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Computation of Weighted Root Mean Square of Path Length Changes Caused by the Deformations and Imperfections of Rotational Paraboloidal Antennas

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ABSTRACT

This work concerns the rotational paraboloidal reflector of the Advanced Antenna System of the NASA/JPL Deep Space Instrumentation Facility. Since the reflector itself and its supports are both flexible, the displacements under loads are partially caused by the displacements of the supports and partially by the distortions of the reflector. The former causes the dislocation of the focal point, which can be corrected by physically moving the focal equipment, whereas the latter is an intrinsic characteristic of the antenna, which causes interference of the radio waves. Previous work has shown that the root mean square of the change in the path length over the reflector surface is a measure of the magnitude of the interference taking place for wave lengths much greater than the expected maximum distortions of the surface.

Mathematically this is a problem of eliminating the rigid body motion from the displacements of a surface and of varying a surface parameter such that the remaining distortions will minimize a specified function. This specified function is formed by the sum of the squares of the normal distortions (the projections of the distortions on the undeformed normal of the surface) with certain known weights. This scheme is comparable to the general method of weighted residuals, where the distortions correspond to the residuals. Since the distortions are linear functions of the components of the rigid body motion and the surface parameter, the specified function is a quadric which is always positive. A digital computer program is developed to compute from the displacements both the new orientation of the best fit paraboloid and the associated root mean square.

I. INTRODUCTION

The physical antenna of the Advanced Antenna System of the JPL/NASA Deep Space Instrumentation Facility is a steel space truss structure with a rotational paraboloidal reflector surface of 210 ft D and focal-length-to-diameter ratio of 0.4235. Under different loading conditions the components of the displacement vector are computed by a computer program (Ref. 1). The displacements of the reflector points are caused partially by the distortions of the reflector and partially by the displacements of its supports. The displacements of the supports cause a rigid body translation and rotation of the reflector, resulting in the dislocation of the focal point. In order to determine apriori the location of the focal equipment, the components of the rigid body motion should be computed. The radio waves converging at the newly established focal point are disturbed by the interference resulting from the distortions of the reflector. Since the length of the utilized radio wave is much greater than the expected maximum distortions, the change in path length of the rays can be used as the phase difference.

On the basis of a previous work (Ref. 2), the root mean square of the weighted phase differences of the rays associated with the observed points on the surface (using their corresponding reflector surface areas) is used to describe the over-all interference. Since the Advanced Antenna System operates at the highest possible noise-to-signal ratio, the root mean square should be minimized by varying the focal length and the components of the rigid body motion.

A digital computer program was developed to compute the focal length and the components of the rigid body motion from the cartesian coordinates of the observed points on the deformed reflector, yielding the minimum root mean square.

II. MATHEMATICAL FORMULATION

In the following formulation, the possible constituents of the displacement vectors are analyzed, the change in path length due to the distortions is explained, a scheme to compute the minimum root mean square is given, and, finally, the root mean square is minimized further with respect to the focal length.

A. Rigid Body Components of the Displacement Vector

Let $\mathbf{r}_i$ be the position vector of the ith point on the undeformed reflector surface, and let $\boldsymbol{\rho}_i$ be the position vector of the same point on the deformed surface. In Fig. 1, the components of the deformation are illustrated, in which $\mathbf{t}$ is the translation, $\boldsymbol{\omega}_i$ is the displacement due to rotation, and $\mathbf{d}_i$ the distortion of the ith point. Then it follows from Fig. 1 that

$$\boldsymbol{\rho}_i = \mathbf{r}_i + \mathbf{t} + \boldsymbol{\omega}_i + \mathbf{d}_i, \; i = 1, 2, \ldots, N \qquad (1)$$

where the magnitudes of $\mathbf{t}$ and $\boldsymbol{\omega}_i$ are small in comparison to $\mathbf{r}_i$, and N is the total number of observed points. Referring to the (x, y, z) coordinate system, one can write

$$\mathbf{r}_i = x_i\,\mathbf{i} + y_i\,\mathbf{j} + z_i\,\mathbf{k} \qquad (2)$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors and

$$z_i = \frac{1}{4f}(x_i^2 + y_i^2) \qquad (3)$$

where f is the focal length of the undeformed surface.

Designating the displacement vector of the ith point as $\mathbf{q}_i$, one writes

