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TRANSPORT PROPERTIES OF A TURBULENT LORENTZ GAS*

by

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ABSTRACT

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The fields of a turbulent Lorentz gas with average magnetic field are expressed in terms of fluctuating quantities. The equations of conservation of momentum and matter are used along with Maxwell's equations to obtain a Fourier analyzed momentum equation in terms of the fluctuating velocities alone. Non-linear terms are linearized through the use of a dynamic viscosity-like term after Heisenberg. The study is then restricted to a semi-compressible plasma. It is found that in general six modes of wave motion are possible; three are modified classical modes and the others are mixed or "turbulent modes" which express the cross correlation between the classical modes present in the turbulence. Numerical results are obtained for the indices of refraction and the mobilities for the various modes. One of the most important results of the analysis is the appearance of an enhanced diffusion above a certain critical value of the magnetic field.

author

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CHAPTER I

INTRODUCTION

1.1 Turbulence and Plasma Dynamics

Collective fluctuating phenomena in ionized gases have been known to exist in several situations. It has often been argued, for instance, that ionized matter in interstellar space is in a state of turbulence due to the large Reynolds number associated with the flow fields.[1] In a totally different situation, investigators have reported on the random appearance of the magnetic and velocity fields accompanying a collapsing pinch.[2] It is also known to workers in gas discharges that the density of the plasma and the electromagnetic field radiated from the positive column of the discharge exhibits a predominately statistical character implying that the plasma is essentially in a state of turbulence.[3]

There have been some arguments that the fluctuations encountered in the phenomena described above do not actually correspond to a turbulent state of the plasma as understood in a classical sense but are due rather to the presence of microinstabilities.[4] This fine distinction is felt to be artificial and the point of view adopted here is that the character of the fluctuating field is akin to that of the usually recognized turbulent field.

In order to clarify the nature of these fluctuations some of the characteristic features of a turbulent field will be summarized. Turbulent motion is characterized by the fact that if the velocity, for example, is measured under seemingly identical conditions, the values obtained are found to be a random function of position and time. For the sake of simplicity restriction is made to homogeneous turbulence for which the probabilities associated with random quantities are independent of spatial translation.[5]

It turns out that the velocity and other field quantities are continuous functions of space and time so that they may be Fourier analyzed. This Fourier analysis is a resolution into components of different linear size since the wavelength is a parameter specifying the different Fourier components. Turbulent motion may therefore be thought of as consisting of the superposition of a large number of different sized components which make additive contributions to the field quantity. These components interact with each other in a way demanded by the non-linear terms in the equation of motion. If this equation were linear, the excitation of one of the components would not involve the others; however, it is precisely these non-linear terms which produce the turbulence by requiring coupling between modes.

From probability theory it can be shown that turbulent motion is completely specified by the complete set of averaged products of the field vectors. When the turbulence is homogeneous this average may be a spatial one. Two types of averaged products which are germane to this work are the first order and second order product mean values.

For example, if the field quantity chosen is the velocity $\underline{v}$,

$$\langle v_i(\underline{r})\rangle = \frac{1}{V}\int v_i(\underline{r})\,d\underline{r} \tag{1-1}$$

is the first order product mean value. Here V is the volume over which the average is taken and $\underline{r}$ is the position vector.

The second order product mean value is

$$\langle v_i(\underline{r})\,v_i(\underline{r}-\underline{r}')\rangle = \frac{1}{V}\int v_i(\underline{r})\,v_i(\underline{r}-\underline{r}')\,d\underline{r} \tag{1-2}$$

and is better known as the "velocity correlation". The vector $\underline{r}-\underline{r}'$ is the separation in space between the locations at which the partial products are measured. In (1-2) the partial products are, of course, functions of position and time so that the velocity correlation is in general a function of time.

Throughout this investigation turbulence will be studied as resulting from the dynamics of a plasma fluid so that moments of the Boltzmann-Vlasov equation are involved. This is in contradistinction to a totally different technique in which solutions of the Boltzmann-Vlasov equation exhibiting a random character are considered.

1.2 Critical Survey of Hydromagnetic Turbulence

In early work on hydromagnetic turbulence, the line of approach often consisted in reasoning similar to that of classical hydrodynamic turbulence (Batchelor[6] and Lee[7]). Batchelor, for instance, employed an analogy between the magnetic field and the vorticity field of hydrodynamics obtaining conditions under which growth of the average magnetic field would take place assuming the presence of small spontaneous fields. More recently this problem of the growth of a magentic field in a turbulent conducting fluid has been considered by Pao.[8]

Since 1952 there have been two main lines of attack used in the study of plasma turbulence. In the first, one starts with the equations for the conservation of momentum and matter, assumes homogeneity and isotropy of the field, and derives spectral energy densities for the field in terms of correlation functions. This method will be referred to as the "deductive approach". In the other line of reasoning, certain heuristic assumptions are made to account for the transfer of turbulent energy between different Fourier components of the flow field. This second line of attack will be referred to as the "heuristic approach". The more salient aspects of this recent work are summarized below.

1.2.1 Deductive Approach

One of the first serious attempts to develop a deductive theory of turbulence in a plasma is that of S. Chandrasekhar.[9] Chandrasekhar's theory is based on earlier work concerned with classical turbulence in a incompressible fluid.[10] This theory of plasma turbulence includes the following assumptions:

(1) The conducting fluid is incompressible,

(2) In the spirit of the usual approximations of MHD, charge neutrality prevails and the displacement current is neglected,

(3) The turbulent field is homogeneous,

(4) A stationary state prevails so that energy supplied to maintain turbulence is dissipated thermally,

(5) The assumption of isotropy is made which requires that the time average of any function of the field quantities defined with respect to a particular set of axes is invariant under arbitrary rotations and reflections of the axes of reference,

(6) Correlations are introduced between u_i and h_j, the fluctuating velocity and magnetic field, at two

different points and at two different times. By introducing the time interval into the definition of the correlations, Chandrasekhar is able to account for the phase correlation effects in turbulent motion. The important assumption he makes here is that these correlations depend only on the difference in times t'-t" as far as the time dependence is concerned,

(7) It is assumed that all correlations which include an odd number of components of h_j will vanish identically, and finally

(8) The fourth order products introduced are assumed to be directly related to the second moments as in a normal distribution.

Of the many correlations introduced in Chandrasekhar's theory, of special interest in the present context are

$$Q_{ij} = \overline{u_i(\underline{r}',t')\,u_j(\underline{r}'',t'')} \tag{1-3}$$

and

$$H_{ij} = \overline{h_i(\underline{r}',t')\,h_j(\underline{r}'',t'')}. \tag{1-4}$$

The importance of the quantities Q_{ij} and H_{ij} is due to the fact that their Fourier transforms describe how the energy associated with each velocity or magnetic field component is distributed over the various wave-numbers and frequencies in a harmonic resolution of the turbulent fields.[5] These quantities are also important in that they provide a measure of the scale of the turbulent fields.

Under the assumptions enumerated above it can be shown that

$$Q_{ij} = \nabla \times Q(r,t)\, \epsilon_{ijl} (\underline{r}'' - \underline{r}')_l \tag{1-5}$$

and

$$H_{ij} = \nabla \times H(r,t)\, \epsilon_{ijl} (\underline{r}'' - \underline{r}')_l , \tag{1-6}$$

where $Q(r,t)$ and $H(r,t)$ are defining scalars of the tensors in question. The quantity ϵ_{ijk} is a unit alternating tensor having the values $\epsilon_{ijk} = 0$ when k, j and k are not all different; $\epsilon_{ijk} = +1$ or -1 when k, j and k are all different and in cyclic or acyclic order respectively.

In terms of Q and H the equation of motion for the fluid becomes

$$\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial t^2} - \sigma^{*2} D_5^2\right) Q$$
$$= -2Q\frac{\partial}{\partial r} D_5 Q - 2H\frac{\partial}{\partial r} D_5 H, \qquad (1\text{-}7)$$

where σ^* is the kinematic viscosity and

$$D_n = \frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r}. \qquad (1\text{-}8)$$

Another equation relating Q and H is developed from the equation involving the magnetic field. In conjunction with (1-7) there results a system connecting Q and H. The consequences of this system are pursued further for the case of zero viscosity and infinite conductivity with Kolmogoroff's law for the defining scalars being confirmed.

Of major interest here is (1-7) which represents the equation of motion. Chandrasekhar extended the Heisenberg approach to hydromagnetic turbulence in view of the symmetry of Q and H in this equation.

1.2.2 Heuristic Approach

1.2.2.1 <u>Work of Chandrasekhar.</u> In a subsequent paper on hydromagnetic turbulence Chandrasekhar points out that the equation

$$\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial t^2} - \sigma^{*2} D_5^2\right) Q = -2Q\frac{\partial}{\partial r} D_5 Q \qquad (1\text{-}9)$$

in the framework of ordinary hydrodynamics is, in the framework of hydromagnetics, replaced by (1-7). Now the equation for the time rate of change of the spectrum of kinetic energy in ordinary hydrodynamics is

$$\frac{1}{2}\frac{\partial F(k)}{\partial t} = \int_0^k Q(v,k';v,k)\,dk' - \int_k^\infty Q(v,k;v,k'')\,dk'' - \sigma^* F(k) k^2, \qquad (1\text{-}10)$$

where F(k) is the spectral function giving the energy density in k-space involved in Fourier components (or eddies) of wave-number <u>k</u>. Q(v,k'; v,k) is the transition probability such that the first integral gives the energy contributed by larger eddies and the second integral gives the energy lost to smaller eddies. Since Q and H appear symmetrically in (1-7) and in view of (1-9) and (1-10) Chandrasekhar extended (1-10) to hydromagnetics by the relation

$$\frac{1}{2}\frac{\partial F(k)}{\partial t} = \int_0^k Q(v,k';v,k)\,dk' + \int_0^k Q(h,k';v,k)\,dk' - \int_k^\infty Q(v,k;v,k'')\,dk'' - \int_k^\infty Q(v,k;h,k'')\,dk'' - \sigma^* F(k)k^2, \tag{1-11}$$

where Q(h,k'; v,k) and Q(v, k; h,k") are transition probabilities between the velocity and magnetic turbulent fields.

Letting G(k) be the spectrum of the turbulent magnetic field, Chandrasekhar writes the following

$$\frac{1}{2}\frac{\partial G(k)}{\partial t} = \int_0^k Q(v,k';h,k)\,dk' - \int_k^\infty Q(h,k;v,k'')\,dk'' - \lambda^* G(k)k^2, \tag{1-12}$$

where the integrals are representative of eddy energy transfer as above. The last term represents losses due to Joule dissipation of the magnetic energy associated with the kth mode and λ* stands for the resistivity. Equation (1-10) is recognized as the extension to hydromagnetics of the equation giving the time rate of change of the magnetic energy density in ordinary hydrodynamics of the equation giving the time rate of change of the magnetic energy density in ordinary hydrodynamics which does not involve the transition integrals.

The assumption is made that the velocity-velocity transition probability is given by the Heisenberg form

$$Q(v, k'; v, k'') = F(k')k'^2 \sigma(k''), \tag{1-12a}$$

where

$$\sigma(k) = K\sqrt{\frac{F(k)}{k^3}} \,. \tag{1-13}$$

Due to the fact that Q and H appear symmetrically in (1-7) the transition probability involving h_j and v_i is written in the form

$$Q(h, k; v, k'') = G(k')k'^2 \lambda(k''), \tag{1-14}$$

where

$$\lambda(k) = K\sqrt{\frac{G(k)}{k^3}} \,, \tag{1-15}$$

with the assumption that K is the same numerical constant as in (1-13). By integrating over k and using both (1-10) and (1-11) the generalization to hydromagnetics of the well-known equation of Heisenberg is obtained and is

$$
\begin{aligned}
-\frac{1}{2}\frac{\partial}{\partial t}\int_0^k dk' \left[F(k')+G(k')\right] \\
= \int_0^k dk' \left[(\sigma^* + \sigma_k) F(k') \right. \\
\left. + (\lambda^* + \lambda_k) G(k')\right] k'^2 .
\end{aligned}
\qquad (1\text{-}16)
$$

Another result of Chandrasekhar which he arrives at by assuming stationary turbulence, infinite conductivity and zero viscosity is that two turbulent "modes" are possible with different values of the ratio of magnetic to kinetic energy for the same wave number.

