NASA TM X-55143

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IN AN ANISOTROPIC PLASMA

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OCTOBER 1964

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Introduction

The purpose of the present paper is to investigate the Kelvin-Helmholtz instability, arising due to a tangential discontinuity of velocities between two streams of a homogenous, nondissipative, anisotropic plasma. This problem is of interest in a variety of astro- and geophysical situations, e.g. the stability of the magnetospheric boundary, coronal streamers moving through the solar wind. The instability of the magnetospheric boundary, if ascertained, would help in understanding of the phenomena of magnetic storms, aurorae and the formation of radiation belts. Again the solar wind may possess a fine structure resulting in a tangential discontinuity of velocity between adjacent streams. The instability of this fine structure would result in production of irregularities of plasma properties and the magnetic field in the incoming solar wind and this may be the cause of the irregularity in the direction and the magnitude of the solar wind magnetic field vector as given by Explorer 10 and 12 and Imp 1 measurements (Heppner, Ness et. al, 1963 and Cahill and Amazeen, 1963, Ness et. al, 1964)

Previous investigations on Kelvin-Helmholtz instability in hydro-magnetics were concerned with the instability due to a tangential discontinuity in velocities present in _incompressible_ fluids (Chandrasekhar 1961), although the effects of compressibility has also been incorporated recently (Fejer, 1964; Talwar, 1964; Parker, 1964; Sen, 1964). A general result of these investigations is that the instability of the vortex sheet is suppressed by a strong enough magnetic field. These investigations

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made use of collision-dominated, single fluid hydromagnetic equations with scalar gas pressure approximation. The latter assumption is not likely to hold true in dilute plasmas such as the coronal streamers and the solar wind. In fact, the plasma pressure may be anisotropic due to infrequent collisions, in particular having two scalar components, one parallel and the other perpendicular to the local interplanetary magnetic field. In this paper we shall make use of the modified single fluid hydromagnetic equations as given by Chew, Goldberger, and Low (1956) where the heat flow is neglected, and the independent components of anisotropic pressure are determined by the two adiabatic equations of state instead of the single equation of state of the ideal fluid theory. The results obtained should, therefore, be applicable to situations intermediate between those of weak collisions and strong collisions, i.e. the situations in which collisions are not sufficiently strong to keep the pressure a scalar but sufficiently strong to prevent the heat flow and other transport processes.

Equations of the Problem

Consider a system of Cartesian axes with the Z-direction as vertical. Suppose that a plane surface of discontinuity of tangential velocity exists at the interface ($Z=0$) between the two semi-infinite regions of a single homogenous, non-dissipative plasma permeated with a uniform magnetic field B_0 along the x-direction. Let U_1, U_2 be the uniform velocities along x-direction in the two regions $Z < 0$, and $Z > 0$ respectively, so that there is no initial electric field in the medium.

The initial steady state of the configuration requires

$$\nabla \cdot \overleftrightarrow{p} = 0 \tag{1}$$

for either region. Here $\overleftrightarrow{p}$ denotes the tensorial plasma pressure having components p_{11} and $p_{\perp}$ along and normal to the ambient magnetic field. These components are constants, as follows from equation (1).

To investigate the stability of the initial state, we impart a perturbation velocity $\underline{u}$ having components u, v, w in the x, y, z directions. Let the corresponding perturbations in other physical parameters be

$$\delta\rho, \ \delta p_{11}, \ \delta p_{\perp}, \ \delta\underline{B} \ .$$

Assuming the perturbations to be of the first order of smallness and of the form,

$$(\text{Some function of Z}) \cdot \exp. \ [ikx + nt] \tag{2}$$

we write the linearized perturbation equations as,

$$\rho_0 \left(n + ik U_0 \right) u = - \left(\nabla \cdot \overleftrightarrow{\delta p} \right)_x \tag{3}$$

$$\rho_0 \left(n + ik U_0 \right) v = - \left(\nabla \cdot \overleftrightarrow{\delta p} \right)_y + \frac{ik B_0}{4\pi} \delta B_y \tag{4}$$

$$\rho_0 \left(n + ik U_0 \right) w = - \left(\nabla \cdot \overleftrightarrow{\delta p} \right)_z - \frac{B_0}{4\pi} \left(D \delta B_x - ik \delta B_z \right) \tag{5}$$