$$\mathbf{q}_i = \mathbf{t} + \boldsymbol{\omega}_i + \mathbf{d}_i \qquad (4)$$

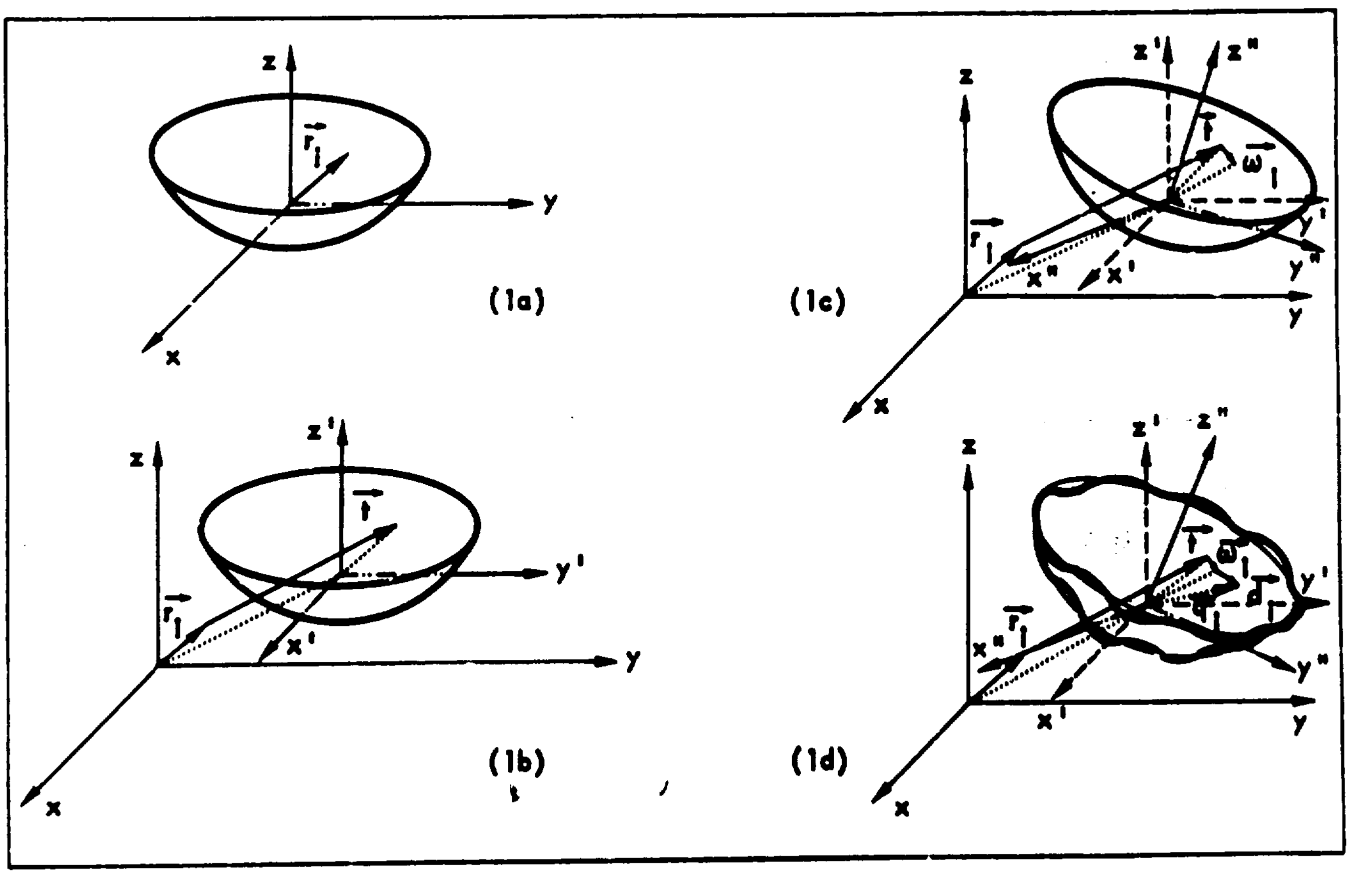

Fig. 1. Components of the displacement vector

and

$$\mathbf{q}_i = u_i\,\mathbf{i} + v_i\,\mathbf{j} + w_i\,\mathbf{k} \tag{5}$$

where u_i, v_i, and w_i are the components of the displacement vector. When the loads are specified, u_i, v_i, and w_i are obtained from the structure analysis (Ref. 1). Using Eqs. (1 and 4), one writes

$$\boldsymbol{\rho}_i = \mathbf{r}_i + \mathbf{q}_i \tag{6}$$

or

$$\boldsymbol{\rho}_i = (x_i + u_i)\,\mathbf{i} + (y_i + v_i)\,\mathbf{j} + (z_i + w_i)\,\mathbf{k} \tag{7}$$

In the cartesian coordinate system $\mathbf{t}$ can be written

$$\mathbf{t} = u_0\,\mathbf{i} + v_0\,\mathbf{j} + w_0\,\mathbf{k} \tag{8}$$

The displacement caused by the rotation, $\boldsymbol{\omega}_i$, can be expressed as

$$\boldsymbol{\omega}_i = \mathbf{r}_{r_i} - \mathbf{r}_i \tag{9}$$

where $\mathbf{r}_{r_i}$ is the rotated position vector. Defining θ, ϕ, and ψ as the rotation angles about the z axis, the rotated x axis (the x axis after the θ rotation), and the rotated rotated y axis (the y axis after both the θ and the ϕ rotations), respectively, then $\mathbf{r}_{r_i}$ can be written (see Appendix A):

$$\mathbf{r}_{r_i} = [\mathbf{i}\ \mathbf{j}\ \mathbf{k}]\begin{bmatrix} 1 & \theta & -\psi \\ -\theta & 1 & \phi \\ \psi & -\phi & 1 \end{bmatrix}^T \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} \tag{10}$$

Equation (2) can be rewritten as

$$\mathbf{r} = [\mathbf{i}\ \mathbf{j}\ \mathbf{k}]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} \tag{11}$$

Substituting $\mathbf{r}_{r_i}$ and $\mathbf{r}_i$ from Eqs. (10 and 11), respectively, into Eq. (9), one obtains

$$\boldsymbol{\omega}_i = \mathbf{r}_{r_i} - \mathbf{r}_i = [\mathbf{i}\ \mathbf{j}\ \mathbf{k}]\begin{bmatrix} 0 & -\theta & \psi \\ \theta & 0 & -\phi \\ -\psi & \phi & 0 \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} \tag{12}$$

or

$$\boldsymbol{\omega}_i = (-\theta y_i + \psi z_i)\,\mathbf{i} + (\theta x_i - \phi z_i)\,\mathbf{j} + (-\psi x_i + \phi y_i)\,\mathbf{k} \tag{13}$$

Defining $\mathbf{d}_i$ as

$$\mathbf{d}_i = d_{x_i}\,\mathbf{i} + d_{y_i}\,\mathbf{j} + d_{z_i}\,\mathbf{k} \tag{14}$$

and substituting $\boldsymbol{\rho}_i$, $\mathbf{r}_i$, $\mathbf{t}$, $\boldsymbol{\omega}_i$, and $\mathbf{d}_i$ from Eqs. (7, 2, 8, 13, and 14), respectively, into Eq. (1), one obtains the identity,

$$\begin{aligned}(x_i + u_i)\,\mathbf{i} + (y_i + v_i)\,\mathbf{j} + (z_i + w_i)\,\mathbf{k} &= (x_i + u_0 - \theta y_i + \psi z_i + d_{x_i})\,\mathbf{i} + (y_i + v_0 + \theta x_i - \phi z_i + d_{y_i})\,\mathbf{j} \\ &\quad + (z_i + w_0 - \psi x_i + \phi y_i + d_{z_i})\,\mathbf{k}\end{aligned} \tag{15}$$

which leads to

$$d_{x_i} = u_i - (u_0 - \theta y_i + \psi z_i) \tag{16a}$$

$$d_{y_i} = v_i - (v_0 + \theta x_i - \phi z_i) \tag{16b}$$

$$d_{z_i} = w_i - (w_0 - \psi x_i + \phi y_i) \tag{16c}$$

B. Change in Path Length Due to $\mathbf{d}_i$

In Fig. 2 the change in path length, λ_i, for the ith ray caused by the distortion is illustrated. From Fig. 2 one writes for small distortions

$$\lambda_i = 2\,(\mathbf{n}_i \cdot \mathbf{d}_i)\,(\mathbf{n}_i \cdot \mathbf{k}) \tag{17}$$

where $\mathbf{n}_i$ is the unit normal of the undeformed surface at the ith point and $\mathbf{k}$ is the unit vector along the axis of the undeformed paraboloid. The expression for $\mathbf{n}_i$ is

$$\mathbf{n}_i = n_{1_i}\,\mathbf{i} + n_{2_i}\,\mathbf{j} + n_{3_i}\,\mathbf{k} \tag{18a}$$

where

$$n_{1_i} = \frac{\left(\frac{\partial F}{\partial x}\right)_i}{\left[\left(\frac{\partial F}{\partial x}\right)_i^2 + \left(\frac{\partial F}{\partial y}\right)_i^2 + \left(\frac{\partial F}{\partial z}\right)_i^2\right]^{\frac{1}{2}}} \tag{18b}$$

$$n_{2_i} = \frac{\left(\frac{\partial F}{\partial y}\right)_i}{\left[\left(\frac{\partial F}{\partial x}\right)_i^2 + \left(\frac{\partial F}{\partial y}\right)_i^2 + \left(\frac{\partial F}{\partial z}\right)_i^2\right]^{\frac{1}{2}}} \tag{18c}$$

$$n_{3_i} = \frac{\left(\frac{\partial F}{\partial z}\right)_i}{\left[\left(\frac{\partial F}{\partial x}\right)_i^2 + \left(\frac{\partial F}{\partial y}\right)_i^2 + \left(\frac{\partial F}{\partial z}\right)_i^2\right]^{\frac{1}{2}}} \tag{18d}$$