It might be mentioned here that a few other studies exist concerning the energy spectrum in magneto-fluid turbulence such as those of Deissler[12] and Tatsumi.[13] Tatsumi feels that Chandrasekhar's treatment of the magnetic-velocity transition probability is questionable since the magnetic field has no means of adjustment within its own spectrum as does the velocity field due to the linearity of Maxwell's equations. Kraichnan[14] has also criticized Chandrasekhar's assumption (8) connecting the fourth-order and second-order moments showing that there results no transfer of energy between eddies and the creation of energy by each mode.

It should be pointed out that one of the most restrictive assumptions made in all of the above mentioned treatments is that of the incompressibility of the conducting fluid. In effect this procedure removes all possible fluctuation of the charge density which is unrealistic in most cases. In the presence of an average magnetic field the assumption of isotropy which is usually made is also to be questioned.

1.2.2.2 Work of Yoshikawa. Taking compressibility and anisotropy effects into account Yoshikawa and Rose developed a theory of plasma turbulence in an attempt to explain the anomalous diffusion of the magnetic field.[15,16] Classical theory based on the linearized Boltzmann-Vlasov equation predicts a $1/B^2$ dependence of the diffusion on the magnetic field, B. On the other hand, it has been observed that as a result of the collective random oscillations in a plasma the diffusion shows a 1/B dependence.[17] The term "anomalous" diffusion is frequently used in the literature for the latter effect.

The assumptions made by Yoshikawa and Rose in their treatment are as follows:

(1) The plasma is macroscopically homogeneous and subject to homogeneous turbulence,

(2) A small pressure gradient is introduced,

(3) A uniform average electric field is introduced,

(4) The ions are assumed immobile,

(5) The effect of the fluctuation of the magnetic field is neglected which essentially means that the kinetic pressure is much smaller than the magnetic pressure,

(6) Temperature is assumed independent of position,

(7) The inertia term of the momentum equation is neglected which is justified by assuming the drift velocity is much smaller than the average thermal velocity,

(8) Cross terms in the equation for the kth harmonic are neglected,

(9) The plasma is considered to be in a steady state so that partials with respect to time vanish, and

(10) Isotropy is introduced in order to obtain results from the complex system of equations obtained.

Under these assumptions the momentum equation for the field quantities becomes

$$en\underline{E} + en\underline{E}' + e(\underline{\Gamma}_e \times \underline{B}) = -m\nu_c \underline{\Gamma}_e, \qquad (1\text{-}17)$$

where e is the charge of the electron, n the number density, $\underline{E}$ the fluctuating electric field, $\underline{E}'$ the effective average field which includes a contribution from the density gradient, $\underline{\Gamma}_e$ the drift current which is the product of the number density and the velocity, m the mass of the electron and ν_c the Coulomb collision frequency.

Equation (1-17) is the Fourier analyzed by means of expanding all fluctuating quantities in Fourier series; i.e.,

$$\underline{E} = \Sigma' \underline{E}_k e^{i\underline{k}\underline{r}}, \tag{1-18}$$

$$\underline{\Gamma}_e = \underline{\Gamma}_o + \Sigma' \underline{\Gamma}_k e^{i\underline{k}\underline{r}} \tag{1-19}$$

and

$$n = n_o + \Sigma' n_k e^{i\underline{k}\underline{r}}, \tag{1-20}$$

where the subscript k refers to the kth harmonic, the zero subscript indicates average values and the prime signifies that the value k = 0 is excluded from the sum. Due to the independence of the various Fourier harmonics ther results for the zeroth harmonic

$$e\Sigma' n_k E_{-k} + e n_o \underline{E}' + e(\underline{\Gamma}_o \times \underline{B}) =$$
$$-m\nu_c \Sigma' \left(\frac{n_k}{n_o}\right)\Gamma_{-k} - m\nu_o \underline{\Gamma}_o, \tag{1-21}$$

and for the kth harmonic

$$e n_0 \underline{E}_k + e n_k \underline{E}' + e(\underline{\Gamma}_k \times \underline{B}) \\ + (\text{cross terms}) \\ = m \nu_0 \underline{\Gamma}_k - m \nu_0 \left(\frac{n_k}{n_0}\right) \underline{\Gamma}_0 . \qquad (1\text{-}22)$$

In accordance with assumption (8) the cross terms in (1-22) are to be neglected. Now in conjunction with (1-21) and the Maxwell equations

$$\nabla \times \underline{E} = 0 \qquad (1\text{-}23)$$

and

$$\nabla \cdot \underline{\Gamma}_e = 0, \qquad (1\text{-}24)$$

it is possible to write $\underline{E}_k$ and $\underline{\Gamma}_k$ in terms of n_k, $\underline{E}'$, and $\underline{\Gamma}_0$. Writing $\underline{\Gamma}_0$ as a vector with components

$$\underline{\Gamma}_0 = (\Gamma_x, \Gamma_y, \Gamma_z), \qquad (1\text{-}25)$$

and after some manipulations which include the assumption of isotropy, Yoshikawa obtains for the x-component of the drift current

$$\Gamma_x = -\frac{1}{4}\pi\left(\frac{n_0 E'}{B}\right)S - \frac{m\nu_0}{e}\frac{n_0 E'}{B^2}, \qquad (1\text{-}26)$$

where S is the mean square deviation of the density fluctuation to be obtained experimentally. Nctice that for large magnetic fields Γ_x is proportional to 1/B in accordance with anomalous diffusion. It is to be emphasized that (1-26) represents an average drift current as is clear from the fact that it is essentially obtained from the zero-order or average-value equation (1-21).

1.3 Purpose of the Present Study

The aim of this investigation is to derive the governing relation for transport processes in a turbulent plasma. The argument begins with the conservation equations for momentum and mass which is the traditional approach except that account is taken of compressibility and of magnetic field fluctuations. In deriving the master equations terms up to second order are retained. Unlike the work of Yoshikawa and Rose, this treatment is general enough to include the case of no average drift in the plasma for which the second order terms involving cross coupling of the fluctuating momentum, electric, and magnetic fields becomes important. The advantage of this added complication is that the results can be applied to the case of the hot plasma whereas the predictions of the Yoshikawa and Rose study are not applicable at high temperatures in general.

This study is concerned with a turbulent Lorentz gas which may be thought of as a gas of electrons in a neutralizing field of positive ions which are assumed immobile. Most of the assumptions made by Denisse and Delcroix[18] in treating the linear equations of motion will be made and are discussed in the next chapter. The difference here, of course, is that the non-linear terms of the momentum equation are retained and the motion of the ions is neglected. Neglect of ionic motion is justified for turbulent waves of frequency much higher than the ion-cyclotron frequency.

The assumptions upon which the theory is based are as follows:

(1) The effect of the non-linear terms in the momentum equation on each Fourier component is dependent on a dynamic viscosity which is independent of the frequency or wave number of the component. The way that the dynamic viscosity enters into the equations for turbulent motion is slightly more general than its use by Heisenberg since it appears in an anisotropic way. In chapter two Tchen's approach to turbulent energy transfer is summarized, two of his intermediate equations being used to introduce the dynamic viscosity

into the momentum equations. The essence of the heuristic assumption made is that the wave numbers involved in producing gradients are distinct in function from the wave numbers involved in the turbulent viscosity.

(2) Energy transfer between eddies is assumed to take place locally; i.e., it involves wave numbers of approximately the same value. For a discussion of this point in the hydrodynamic case reference may be made to a paper by Tanenbaum.[19]

(3) Third and higher terms in the turbulent quantities are neglected.

The approach taken in this study is rather general in the sense that it may be extended to include the two fluid case or situations of even greater complexity. Since provision is made for anisotropy it is possible to be relatively confident of the results in the presence of an average magnetic field. In these two respects it is felt that the theory is an improvement over that of Yoshikawa since his method seems to rely on the simple form of the kth order equation and in addition isotropy is assumed rather early in the development. The approximations made in the present work are introduce in a straightforward manner so that perhaps the development is a little clearer than in Yoshikawa's case. The two terms dropped by Yoshikawa are retained;

i.e., the inertia term and the term due to fluctuations of the magnetic fields.

In comparing the present approach with that of Chandrasekhar it is to be noted that compressibility is taken into account which greatly extends the generality of the theory. During the development of the theory both the electric and magnetic fields are given explicitly as functions of the fluctuating velocities alone. This result should aid in future analysis of energy spectra and makes the Chandrasekhar method of replacing non-linear terms in the magnetic field by an effective resistivity suspect in view of the way the velocity correlations enter into these terms.

One limiting feature of the approach used here is that it is applicable only to Fourier components of frequency higher than the cyclotron frequency of the ions not only because of the neglect of ionic motion, but also due to the fact that assumption (1) above does not hold for low frequencies. As a consequence the equivalent of the kth order equation of Yoshikawa is employed rather than the zeroth order equation he used to study D.C. diffusion.

1.4 Procedure to be Followed in the Present Study

In chapter two the analysis of a turbulent Lorentz gas is begun by writing the momentum equation including all non-linear contributions. The field quantities are expressed as an average plus a fluctuating part, the fluctuating part being expressible in terms of Fourier integrals. Proceeding in this way it is possible to write the momentum equation in terms of fluctuating quantities carrying out a similar procedure for Maxwell's equations. In the second section of chapter two it is indicated how Heisenberg came to replace the inertial non-linear term in the hydrodynamic energy equation by a contribution involving a dynamic viscosity. It is outlined how Tchen went further in studying the Heisenberg and Obukhov theories of turbulence and developed relations between the velocity convolution integrals and the dynamic viscosity. To close the second chapter it is shown how the results of Tchen which are concerned with hydrodynamics are to be applied in the hydromagnetic situation studied here.

Since Tchen's integrals provide a means of linearizing the velocity convolutions, in the first section of chapter three all fluctuating quantities are expressed in terms of the velocities. An iterative procedure is employed in which it is assumed that the first order fluctuations in the density are much larger than those of second order. As an initial use of the integrals developed by Tchen, a relation is obtained giving a measure of the validity of the iterative procedure. At the end of the first section, a summary of the Fourier components of the fluctuating field quantities in terms of the velocities is given.

The task undertaken in the second section of chapter three is to develop a linearized form of the complete momentum equation. This is done by expressing the non-linear terms of the momentum equation in terms of the fluctuating velocities. These non-linear terms are then linearized by the use of Tchen's integrals. The general momentum equation obtained for the Fourier quantities is valid for a compressible plasma with turbulence subject to the validity of the iteration performed in the previous section of chapter three. Since the general momentum equation is rather complex, a simplification is made to include only a semi-compressible Lorentz gas; that is to say, all quantities of second order in the fluctuating densities are dropped. In this way the Fourier momentum equation for a semi-compressible Lorentz gas is obtained.

In chapter four the mobility tensor and dispersion relations are obtained from the expressions dealing with the semi-compressible plasma. Since the general results are rather complex, detailed study is limited to two degenerate types of propagation; along the average magnetic field and perpendicular to the average magnetic field. Modes corresponding to modified non-turbulent waves, as well as mixed modes; i.e., modes due strictly to the appearance of the dynamic viscosity terms are analyzed under certain simplifying conditions. Equations yielding indices of refraction, mobilities, and related quantities under the assumed conditions are developed. The chapter ends with a summary of some of the salient aspects of the degenerate modes studied.

