head Wind	-.34235	- .4055	-5.3173	-5.40216
7	Left Cross Wind	-.10434	- .1788	-1.5250	-2.33100
8	Right Cross Wind	.09092	.2155	1.4240	2.87015
9	-1% $\dot{W}$	4.76116	1.1588	48.5244	-4.54684
10	+1% $\dot{W}$	-3.31692	-.6959	-39.7298	3.74070
11	+1% F	1.19032	1.9245	29.8569	29.73483
12	-1% F	-1.22157	-1.9050	-30.7597	-29.22741

TABLE 3.2

COEFFICIENTS DETERMINED FROM INITIAL CONDITIONS

Example No.	ΔC_0	ΔC_2	b_0	b_1	b_2
1	- .1464	.00400	11.1381	-4.1877	.4197
2	-2.0636	.12186	11.9115	-4.5439	.4616
3	1.7735	-.11491	10.3634	-3.8305	.3777
4	-1.9224	.11798	11.8556	-4.5206	.4589
5	1.6001	-.12227	10.4293	-3.8516	.3801
6	- .5977	.03329	11.3205	-4.2725	.4297
7	- .3295	.01634	11.2122	-4.2224	.4238
8	.0702	-.01088	11.0504	-4.1465	.4148
9	.6194	-.02062	10.8343	-4.0593	.4047
10	- .7151	.01946	11.3631	-4.2814	.4306
11	2.2337	-.15007	10.1762	-3.7408	.3671
12	-2.5718	.15881	12.1178	-4.6418	.4731

Section 3. Results of Application

Tables 3.3 and 3.4, respectively, show the deviations in Δr and $\Delta\theta$ obtained from these examples. Example No. 4 which resulted from an engine out at 100 seconds in the first stage had a rather large out-of-plane position and velocity deviation which was not assumed to exist in the equations for which the function was derived. In spite of this, the results reflect the type of accuracy that can be expected when mission accomplishment is mathematically imposed as a criterion for determining the coefficients of the guidance function.

Table 3.5 shows the additional fuel required beyond that required for the calculus of variations solution to the same set of conditions. No calculus of variations solution was available for the case in which F and $\dot{W}$ were each -5% throughout the second stage so that a fuel comparison could not be made for this example.

TABLE 3.3

Δr (Meters)

Perturbations

1st Stage \ 2nd Stage	None	-1% f, $\dot{W}$	+1% f, $\dot{W}$	-5% f, $\dot{W}$
None	-1	53	-40	23
+5000 lb	35	95	-27	-157
-5000 lb	25	55	28	151
Engine out at 100 sec	179	175	179	-326
Tail Wind	*	73	7	201
Head Wind	-2	60	-44	-12
Left Cross Wind	-7	54	-43	7
Right Cross Wind	*	46	-42	32
-1% $\dot{W}$	*	61	50	*
+1% $\dot{W}$	*	144	-9	*
+1% f	*	57	43	*
-1% f	*	92	-21	*

*Due to underestimation of required computer time, these cases were not completed. It was felt that their inclusion would not significantly change the results.

TABLE 3.4

$\Delta\theta$ (Degrees)

Perturbations: 1st Stage \ 2nd Stage	None	-1% f, $\dot{W}$	+1% f, $\dot{W}$	-5% f, $\dot{W}$
None	-.001	.012	-.027	-.004
+5000 lb	.004	.019	-.026	.020
-5000 lb	-.003	.006	-.023	-.023
Engine 2 out at 100 sec	-.001	.013	-.029	.008
Tail Wind	-.006	.006	-.030	-.015
Head Wind	*	.013	-.026	.000
Left Cross Wind	-.001	.012	-.027	-.002
Right Cross Wind	-.002	.011	-.027	-.005
-1% $\dot{W}$	*	.009	-.021	*
+1% $\dot{W}$	*	.014	-.032	*
+1% f	*	.005	-.023	*
-1% f	*	.021	-.025	*

TABLE 3.5

Fuel Loss (lbs)

Perturbations

1st Stage \ 2nd Stage	None	-1% f, $\dot{W}$	+1% f, $\dot{W}$
None	5	5	5
+5000 lb	7	7	6
-5000 lb	4	4	4
Engine 2 out at 100 sec	2	1	2
Tail Wind	5	4	5
Head Wind	*	5	5
Left Cross Wind	5	5	5
Right Cross Wind	5	5	5
-1% $\dot{W}$	*	5	5
+1% $\dot{W}$	*	7	7
+1% f	*	4	4
-1% f	*	7	6

Note: Calculus of variations solutions for the -5% f, $\dot{W}$ case were not available to compare fuel consumption.