and

$$F = 4fz - (x^2 + y^2) \tag{19}$$

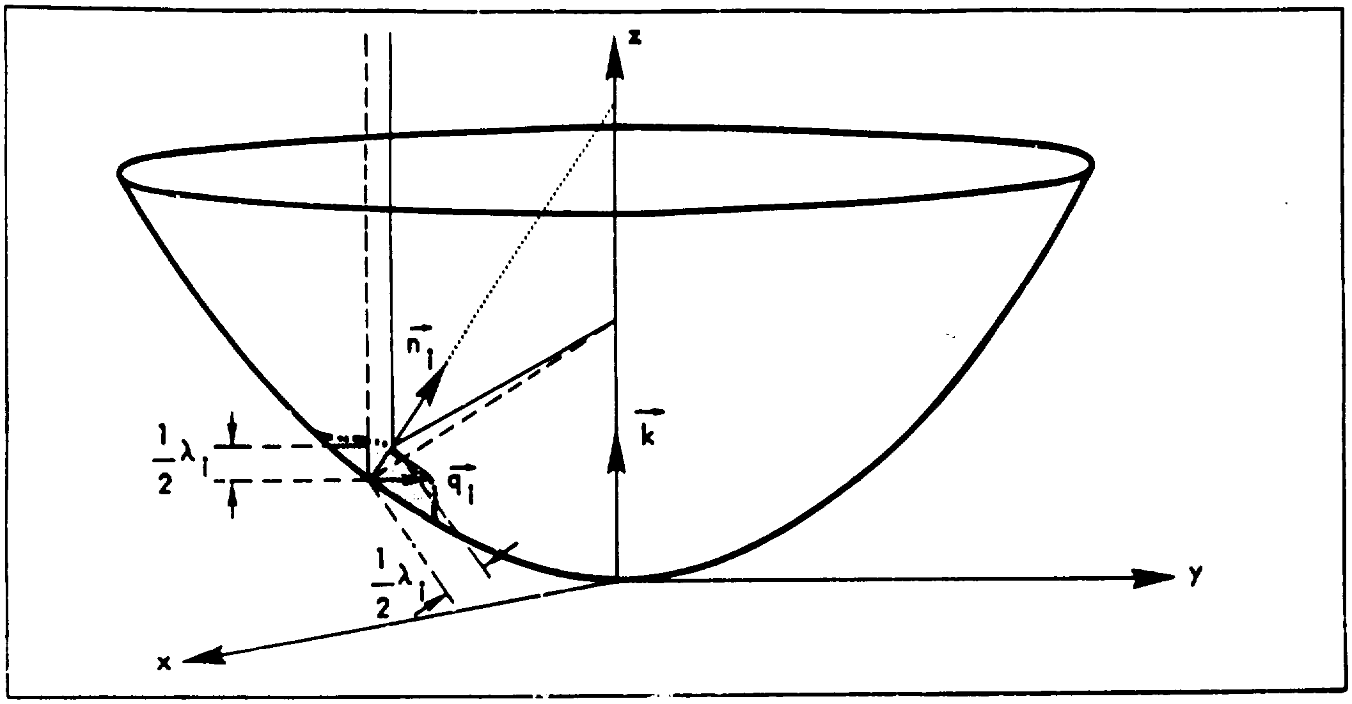

Fig. 2. Change in path len... due to distortion of the reflector

C. Minimization of the Root Mean Square with Respect to the Components of the Rigid Body Motion

The root mean square (rms) of the path length changes can be defined as

$$\text{rms} = \sqrt{\frac{\sum_{i=1}^{N} A_i \lambda_i^2}{\sum_{i=1}^{N} 4(\mathbf{n}_i \cdot \mathbf{k})^2 A_i}} \qquad (20)$$

where A_i is the area of the reflector associated with the ith point. Defining W_i as

$$W_i = 4(\mathbf{n}_i \cdot \mathbf{k})^2 A_i \qquad (21)$$

and using Eqs. (14 and 18), Eq. (20) can be rewritten as

$$\text{rms} = \sqrt{\frac{\sum_{i=1}^{N} W_i (n_{1_i} d_{x_i} + n_{2_i} d_{y_i} + n_{3_i} d_{z_i})^2}{\sum_{i=1}^{N} W_i}} \qquad (22)$$

Since W_i weights are positive quantities for all values of i, the rms will be minimum when the numerator under the radical sign is minimum; that is, when

$$S = \sum_{i=1}^{N} W_i (n_{1_i} d_{x_i} + n_{2_i} d_{y_i} + n_{3_i} d_{z_i})^2 \qquad (23)$$

is minimum.

S can be minimized by varying it with respect to u_0, v_0, w_0, θ, ϕ, and ψ yielding

$$\frac{\partial S}{\partial u_0} = \sum_{i=1}^{N} 2 W_i (n_{1_i} d_{x_i} + n_{2_i} d_{y_i} + n_{3_i} d_{z_i}) \left(n_{1_i} \frac{\partial d_{x_i}}{\partial u_0} + n_{2_i} \frac{\partial d_{y_i}}{\partial u_0} + n_{3_i} \frac{\partial d_{z_i}}{\partial u_0} \right) = 0 \qquad (24a)$$

$$\frac{\partial S}{\partial v_0} = \sum_{i=1}^{N} 2 W_i (n_{1_i} d_{x_i} + n_{2_i} d_{y_i} + n_{3_i} d_{z_i}) \left(n_{1_i} \frac{\partial d_{x_i}}{\partial v_0} + n_{2_i} \frac{\partial d_{y_i}}{\partial v_0} + n_{3_i} \frac{\partial d_{z_i}}{\partial v_0} \right) = 0 \qquad (24b)$$

$$\frac{\partial S}{\partial w_0} = \sum_{i=1}^{N} 2 W_i (n_{1_i} d_{x_i} + n_{2_i} d_{y_i} + n_{3_i} d_{z_i}) \left(n_{1_i} \frac{\partial d_{x_i}}{\partial w_0} + n_{2_i} \frac{\partial d_{y_i}}{\partial w_0} + n_{3_i} \frac{\partial d_{z_i}}{\partial w_0} \right) = 0 \qquad (24c)$$

$$\frac{\partial S}{\partial \theta} = \sum_{i=1}^{N} 2\, W_i \,(n_{1_i} d_{x_i} + n_{2_i} d_{y_i} + n_{3_i} d_{z_i}) \left(n_{1_i} \frac{\partial d_{x_i}}{\partial \theta} + n_{2_i} \frac{\partial d_{y_i}}{\partial \theta} + n_{3_i} \frac{\partial d_{z_i}}{\partial \theta} \right) = 0 \qquad (24d)$$

$$\frac{\partial S}{\partial \phi} = \sum_{i=1}^{N} 2\, W_i \,(n_{1_i} d_{x_i} + n_{2_i} d_{y_i} + n_{3_i} d_{z_i}) \left(n_{1_i} \frac{\partial d_{x_i}}{\partial \phi} + n_{2_i} \frac{\partial d_{y_i}}{\partial \phi} + n_{3_i} \frac{\partial d_{z_i}}{\partial \phi} \right) = 0 \qquad (24e)$$

$$\frac{\partial S}{\partial \psi} = \sum_{i=1}^{N} 2\, W_i \,(n_{1_i} d_{x_i} + n_{2_i} d_{y_i} + n_{3_i} d_{z_i}) \left(n_{1_i} \frac{\partial d_{x_i}}{\partial \psi} + n_{2_i} \frac{\partial d_{y_i}}{\partial \psi} + n_{3_i} \frac{\partial d_{z_i}}{\partial \psi} \right) = 0 \qquad (24f)$$