In chapter five the results of chapter four are put into a more convenient form for numerical calculation by the definition of certain non-dimensional quantities. For the purpurposes of comparison, appendix C may be consulted which gives relations for the indices of refraction and for mobilities of the non-turbulent case. In the final section of the chapter the results of the numerical computation are displayed in graphical form.

CHAPTER II

PLASMA EQUATIONS FOR FLUCTUATING QUANTITIES

2.1 Introduction

It was stated in the previous chapter that the present study would be restricted to the case of the Lorentz gas. This restriction is not a severe one since there exists a host of examples of great practical importance to plasma physics where the Lorentz gas model is quite adequate in describing the physics of the situation. The onset of turbulent diffusion in a confined hot plasma or in the current sheet in a coaxial gun[20] can be accounted for almost completely by the electron gas behavior. Another instance in which electrons dominate the dynamics of the phenomenon refers to the so-called micro-instabilities in a plasma when the characteristic length of the phenomenon under investigation is much smaller than the ion-Larmor radius.

In the approach used here, being a macroscopic one, it is implied throughout that the Larmor radius of the electron and the Debye length are much smaller than the characteristic length L associated with the plasma field. It is also assumed that the electron temperature does not vary appreciably over the length L.

2.2 Conservation Relations and Fluctuating Quantities

The starting point of the argument will be the conservation relations for the momentum and charged particles in the plasma. All field parameters are to be expressed in terms of an average value plus a fluctuating term.

The momentum equation for the electrons which is obtained from the first moment of the Boltzmann-Vlasov equation is[21]

$$\rho \frac{\partial v_i}{\partial t} + \rho v_\ell \frac{\partial v_i}{\partial x_\ell} = - \nabla \cdot \underline{\underline{\Psi}} + \frac{\rho}{m} e \left[E_i + (\underline{v} \times \underline{B})_i \right] + P_{ei} \, . \tag{2-1}$$

In the above equation the usual summation convention is used for repeated indices; also ρ stands for the electron density, m the mass and e the charge of the electron, $\underline{v}$ the macroscopic velocity of the electrons, $\underline{\underline{\Psi}}$ the stress tensor, $\underline{E}$ and $\underline{B}$ the electric and magnetic fields experienced by the gas which include the constant applied electric and magnetic fields $\underline{\bar{E}}$ and $\underline{\bar{B}}$, P_{ei} the momentum transferred to the electrons per unit volume per unit time as a result of collisions with ions in the case of a strongly ionized gas or with neutrals in the case of a weakly ionized gas.

In evaluating the collision term, P_{ei}, account must be taken of the long-range nature of the Coulomb force. If shielding effects are neglected, the collision cross-section turns out to be infinite. In order to obtain finite results, Spitzer and Cohen took electron and ion correlations into account by making use of an effective cutoff distance, h, for the Coulomb force.[22] The quantity h is referred to as the Debye length. A more careful analysis making use of the Fokker-Planck equation with arbitrary electron and ion distribution functions expanded in terms of Legendre polynomials was made by Rosenbluth et al.[23] This analysis showed that the Spitzer and Cohen formulation is equivalent to retaining the first two terms of the expanded distribution functions. From the Spitzer approximation it follows[24] that the collision frequency is a scalar proportional to w where w is the thermal velocity of the electrons (not to be confused with the turbulent fluctuating velocities which are on a macroscopic scale) so that

$$P_{ei} = -\rho \nu v_i . \tag{2-2}$$

The usual assumption that the stress tensor may be replaced by a pressure is now made. It is further assumed that the perturbations caused by the turbulence in the electron gas are adiabatic leading to the acoustic approximation for the pressure.

Making use of the above approximations, (2-1) becomes

$$\rho \frac{\partial v_i}{\partial t} + \rho v_\ell \frac{\partial v_i}{\partial x_\ell} = -a^2 \frac{\partial \rho}{\partial x_i} + \frac{e}{m}\rho\left[E_i + (\underline{v} \times \underline{B})_i\right] - \rho \nu v_i , \tag{2-3}$$

where a is the velocity of sound which is considered to be a constant assuming that any temperature gradients have characteristic lengths much larger than the space scale of the turbulence.

For a fully ionized gas consisting of protons and electrons the current density is

$$\underline{J} = (n\underline{v} - N_I \underline{v}_I)e , \tag{2-4}$$

and Maxwell's equations are

$$\nabla \cdot \underline{E} = \frac{(n - N_I)e}{\epsilon_0} , \tag{2-5}$$

$$\nabla \times \underline{E} = -\partial \underline{B}/\partial t , \tag{2-6}$$

$$\nabla \times \underline{B} = \mu \underline{J} + \epsilon_0 \mu\, \partial \underline{E}/\partial t \tag{2-7}$$

and

$$\nabla \cdot \underline{B} = 0. \tag{2-8}$$

In the above equations, N_I is the ion number density, n the electron number density, $\underline{v}_I$ the macroscopic ion velocity, μ the permeability of free space, and ϵ_o the dielectric constant for free space.

The conservation of mass for the electrons is expressed by

$$\rho \, \nabla \cdot \underline{v} + \underline{v} \cdot \nabla \rho + \frac{\partial \rho}{\partial t} = 0, \tag{2-9}$$

where

$$\rho = m n. \tag{2-10}$$

It will also be convenient to make use of the wave equation formed by taking the curl of (2-6) and using (2-7) so that

$$\nabla \times \nabla \times \underline{E} = -\mu \frac{\partial \underline{J}}{\partial t} - \epsilon_o \mu \frac{\partial^2 \underline{E}}{\partial t^2}. \tag{2-11}$$

Now each quantity is written as an average plus a fluctuating part; thus, taking the Fourier transform of the quantity

A there results

$$A = \int_{-\infty}^{\infty} A(\underline{k}, \omega) e^{i(\underline{k}\underline{r} - \omega t)} d\omega \, d\underline{k}, \tag{2-12}$$

where $A(\underline{k}, \omega)$ is the Fourier coefficient in question. The limits of integration are interpreted to mean that integration takes place over time scale T and space scale V large compared with the scales of the turbulence.

Equation (2-12) may be written in the form

$$A = \bar{A} + \tilde{A}, \tag{2-13}$$

where

$$\bar{A} = \frac{1}{16VT} \int_{VT} A(\underline{r}, t) \, d\underline{r} \, dt \tag{2-14}$$

and

$$\tilde{A} = \int_{-\infty}^{\infty'} A(\underline{k}, \omega) e^{i(\underline{k}\underline{r} - \omega t)} d\omega \, d\underline{k}. \tag{2-15}$$

The primes indicate that the values $\underline{k} = 0$ and $\omega = 0$ are excluded.

In a frame for which $\bar{\underline{v}} = 0$ the momentum equation becomes

$$\rho\frac{\partial\tilde{v}_i}{\partial t}+\bar{\rho}\, v_\ell\frac{\partial v_i}{\partial x_\ell}=-a^2\frac{\partial\tilde{\rho}}{\partial x_i}$$
$$+\frac{e}{m}\rho\left[\tilde{E}_i+(\underline{\tilde{v}}\times\underline{\bar{B}})_i\right]$$
$$+\frac{e}{m}\bar{\rho}(\underline{\tilde{v}}\times\underline{\tilde{B}})_i+\frac{e}{m}\tilde{\rho}\,\bar{E}_i$$
$$-\rho\bar{\nu}\tilde{v}_i-\bar{\rho}\tilde{\nu}\tilde{v}_i\,, \tag{2-16}$$

where third order terms have been ignored and terms belonging to the zero-order or average value equation have been left out. In the subsequent discussion $\bar{\underline{E}}$ is assumed to be zero in the new frame.

In the new frame the equation for the conservation of electrons becomes

$$\rho\,\nabla\cdot\underline{\tilde{v}}+\underline{\tilde{v}}\cdot\nabla\rho+\frac{\partial\rho}{\partial t}=0\,. \tag{2-17}$$

In accordance with the assumption that the ions are stationary $\underline{\tilde{v}}_I = 0$ and $\tilde{N}_I = 0$ so that Maxwell's equations for the fluctuating quantities are

$$\nabla\cdot\underline{\tilde{E}}=\frac{\tilde{\rho}}{m}\frac{e}{\epsilon_0}\,, \tag{2-18}$$

$$\nabla\times\underline{\tilde{E}}=-\partial\underline{\tilde{B}}/\partial t\,, \tag{2-19}$$

$$\nabla \times \underline{\tilde{B}} = \frac{e\mu\rho}{m}\underline{\tilde{N}} + \epsilon_0 \mu \, \partial\underline{\tilde{E}}/\partial t, \tag{2-20}$$

and

$$\nabla \cdot \underline{\tilde{B}} = 0. \tag{2-21}$$

Making use of the vector identity

$$\nabla \times \nabla \times \underline{\tilde{E}} = \nabla(\nabla \cdot \underline{\tilde{E}}) - \nabla^2 \underline{\tilde{E}}, \tag{2-22}$$

the wave equation (2-11) becomes

$$\nabla\tilde{\rho} - \frac{m\epsilon_0}{e}\nabla^2\underline{\tilde{E}} = -\mu\epsilon_0\frac{\partial}{\partial t}(\rho\underline{\tilde{N}}) - \frac{m\epsilon_0^2\mu}{e}\frac{\partial^2\underline{\tilde{E}}}{\partial t^2}. \tag{2-23}$$

The procedure to be followed now is to extract the equations for the kth and ωth Fourier component from the above, solve for all fluctuating quantities in terms of the turbulent velocities and substitute the results into the momentum equation. In order to linearize the resulting momentum equation use will be made of the Heisenberg approximation as developed by Tchen. It is convenient to discuss this approximation at this point; accordingly a short digression is made in the next section summarizing the reasoning leading to the concept of dynamic viscosity as introduced by Heisenberg and extended by Tchen.

2.3 Discussion of Heisenberg's Dynamic Viscosity

In order to see how Heisenberg arrived at his concept of an effective dynamic viscosity to account for turbulent energy transfer the discussion, for the present, will be limited to a neutral incompressible fluid in a state of turbulence. It is assumed that Navier-Stokes equation[25] of hydrodynamics still holds when the fluid is in a state of turbulence so that

$$\frac{\partial v_i}{\partial t} + v_l \frac{\partial v_i}{\partial x_l} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \sigma^* \frac{\partial^2 v_i}{\partial x_l \partial x_l}, \qquad (2\text{-}24)$$

where σ^* is the kinematic viscosity and P the pressure.

Writing $\underline{v}_i$ as an average plus a fluctuating part as was first done by Reynolds, and in a frame for which $\overline{\underline{v}}_i$ is zero there follows

$$\frac{\partial \tilde{v}_i}{\partial t} + \tilde{v}_l \frac{\partial \tilde{v}_i}{\partial v_l} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} + \sigma^* \frac{\partial^2 \tilde{v}_i}{\partial x_l \partial x_l}. \tag{2-25}$$

Multiplying through by $\tilde{v}_i$ and using the summation convention, the energy equation is obtained and is

$$\frac{1}{2}\frac{\partial}{\partial t}(\tilde{v}_i \tilde{v}_i) + \frac{1}{2}\tilde{v}_l \frac{\partial}{\partial x_l}(\tilde{v}_i \tilde{v}_i = -\frac{1}{\rho}\tilde{v}_i \frac{\partial \tilde{p}}{\partial x_i} + \sigma^* \tilde{v}_i \frac{\partial^2 \tilde{v}_i}{\partial x_l \partial x_l}. \tag{2-26}$$

As is easily seen from (2-25) the second term in (2-26) represents the non-linear turbulent energy transfer. After Fourier analyzing and obtaining the relation for the average energy contributed by the kth Fourier mode it will be shown how Heisenberg replaced the second non-linear term by an equivalent dynamic viscosity term.