It is interesting that the only requirements which would keep fuel consumption small was the fact that $\bar{X}$ was fitted to the value of X obtained from the calculus of variations solution for the standard trajectory and the ΔX_0 was chosen of a form which would fit well the deviations encountered by calculus of variations solutions with nonstandard initial conditions. Aside from these two requirements, no further attempt was made to minimize fuel. Nevertheless, Table 3.5 indicates that this appears to be sufficient for negligible fuel loss.

To illustrate the difference in some of these nonstandard trajectories, it should be pointed out that example No. 4 which resulted from a first stage engine out at 100 seconds with second stage perturbations of -5% thrust deviation and simultaneously -5% flow rate deviation required 30 seconds additional burning time and displaced the cutoff point 136 km. The radius error was -326 m and cutoff angle error +.008 degrees both of which appear quite acceptable.

Section 4. Further Applications.

The guidance function derived and applied in this report was presented primarily to illustrate the effectiveness of obtaining an explicit solution to the linearized differential equations of motion. It should be emphasized that to linearize the differential equations, it is not necessary to linearize every parameter. It is sufficient that the state variables be linear leaving the forcing functions to whatever form they may have. This same procedure can be applied to other differential equations including calculus of variations equations or equations of motion in a different coordinate system. The linearized equations are always sufficiently accurate in some neighborhood of the standard solution. Whether this neighborhood is sufficiently large to include all expected deviations cannot be stated in general but must be specifically investigated for the particular differential equations under consideration. The differential equations of motion are readily adaptable to such analysis.

In addition to the parameters included in the analysis of Chapter I, any of a number of other parameters could have been included and their effect on cutoff deviations determined.

Section 5. Conclusions

The results of this investigation illustrate the accuracy of the solution to the linearized differential equations of motion as well as the value and means of obtaining the solution in explicit form. In addition to the insight it provides, it is a powerful tool for mathematically imposing the mission criteria in the determination of the coefficients of a guidance function. Further applications of the explicit solution to a linearized set of differential equations to the analysis of a nonlinear system are restricted only by the differential equations themselves and the imagination and ingenuity of the analyst.

APPENDIX I

$U(t_n, t)$

$(t_n = 620.68 \text{ sec.})$

$$U(t_n, t_n) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$U(t_n, 600) = \begin{bmatrix} 1 & 0 & .02608 & 0 \\ 0 & 1 & 0 & .02608 \\ -.01886 & .02833 & 1 & 0 \\ .02833 & .04812 & 0 & 1 \end{bmatrix}$$

$$U(t_n, 560) = \begin{bmatrix} .99917 & .00104 & .06068 & .00000 \\ .00104 & 1.00200 & .00000 & .06068 \\ -.05890 & .07863 & .99925 & .00113 \\ .07863 & .14472 & .00113 & 1.00192 \end{bmatrix}$$

$$U(t_n, 520) = \begin{bmatrix} .99647 & .00366 & .10065 & .00004 \\ .00366 & 1.00808 & .00004 & .10076 \\ -.10332 & .12191 & .99689 & .00428 \\ .12186 & .24516 & .00428 & 1.00771 \end{bmatrix}$$

$$U(t_n, 480) = \begin{bmatrix} .99177 & .00760 & .14051 & .00019 \\ .00760 & 1.01846 & .00019 & .14108 \\ -.14971 & .16133 & .99276 & .00916 \\ .16106 & .34912 & .00915 & 1.01752 \end{bmatrix}$$

$$U(t_n, 440) = \begin{bmatrix} .98488 & .01240 & .18018 & .00049 \\ .01239 & 1.03332 & .00049 & .18182 \\ -.19814 & .19605 & .98677 & .01561 \\ .19521 & .45658 & .01577 & 1.03148 \end{bmatrix}$$

$$U(t_n, 400) = \begin{bmatrix} .97571 & .01771 & .21958 & .00099 \\ .01767 & 1.05283 & .00099 & .22315 \\ -.24801 & .22654 & .97884 & .02345 \\ .22453 & .56761 & .02340 & 1.04974 \end{bmatrix}$$

$$U(t_n, 360) = \begin{bmatrix} .96418 & .02326 & .25861 & .00170 \\ .02315 & 1.07716 & .00170 & .26526 \\ -.29892 & .25337 & .96892 & .03251 \\ .24933 & .68251 & .03238 & 1.07244 \end{bmatrix}$$