These can be written as

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} & b_{56} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta \\ \phi \\ \psi \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix} \qquad (25)$$

where the coefficient matrix is a positive definite symmetric matrix, the entries of which are:

$$b_{11} = \sum_{i=1}^{N} n_{1_i}^2 W_i \qquad (26a)$$

$$b_{12} = \sum_{i=1}^{N} n_{1_i} n_{2_i} W_i \qquad (26b)$$

$$b_{13} = \sum_{i=1}^{N} n_{1_i} n_{3_i} W_i \qquad (26c)$$

$$b_{14} = \sum_{i=1}^{N} (x_i n_{1_i} n_{2_i} W_i - y_i n_{1_i}^2 W_i) \qquad (26d)$$

$$b_{15} = \sum_{i=1}^{N} (y_i n_{1_i} n_{3_i} W_i - z_i n_{1_i} n_{2_i} W_i) \qquad (26e)$$

$$b_{16} = \sum_{i=1}^{N} (z_i n_{1_i}^2 W_i - x_i n_{1_i} n_{3_i} W_i) \qquad (26f)$$

$$b_{22} = \sum_{i=1}^{N} n_{2_i}^2 W_i \qquad (26g)$$

$$b_{23} = \sum_{i=1}^{N} n_{2_i} n_{3_i} W_i \qquad (26h)$$

$$b_{24} = \sum_{i=1}^{N} (x_i n_{2_i}^2 W_i - y_i n_{1_i} n_{2_i} W_i) \qquad (26i)$$

$$b_{25} = \sum_{i=1}^{N} (y_i n_{2_i} n_{3_i} W_i - z_i n_{2_i}^2 W_i) \qquad (26j)$$

$$b_{26} = \sum_{i=1}^{N} (z_i n_{1_i} n_{2_i} W_i - x_i n_{3_i} n_{2_i} W_i) \qquad (26k)$$

$$b_{33} = \sum_{i=1}^{N} n_{3_i}^2 W_i \qquad (26l)$$

$$b_{34} = \sum_{i=1}^{N} (x_i n_{2_i} n_{3_i} W_i - y_i n_{1_i} n_{3_i} W_i) \qquad (26m)$$

$$b_{35} = \sum_{i=1}^{N} (y_i n_{3_i}^2 W_i - z_i n_{2_i} n_{3_i} W_i) \qquad (26n)$$

$$b_{36} = \sum_{i=1}^{N} (z_i n_{1_i} n_{3_i} W_i - x_i n_{3_i}^2 W_i) \qquad (26o)$$

$$b_{44} = \sum_{i=1}^{N} \left(W_i \,(n_{1_i} y_i - n_{2_i} x_i)^2 \right) \qquad (26p)$$

$$b_{45} = \sum_{i=1}^{N} \left(W_i \,(n_{1_i} y_i - n_{2_i} x_i)(n_{2_i} z_i - n_{3_i} y_i) \right) \qquad (26q)$$

$$b_{46} = \sum_{i=1}^{N} \left(W_i \,(n_{1_i} y_i - n_{2_i} x_i)(n_{3_i} x_i - n_{1_i} z_i) \right) \qquad (26r)$$

$$b_{55} = \sum_{i=1}^{N} \left(W_i \,(n_{2_i} z_i - n_{3_i} y_i)^2 \right) \qquad (26s)$$

$$b_{56} = \sum_{i=1}^{N} \left(W_i \,(n_{2_i} z_i - n_{3_i} y_i)(n_{3_i} x_i - n_{1_i} z_i) \right) \qquad (26t)$$

$$b_{66} = \sum_{i=1}^{N} \left(W_i \,(n_{3_i} x_i - n_{1_i} z_i)^2 \right) \qquad (26u)$$

The entries of the right-hand side vector are:

$$c_1 = \sum_{i=1}^{N} (W_i n_{1_i})\,(n_{1_i} u_i + n_{2_i} v_i + n_{3_i} w_i) \qquad (27a)$$

$$c_2 = \sum_{i=1}^{N} (W_i n_{2_i})\,(n_{1_i} u_i + n_{2_i} v_i + n_{3_i} w_i) \qquad (27b)$$

$$c_3 = \sum_{i=1}^{N} (W_i n_{3_i})\,(n_{1_i} u_i + n_{2_i} v_i + n_{3_i} w_i) \qquad (27c)$$

$$c_4 = -\sum_{i=1}^{N} W_i \,(n_{1_i} y_i - n_{2_i} x_i)\,(n_{1_i} u_i + n_{2_i} v_i + n_{3_i} w_i) \qquad (27d)$$

$$c_5 = -\sum_{i=1}^{N} W_i \left(n_{2_i} z_i - n_{3_i} y_i\right)\left(n_{1_i} u_i + n_{2_i} v_i + n_{3_i} w_i\right) \tag{27e}$$

$$c_6 = -\sum_{i=1}^{N} W_i \left(n_{3_i} x_i - n_{1_i} z_i\right)\left(n_{1_i} u_i + n_{2_i} v_i + n_{3_i} w_i\right) \tag{27f}$$

The solution of Eq. (25) yields the components of the rigid body motion which minimize the rms.

D. Minimization of the Root Mean Square with Respect to the Focal Length

After the components of the rigid body motion have been eliminated from the displacement vector, the distortions can be further minimized with respect to the focal length. Referring to Fig. 3, if the new focal length is f', the new distortion vector becomes

$$\begin{Bmatrix} d'_{x_i} \\ d'_{y_i} \\ d'_{z_i} \end{Bmatrix} = \begin{Bmatrix} d_{x_i} \\ d_{y_i} \\ d_{z_i} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \dfrac{x_i^2 + y_i^2}{4}\left(\dfrac{1}{f} - \dfrac{1}{f'}\right) \end{Bmatrix} \tag{28}$$

Defining

$$\xi = \frac{f}{f'} \tag{29}$$

and using

$$z_i = \frac{x_i^2 + y_i^2}{4f} \tag{30}$$

Eq. (28) can be written as

$$\mathbf{d}' = d_{x_i}\mathbf{i} + d_{y_i}\mathbf{j} + \left[d_{z_i} + z_i(1-\xi)\right]\mathbf{k} \tag{31}$$

Using the components of $\mathbf{d}'$ in Eq. (23), one writes

$$S = \sum_{i=1}^{N} W_i \left\{n_{1_i} d_{x_i} + n_{2_i} d_{y_i} + n_{3_i}\left[d_{z_i} + z_i(1-\xi)\right]\right\}^2 \tag{32}$$

The minimization of S with respect to f may be realized if one writes

$$\frac{dS}{d\xi} = \sum_{i=1}^{N} 2W_i \left\{n_{1_i} d_{x_i} + n_{2_i} d_{y_i} + n_{3_i} \times \left[d_{z_i} + z_i(1-\xi)\right]\right\}\left(-n_{3_i} z_i\right) = 0 \tag{33}$$

