Construction of Nonlinear Programming Test Problems*

by

J. B. Rosen and S. Suzuki

In order to test a nonlinear programming algorithm it is very useful to be able to construct test problems with known optimum solutions. The purpose of this note is to describe a simple procedure for constructing such test problems. We will describe the procedure for a concave maximization problem subject to concave constraints.

The concave maximization problem is

$$\max_x \; \{\varphi(x) \,|\, h_i(x) \geq 0, \; i = 1, 2, \ldots, k\},$$

where $x \in E^m$, and $\varphi(x)$ and $h_i(x)$ are real valued concave functions of x. The procedure will be described for $\varphi(x) = \theta(x) + c'x$, and $h_i(x) = q_i(x) + b_i$, $i = 1, \ldots k$, where $\theta(x)$ and $q_i(x)$, $i = 1, \ldots k$ are any selected differentiable concave functions of x, c is a vector $\in E^m$ and the b_i are scalars.

<u>Step I</u>

Choose any $x^o \in E^m$ as a desired optimum point, and any set of $u_i^o \geq 0$, $i = 1, \ldots k$, as the corresponding optimum dual solution. That is, we first specify the primal and dual solution to the problem.

---

*Prepared under NASA grant Ns G 565 at Stanford University.

Step II

Choose b_i, $i = 1, \dots k$, so that $h_i(x^o) = 0$ for $u_i > 0$ and $h_i(x^o) \geq 0$ for $u_i = 0$. Note that $u_i > 0$ means that the i^{th} constraint is active.

Step III

Let

$$c = - \nabla \theta(x^o) - \sum_{i=1}^{k} u_i^o \nabla q_i(x^o)$$

This choice satisfies the Kuhn-Tucker condition $\nabla \varphi(x^o) + \sum_{i=1}^{k} u_i^o \nabla h_i(x^o) = 0$, and therefore ensures that x^o is an optimum solution to the concave programming problem.

We will illustrate this procedure by applying it to the quadratic problem where $\theta(x) = x' Q_o x$, $q_i(x) = x' Q_i x + a_i' x$, and the Q_i, $i = 0, 1, \dots k$, are negative semi-definite matrices.

Example (quadratic problem with four variables and three constraints)

Let

$$Q_o = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad a_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Step I

Let $x^o = \begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix}$ and $u^o = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

Step II

$h_1(x^o) = x^{o\prime} Q_1 x^o + a_1' x^o + b_1 = -8 + b_1$

Since $u_1 > 0$, $b_1 = 8$.

$h_2(x^o) = x^{o\prime} Q_2 x^o + a_2' x^o + b_2 = -9 + b_2$

Since $u_2 = 0$, we choose $b_2 = 10$ so that $h_2(x^o) = 1 > 0$.

$h_3(x^o) = x^{o\prime} Q_3 x^o + a_3' x^o + b_3 = -5 + b_3$

Since $u_3 > 0$, $b_3 = 5$.

Step III

$$c = \begin{pmatrix} 2x_1^o \\ 2x_2^o \\ 4x_3^o \\ 2x_4^o \end{pmatrix} + \begin{pmatrix} 2x_1^o + 1 \\ 2x_2^o - 1 \\ 2x_3^o + 1 \\ 2x_4^o - 1 \end{pmatrix} +2 \begin{pmatrix} 4x_1^o + 2 \\ 2x_2^o - 1 \\ 2x_3^o \\ 0 \quad - 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 21 \\ -7 \end{pmatrix}$$

The constructed problem is:

minimize $\varphi = -x_1^2 - x_2^2 - 2x_3^2 - x_4^2 + 5x_1 + 5x_2 + 21x_3 - 7x_4$

subject to

$$-x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_1 + x_2 - x_3 + x_4 + 8 \geq 0$$

$$-x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 + x_1 + x_4 + 10 \geq 0$$

$$-2x_1^2 - x_2^2 - x_3^2 - 2x_1 + x_2 + x_4 + 5 \geq 0$$

and has as its optimum function value $\varphi(x^o) = 44$.

Computer Science Division
Stanford University
August 12, 1964