In order to obtain the average energy equation a relation is needed for the contribution to the equation of a product of two turbulent quantities A and B. For simplicity tildas and explicit notation for the time dependence are omitted.

The definition for the space average of the product of the two quantities A($\underline{r}$) and B($\underline{r}$) is given by

$$\langle A(\underline{r})B(\underline{r})\rangle = \int_{-\infty}^{\infty} A(\underline{r})B(\underline{r})d\underline{r} \Big/ \int_{-\infty}^{\infty} d\underline{r} \; . \tag{2-27}$$

As before, the limits of space integration are taken to mean integration over a volume large with respect to the characteristic turbulent lengths.

The Fourier transforms for the quantities A and B are

$$A(\underline{r}) = \int A(\underline{k})\, e^{i\underline{k}\,\underline{r}}\, d\underline{k} \tag{2-28}$$

and

$$B(\underline{r}) = \int B(\underline{k}')\, e^{i\underline{k}'\underline{r}}\, d\underline{k}' \; . \tag{2-29}$$

Substituting these quantities into (2-27) there results

$$\langle A(\underline{r})B(\underline{r})\rangle = \frac{1}{8V}\iiint A(\underline{k})B(\underline{k}')\, e^{i(\underline{k}+\underline{k}')\underline{r}}\, d\underline{k}\, d\underline{k}'\, d\underline{r} \; . \tag{2-30}$$

Now noting that

$$\int_{-\infty}^{\infty} e^{i(\underline{k}+\underline{k}')\underline{r}} \, d\underline{r} = 8\pi^3 \, \delta(\underline{k}+\underline{k}'), \tag{2-31}$$

where δ is the three dimensional Dirac Delta function and integrating over $d\underline{r}$ and $d\underline{k}'$, (2-30) becomes

$$\langle A(\underline{r}) B(\underline{r}) \rangle = \frac{\pi^3}{V} \int_{-\infty}^{\infty} A(\underline{k}) B(-\underline{k}) \, d\underline{k} \, . \tag{2-32}$$

By the Convolution Theorem (see Appendix B), the k'th Fourier component of the velocity cross products involved in the inertia term is

$$\tilde{v}_\ell \tilde{v}_i(\underline{k}') = \int_{-\infty}^{\infty} v_\ell(\underline{k}'-\underline{k}) \, v_i(\underline{k}) \, d\underline{k} \, . \tag{2-33}$$

To obtain the average kinetic energy (2-33) is used with i = j and with k' = 0 so that

$$\tilde{v}_i \tilde{v}_i(0) = \int_{-\infty}^{\infty} v_i(-\underline{k}) \, v_i(\underline{k}) \, d\underline{k} \, . \tag{2-34}$$

The contribution from the kth component to the average energy is then,

$$(\tilde{v}_i \cdot \tilde{v}_i)_k(0) = 2 v_i(\underline{k}) \, v_i(-\underline{k}) \, . \tag{2-35}$$

In a similar way the other terms of the energy equation become

$$\left(\sigma^* \tilde{v}_i \frac{\partial^2 \tilde{v}_i}{\partial x_l \partial x_l}\right)_k (0)$$

$$= -2\sigma^* k^2 v_i(\underline{k})\, v_i(-\underline{k}) \tag{2-36}$$

and

$$\left(\frac{\partial \tilde{v}_i}{\partial x_l}\tilde{v}_l \tilde{v}_i\right)_k (0)$$

$$= i k_l v_l(\underline{k}) \int_{-\infty}^{\infty} v_i(\underline{k}')\, v_l(-\underline{k}-\underline{k}')\, d\underline{k}'$$

$$- i k_l v_l(-\underline{k}) \int_{-\infty}^{\infty} v_i(\underline{k}')\, v_l(\underline{k}-\underline{k}')\, d\underline{k}' . \tag{2-37}$$

The pressure term does not appear since if the pressure is written as

$$\tilde{p}(\underline{r}) = \int_{-\infty}^{\infty} P(\underline{k})\, e^{i\underline{k}\underline{r}}\, d\underline{k}, \tag{2-38}$$

and use is made of the incompressibility condition

$$\frac{\partial v_j}{\partial x_j} = 0, \tag{2-39}$$

that is

$$k_j v_j(\underline{k}) = 0, \tag{2-40}$$

it is seen that

$$P(\underline{k})k_j v_j = 0, \tag{2-41}$$

indicating that the pressure term does not contribute to the average energy.

Assembling all of the various terms involved, the equation for the average energy of the kth mode is

$$\begin{aligned}\frac{\partial}{\partial t} v_i(\underline{k}) v_i(-\underline{k}) = &- 2\sigma^* k^2 v_i(\underline{k}) v_i(-\underline{k}) \\ &- \int_{-\infty}^{\infty} d\underline{k}' \, q(\underline{k}, \underline{k}'),\end{aligned} \tag{2-42}$$

where

$$\begin{aligned} q(\underline{k},\underline{k}') = {} & \\ & i k_l v_l(\underline{k}) \int_{-\infty}^{\infty} v_i(\underline{k}') v_l(-\underline{k}-\underline{k}') \, d\underline{k}' \\ & - i k_l v_l(-\underline{k}) \int_{-\infty}^{\infty} v_i(\underline{k}') v_l(\underline{k}-\underline{k}') \, d\underline{k}'. \end{aligned} \tag{2-43}$$

To indicate how the Heisenberg assumption is made it is found convenient to define a spectral function F(k). This is done as follows:

The average value of v_1^2 is given by

$$\langle v_i^2 \rangle = \int_{-\infty}^{\infty} v_i^2(\underline{r})\,d\underline{r} \Big/ \int_{-\infty}^{\infty} d\underline{r}\,, \tag{2-44}$$

which according to (2-32) is

$$\langle v_i^2 \rangle = \frac{\pi^3}{V}\int_{-\infty}^{\infty} v_i(\underline{k})\, v_i(-\underline{k})\, d\underline{k}\,. \tag{2-45}$$

Now instead of integrating over the whole of k-space the integration can be limited to a spherical shell of radius between $\underline{k}$ and $\underline{k} + d\underline{k}$ and a spectral function $F(\underline{k})$ defined by

$$2\cdot F(\underline{k}) = \frac{\pi^3}{V}\int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \sin\theta\; k^2 v_i(\underline{k})\, v_i(-\underline{k})\,, \tag{2-46}$$

where ϕ and θ are the azimuthal and polar angles in spherical coordinates. For isotropic turbulence, $F(\underline{k})$ is a function of the magnitude of $\underline{k}$ only so that

$$F(k) = \frac{\pi^3}{V} 2\pi k^2 v_i(\underline{k})\, v_i(-\underline{k})\,. \tag{2-47}$$

It is clear that the spectral function $F(\underline{k})$ represents the energy density of the kth mode in k-space since the average kinetic energy is

$$\frac{1}{2}\rho\langle v_i^2 \rangle = \rho\int_0^{\infty} d\underline{k}\, F(k)\,. \tag{2-48}$$

Using the function F(k) in (2-42) and integrating from 0 to k, an equation for the total energy contributed by modes $\leq$ k results and is

$$-\frac{\partial}{\partial t}\int_0^k F(k)\,d\underline{k} = 2\sigma^*\int_0^k dk'\,k'^2 F(k') + W_k, \qquad (2\text{-}49)$$

where

$$W_k = \frac{\pi^3}{V}\int_0^k d\underline{k}\int_{-\infty}^{\infty} dk'\,\frac{1}{2}\,q(\underline{k},\underline{k}'). \qquad (2\text{-}50)$$

The quantity W_k is the transfer function and represents the transfer of energy from wave numbers smaller than k to those larger than k. Heisenberg's assumption was to write W_k in the form

$$W_k = 2\sigma_k\int_0^k d\underline{k}'\,k'^2 F(k'), \qquad (2\text{-}51)$$

where σ_k is a dynamic viscosity analogous to σ^*, the kinematic viscosity. The quantity σ_k is given by

$$\sigma_k = K\int_k^{\infty} d\underline{k}'\left[\frac{F(k')}{k'^3}\right]^{1/2}, \qquad (2\text{-}52)$$

where K is a constant.

By inspecting the form of q(k,k'), it is seen that the integrands $\int_{-\infty}^{\infty} d\underline{k}' v_i(\underline{k}') v_j(\underline{k}'' - \underline{k}')$ and $\int_{-\infty}^{\infty} d\underline{k}' v_i(-\underline{k}') v_j(\underline{k}' - \underline{k}'')$ are involved in the transfer function. Tchen[26] has shown that by assuming

$$\int_{k}^{\infty} d\underline{k}' v_i(\underline{k}') v_j(\underline{k} - \underline{k}') = -i k_j v_i(\underline{k}) \sigma_k \tag{2-53}$$

and

$$\int_{k}^{\infty} d\underline{k}' v_i(-\underline{k}') v_j(\underline{k}' - \underline{k}) = i k_j v_i(-\underline{k}) \sigma_k , \tag{2-54}$$

one obtains just the Heisenberg form of W_k. These integrals represent the phase correlations between the Fourier coefficients. The fact that the integrals are of this form is shown by Tchen to follow from a statistical treatment of the transport processes. An indication of the arguments used by Tchen is given below.

Let the mean value of the displacement of a fluid element be $\langle \underline{\ell} \rangle$ and the mean value of the second power of the displacement be $\langle \ell^2 \rangle$. Following Kolmogoroff, Tchen assumes that the ratios $\langle \underline{\ell} \rangle / \tau$ and $\langle \ell^2 \rangle / \tau$ tend to a constant independent of τ as τ approaches zero. The quantity τ is the time elapsed as the fluid makes the displacement involved. Using these assumptions and the equations of motion it is shown that

$$\int_{-\infty}^{\infty} d\underline{k}' v_j(\underline{k}'-\underline{k}) v_i(-\underline{k}')$$

$$= i\int_{-\infty}^{\infty} d\underline{k}' k_j' v_i(-\underline{k}')\sigma(\underline{k}'-\underline{k}) \qquad (2\text{-}55)$$

and

$$\int_{-\infty}^{\infty} d\underline{k}' v_j(\underline{k}-\underline{k}') v_i(\underline{k}')$$

$$= -i\int_{-\infty}^{\infty} d\underline{k}' k_j' v_i(\underline{k}')\sigma(\underline{k}-\underline{k}'), \qquad (2\text{-}56)$$

where $\sigma(\underline{k}-\underline{k}')$ is the dynamic viscosity corresponding to the $\underline{k}'$-$\underline{k}$th mode.

In equation (2-49) the term W_k involves a transfer of momentum from the k'th to the kth mode; thus, there is a sort of collision between eddies of characteristic size 1/k' and 1/k. The right hand sides of (2-55) and (2-56) indicate that this process may be thought of as consisting of a dynamic viscosity and an eddy gradient. It may be that these functions are distinct so that the gradient forming eddies are independent of the eddies involved in the dynamic viscosity; in this way the dynamic viscosity may be written as

$$\sigma(\underline{k}-\underline{k}') = |\sigma|\, \delta(\underline{k}-\underline{k}'), \qquad (2\text{-}57)$$

which leads to

$$\int_{-\infty}^{\infty} d\underline{k}'\, v_j(\underline{k}'-\underline{k})\, v_i(-\underline{k}') = i k_j v_i(-\underline{k})\sigma \qquad (2\text{-}58)$$

and

$$\int_{-\infty}^{\infty} d\underline{k}'\, v_j(\underline{k}-\underline{k}')\, v_i(\underline{k}') = -i k_j v_i(\underline{k})\sigma . \qquad (2\text{-}59)$$

where the term "dynamic viscosity" will henceforth refer to the quantity σ .