$$\left(n + ik U_0 \right) \delta\rho = - \rho_0 \nabla \cdot \underline{u} \tag{6}$$

$$\left(n + ik U_0 \right) \delta B_x = ik B_0 u - B_0 \nabla \cdot \underline{u} \tag{7}$$

$$\left(n+ikU_0\right)\delta B_y = ikB_0 v \tag{8}$$

$$\left(n+ikU_0\right)\delta B_z = ikB_0 w \tag{9}$$

$$ik\delta B_x + D\delta B_z = 0 \tag{10}$$

$$\frac{\delta p_{\shortparallel}}{p_{\shortparallel}} = 3\frac{\delta\rho}{\rho_0} - 2\frac{\delta B_x}{B_0} \tag{11}$$

$$\frac{\delta p_{\perp}}{p_{\perp}} = \frac{\delta B_x}{B_0} + \frac{\delta\rho}{\rho_0} \tag{12}$$

Using the relation

$$p_{ij} = p_{\shortparallel}n_i n_j + p_{\perp}\left(\delta_{ij} - n_i n_j\right) \quad \text{with} \quad n_{ij} = \frac{B_{i,j}}{|B|} \tag{13}$$

We can rewrite the equations (3) - (5) as,

$$\rho_0 n' u = -ik\delta p_{\shortparallel} - (p_{\shortparallel}-p_{\perp})\frac{D\delta B_z}{B_0} \tag{14}$$

$$\rho_0 n' v = \frac{ikB_0}{4\pi}\delta B_y\left[1 - \frac{4\pi(p_{\shortparallel}-p_{\perp})}{B_0^2}\right] \tag{15}$$

$$\rho_0 n' w = -D\delta p_{\perp} - \frac{ik}{B_0}\left[(p_{\shortparallel}-p_{\perp}) + \frac{B_0^2}{4\pi k^2}(D - k^2)\right]\delta B_z \tag{16}$$

where we have put n' for $(n + ikU_0)$.

With the help of equations (6) - (12) we can rewrite the equation (16) as

$$\left(S_\perp^2 + V_0^2\right)\overset{2}{D}w - k^2\left[\frac{n'^2}{k^2} - \left(S_{||}^2 - S_\perp^2 - V_0^2\right)\right]w = -S_\perp^2 D\nabla\cdot\underline{u} \tag{17}$$

where we have written

$$S_{\perp,||}^2 = \frac{p_{\perp,||}}{\rho_0} \quad \text{and} \quad V_0^2 = \frac{B_0^2}{4\pi\rho_0} \tag{18}$$

It may be noted that the equation (15) yields a mode uncoupled with the equations (14) and (16), described by,

$$n = -ikV_0 \pm k\left[S_{||}^2 - S_\perp^2 - V_0^2\right]^{1/2} \tag{19}$$

which yields the usual 'Fire-hose' instability condition for a uniform static anisotropic plasma.

The expression for $\nabla\cdot\underline{u}$ can be obtained from equation (14) using the equations (6) - (12), and is written as,

$$\nabla\cdot\underline{u}\left(n'^2 + 3k^2 S_{||}^2\right) = \left[n'^2 + k^2\left(3S_{||}^2 - S_\perp^2\right)\right]Dw \tag{20}$$

Thus the perturbation equation (17) for the vertical component **w** of the velocity perturbation vector can be finally written as,

$$\left[\left(n'^2 + 3k^2 S_{||}^2\right)\left(2S_\perp^2 + V_0^2\right) - k^2 S_\perp^4\right]\overset{2}{D}w \tag{21}$$

$$- \left[n'^2 - k^2\left(S_{||}^2 - S_\perp^2 - V_0^2\right)\right]\left(n'^2 + 3k^2 S_{||}^2\right)w = 0$$

Boundary Conditions and Dispersion Relation

For a configuration of two semi-infinite streams slipping past each other at the horizontal interface $Z=0$, the respective solutions vanishing at $Z = \pm \infty$, of equation (21) are written as,

$$w_1(z) = A_1\, e^{m_1 z} \qquad (z < 0)$$

$$w_2(z) = A_2\, e^{-m_2 z} \qquad (z > 0)$$

(22)

where m_i ($i=1, 2$) supposed to be having positive real part, are given by,

$$m_i^2 = k^2 \frac{\left[n_i'^2 + 3k^2 s_{\parallel}^2 \right]\left[\dfrac{n_i'^2}{k^2} - \left(s_{\parallel}^2 - s_{\perp}^2 - V_0^2 \right) \right]}{\left[\left(n_i'^2 + 3k^2 s_{\parallel}^2 \right)\left(2 s_{\perp}^2 + V_0^2 \right) - k^2 s_{\perp}^4 \right]}$$

