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ON COUNTING TOPOLOGIES*

by

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(PRINCIPAL INVESTIGATOR: T. N. BHARGAVA)

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* Some of these results were presented jointly with T. N. Bhargava as part of an invited paper at the European meetings of Institute of Mathematical Statistics held at Berne, Switzerland in September, 1964.

Let $\Omega_n = (1,2,\ldots n)$. A collection of sets $\{0\} = S$ of Ω_n forms a topology if:

(i) $\phi, \Omega_n \in S$, ϕ = null set.

(ii) 0_1, 0_2 $\in S \Rightarrow 0_1 \cap 0_2$ and $0_1 \cup 0_2$ $\in S$

My purpose is to count the number f(n) of distinct topologies on Ω_n i e. the number of distinct collections S formed from subsets on Ω_n which satisfy (i) and (ii).

It is convenient to have the following alternative equivalent way of getting a topology on Ω_n. Each topology is uniquely induced by a function F: $\Omega_n \to P(\Omega_n)$ where $P(\Omega_n)$ = the set of all subsets of Ω_n and F satisfies the following:

(1) $i \in F(i)$ $\quad \forall i \in \Omega_n$

(2) $\forall i, j \in \Omega_n$, $j \in F(i) \Rightarrow F(j) \subset F(i)$

Given $A \subset \Omega_n$, define $F(A) = \cup\{F(i): i \in A\}$, $F(\phi) = \phi$ then F can be considered as the closure of A and it is easy to verify that if F satisfies (1) and (2) then this operation satisfies the Kuratowski closure postulates and hence defines a genuine topology on Ω_n. Hence the problem can be reformulated as counting the number of distinct mappings F from Ω_n to $P(\Omega_n)$ satisfying (1) and (2).

I shall change this formulation again to an equivalent one involving relations on Ω_n (or directed graphs or digraphs with points of Ω_n as vertices).

A relation on Ω_n is a subset R of $\Omega_n \times \Omega_n$. Given a function $F:\Omega_n \to P(\Omega_n)$ a relation R can be defined as follows:

$(i,j,) \in R \Leftrightarrow j \in F(i)$

The properties (1) and (2) above of F are equivalent then to the following properties of the corresponding relation R:

(3) $(i,i) \in R$ (reflexive)

(4) $(i,j) \in R,\ (j,k) \in R \Longrightarrow (i,k) \in R$ (transitive)

Hence the question is one of counting all reflexive and transitive relations on Ω_n.

A further reduction is possible. One may consider only those relations which are anti-symmetric i.e.

(5) $i \neq j,\ (i,j,) \in R \Longrightarrow (j,i) \notin R$

Let the number of such relations on Ω_n (usually called partial-orders) be denoted by $f_o(n)$. Clearly

$$\text{(6)} \quad f(n) = \sum_{k=1}^{n} \pi_{n,k} \quad f_o(k)$$

where

$\pi_{n,k}$ = the number of partitions of Ω_n into k disjoint non-empty subsets.

= the number of distinct functions from $\Omega_n \to \Omega_k$ (onto).

It is also easy to show that each partial-order on Ω_n uniquely induces a T_o-topology on Ω_n. Hence $f_o(n)$, the number of partial-orders, is the number of T_o-topologies on Ω_n. At this point, I make the trivial remark that the number of T_1-topologies (and a fortiori better-separated topologies) on Ω_n is exactly one.

For the sake of clarifying what is known (to the best of my knowledge) in this area, I add the following information.

Let R_n = set of all relations on Ω_n.

R_n^1 = set of reflexive relations on Ω_n

R_n^2 = set of transitive relations on Ω_n

R_n^3 = set of symmetric relations of Ω_n

R_n^4 = set of anti-symmetric relations on Ω_n

If $A \subset R_n$ let $\mu(A)$ = the number of elements in A.

Then

$$\mu(R_n) = 2^{n^2}, \qquad \mu(R_n^1) = 2^{n(n-1)}$$

$$\mu(R_n^2) = \qquad \mu(R_n^3) = 2^{\frac{n(n+1)}{2}}$$

$$\mu(R_n^1 \cap R_n^2) = f\ (n) \quad \mu(R_n^1 \cap R_n^2 \cap R_n^4) = f_1(n)$$

$$\mu(R_n^1 \cap R_n^2 \cap R_n^3) = B_n \quad = \text{exponential numbers} .$$

The only nontrivial assertion above is the concerning exponential numbers. For a recent report on these see Rota (American Math. Monthly, 1964). I shall content myself with the observation that B_n also equals the number of distinct algebras (or σ-algebras or Borel-fields) formed out of the subsets of Ω_n.