$$U(t_n, 320) = \begin{bmatrix} .95028 & .02880 & .29718 & .00263 \\ .02855 & 1.10648 & .00263 & .30835 \\ -.35047 & .27702 & .95696 & .04264 \\ .26978 & .80158 & .04235 & 1.09974 \end{bmatrix}$$

$$U(t_n, 280) = \begin{bmatrix} .93395 & .03412 & .33519 & .00378 \\ .03362 & 1.14106 & .00377 & .35261 \\ -.40248 & .29797 & .94294 & .05372 \\ .28603 & .92547 & .05314 & 1.13180 \end{bmatrix}$$

$$U(t_n, 240) = \begin{bmatrix} .91513 & .03904 & .37255 & .00514 \\ .03813 & 1.18117 & .00511 & .39825 \\ -.45485 & .31671 & .92684 & .06564 \\ .29820 & 1.05475 & .06458 & 1.16882 \end{bmatrix}$$

$$U(t_n, 200) = \begin{bmatrix} .89380 & .04336 & .40916 & .00670 \\ .04182 & 1.22718 & .00664 & .44550 \\ -.50739 & .33355 & .90865 & .07831 \\ .30618 & 1.19028 & .07651 & 1.21101 \end{bmatrix}$$

$$U(t_n, 160) = \begin{bmatrix} .86987 & .04693 & .44491 & .00843 \\ .04446 & 1.27957 & .00831 & .49459 \\ -.56010 & .34893 & .88835 & .09165 \\ .30993 & 1.33309 & .08876 & 1.25862 \end{bmatrix}$$

$$U(t_n, t_o) = \begin{bmatrix} .86115 & .04792 & .45637 & .00905 \\ .04503 & 1.29901 & .00890 & .51145 \\ -.57740 & .35385 & .88097 & .09625 \\ .31006 & 1.38267 & .09284 & 1.27619 \end{bmatrix}$$

APPENDIX II

$\bar{U}(t_j)$

t_j	$\bar{U}_{11}$	$\bar{U}_{12}(10^{-2})$	$\bar{U}_{13}(10^{-4})$	$\bar{U}_{14}$
160	.8296	-1.259	-.4212	12.69
200	.8888	-1.548	-.4512	14.64
240	.7704	-1.549	-.3911	13.67
280	.6480	-1.534	-.3290	12.58
320	.5228	-1.501	-.2654	11.37
360	.3968	-1.447	-.2015	10.06
400	.2768	-1.367	-.1385	8.65
440	.1556	-1.258	-.0790	7.17
480	.0508	-1.106	-.0258	5.62
520	-.0308	- .904	.0156	4.02
560	-.0748	- .630	.0380	2.41
600	-.0533	- .254	.0271	.81
600*	-.0426	- .203	.0216	.81

The elements of $\bar{U}(t_j)$ were determined by multiplying the elements of $U(t_n, t_j)\ H(t_j)$ by Δt_j. Except for t = 160 and t = 600, Δt_j = 40 sec was used. For t = 160, Δt_j = 33.19 sec. and for t = 600, Δt_k = 40.68. For t = 600, Δt_j = 32.5 sec was used for the first three columns since their multiplier is a power of ΔX which, for the guidance function, was defined to be zero for the last 8.18 seconds. These values should be used instead of those at 600 in evaluating the integral for the guidance function.

t_j	$\bar{U}_{21}$	$\bar{U}_{22}(10^{-2})$	$\bar{U}_{23}(10^{-4})$	$\bar{U}_{24}$
160	-1.585	-.895	.805	9.014
200	-1.900	-.897	.965	8.475
240	-1.874	-.762	.952	6.730
280	-1.834	-.630	.931	5.168
320	-1.775	-.500	.901	3.791
360	-1.696	-.374	.861	2.603
400	-1.590	-.254	.807	1.610
440	-1.452	-.144	.737	.821
480	-1.272	-.048	.646	.242
520	-1.037	.026	.526	-.117
560	- .723	.065	.367	-.249
600	- .296	.046	.150	-.147
600*	- .237	.037	.120	-.147

*See Footnote on Page 83.

t_j	$\bar{U}_{31}$	$\bar{U}_{32}(10^{-2})$	$\bar{U}_{33}(10^{-4})$	$\bar{U}_{34}$
160	1.414	-2.676	-.718	26.960
200	1.701	-3.563	-.864	33.676
240	1.666	-3.951	-.846	34.879
280	1.598	-4.390	-.811	36.005
320	1.486	-4.888	-.754	37.036
360	1.319	-5.457	-.670	37.944
400	1.081	-6.117	-.549	38.708
440	.749	-6.898	-.380	39.338
480	.290	-7.816	-.147	39.690
520	-.347	-8.949	.176	39.855
560	-1.242	-10.383	.631	39.754
600	-2.583	-12.286	1.311	39.364
600*	-2.064	-9.816	1.048	39.364