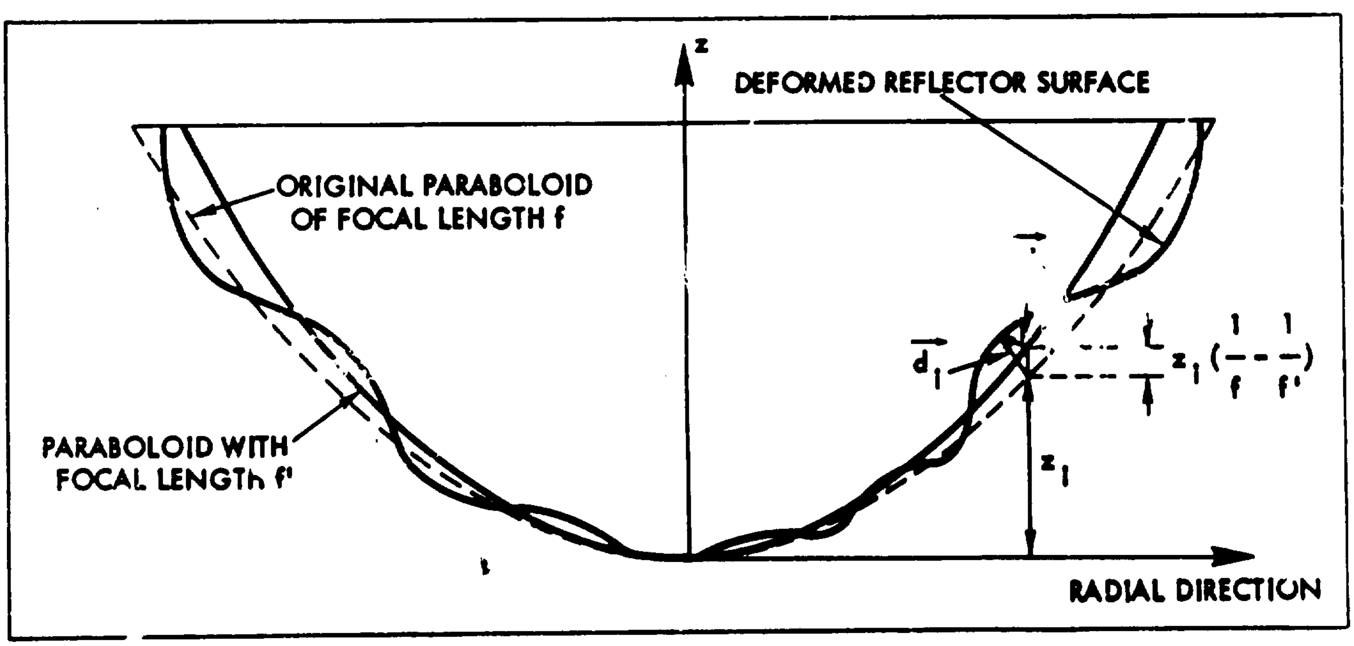

Fig. 3. The minimization of path length change by the variation of focal length

This leads to

$$\xi = \frac{\sum_{i=1}^{N} W_i \left[n_{3_i} z_i \left(n_{1_i} d_{x_i} + n_{2_i} d_{y_i} \right) + n_{3_i}^2 z_i \left(d_{z_i} + z_i \right) \right]}{\sum_{i=1}^{N} n_{3_i}^2 z_i^2 W_i} \tag{34}$$

and it follows that

$$f' = \frac{f \sum_{i=1}^{N} W_i \left(n_{3_i}^2 z_i^2 \right)}{\sum_{i=1}^{N} W_i \left[n_{3_i} z_i \left(n_{1_i} d_{x_i} + n_{2_i} d_{y_i} \right) + n_{3_i}^2 z_i \left(d_{z_i} + z_i \right) \right]} \tag{35}$$

Using ξ from Eq. (34) in Eq. (32), one obtains the minimum rms from Eq. (22).

III. FORMULATION OF NUMERICAL SOLUTION

A digital computer program was developed to compute the entries of an augmented matrix associated with Eq. (25) from the x_i, y_i, z_i, u_i, v_i, and w_i of the observed points on the surface, and the solution of this set was obtained. Having this solution, the displacements of the rigid body motion were purified and the remaining distortions were processed with respect to the focal length in order to obtain the minimum rms. The flow chart of the program is given in Fig. 4. The listing of this program and instructions for its use are given in Appendix E.

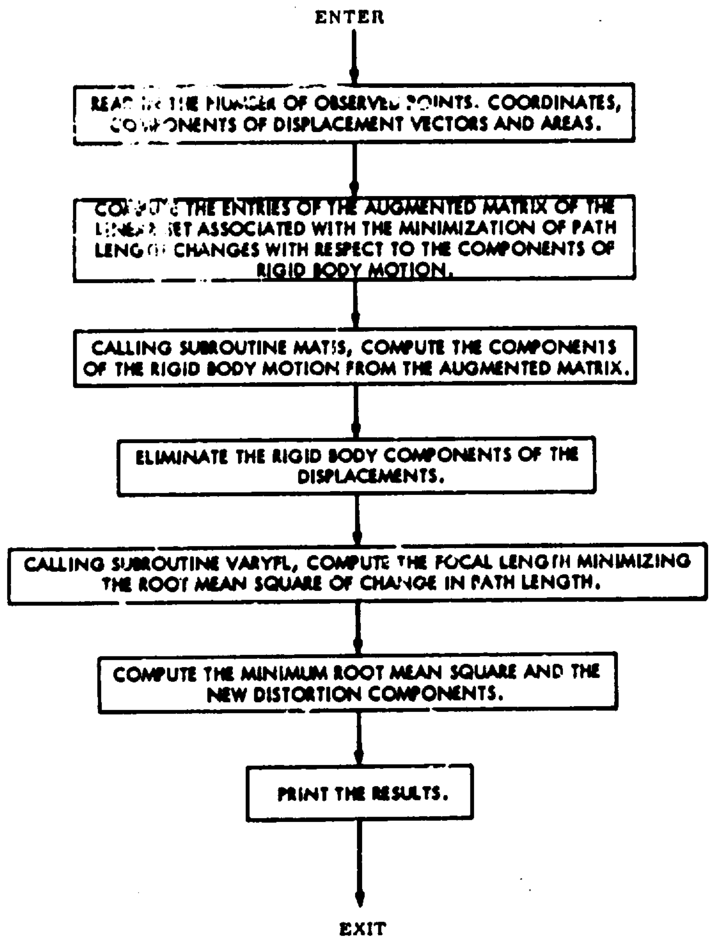

Fig. 4. Flow chart of the program

NOMENCLATURE

A_i	area of the reflector surface associated with the ith point
b_{ij}, c_i	entries of the augmented matrix related with the minimization of rms with respect to the components of rigid body motion
$\mathbf{d}_i, d_{x_i}, d_{y_i}, d_{z_i}$	distortion vector and its cartesian components (before minimization with respect to focal length)
$\mathbf{d}'_i, d'_{x_i}, d'_{y_i}, d'_{z_i}$	distortion vector and its cartesian components (after minimization with respect to focal length)
f, f'	original and rms minimizing focal lengths
$F(x, y, z)$	function representing closed form equation of the reflector surface
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors of the fixed cartesian coordinate system
$\mathbf{n}_i, n_{1_i}, n_{2_i}, n_{3_i}$	unit normal vector and its cartesian components of the reflector surface (heading towards the center of curvature)
N	total number of observed points on the reflector surface
$\mathbf{q}_i, u_i, v_i, w_i$	displacement vector and its cartesian components
$\mathbf{r}_i, x_i, y_i, z_i$	position vector of undeformed reflector and its cartesian components
$\mathbf{r}_{r_i}$	Rotated $\mathbf{r}$ vector
rms	Weighted root mean square of the path length changes
S	part of the rms expression containing the minimization variables
$\mathbf{t}, u_0, v_0, w_0$	translation vector and its cartesian components
W_i	weights of the normal components of the reflector distortions
θ, ϕ, ψ	rotations about z-axis, rotated x-axis, and rotated rotated y-axis
$\boldsymbol{\rho}_i$	position vector of the deformed reflector points
$\boldsymbol{\omega}_i$	rotation component of the displacement vector $\mathbf{q}_i$
λ_i	change in the path length of the ith ray because of distortions
ϵ	ratio of the focal lengths (original to minimizing)

REFERENCES

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2. Ruze, John, "Physical Limitations on Antenna," *Technical Report 248*, October, 1952, ASTIA/AD No. 6235, Research Laboratory of Electronics, M. I. T., Cambridge, Massachusetts.