(23)

At the common interface we need to satisfy the following boundary conditions:

(i) The normal component of velocity is continuous, leading to,

$$w_1 - ikU_1 \xi = w_2 - ikU_2 \xi = \frac{\partial \xi}{\partial t}$$

(24)

where ξ denotes the small displacement of the interface. Using (22) we get

$$A_2 = A_1 \frac{n_2'}{n_1'}$$

(25)

(ii) The normal component of the magnetic field is zero in each

region. This requirement follows, as can be easily verified.

(iii) The normal stress should be continuous across the interface.

This requires that,

$$\delta p_{\perp}^{(1)} - \delta p_{\perp}^{(2)} + \frac{B_o}{4\pi}\left[\delta B_{x_1} - \delta B_{x_2}\right] = o \tag{26}$$

Making use of equations (9), (10), (12), (14) and (25) in equation

(26), we obtain

$$\frac{m_1}{m_2} = - \frac{\left(n_1'^2 + 3k^2 s_{\parallel}^2\right)\left[\left(V_o^2 + 2s_{\perp}^2\right)\left(n_2'^2 + 3k^2 s_{\parallel}^2\right) - k^2 s_{\perp}^4\right]}{\left(n_2'^2 + 3k^2 s_{\parallel}^2\right)\left[\left(V_o^2 + 2s_{\perp}^2\right)\left(n_1'^2 + 3k^2 s_{\parallel}^2\right) - k^2 s_{\perp}^4\right]} \tag{27}$$

Substituting for m_1, m_2 from equation (23) and simplifying, we finally

obtain the dispersion relation in the form

$$\left(V_o^2 + 2s_{\perp}^2\right)n_1'^2 n_2'^2 + 3k^2 s_{\parallel}^2\left(V_o^2 + 2s_{\perp}^2\right)\left(n_1'^2 + n_2'^2\right)$$

$$+ k^4\left[\left(4s_{\parallel}^2 - s_{\perp}^2 - V_o^2\right)\left\{3s_{\parallel}^2\left(V_o^2 + 2s_{\perp}^2\right) - s_{\perp}^4\right\}\right.$$

$$\left. - 3s_{\parallel}^2\left(V_o^2 + 2s_{\perp}^2\right)\left(s_{\parallel}^2 - s_{\perp}^2 - V_o^2\right)\right] = o \tag{28}$$

here

$$n_1' = ik\left(U_1 - U_p\right) \qquad\qquad U_p = \frac{in}{k} = \text{Phase velocity} \tag{29}$$

$$n_2' = -ik\left(U_2 + U_p\right)$$

Taking $U_1 = -U_2 = U$, the equation (28) reduces to,

$$U_p^4 - 2U_p^2\left[U^2 + 3S_{||}^2\right] + \left[U^4 + 3S_{||}^2\left(3S_{||}^2 - 2U^2\right) - \frac{S_\perp^4\left(4S_{||}^2 - S_\perp^2 - V_0^2\right)}{\left(V_0^2 + 2S_\perp^2\right)}\right] = 0 \qquad (30)$$

The equation (30) determines the stability of the configuration.
For stability we require all roots U_p of the above equation to be real.
If U_p^2 is negative real, it corresponds to monotonic instability whereas
if U_p^2 is complex, it leads to an overstable situation.