*See Footnote on Page 83.

t_j	$\bar{U}_{41}$	$\bar{U}_{42}(10^{-2})$	$\bar{U}_{43}(10^{-4})$	$\bar{U}_{44}$
160	-3.904	-2.468	1.982	24.868
200	-5.035	-2.657	2.555	25.112
240	-5.396	-2.442	2.739	21.561
280	-5.804	-2.209	2.947	18.121
320	-6.271	-1.950	3.184	14.778
360	-6.815	-1.656	3.460	11.514
400	-7.456	-1.314	3.785	8.312
440	-8.226	- .906	4.176	5.167
480	-9.173	- .405	4.657	2.055
520	-10.369	.229	5.264	- 1.018
560	-11.933	1.063	6.058	- 4.071
600	-14.318	2.216	7.270	- 7.100
600*	-11.439	1.770	5.808	- 7.100

*See Footnote on Page 83.

APPENDIX III

NUMERICAL RESULTS FROM 5 SECOND INTERVALS

Integrand \ Limits	$\int_{t_o}^{t_n}$	$\int_{t_n}^{t_n+\delta}$	$\int_{t_o}^{t_n+\delta}$
f_1	-14.541427	.017713	-14.523714
$f_1\tau$	-29.213315	.082930	-29.130385
$f_1\tau^2$	-80.125334	.388272	-79.737062
$f_1\tau^3$	-251.962610	1.817852	-250.144758
$f_1\tau^4$	-857.620677	8.511001	-849.109676
$f_1\tau^5$	-3076.064786	39.847657	-3036.217129
$f_1\tau^6$	-11,454.32276	186.56275	-11,267.76001
f_2	-.092309	.000024	-.092285
$f_2\tau$	-.142486	.000113	-.142373
$f_2\tau^2$	-.329282	.000531	-.328751
$f_2\tau^3$	-.914112	.002487	-.911625
$f_2\tau^4$	-2.824898	.011646	-2.813252
$f_2\tau^5$	-9.372182	.054524	-9.317658
$f_2\tau^6$	-32.711274	.255278	-32.455996
f_3	.000738	-.000001	.000737
$f_3\tau$	.001483	-.000004	.001479
$f_3\tau^2$	.004068	-.000020	.004048
$f_3\tau^3$	.012792	-.000092	.012700
$f_3\tau^4$	.043541	-.000432	.043109
$f_3\tau^5$	.156170	-.002023	.154147
$f_3\tau^6$	.581531	-.009472	.572059

Integrand \ Limits	$\int_{t_o}^{t_n}$	$\int_{t_n}^{t_n+\delta}$	$\int_{t_n}^{t_n+\delta}$
g_1	.594373	-.023405	.570968
$g_1\tau$	1.824835	-.109580	1.715254
$g_1\tau^2$	6.557068	-.513042	6.044026
$g_1\tau^3$	25.223896	-2.402011	22.821885
$g_1\tau^4$	100.949256	-11.245976	89.703280
$g_1\tau^5$	414.750164	-52.652539	362.097625
$g_1\tau^6$	1736.333387	-246.513923	1489.819464
g_2	.003170	-.000002	.003168
$g_2\tau$	.006143	-.000012	.006131
$g_2\tau^2$	.016530	-.000055	.016475
$g_2\tau^3$	.051330	-.000256	.051074
$g_2\tau^4$	.173072	-.001197	.171875
$g_2\tau^5$	.615988	-.005604	.610384
$g_2\tau^6$	2.278515	-.026239	2.252276
g_3	-.000030	.000001	-.000029
$g_3\tau$	-.000093	.000006	-.000087
$g_3\tau^2$	-.000333	.000026	-.000307
$g_3\tau^3$	-.001281	.000122	-.001159
$g_3\tau^4$	-.005125	.000571	-.004554
$g_3\tau^5$	-.021057	.002673	-.018384
$g_3\tau^6$	-.088153	.012516	-.075637

$$U_1 = \begin{bmatrix} .349481 & 1.259049 & .170980 & .491843 \end{bmatrix}$$

$$U_2 = \begin{bmatrix} -.008175 & -.007129 & -.005744 & -.008092 \end{bmatrix}$$

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GUIDANCE APPLICATIONS OF LINEAR ANALYSIS

By Lyle R. Dickey

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