APPENDIX A

Coordinate Transformation with Successive Rotations

Referring to Fig. A-1, a vector $\mathbf{r}$ can be written in the (x, y, z) coordinate system

$$\mathbf{r} = [x\ y\ z]\begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix} \tag{A-1}$$

and in the (x', y', z') coordinate system

$$\mathbf{r} = [x'\ y'\ z']\begin{Bmatrix} \mathbf{i}' \\ \mathbf{j}' \\ \mathbf{k}' \end{Bmatrix} \tag{A-2}$$

The relation between the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of the (x, y, z) coordinate system and the $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ of the (x', y', z') coordinate system is

$$\begin{Bmatrix} \mathbf{i}' \\ \mathbf{j}' \\ \mathbf{k}' \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix} \tag{A-3}$$

where the rows of the coefficient matrix are the direction cosines of the rotated axes in the original (x, y, z) coordinate system.

Substituting the $\mathbf{i}', \mathbf{j}', \mathbf{k}'$, of Eq. (A-3) into Eq. (A-2) and equating the latter to Eq. (A-1), one obtains

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{A-4}$$

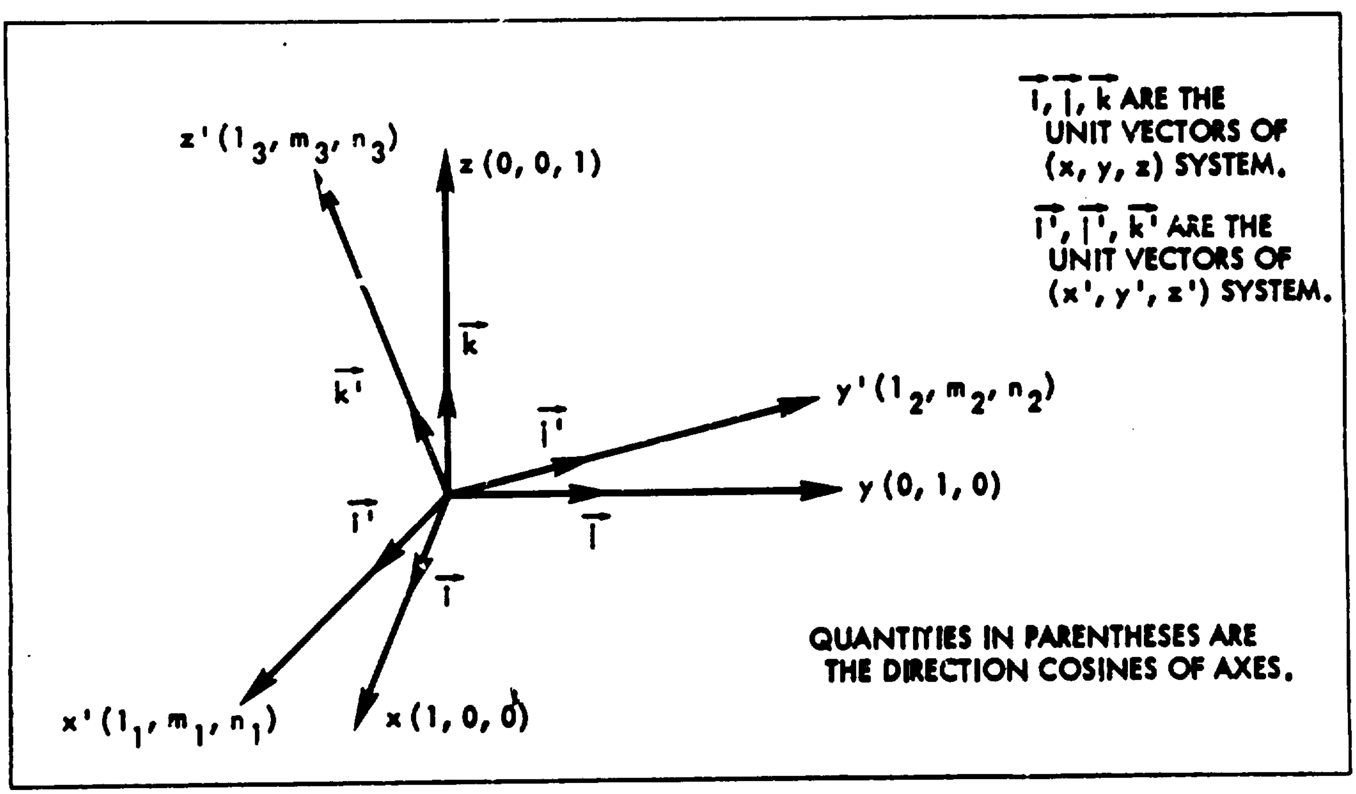

Fig. A-1. Rotation of coordinate system

Let $[Q_1]$ represent the coefficient matrix of (A-4). If the (x', y', z') coordinate system is subjected to another rotation, the components of $\mathbf{r}$ in this new system are

$$\begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix} = [Q_2]\,[Q_1] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{A-5}$$

Similarly, for a third rotation the components of $\mathbf{r}$ are

$$\begin{Bmatrix} x''' \\ y''' \\ z''' \end{Bmatrix} = [Q_3]\,[Q_2]\,[Q_1] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{A-6}$$

For the case discussed in the text, rotation of θ about the z axis yields

$$[Q_1] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A-7}$$

rotation of ϕ about the x' axis yields

$$[Q_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \tag{A-8}$$

and rotation of ψ about the y'' axis yields

$$[Q_3] = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix} \tag{A-9}$$

Then it follows from Eq. (A-6) that the total rotation matrix is

$$[Q] = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A-10}$$

For small rotations, one can truncate the Taylor expansion series for sine and cosine after the first power term. Maintaining the first power approximation, Eq. (A-10) can be rewritten,

$$[Q] = \begin{bmatrix} 1 & \theta & -\psi \\ -\theta & 1 & \phi \\ \psi & -\phi & 1 \end{bmatrix} \tag{A-11}$$

which is identical with the square matrix appearing in Eq. (10) of the text.

APPENDIX B

Computer Program

This program, written in Fortran for the 32K IBM 7090, can handle 1000 observed points on the reflector surface. This program requires the x_i, y_i, z_i coordinates, the u_i, v_i, w_i components of the displacement vectors, and the reflector surface area associated with each observed point. The input should be compatible with the following Fortran statements:

```
    READ INPUT TAPE 5, 1, NP, F, (X(I), Y(I),
    Z(I), U(I), V(I), W(I), A(I), K, I=1, NP)
1   FORMAT (I10, E15. 5, /, (7F10.5,I10))
```

where NP is the total number of observed points, F is the original focal length, X, Y, and Z are the coordinates of the observed points, U, V, and W are the components of the displacement vector, A is the reflector surface area associated with each observed point, and K is the sequence number of the input card.