The two integrals (2-58) and (2-59) become (2-53) and (2-54) for the k, ∞ part of the spectrum of σ .

2.4 Use of Dynamic Viscosity in the Present Context

It will be seen in the following chapter that the non-linear terms of equation (2-16) may be expressed in terms of velocity correlations. These velocity correlations are conveniently given by equations (2-58) and (2-59) above; accordingly, it is assumed that these equations remain valid in the case of the Lorentz gas. To put these equations in a form consistent with the analysis to follow, they are Fourier analyzed with respect to time yielding

$$\int_{-\infty}^{\infty} d\omega' d\underline{k}' v_j(\omega'-\omega, \underline{k}'-\underline{k}) v_i(-\omega', -\underline{k}')$$
$$= i k_j v_i(-\omega, -\underline{k})\sigma \tag{2-60}$$

and

$$\int_{-\infty}^{\infty} d\omega' d\underline{k}' v_j(\omega-\omega', \underline{k}-\underline{k}') v_i(\omega', \underline{k}')$$
$$= -i k_j v_i(\omega, \underline{k})\sigma, \tag{2-61}$$

which then become the fundamental equations for the development of the theory in the present work.

CHAPTER III

EQUATIONS INVOLVING FLUCTUATING VELOCITY FIELDS

3.1 Fluctuating Quantities in Terms of Velocities

The equations relating the various fluctuating field quantities to the velocity contain linear or first order terms, and bilinear or second order terms. Assuming that terms of second order are less than those of first order, an iterative procedure is used in this section to obtain the fluctuating field quantities in terms of the velocities alone. The conditions for which the second order contributions are less than those of first order will also be investigated.

3.1.1 First and Second Order Density Expressions

Assuming that $\partial \bar{\rho}/\partial t = 0$ and that the gradient of the average density is small; i.e., that the characteristic length over which the average density changes is large with respect to the size of the turbulence so that the plasma is homogeneous on the average, the quantity $\nabla \bar{\rho}$ may be neglected and the relation for the conservation of mass becomes

$$\bar{\rho}\, \nabla \cdot \underline{\tilde{v}} + \nabla \cdot (\tilde{\rho}\, \underline{\tilde{v}}) + \frac{\partial \tilde{\rho}}{\partial t} = 0 \,. \qquad (3\text{-}1)$$

Using the convolution theorem, the Fourier analog of (3-1) is

$$i k_j v_j \bar{\rho} + i k_\ell \int \rho' v_\ell'' \, dk' dw' = i w \rho , \tag{3-2}$$

where the notations

$$v_j = v_j(\underline{k}, w) , \tag{3-3}$$

$$v_j' = v_j(\underline{k}', w') , \tag{3-4}$$

and

$$v_j'' = v_j(\underline{k} - \underline{k}', w - w') \tag{3-5}$$

have been used.

To solve equation (3-2) for ρ in terms of the velocities alone an iterative procedure is used. Neglecting the convolution term the first order approximation to ρ is obtained and is

$$\rho^{(1)} = \bar{\rho} \, \frac{k_j}{w} v_j . \tag{3-6}$$

Using this value for ρ in the convolution term, the second order contribution to ρ is obtained and is

$$\rho^{(2)} = \bar{\rho}\,\frac{k_\ell}{\omega}\int \frac{k_j'}{\omega'}\, v_j'\, v_i'' . \qquad (3\text{-}7)$$

To evaluate (3-7) the assumption of local transfer is made so that only eddies with size and time characteristics close to those of the k, ω th eddy are involved in the contributions to the convolution integral. An examination of the Tchen integrals (2-60) and (2-61) support this assumption since the velocity phase correlation seems to be such that the integrand is a peaked function behaving much like a Dirac Delta function. Under this local transfer assumption and allowing for a factor A_1' of order unity, $\rho^{(2)}$ may be written as

$$\rho^{(2)} = A_1'\,\frac{k_\ell}{\omega}\,\frac{k_j}{\omega}\,\bar{\rho}\int v_j' v_\ell'' . \qquad (3\text{-}8)$$

Now making use of the Tchen integral (2-61) the final expression for the second order contribution to the density is

$$\rho^{(2)} = -i A_1'\,\bar{\rho}\,\frac{k_j}{\omega}\,\frac{k^2\sigma}{\omega}\, v_j' , \qquad (3\text{-}9)$$

where

$$k^2 = k_j k_j . \qquad (3\text{-}10)$$

In order that the above iterative procedure be valid, it must be that

$$|\rho^{(2)}|/|\rho^{(1)}| < 1, \tag{3-11}$$

leading to the condition

$$A_1'\sigma k^2 < \omega. \tag{3-12}$$

3.1.2 First and Second Order Electric Field Expressions

Turning now to the wave equation (2-23) and Fourier analyzing there results

$$\frac{e}{m}\frac{1}{\omega_p^2}\left[k^2 - \frac{\omega^2}{c^2}\right]E_i$$
$$= i\frac{\omega}{c^2}\left[v_i + \frac{1}{\bar\rho}\int \rho'' v_i'\right] - i k_i \frac{\rho}{\bar\rho}, \tag{3-13}$$

where ω_p is the plasma frequency given by

$$\omega_p^2 = \frac{e^2\bar\rho}{m^2\epsilon_0}, \tag{3-14}$$

and the substitution

$$\mu \epsilon_0 = \frac{1}{c^2} \tag{3-15}$$

has been made, c being the velocity of light in free space.

Again using the iteration procedure in conjunction with equation (3-13), there results

$$E_i = E_i^{(1)} + E_i^{(2)}, \tag{3-16}$$

where

$$E_i^{(1)} = i\,\frac{m}{e}\,\frac{\omega_p^2}{\omega^2}\,\beta\,\Big[\omega v_i - k_i\,\frac{k_j}{\omega}\,c^2 v_j\Big] \tag{3-17}$$

and

$$E_i^{(2)} = i\,\frac{m}{e}\,\frac{\omega_p^2}{\omega^2}\,\beta\Big[\frac{\omega}{\bar\rho}\int \rho'' v_i' - k_i c^2\,\frac{\rho^{(2)}}{\bar\rho}\Big], \tag{3-18}$$

with

$$\beta = \Big[\frac{k^2 c^2}{\omega^2} - 1\Big]^{-1}, \tag{3-19}$$

which is a dimensionless constant involving the phase velocity of the Fourier component and the velocity of light.

Using (2-61) and (3-7) the second order expression for the electric field (3-18) becomes

$$E_i^{(2)} = -\frac{m}{e}\frac{\omega_p^2}{\omega^2} A_1' \sigma k_j v_j k_i \,. \qquad (3\text{-}20)$$

For the above iteration to be valid it must be that

$$|E_i^{(2)}| / |E_i^{(1)}| < 1 \qquad (3\text{-}21)$$

or

$$\left| A_1' \frac{\sigma k_i k_j}{\omega} \frac{v_j'}{v_i} \frac{\left[1 - c^2 \frac{k^2}{\omega^2}\right]}{\left[1 - c^2 \frac{k_i k_j}{\omega^2} \frac{v_j'}{v_i}\right]} \right| < 1, \qquad (3\text{-}22)$$

there being no sum on i.

If the axes are rotated so that the propagation vector is along one of the coordinate axes, the above condition becomes the inequality (3-12).

3.1.3 First and Second Order Magnetic Field Expressions

The next task is to solve for B_1 in terms of the velocities.

From the Fourier analysis of (2-20) there results

$$i\underline{k}\times\underline{B} = \frac{e\mu\bar{\rho}}{m}\underline{v} + \frac{e\mu}{m}\int\rho''\underline{v}'d\underline{k}'d\omega' - \epsilon_0\mu i\omega\underline{E}\,, \tag{3-23}$$

also the divergence of B is zero so that

$$k_j B_j = 0. \tag{3-24}$$

After inserting the values for the plasma frequency and the velocity of light in the three equations (3-23) it follows that

$$k_y B_z - k_z B_y = -\frac{1}{c^2}[x]\,, \tag{3-25}$$

$$k_z B_x - k_x B_z = -\frac{1}{c^2}[y] \tag{3-26}$$

and

$$k_x B_y - k_y B_x = -\frac{1}{c^2}[z]\,, \tag{3-27}$$

where

$$[x] = \left[\frac{im\omega_p^2}{e}\left(v_x + \int\frac{\rho''}{\bar{\rho}}v_x'\right) + \omega E_x\right] \tag{3-28}$$

and so forth.

Using (3-24) to eliminate B_z , and eliminating B_x between (3-25) and (3-27) it is found that

$$B_y = \frac{1}{k^2c^2}\left(k_z[x] - k_x[z]\right). \tag{3-29}$$

As before the first and second order contributions may be separated so that

$$B_y = B_y^{(1)} + B_y^{(2)}, \tag{3-30}$$

where

$$B_y^{(1)} = \frac{1}{k^2c^2}\left(k_z[x]^{(1)} - k_x[z]^{(1)}\right) \tag{3-31}$$

and

$$B_y^{(2)} = \frac{1}{k^2c^2}\left(k_z[x]^{(2)} - k_x[z]^{(2)}\right), \tag{3-32}$$

with

$$[x]^{(1)} = \left[\frac{im\omega_p^2}{e}v_x + \omega E_x^{(1)}\right] \tag{3-33}$$

and

$$[x]^{(2)} = \left[\frac{i m \omega_p^2}{e} \int \frac{\rho''}{\bar{\rho}} v_x' + \omega E_x^{(2)} \right] . \tag{3-34}$$

It is easily seen that the conditions that

$$|B_i^{(2)}| / |B_i^{(1)}| < 1 \tag{3-35}$$

are that (3-20) holds and that

$$|[x_j]^{(2)}| / |[x_j]^{(1)}| . \tag{3-36}$$

These conditions will obtain when (3-12) applies so that (3-12) is the condition for the validity of the iteration procedure in all cases.

Expressions for all fluctuating quantities in terms of the fluctuating velocities have now been obtained. Assuming local transfer and using Tchen's integrals these quantities become

$$\rho^{(1)} = \bar{\rho} \frac{k_j'}{\omega} v_j' , \tag{3-37}$$

$$\rho^{(2)} = -i A_1' \bar{\rho} \frac{k^2 \sigma}{\omega} \frac{k_j' v_j'}{\omega} , \tag{3-38}$$

$$E_i^{(1)} = i\,\frac{m}{e}\,\frac{\omega_p^2}{\omega^2}\,\beta\Big[\omega v_i - c^2 k_i\,\frac{k_j v_j}{\omega}\Big], \tag{3-39}$$

$$E_i^{(2)} = -\frac{m}{e}\,\frac{\omega_p^2}{\omega^2}\,A_1'\sigma\, k_j v_j k_i\,, \tag{3-40}$$

$$B_y^{(1)} = \frac{1}{k^2c^2}\left(k_z[x]^{(1)} - k_x[z]^{(1)}\right) \tag{3-41}$$

and

$$B_y^{(2)} = \frac{1}{k^2c^2}\left(k_z[x]^{(2)} - k_x[z]^{(2)}\right), \tag{3-42}$$

where

$$[x]^{(1)} = \Big[\frac{i m\omega_p^2}{e}\,v_x + \omega E_x^{(1)}\Big] \tag{3-43}$$

and

$$[x]^{(2)} = \Big[\omega E_x^{(2)} - i\,\frac{m\omega_p^2}{e}\,A_1'\sigma\,\frac{k_j v_j}{\omega}\,k_x\Big]; \tag{3-44}$$

the other components of B being obtained by a cyclic permutation of x, y, and z.