I shall now begin to estimate the function $f_o(n)$, the number of T_o-topologies on $\Omega_n = (1,2,\ldots n)$ or equivalently the number of partial orders on Ω_n.

For this purpose, I shall use the familiar device of the so-called "Hasse diagrams" for partial orders (See Birkhoff, "Lattice Theory"). For my purposes here, I shall have to make use of a slightly more elaborate description of the diagram than is usually given.

I shall write $\leq$ for the partial order relation and $x < y$ if $x \leq y$ and $x \neq y$

Let a partial order $\leq$ be given on Ω_n. Define, for any $x \in \Omega_n$

$$d(x) = \text{length of maximal chain up to } x$$

$$= \max. \{k \mid \exists\ x_o < x_1 < \ldots < x_{k-1} < x\}$$

with the convention that $d(x) = 0$ if x is minimal or if x is unrelated to any other element. The following lemma is obvious:

Lemma: $d(x) = d(y) \Rightarrow x$ and y are not comparable.

I shall write $b \succ a$ (b covers a) iff $a < b$ and there is no x such that $a < x < b$.

Let $S_j = \{x \mid d(x) = j\}$ $0 \le j \le k$, $k = \max\limits_{1 \le x \le n} d(x)$.

Clearly $0 \le k \le (n-1)$ and S_j's are non-empty with $\Omega_n = \bigcup\limits_{j=o}^{k} S_j$.

I now form a graph in the following manner. Place points of the set S_j in one row, called j^{th} row and for convenience arrange them so that S_j is above S_i if $j>i$. Join $a \in S_i$ with $b \in S_j$ $(j>i)$ iff $b \succ a$. Two points of the same set S_i are not to be joined.

Such a graph is completely characterized by the following description. Points $\{1,2,...n\}$ are arranged in $(k+1)$ rows $0 \le k \le n-1$ called o^{th}, 1^{st} ... k^{th} row. None of the rows are empty. Each point of the i^{th} row $1 \le i \le k$ has to be connected by an edge to some point of the $(i-1)^{th}$ row. A point of the i^{th} row $0 \le i \le k$ may be connected to a point of the j^{th} row, $j>i$, only if there is no point of an intermediate row which is connected to both of them. No two points of the same row are connected.

For convenience, I shall introduce some further notation. Let $P_k(n)$ = the number of partial orders on Ω_n with $\max\limits_{1 \le x \le n} d(x) = k$

Then

$$f_o(n) = \sum_{k=0}^{n-1} P_k(n)$$

Clearly $P_o(n) = 1$

$$P_1(n) = \sum_{r=1}^{n-1} \binom{n}{r} (2^{n-r} - 1)^r$$

I obtain a lower bound for $f_o(n)$ by obtaining a lower bound for $P_1(n)$.

$$P_1(n) > \begin{cases} \binom{n}{\frac{n}{2}} (2^{\frac{n}{2}} - 1)^{\frac{n}{2}} & \text{(n even)} \\ \\ \binom{n}{\frac{n-1}{2}} (2^{\frac{n}{2}+\frac{1}{2}} - 1)^{\frac{n}{2}-\frac{1}{2}} & \text{(n odd)} \end{cases}$$

In either case

$$P_1(n) > \text{constant } 2^{\frac{n^2}{4}} \binom{n}{\frac{n}{2} \text{ or } \frac{n-1}{2}}$$

where the constant is between 0 and 1.

Since $f_o(n) > P_o(n) + P_1(n)$

it follows that

$$f_o(n) > 2^{\frac{n^2}{4}} \text{ and } \frac{f_o(n)}{2^{\frac{n^2}{4}}} \to \infty$$

A trivial upper bound for $f_o(n)$ is $2^{n(n-1)}$ (obtained from reflexivity of the partial order). Thus

Theorem:-

$$2^{\frac{n^2}{4}} < f_o(n) < 2^{n(n-1)}$$

And

$$\frac{f_o(n)}{2^{n^2}} \to 0 \qquad \frac{f_o(n)}{2^{\frac{n^2}{4}}} \to +\infty$$

The same is true for f(n).