The output of this program is the root mean square of the changes in path length, the orientation and focal length of the best fit paraboloid, and the components of the associated distortions. No tapes or sense switch settings are required; however, one may bypass minimization with respect to the focal length by setting sense switch 1 down. The program given on the following pages does not minimize the root mean square with respect to the rotation about the axis of the reflector.

```
*       XEQ
*       LIST8
*       LABEL
C       BEST FIT PARABOLOID WITH MINIMUM PATH LENGTH IN LEAST SQUARE SENSE
        DIMENSION X(1000),Y(1000),Z(1000),U(1000),V(1000),W(1000),B(6,6),
       1C(6),A(1000),D1(1000),D2(1000),D3(1000),R(6),WT(1000),BI(6,6),XX(6
       2)
        READ INPUT TAPE 5,1,NP,F,(X(I),Y(I),Z(I),U(I),V(I),W(I),A(I),K,I=1
       1,NP)
        WRITE OUTPUT TAPE 6,2,(X(I),Y(I),Z(I),U(I),V(I),W(I),A(I),I,I=1,N
       1P)
        DO 20 I=1,6
        C(I)=0.
        DO 19 J=1,6
   19   B(I,J)=0.
   20   CONTINUE
        S=0.
        SW=0.
        DO 30 I=1,NP
        T=SQRTF (4.*(X(I)**2+Y(I)**2+4.*F**2))
        D1(I)=-2.*X(I)/T
        D2(I)=-2.*Y(I)/T
        D3(I)=4.*F/T
        WT(I)=4.*D3(I)**2*A(I)
        B(1,1)=B(1,1)+D1(I)**2*WT(I)
        B(1,2)=B(1,2)+D1(I)*D2(I)*WT(I)
        B(1,3)=B(1,3)+D1(I)*D3(I)*WT(I)
        B(1,4)=B(1,4)+(X(I)*D1(I)*D2(I)-Y(I)*D1(I)**2)*WT(I)
        B(1,5)=B(1,5)+(Y(I)*D1(I)*D3(I)-Z(I)*D1(I)*D2(I))*WT(I)
        B(1,6)=B(1,6)+(Z(I)*D1(I)**2-X(I)*D1(I)*D3(I))*WT(I)
        B(2,2)=B(2,2)+D2(I)**2*WT(I)
        B(2,3)=B(2,3)+D2(I)*D3(I)*WT(I)
        B(2,4)=B(2,4)+(X(I)*D2(I)**2-Y(I)*D1(I)*D2(I))*WT(I)
        B(2,5)=B(2,5)+(Y(I)*D2(I)*D3(I)-Z(I)*D2(I)**2)*WT(I)
        B(2,6)=B(2,6)+(Z(I)*D2(I)*D 1(I)-X(I)*D3(I)*D2(I))*WT(I)
        B(3,3)=B(3,3)+D3(I)**2*WT(I)
        B(3,4)=B(3,4)+(X(I)*D2(I)*D3(I)-Y(I)*D1(I)*D3(I))*WT(I)
        B(3,5)=B(3,5)+(Y(I)*D3(I)**2-Z(I)*D2(I)*D3(I))*WT(I)
        B(3,6)=B(3,6)+(Z(I)*D1(I)*D3(I)-X(I)*D3(I)**2)*WT(I)
        B(4,4)=B(4,4)+WT(I)*(D1(I)*Y(I)-D2(I)*X(I))**2
        B(4,5)=B(4,5)+WT(I)*(D1(I)*Y(I)-D2(I)*X(I))*(D2(I)*Z(I)-D3(I)*Y(I)
       1)
        B(4,6)=B(4,6)+WT(I)*(D1(I)*Y(I)-D2(I)*X(I))*(D3(I)*X(I)-D1(I)*Z(I)
       1)
        B(5,5)=B(5,5)+WT(I)*(D2(I)*Z(I)-D3(I)*Y(I))**2
        B(5,6)=B(5,6)+WT(I)*(D2(I)*Z(I)-D3(I)*Y(I))*(D3(I)*X(I)-D1(I)*Z(I)
       1)
        B(6,6)=B(6,6)+WT(I)*(D3(I)*X(I)-D1(I)*Z(I))**2
        Q=D1(I)*U(I)+D2(I)*V(I)+D3(I)*W(I)
        C(1)=C(1)+Q*WT(I)*D1(I)
        C(2)=C(2)+Q*WT(I)*D2(I)
        C(3)=C(3)+Q*WT(I)*D3(I)
```

```
      C(4)=C(4)-Q*WT(I)*(D1(I)*Y(I)-D2(I)*X(I))
      C(5)=C(5)-Q*WT(I)*(D2(I)*Z(I)-D3(I)*Y(I))
   30 C(6)=C(6)-Q*WT(I)*(D3(I)*X(I)-D1(I)*Z(I))
      DO 40 I=1,6
      DO 40 J=I,6
   40 B(J,I)=B(I,J)
      B(4,1)=0.
      B(4,2)=0.
      B(4,3)=0.
      B(4,4)=1.0
      B(4,5)=0.
      B(4,6)=0.
      C(4)=0.
      M=6
      N=1
      D=0.
      WRITE OUTPUT TAPE 6,9, ((B(I,J),J=1,6),I=1,6)
      WRITE OUTPUT TAPE 6,11, (C(I),I=1,6)
      CALL MATIS (B,M.R,N,D,BI)
      IF (ABSF(D)-.0000001) 99,99,98
   98 DO 50 I=1,6
      XX(I)=0.
      DO 49 K=1,6
   49 XX(I)=XX(I)+BI(I,K)*C(K)
   50 CONTINUE
      DO 60 I=1,NP
      U(I)=U(I)-(XX(1)-XX(4)*Y(I)+XX(6)*Z(I))
      V(I)=V(I)-(XX(2)+XX(4)*X(I)-XX(5)*Z(I))
   60 W(I)=W(I)-(XX(3)-XX(6)*X(I)+XX(5)*Y(I))
      WRITE OUTPUT TAPE 6,3,(U(I),V(I),W(I),I,I=1,NP)
      IF (SENSE SWITCH 1) 70,69
 69   WRITE OUTPUT TAPE 6,7
      CALL VARYFL (X,Y,Z,U,V,W,A,WT,D1,D2,D3,NP,F)
      GO TO 81
 70   WRITE OUTPUT TAPE 6,6
 81   DO 80 I=1,NP
      SW=SW+WT(I)
   80 S=S+WT(I)*(D1(I)*U(I)+D2(I)*V(I)+D3(I)*W(I))**2
      RMS=SQRTF (S/SW)
      WRITE OUTPUT TAPE 6,4,RMS,(XX(I),I=1,6),F
      WRITE OUTPUT TAPE  6,8,(D1(I),D2(I),D3(I),WT(I),I,I=1,NP)
      WRITE OUTPUT TAPE 6,10,  ((BI(I,J),J=1,6),I=1,6)
8     FORMAT (1H1,/,23X,2HN1,23X,2HN2,23X,2HN3,23X,2HWT,19X,1HI,//,
     1(4E25.9,I20))
9     FORMAT (19H1COEFFICIENT MATRIX,//,(6E20.9))
10    FORMAT (12H1THE INVERSE,//,(6E20.9))
11    FORMAT (9H1C MATRIX,//,(E50.9))
      CALL EXIT
 99   WRITE OUTPUT TAPE 6,5
      CALL EXIT
    1 FORMAT (I10,E15.5,/,(7F10.5,I10))
    2 FORMAT (88H1BEST FIT PARABOLOID WITH MINIMUM PATH LENGTH IN LEAST
```

```
     1SQUARE SENSE, UTKU-BARONDESS, JPL,///,3X,13H X COORDINATE,3X, 13H
     2Y COORDINATE,3X,13H Z COORDINATE,3X,13H U DEFLECTION,3X,13H V DEFL
     3ECTION,3X,13H W DEFLECTION,7X,5H AREA,4X,8HPOINT NO,//,(7E16.5,I8)
     4)
   3 FORMAT (66H1DISTORTIONS AFTER MINIMIZATION WITH RESPECT TO RIGID
     1BODY MOTION,//,12X,13H U DISTORTION,12X,13H V DISTORTION,12X,13H W
     2 DISTORTION,3X,13H POINT NUMBER,//,(3E25.9,I16))