3.2 The Generalized Momentum Equation

The procedure now will be to use the expressions obtained in the previous section to linearize the momentum equation so that all quantities may be expressed in terms of the Fourier components of the fluctuating velocities; thus, the momentum equation (2-16) is

$$\rho \frac{\partial \tilde{v}_i}{\partial t} + \bar{\rho}\, \tilde{v}_\ell \frac{\partial \tilde{v}_i}{\partial x_\ell} = -a^2 \frac{\partial \tilde{\rho}}{\partial x_i}$$
$$+ \frac{e}{m} \rho \left[\tilde{E}_i + (\tilde{v} \times \bar{B})_i \right]$$
$$+ \frac{e}{m} \bar{\rho} (\tilde{v} \times \tilde{B})_i - \rho \bar{\nu} \tilde{v}_i - \bar{\rho} \tilde{\nu} \tilde{v}_i \,. \qquad (3\text{-}45)$$

Now setting $\nu \propto \rho$ so that $\tilde{\nu} \propto \tilde{\rho}$ in accordance with the form of the collision term for a fully ionized gas there results

$$\frac{\tilde{\nu}}{\bar{\nu}} = \frac{\tilde{\rho}}{\bar{\rho}} \,. \qquad (3\text{-}46)$$

Since

$$\rho = \bar{\rho} + \tilde{\rho} \,, \qquad (3\text{-}47)$$

the momentum equation (3-45) becomes

$$\frac{\rho}{\bar{\rho}}\frac{\partial \tilde{v}_i}{\partial t} + \tilde{v}_\ell \frac{\partial \tilde{v}_i}{\partial x_\ell} = -\frac{a^2}{\bar{\rho}}\frac{\partial \tilde{\rho}}{\partial x_i}$$
$$+ \frac{e}{m}\frac{\rho}{\bar{\rho}}\left[\tilde{E}_i + (\tilde{\underline{v}} \times \bar{\underline{B}})_i\right]$$
$$+ \frac{e}{m}(\tilde{\underline{v}} \times \tilde{\underline{B}})_i - 2\bar{\nu}\frac{\tilde{\rho}}{\bar{\rho}}\tilde{v}_i - \bar{\nu}\tilde{v}_i \, . \tag{3-48}$$

The second order terms in (3-48) which must be linearized are

$$T_1(\underline{r},t) = \frac{\tilde{\rho}}{\bar{\rho}}\frac{\partial \tilde{v}_i}{\partial t} + \tilde{v}_\ell \frac{\partial \tilde{v}_i}{\partial x_\ell} \, , \tag{3-49}$$

$$T_2(\underline{r},t) = \frac{e}{m}\frac{\tilde{\rho}}{\bar{\rho}}\tilde{E}_i \, , \tag{3-50}$$

$$T_3(\underline{r},t) = \frac{e}{m}\frac{\tilde{\rho}}{\bar{\rho}}(\tilde{\underline{v}} \times \bar{\underline{B}})_i \, , \tag{3-51}$$

$$T_4(\underline{r},t) = \frac{e}{m}(\tilde{\underline{v}} \times \tilde{\underline{B}})_i \tag{3-52}$$

and

$$T_5(\underline{r},t) = -2\bar{\nu}\frac{\tilde{\rho}}{\bar{\rho}}\tilde{v}_i \, . \tag{3-53}$$

To first order, the continuity equation is

$$\bar{\rho}\frac{\partial v_j'}{\partial x_j'} + \frac{\partial \tilde{\rho}}{\partial t} = 0 \, , \tag{3-54}$$

so that

$$T_1(\underline{r},t) = \frac{\partial}{\partial x_j}(\tilde{v}_i \tilde{v}_j) + \frac{1}{\rho}\frac{\partial}{\partial t}(\tilde{\rho}\tilde{v}_i) , \qquad (3\text{-}55)$$

where the identity

$$\frac{\partial}{\partial x_j}(\tilde{v}_i \tilde{v}_j) = \tilde{v}_j \frac{\partial \tilde{v}_i}{\partial x_j} + \tilde{v}_i \frac{\partial \tilde{v}_j}{\partial x_j} \qquad (3\text{-}56)$$

has been used.

Fourier analyzing and using the value for $\rho^{(1)}$ from the previous section it is seen that

$$T_1(\underline{k},\omega) = i k_j \int v_i'' v_j' - i\omega \int v_i'' \frac{k_j'}{\omega'} v_j' , \qquad (3\text{-}57)$$

where $T_1(\underline{k}, \omega)$ is the $\underline{k}, \omega$ Fourier transform of $T_1(\underline{r},t)$.

Making the local transfer assumption as was done in evaluating (3-7), the integral

$$\int v_i'' k_j' v_j' = A_1' k_j \int v_i'' v_j' . \qquad (3\text{-}58)$$

It is to be noted that the constant A_1' depends on the ratio $|\rho^{(2)}|/|\rho^{(1)}|$ and is zero for a semi-compressible plasma for which $\rho^{(2)}$ may be neglected.

In accordance with the above and making use of the Tchen integral (2-61) the inertia term becomes

$$T_1(\underline{k},\omega) = A_1 k^2 \sigma N_i \,, \tag{3-59}$$

where

$$A_1 = 1 - A_1' \,. \tag{3-60}$$

For a semi-compressible gas the quantity $A_1 = 1$, otherwise $A_1 < 1$.

Turning now to $T_2(\underline{r},t)$ and using the Fourier expression for the first order fluctuating electric field obtained from (3-39) there results

$$T_2(\underline{k},\omega) = i \int \frac{k_j''}{\omega''} N_j'' \frac{\omega_p^2}{\omega'^2} \beta' \Big[\omega' N_i' - c^2 k_i' \frac{1}{\omega'} k_l' N_l' \Big] \,. \tag{3-61}$$

Assuming local transfer, the quantity β/ω^2 may be brought outside the convolution integrals so that

$$I_1 = \int \frac{k_j - k_j'}{\omega - \omega'} \omega' N_i' N_j'' \tag{3-62}$$

and

$$I_2 = \int \frac{k_j - k_j'}{\omega - \omega'} \omega' \left[-c^2 \frac{k_i'}{\omega'} \frac{k_\ell'}{\omega'} \nu_\ell' \right] \nu_j'' \tag{3-63}$$

remain to be evaluated.

Making use of relations of the type

$$\int \frac{1}{\omega'} \nu_j'' \nu_i' = \int \frac{1}{\omega - \omega'} \nu_j' \nu_i'' \tag{3-64}$$

and the assumption of local transfer, (3-62) and (3-63) may be written

$$I_1 = A_2 k_j \int \nu_i' \nu_j'' \tag{3-65}$$

and

$$I_2 = -A_3 c^2 k_j \frac{k_i}{\omega} \frac{k_\ell}{\omega} \int \nu_\ell' \nu_j'' , \tag{3-66}$$

where A_2 and A_3 are constants to be determined experimentally. Both A_2 and A_3 are zero for a semi-compressible plasma since the integrands come ultimately from second order terms in the density fluctuation. In this way the term $T_2(k, \omega)$ becomes

$$T_2(\underline{k}, \omega) = \frac{\omega_p^2}{\omega^2} \beta k^2 \sigma \left[A_2 \nu_i - A_3 c^2 \frac{k_i}{\omega} \frac{k_\ell}{\omega} \nu_\ell \right]. \tag{3-67}$$

Taking the average magnetic field to be in the z-direction, the second order contribution due to the average magnetic field is

$$T_3(\underline{k},\omega) = \omega_b \int \frac{k_j'}{\omega'} \left[v_y'' \hat{a}_x - v_x'' \hat{a}_y \right] \cdot \hat{a}_i v_j' \quad , \tag{3-68}$$

where

$$\omega_b = \frac{|\bar{B}|e}{m} \tag{3-69}$$

is the cyclotron frequency. The quantities $\hat{a}_x$, $\hat{a}_y$, and $\hat{a}_z$ are the unit vectors in the directions indicated.

Proceeding as before

$$T_3(\underline{k},\omega) = -i A_1' \frac{k^2\sigma}{\omega} \omega_b \left[v_y \hat{a}_x - v_x \hat{a}_y \right] \cdot \hat{a}_i \, . \tag{3-70}$$

The velocity-magnetic field second order term is of a rather complex nature consisting of contributions of the type

$$T_{4ij\ell}(\underline{k},\omega)$$
$$= \frac{e}{m}\int \nu_i'' \frac{1}{c^2}\frac{k_j'}{k'^2}\left[\frac{i m \omega_p^2}{e}\nu_\ell' + \omega' E_\ell'^{(1)}\right], \quad (3\text{-}71)$$

where the subscript i indicates the component of the contribution. The other subscripts j and l take on different values depending on which part of the cross product is under consideration. Using local transfer and Tchen's integrals these contributions become

$$T_{4ij\ell}(\underline{k},\omega) = A_4 \frac{\sigma}{c^2}\frac{k_j}{k^2}\omega_p^2\Big[(1+\beta)k_i\nu_\ell - \beta c^2 A_5 \frac{k_\ell}{\omega}\frac{k_m}{\omega}k_i\nu_m\Big]. \quad (3\text{-}72)$$

The evaluation of the collision second order term is straightforward and is

$$T_5(\underline{k},\omega) = 2A_1'' i \frac{k^2}{\omega}\bar{\nu}\sigma\nu_i . \quad (3\text{-}73)$$

Having evaluated the second order terms of the Fourier analyzed momentum equation for a compressible plasma subject to the condition (3-12) it is now possible to write down the linearized from of the momentum equation including the effects of turbulence; accordingly,

$$\left[-i\omega+\bar{\nu}+A_1k^2\sigma-2A_1'i\frac{k^2}{\omega}\bar{\nu}\sigma\right]v_i$$
$$=-ia^2k_i\frac{k_j}{\omega}v_j-i\frac{a^2}{\bar{\rho}}k_i\rho^{(2)}$$
$$+\frac{e}{m}\left[E_i^{(1)}+E_i^{(2)}\right]$$
$$+\frac{\omega_p^2}{\omega^2}\beta k^2\sigma\left[A_2v_i-A_3c^2\frac{k_i}{\omega}\frac{k_\ell}{\omega}v_\ell\right]$$
$$+\omega_b\left[v_y\hat{a}_x-v_x\hat{a}_y\right]\cdot\hat{a}_i$$
$$-iA_1'\frac{k^2\sigma}{\omega}\omega_b\left[v_y\hat{a}_x-v_x\hat{a}_y\right]\cdot\hat{a}_i$$
$$+T_{4i}(\underline{k},\omega)\;, \tag{3-74}$$

where $T_{4i}(\underline{k},\omega)$ is given by a sum over terms of the type (3-72), and the field quantities other than the velocity are given by equations (3-37) to (3-40).