Discussion of Results

It follows from equation (30) that the discriminant is negative
leading to complex U_p^2 provided

$$U^2 < \frac{S_\perp^4\left(S_\perp^2 + V_0^2 - 4S_{||}^2\right)}{12S_{||}^2\left(V_0^2 + 2S_\perp^2\right)} \qquad (31)$$

Alternatively we may argue that the configuration of two slipping
streams of an anisotropic plasma can show instability through overstability
(growing wave instability) only if the relative speed 2U is less than
a certain critical value $\frac{S_\perp^2}{\sqrt{3}\,S_{||}}$, and the prevailing magnetic field
is sufficiently strong as to satisfy the condition,

$$\left(V_0^2 + 2S_\perp^2\right) > \frac{S_\perp^4\left(S_\perp^2 + 4S_{||}^2\right)}{S_\perp^4 - 12S_{||}^2U^2} \qquad (32)$$

If $U^2 > \frac{S_\perp^4}{12S_{||}^2}$, there is no overstability possible and the con-
figuration is either stable or monotonically unstable, depending upon
the strength of the magnetic field. The conditions are written as:

$$\left(V_0^2 + 2S_\perp^2\right) < \frac{S_\perp^4\left(4S_{\parallel}^2 + S_\perp^2\right)}{\left[\left(U^2 - 3S_{\parallel}^2\right)^2 + S_\perp^4\right]} \qquad \text{(unstable)}$$

$$> \frac{S_\perp^4\left(4S_{\parallel}^2 + S_\perp^2\right)}{\left[\left(U^2 - 3S_{\parallel}^2\right)^2 + S_\perp^4\right]} \qquad \text{(stable)} \qquad (33)$$

It may be noted that the denominator on the right hand side in equation (33) is always larger than the denominator in equation (32) so that the critical value of $\left(V_0^2 + 2S_\perp^2\right)$ required for overstability to manifest is larger than the corresponding value for monotonic instability. Thus we may summarize the conclusion by saying that as the relative speed between the two streams of an anisotropic plasma is increased from a non-zero low value, the configuration; stable, unstable (monotonic or overstable) depending upon the strength of the prevailing magnetic field, shows no overstability for relative speeds beyond the critical value $\frac{S_\perp^2}{\sqrt{3}\,S_{\parallel}}$. Thereafter the configuration will be stable or unstable according to equation (33), for all modes of disturbance and the instability cannot be suppressed completely for any relative streaming speed.

Figure 1 gives a plot of the critical value V_* against U (both measured in units of $S_\perp$) as defined by equations (32) and (33) for $\delta = 0.5$, 1.0, 1.5 where $\delta \left(= \frac{S_{\parallel}}{S_\perp}\right)$ measures the anisotropy of pressure in the medium. The curves a and b give the critical value for a transition from overstable to stable regimes for $\delta = 0.5$ and 1.0 whereas the curves c, and d (On different abscissae scale) demarcate stable-unstable regimes for $\delta = 1$, 1.5. For $\delta = 0.5$, the system is either stable or overstable

even in the absence of magnetic field. It may be mentioned that the conditions (32) and (33) depend substantially on the anisotropy in the medium. We may conclude that the supersonic solar wind velocities past the magnetosphere precludes the possibility of growing wave instability unless the anisotropy is very large. The magnetospheric boundary (more specially the tail, if open) may, however, suffer from monotonic instability (even though the anisotropy is not appreciable) in regions and at times when the relative slippage velocity is low. It is quite likely, although the observational information is not yet available, that the coronal streamers may, at times, satisfy the requirement for overstability or monotonic instability and thus produce some kind of disorderly state in the solar wind.

Acknowledgments

REFERENCES

Cahill, L. J., and Amazeen, P. G., (1963), J. Geophys. Research 68, 1835.

Chandrasekhar, S. (1961) Hydrodynamic and Hydromagnetic Stability (Clarendon Press).

Chew, G. F., Goldberger, M., Low, F. (1956) Proc. Roy. Soc. A 236, 112.

Fejer, J. A. (1964) Phys. Fluids 7, 499.

Heppner, J. P., Ness, N.F., Scearce, C. S. and Skillman, T. L. (1963) Journal Geophysical Research 68, 1.

Ness, N. F., Scearce, D. S., and Seek, J. B. (1964) J. Geophys. Res. 69, 3531

Parker, E. N. (1964) Ap. J. 139, 690.

Sen, A. K. (1964) Phys. Fluids. 7, 1293.

Talwar, S. P., (1964) Journal Geophys. Research 69, 2707.

Abstract

The instability arising due to a tangential discontinuity of velocity in an unbounded, anisotropic plasma subject to a uniform magnetic field is investigated using Chew, Goldberger, and Low approximation. Formal conditions of instability are written, and the bearing of the results on the stability of the magnetospheric boundary, and the coronal streamers is discussed.