   4 FORMAT (24H1THE ROOT-MEAN SQUARE IS,E25.9,//, 34H X COORDINATE OF
     1THE APEX POINT IS,E25.9,//, 34H Y COORDINATE OF THE APEX POINT IS,
     2E25.9,//,34H Z COORDINATE OF THE APEX POINT IS,E25.9,//,25H ROTATI
     3ON ABOUT Z AXIS IS,E25.9,//,33H ROTATION ABOUT ROTATED X AXIS IS,E
     425.9,//,41H ROTATION ABOUT ROTATED ROTATED Y AXIS IS,E25.9,//,47H
     5THE FOCAL LENGTH OF THE BEST FIT PARABOLOID IS,E25.9)
5     FORMAT ( 57H1THE COEFFICIENT MATRIX IS SINGULAR, SOLUTION IS DELET
     1ED.)
6     FORMAT ( 91H1THE PROGRAM IS NOT REQUESTED TO DO MINIMIZATION WITH
     1RESPECT TO FOCAL LENGTH, SS1 IS DOWN.)
 7    FORMAT (85H1THE PROGRAM IS REQUESTED TO DO MINIMIZATION WITH RESPE
     1CT TO FOCAL LENGTH, SS1 IS UP.)
      END
```

```
*      JPL,J007000,01113330    MATIS    6X6    UTKU-BARONDESS
*      LABEL
*      LIST8
*      FAP
*                SUBROUTINE MATIS
*                THIS SUBROUTINE IS IDENTICAL WITH MATIV ON JPL LIBRARY
*                TAPE (AUGUST 1962) WHICH IS REASSEMBLED FOR 6X6 ARRAYS.
       TTL       MATRIX INVERSION
       LBL       MATIS ,X
       ENTRY     MATIS
*                SUBROUTINE MATIS (A,N,B,M.DETERM,C)
*                THIS SUBROUTINE SAVES MATRTX A
 MATIS SXD BOY,4
       CLA 1,4
       ADD ONE
       STA Z
       CLA 2,4
       STO ZZ+2
       CLA 3,4
       STO ZZ+3
       CLA 4,4
       STO ZZ+4
       CLA 5,4
       STO ZZ+5
       CLA 6,4
       STO ZZ+1
       ADD ONE
       STA Y
       AXT 36,4
     Z CLA *,4
     Y STO *,4
       TIX *-2,4,1
    ZZ CALL MATIN
       TSX *
       TSX *
       TSX *
       TSX *
       TSX *
       LXD BOY,4
       TRA      7,4
   ONE DEC 1
   BOY
       END
```

```
      LABEL
      LIST8
      SUBROUTINE VARYFL (X,Y,Z,U,V,W,A,WT,D1,D2,D3,NP,F)
      DIMENSION X(1000),Y(1000),Z(1000),U(1000),V(1000),W(1000),A(1000)
     1,WT(1000),D1(1000),D2(1000),D3(1000)
      S=0.
      B=0.
      DO 20 I=1,NP
      AX=4.*A(I)*D3(I)**3*Z(I)
      P=D1(I)*U(I)+D2(I)*V(I)+D3(I)*W(I)
      S=S+AX*P
      B=B+AX*Z(I)*D3(I)
   20 CONTINUE
      R=S/B
      DO 30 I=1,NP
   30 W(I)=W(I)-Z(I)*R
      FN=F/(1.+R)
      WRITE OUTPUT TAPE 6,1,F,FN,(W(I),W(I+1),W(I+2),W(I+3),W(I+4),W(I+5
     1),W(I+6),W(I+7),W(I+8),W(I+9),I,I=1,NP,10)
      F=FN
      RETURN
    1 FORMAT (29H1THE ORIGINAL FOCAL LENGTH IS,E25.9,//,24H THE NEW FOCA
     1L LENGTH IS,E25.9,////,56H THE NEW DISTORTIONS IN DIRECTION OF PAR
     2ABOLOID AXIS ARE,//,(10F11.6,I10))
      END
```

```
*      JPL,J007000,01113330   MATIN 6X6     UTKU-BARONDESS
*      LABEL
*      LIST8
       SUBROUTINE MATIN (A,N,B,M,DETERM)
C
C      THIS SUBROUTINE IS IDENTICAL WITH MATINV ON JPL LIBRARY TAPE
C      (AUGUST 1962) WHICH IS RECOMPILED DELETING COMMON STATEMENT AND
C      CHANGING DIMENSION STATEMENT FOR 6X6 ARRAYS.
C      MATRIX INVERSION WITH ACCOMPANYING SOLUTION OF LINEAR EQUATIONS
C
       DIMENSION IPIVOT(6),A(6,6),B(6,1),INDEX(6,2),PIVOT(6)
       EQUIVALENCE (IROW,JROW), (ICOLUM,JCOLUM), (AMAX, T, SWAP)
C
C      INITIALIZATION
C
   10 DETERM=1.0
   15 DO 20 J=1,N
   20 IPIVOT(J)=0
   30 DO 550 I=1,N
C
C      SEARCH FOR PIVOT ELEMENT
C
   40 AMAX=0.0
   45 DO 105 J=1,N
   50 IF (IPIVOT(J)-1) 60, 105, 60
   60 DO 100 K=1,N
   70 IF (IPIVOT(K)-1) 80, 100, 740
   80 IF (ABSF(AMAX)-ABSF(A(J,K))) 85, 100, 100
   85 IROW=J
   90 ICOLUM=K
   95 AMAX=A(J,K)
  100 CONTINUE
  105 CONTINUE
  110 IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1
C
C      INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL
C
  130 IF (IROW-ICOLUM) 140, 260, 140
  140 DETERM=-DETERM
  150 DO 200 L=1,N
  160 SWAP=A(IROW,L)
  170 A(IROW,L)=A(ICOLUM,L)
  200 A(ICOLUM,L)=SWAP
  205 IF(M) 260, 260, 210
  210 DO 250 L=1, M
  220 SWAP=B(IROW,L)
  230 B(IROW,L)=B(ICOLUM,L)
  250 B(ICOLUM,L)=SWAP
  260 INDEX(I,1)=IROW
  270 INDEX(I,2)=ICOLUM
  310 PIVOT(I)=A(ICOLUM,ICOLUM)
```

```
  320 DETERM=DETERM*PIVOT(I)
C
C     DIVIDE PIVOT ROW BY PIVOT ELEMENT
C
  330 A(ICOLUM,ICOLUM)=1.0
  340 DO 350 L=1,N
  350 A(ICOLUM,L)=A(ICOLUM,L)/PIVOT(I)
  355 IF(M) 380, 380, 360
  360 DO 370 L=1,M
  370 B(ICOLUM,L)=B(ICOLUM,L)/PIVOT(I)
C
C     REDUCE NON-PIVOT ROWS
C
  380 DO 550 L1=1,N
  390 IF(L1-ICOLUM) 400, 550, 400
  400 T=A(L1,ICOLUM)
  420 A(L1,ICOLUM)=0.0
  430 DO 450 L=1,N
  450 A(L1,L)=A(L1,L)-A(ICOLUM,L)*T
  455 IF(M) 550, 550, 460
  460 DO 500 L=1,M
  500 B(L1,L)=B(L1,L)-B(ICOLUM,L)*T
  550 CONTINUE
C
C     INTERCHANGE COLUMNS
C
  600 DO 710 I=1,N
  610 L=N+1-I
  620 IF (INDEX(L,1)-INDEX(L,2)) 630, 710, 630
  630 JROW=INDEX(L,1)
  640 JCOLUM=INDEX(L,2)
  650 DO 705 K=1,N
  660 SWAP=A(K,JROW)
  670 A(K,JROW)=A(K,JCOLUM)
  700 A(K,JCOLUM)=SWAP
  705 CONTINUE
  710 CONTINUE
  740 RETURN
      END
```