3.3 The Semi-compressible Approximation

At this point the assumption will be made that the plasma is semi-compressible. By the term "semi-compressible" it is to be understood that the ratio of the fluctuating density to the average density is small enough so that second order terms involving density fluctuations may be neglected. Under this condition there results

$$
\begin{aligned}
&A_1 \to 1 \quad A_1' \to 0 \quad \rho^{(2)} \to 0 \\
&E_i^{(2)} \to 0 \quad T_2(\underline{k},\omega) \to 0 \\
&T_3(\underline{k},\omega) \to 0 \quad T_5(\underline{k},\omega) \to 0,
\end{aligned} \tag{3-75}
$$

and the momentum equation (3-74) becomes

$$
\begin{aligned}
&[-i\omega + \bar{\nu} + k^2\sigma]\, v_i \\
&\qquad = -i a^2 k_i \frac{1}{\omega} k_j v_j + \frac{e}{m} E_i^{(1)} \\
&\qquad + \omega_b [v_y \hat{a}_x - v_x \hat{a}_y] \cdot \hat{a}_i \\
&\qquad + T_{4i}(\underline{k},\omega).
\end{aligned} \tag{3-76}
$$

The quantity $T_{4i}(\underline{k},\omega)$ involves contributions from the second order term $\frac{e}{m}(\underline{\tilde{v}} \times \underline{\tilde{B}})$. The x-component of this second order term is

$$T_{4x}(\underline{k},\omega) = \frac{e}{m}(\underline{\tilde{v}} \times \underline{\tilde{B}})_x \ . \tag{3-77}$$

From (3-41) and (3-43) the y-component of the fluctuating magnetic field is

$$B_y^{(1)} = i\,\frac{m}{e}\,\frac{\omega_p^2}{k^2c^2}(1+\beta)\left[k_z v_x - k_x v_z\right]. \tag{3-78}$$

Obtaining B_z by a cyclic permutation of the subscripts and assuming local transfer, (3-77) becomes

$$T_{4x}(\underline{k},\omega) = i\,\frac{\omega_p^2}{k^2c^2}\,\sigma(1+\beta)\,Q_x\,, \tag{3-79}$$

where

$$Q_x = \frac{1}{\sigma}\int\left[v_y'' k_x' v_y' - v_y'' k_y' v_x' - v_z'' k_z' v_x' + v_z'' k_x' v_z'\right]. \tag{3-80}$$

Now from a Fourier analysis of the identity

$$\frac{\partial}{\partial x}\left[v_i(\underline{r},t)\,v_i(\underline{r},t)\right] = 2\left[v_i(\underline{r},t)\,\frac{\partial v_i(\underline{r},t)}{\partial x}\right] \tag{3-81}$$

where there is no sum on the index i, and remembering that exponentials are involved in the Fourier analysis, it follows from (2-61) that

$$\int v_y'' k_x' v_y' = -\tfrac{1}{2}\,i\,k_x k_y \sigma v_y\,. \tag{3-82}$$

Using this result and (2-61) in (3-80) the quantity

$$Q_x = i\left[-\tfrac{1}{2}k_x k_y v_y + (k_y^2+k_z^2)v_x - \tfrac{1}{2}k_x k_z v_z\right]. \tag{3-83}$$

From the above, the linearized form of (3-79) is

$$T_{4x}(\underline{k},\omega) = -\frac{\omega_p^2}{k^2c^2}\sigma(1+\beta)\left[-\tfrac{1}{2}k_x k_y v_y + (k_y^2+k_z^2)v_x - \tfrac{1}{2}k_x k_z v_z\right], \tag{3-84}$$

so that the full semi-compressible momentum equation for the x-component is

$$\begin{aligned}[-i\omega + \bar{\nu} + k^2\sigma]v_x &= -ia^2 k_x \frac{k_j v_j}{\omega} + \frac{e}{m}E_x^{(1)} \\ &+ \omega_b[v_y\hat{a}_x - v_x\hat{a}_y]\cdot\hat{a}_x \\ &- \frac{\omega_p^2}{k^2c^2}\sigma(1+\beta)\left[-\tfrac{1}{2}k_x k_y v_y + (k_y^2+k_z^2)v_x - \tfrac{1}{2}k_x k_z v_z\right],\end{aligned} \tag{3-85}$$

the other components being obtained by a cyclic permutation of the subscripts. The remainder of this study will be based on the three momentum equations of the type (3-85).

CHAPTER IV

TRANSPORT PHENOMENA IN A SEMI-COMPRESSIBLE PLASMA

4.0 Introduction

It is usual to employ a particle description to derive theoretically the transport properties of an ionized gas. In the case of a Lorentz gas without turbulence; i.e., ignoring non-linear terms, it is possible to show that a macroscopic point of view may be taken in obtaining expressions for such transport quantities as the conductivity, mobility, and the diffusion coefficient. Ohm's law for fluctuating quantities, for example, may be written

$$\tilde{J} = \underline{\underline{\sigma}}\,\underline{\tilde{E}}, \tag{4-1}$$

where

$$\underline{\tilde{J}} = \frac{\bar{\rho}}{m} e \underline{\tilde{v}}, \tag{4-2}$$

and where $\underline{\underline{\sigma}}$ is the conductivity tensor.

By inspection of (4-1) and (4-2) it is easily seen that a mobility may be defined according to

$$\underline{\tilde{v}} = \underline{\underline{\mu}}\,\underline{\tilde{E}}, \tag{4-3}$$

and the conductivity tensor expressed as

$$\underline{\underline{\sigma}} = \frac{m}{\rho e}\underline{\underline{\mu}}. \tag{4-4}$$

In the presence of turbulence, however, it is not possible to write Ohm's law as in (4-1) nor is it possible to define a mobility according to (4-3) in view of the non-linearity of the equations of motion. Because of this non-linearity the fluctuations are of a random nature so that the fluctuating field quantities have no unique Fourier decomposition; and the mobility, for instance, which depends on the spectrum, is also of a random nature.

In view of the fact that the Fourier analyzed equations of motion of a turbulent Lorentz gas as developed in previous chapters are linear, it is possible to avoid the above difficulty by defining the mobility according to

$$\underline{v}(\underline{k},\omega) = \underline{\underline{\mu}}^{a}(\underline{k},\omega)\,\underline{E}^{a}(\underline{k},\omega), \tag{4-5}$$

or in component form as

$$v_i(\underline{k},\omega) = \mu^{a}_{ij}(\underline{k},\omega)\,E_j^{(a)}(\underline{k},\omega), \tag{4-6}$$

where the superscript "a" indicates the summation over the various classes of modes of which there are six as will be shown later.

To show how the mobility as defined by (4-6) may be useful it is necessary to make a connection with this definition and known quantities. Taking a case for which one class of modes is dominant and considering two-dimensional turbulence for simplicity, the x-component of (4-6) may be written

$$v_x(\underline{k},\omega) = \mu_{xx}(\underline{k},\omega)E_x(\underline{k},\omega) + \mu_{xy}(\underline{k},\omega)E_y(\underline{k},\omega). \qquad (4\text{-}7)$$

Multiplying (4-7) by its complex conjugate there results

$$\begin{aligned} v_x^2(\underline{k}) = {} & \mu_{xx}^2(\underline{k})E_x^2(\underline{k}) \\ & + \mu_{xx}(\underline{k})\mu_{xy}^*(\underline{k})E_x(\underline{k})E_y^*(\underline{k}) \\ & + \mu_{xx}^*(\underline{k})\mu_{xy}(\underline{k})E_x^*(\underline{k})E_y(\underline{k}) \\ & + \mu_{xy}^2(\underline{k})E_y^2(\underline{k}), \end{aligned} \qquad (4\text{-}8)$$

where the dependence on ω is to be understood.

In this chapter it will be shown that

$$\mu_{ij}(\underline{k},\omega) = \mu_{ij}(\hat{k},\omega) \qquad (4\text{-}9)$$

and that

$$E_y(\underline{k},\omega) = f(\hat{k},\omega) E_x(\underline{k},\omega), \qquad (4\text{-}10)$$

where $f(\hat{k},\omega)$ is determined by the so-called "dispersion" equations.

According to the above, then, (4-8) may be written as

$$\begin{aligned} v_x^2(\underline{k}) = [\,& \mu_{xx}(\hat{k}) \\ &+ \mu_{xx}(\hat{k})\mu_{xy}^*(\hat{k}) f^*(\hat{k}) \\ &+ \mu_{xx}^*(\hat{k})\mu_{xy}(\hat{k}) f(\hat{k}) \\ &+ \mu_{xy}^2(\hat{k}) f^2(\hat{k})] E_x^2(\underline{k}). \end{aligned} \qquad (4\text{-}11)$$

In this way all of the quantities on the right hand side are known except for the electric field term which may be derived theoretically from a statistical analysis.

A further simplification is possible if the electric field term in (4-11) may be written in the form

$$E_x^2(\underline{k}) = g(\hat{k})\, E_x^2(k), \tag{4-12}$$

so that in integrating (4-11) over k-space in spherical co-ordinates (k, θ, ϕ), there results

$$\begin{aligned}\int N_x^2(\underline{k})\, d\underline{k} = \int g(\hat{k}) \big[&\mu_{xx}^2(\hat{k}) \\ &+ \mu_{xx}(\hat{k})\mu_{xy}^*(\hat{k}) f^*(\hat{k}) \\ &+ \mu_{xx}^*(\hat{k})\mu_{xy}(\hat{k}) f(\hat{k}) \\ &+ \mu_{xy}^2(\hat{k}) f^2(\hat{k})\big] E_x^2(k)\, k^2 \\ &\times \sin\phi\, d\theta\, d\phi\, dk \end{aligned} \tag{4-13}$$

or

$$\begin{aligned}\langle N_x^2(\underline{r})\rangle = \Big\{ \int g(\hat{k}) \big[&\mu_{xx}^2(\hat{k}) \\ &+ \mu_{xx}(\hat{k})\mu_{xy}^*(\hat{k}) f^*(\hat{k}) \\ &+ \mu_{xx}^*(\hat{k})\mu_{xy}(\hat{k}) f(\hat{k}) \\ &+ \mu_{xy}^2(\hat{k}) f^2(\hat{k})\big] \sin\phi\, d\theta\, d\phi \Big\} \\ &\times \frac{1}{2\pi} \langle E_x^2(\underline{r})\rangle, \end{aligned} \tag{4-14}$$

where the electric field correlation may be measured in the laboratory.

In certain special cases such as the incompressible case where the fluctuations in the magnetic field are neglected, it may turn out that the mobility and the function $f(\hat{k})$ are independent of $\underline{k}$ so that (4-8) may be integrated immediately with the result

$$
\begin{aligned}
\langle v_x(\underline{r}) v_x(\underline{r}) \rangle = \Big[& \mu_{xx}^2 \\
& + \mu_{xx} \mu_{xy}^* f^* + \mu_{xx}^* \mu_{xy} f \\
& + \mu_{xy}^2 f^2 \Big] \langle E_x(\underline{r}) E_x(\underline{r}) \rangle . \qquad (4\text{-}15)
\end{aligned}
$$

The extension of the above to three dimensional turbulence is straightforward. When more than one class of modes are present there seems to be no reason why cross products between mode classes should appear since there is nothing in the theory to indicate that the contributions to the fluctuations from different classes are correlated. With this remark, the extension to more complicated turbulence is easily made.

The expression for the mobility for different modes may be obtained from the momentum equation by inspection as will be seen. In general the result will depend on the wave-vector. By using the so-called "dispersion" equations, the dependence on the magnitude of the wave-vector may be eliminated and the index of refraction for the various modes obtained. Before proceeding to a detailed analysis of certain degenerate cases, the general form of the momentum and dispersion equations will be considered in the next section.

4.1 General Direction of Propagation

For propagation in the general case the wave-vector components may be written in terms of spherical coordinates so that

$$k_x = k \sin\phi \cos\theta , \tag{4-16}$$

$$k_y = k \sin\phi \sin\theta \tag{4-17}$$

and

$$k_z = k \cos\phi . \tag{4-18}$$

Now neglecting the collision term and slightly restricting the study to waves for which the phase velocity is much greater than the velocity of sound so that the pressure term may be neglected, (3-85) becomes

$$\left[-i\omega + k^2\sigma + \frac{\omega_p^2\sigma}{c^2}(1+\beta)\,\pi_{xx}\right] v_x$$
$$+ \left[-\omega_b + \frac{\omega_p^2\sigma}{c^2}(1+\beta)\,\pi_{xy}\right] v_y$$
$$+ \left[\frac{\omega_p^2\sigma}{c^2}(1+\beta)\,\pi_{xz}\right] v_z$$
$$= \frac{e}{m} E_x , \tag{4-19}$$

$$\left[\omega_b + \frac{\omega_p^2\sigma}{c^2}(1+\beta)\,\pi_{xy}\right] v_x$$
$$+ \left[-i\omega + k^2\sigma + \frac{\omega_p^2\sigma}{c^2}(1+\beta)\,\pi_{yy}\right] v_y$$
$$+ \left[\frac{\omega_p^2\sigma}{c^2}(1+\beta)\,\pi_{yz}\right] v_z$$
$$= \frac{e}{m} E_y , \tag{4-20}$$

and

$$\left[\frac{\omega_p^2\sigma}{c^2}(1+\beta)\,\pi_{xz}\right] v_z$$
$$+ \left[\frac{\omega_p^2\sigma}{c^2}(1+\beta)\,\pi_{yz}\right] v_y$$
$$+ \left[-i\omega + k^2\sigma + \frac{\omega_p^2\sigma}{c^2}(1+\beta)\,\pi_{zz}\right] v_z$$
$$= \frac{e}{m} E_z , \tag{4-21}$$

where

$$\underline{\underline{\Pi}} = \begin{pmatrix} \sin^2\phi\sin^2\theta + \cos^2\phi & -\frac{1}{2}\sin^2\phi\sin\theta\cos\theta & -\frac{1}{2}\sin\phi\cos\phi\cos\theta \\ -\frac{1}{2}\sin^2\phi\sin\theta\cos\theta & \sin^2\phi\cos^2\theta + \cos^2\phi & -\frac{1}{2}\sin\phi\cos\phi\sin\theta \\ -\frac{1}{2}\sin\phi\cos\phi\cos\theta & -\frac{1}{2}\sin\phi\cos\phi\sin\theta & \sin^2\phi \end{pmatrix}. \tag{4-22}$$

To obtain the dispersion equations, use is made of (3-39) which becomes

$$\frac{e}{m} E_i = i \frac{\omega_p^2}{\omega} \beta \left[v_i - \frac{k^2 c^2}{\omega^2} \left(\Delta_{ix} v_x + \Delta_{iy} v_y + \Delta_{iz} v_z \right) \right], \qquad (4\text{-}23)$$

where

$$\underline{\underline{\Delta}} = \begin{pmatrix} \sin^2 \cos^2\theta & \sin^2\phi \cos\theta \sin\theta & \sin\phi \cos\phi \cos\theta \\ \sin^2\phi \sin\theta \cos\theta & \sin^2\phi \sin^2\theta & \sin\phi \cos\phi \cos\theta \\ \sin\phi \cos\phi \cos\theta & \sin\phi \cos\phi \sin\theta & \cos^2\phi \end{pmatrix}. \qquad (4\text{-}24)$$

In this way the dispersion equation involving the x-component of the electric field is

$$\Big[-i\omega + k^2\sigma - i\,\frac{\omega_p^2}{\omega}\beta + \frac{\omega_p^2\sigma}{c^2}(1+\beta)\,\pi_{xx} + i\,\frac{\omega_p^2}{\omega}\beta\,\frac{k^2c^2}{\omega^2}\Delta_{xx}\Big]v_x$$
$$+\Big[-\omega_b + \frac{\omega_p^2\sigma}{c^2}(1+\beta)\,\pi_{xy} + i\,\frac{\omega_p^2}{\omega}\beta\,\frac{k^2c^2}{\omega^2}\Delta_{xy}\Big]v_y$$
$$+\Big[\frac{\omega_p^2\sigma}{c^2}(1+\beta)\,\pi_{xz} + i\,\frac{\omega_p^2}{\omega}\beta\,\frac{k^2c^2}{\omega^2}\Delta_{xz}\Big]v_z = 0, \qquad (4\text{-}25)$$

with similar equations for the y- and z-components.

Because of the obvious complexity of the momentum and dispersion equations in the general case, only two degenerate cases will be considered in detail; Fourier waves propagating along the z-axis, and those propagating along the x-axis.

4.2 Propagation along the z-axis

The first class of component waves which will be studied are those which propagate along the z-axis so that

$$\underline{k} = k\,\hat{a}_z. \qquad (4\text{-}26)$$

Since the collision term and the pressure term are being neglected, (3-85) becomes

$$(\gamma+\delta)v_x - \omega_b v_y = \frac{e}{m}E_x \qquad (4\text{-}27)$$

and the other momentum relations are

$$\omega_b v_x + (\gamma+\delta)v_y = \frac{e}{m}E_y \qquad (4\text{-}28)$$

and

$$\gamma v_z = \frac{e}{m}E_z, \qquad (4\text{-}29)$$

where

$$\gamma = -i\omega + h^2\sigma \qquad (4\text{-}30)$$

and

$$\delta = \frac{\omega_p^2\sigma}{c^2}(1+\beta). \qquad (4\text{-}31)$$

Solving for the velocities in terms of the electric fields in (4-29) to (4-31) inclusive, the following equations are obtained

$$v_x = \frac{e}{m}\frac{1}{(\gamma+\delta)^2+\omega_b^2}\left[(\gamma+\delta)E_x + \omega_b E_y\right], \tag{4-32}$$

$$v_y = \frac{e}{m}\frac{1}{(\gamma+\delta)^2+\omega_b^2}\left[-\omega_b E_x + (\gamma+\delta)E_y\right] \tag{4-33}$$

and

$$v_z = \frac{e}{m}\frac{1}{\gamma}E_z, \tag{4-34}$$

from which the mobility may be obtained by inspection and is

$$\underline{\underline{\mu}} = \frac{e}{m}\begin{pmatrix} \frac{\gamma+\delta}{(\gamma+\delta)^2+\omega_b^2} & \frac{\omega_b}{(\gamma+\delta)^2+\omega_b^2} & 0 \\ \frac{-\omega_b}{(\gamma+\delta)^2+\omega_b^2} & \frac{\gamma+\delta}{(\gamma+\delta)^2+\omega_b^2} & 0 \\ 0 & 0 & \frac{1}{\gamma} \end{pmatrix}. \tag{4-35}$$

In order to eliminate k from the expression for the mobility use is made of the dispersion determinant. This determinant is obtained by substituting

$$E_i^{(1)} = i \frac{m}{e} \frac{\omega_p^2}{\omega} \beta \left[v_i - c^2 \frac{k_i k_j}{\omega^2} v_j \right] \qquad (4\text{-}36)$$

into the three momentum equations (4-27,28,29). The three equations from which the dispersion determinant is formed are

$$(\gamma + \delta - \alpha) v_x - \omega_b v_y = 0, \qquad (4\text{-}37)$$

$$\omega_b v_x + (\gamma + \delta - \alpha) v_y = 0 \qquad (4\text{-}38)$$

and

$$\left(\gamma + i \frac{\omega_p^2}{\omega}\right) v_z = 0, \qquad (4\text{-}39)$$

where

$$\alpha = i \frac{\omega_p^2}{\omega} \beta. \qquad (4\text{-}40)$$

In order for the dispersion equations (4-37,38,39) to be consistent it must be that

$$\begin{vmatrix} \gamma+\delta-\alpha & -\omega_b & 0 \\ \omega_b & \gamma+\delta-\alpha & 0 \\ 0 & 0 & \gamma+i\dfrac{\omega_p^2}{\omega} \end{vmatrix} = 0. \tag{4-41}$$

This dispersion determinant will vanish under several different conditions yielding longitudinal and transverse modes propagating in the z-direction. The nature of these modes will now be investigated more closely.

4.2.1 Longitudinal Modes

Motion for the longitudinal modes is restricted to the z-axis so that $v_z \neq 0$ and $v_x = v_y = 0$; accordingly it follows from (4-39) that

$$\gamma + i\frac{\omega_p^2}{\omega} = 0, \tag{4-42}$$

that is

$$-i\omega + k^2\sigma + i\frac{\omega_p^2}{\omega} = 0. \tag{4-43}$$

Since the cyclotron frequency does not appear in this expression these particular modes are independent of the applied magnetic field and are of the same form when no average magnetic field is present.

Defining the complex index of refraction n_c as

$$n_c = \frac{kc}{\omega}, \tag{4-44}$$

which is the ratio of the velocity of light to the phase velocity of the wave, (4-43) may be written

$$-i\omega + \frac{n_c^2 \omega^2}{\Omega} + i\,\frac{\omega_p^2}{\omega} = 0, \tag{4-45}$$

where

$$\Omega = \frac{\sigma}{c^2} \tag{4-46}$$

is the characteristic turbulent frequency discussed in section 4.4.

Solving for the complex index of refraction there results

$$n_c^2 = i\,\frac{\Omega}{\omega}\left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right]. \tag{4-47}$$

At this point it would be well to recall the condition of the validity of the iteration procedure which is (3-12)

$$A_1' \frac{\omega}{\Omega} |n_c^2| < 1. \tag{4-48}$$

For a strictly semi-compressible plasma it was mentioned in section 3.2 that $A_1' = 0$; however, it is to be expected in an actual case that A_1' will be finite. Defining

$$T = \frac{\omega}{\Omega} |n_c^2|, \tag{4-49}$$

and for a value of A_1' which may be as large as 1/10, a mode for which $T < 1$ may still be considered to be in the semi-compressible region; accordingly for the remainder of this chapter the condition

$$T < 1 \tag{4-50}$$

will be assumed to hold. This condition is, of course, unnecessarily restrictive if $A_1' \ll 1/10$.

From (4-45) in the case of negligible turbulence ($\Omega \to \infty$) it may be noted that

$$\omega = \omega_p, \tag{4-51}$$

which is the condition for the well-known plasma oscillations.

The mobility for the longitudinal waves under consideration in this section is most easily obtained from (4-42) in the form

$$\gamma = -i \frac{\omega_p^2}{\omega}, \tag{4-52}$$

which when substituted in (4-35) results in

$$\mu_{zz} = i \frac{e}{m} \frac{\omega}{\omega_p^2}, \tag{4-53}$$

which is the mobility for these waves. That these modes are strongly damped can be seen from (4-47) and the following discussion.

The complex refractive index may in general be written as

$$n_c = n_R + i n_I, \tag{4-54}$$

where n_R is the refractive index and n_I is the extinction index. The significance of these two quantities may be seen in view of (4-44) by writing the complex wave vector as

$$\underline{k} = \underline{k}_R + i \underline{k}_I, \tag{4-55}$$

where $\underline{k}_R$ is the wave vector and $\underline{k}_I$ the attenuation vector.

Since n_c^2 is imaginary for the longitudinal modes under discussion it follows that

$$|k_R| = |k_I| \tag{4-56}$$

indicating that for these particular modes the amplitude falls to 1/e of its original value in one wave-length. Note that the damping comes only from the inertia term, attenuation of the mode being due to momentum transfer to modes of higher k-value. This type of momentum transfer from larger to smaller eddies is in analogy with the hydrodynamic case for which the attenuation of large eddies is also due to the inertia term. Eventually, of course, a critical eddy size is reached for which the collision term, neglected above, becomes important and serves as the ultimate mechanism for the dissipation of energy by thermalization of the turbulent motion.

4.2.2 Transverse Modes

In this case the motion is transverse to the direction of propagation so that $v_z = 0$ and the velocity vector is confined to the xy-plane. For this situation the dispersion determinant (4-41) yields

$$\gamma + \delta - \alpha = \pm i\omega_b \tag{4-57}$$

or

$$-i\omega + \frac{\omega^2}{\Omega} n_c^2 + \frac{\omega_p^2}{\Omega}(1+\beta) - i\frac{\omega_p^2}{\omega}\beta = \pm i\omega_b, \tag{4-58}$$

which determines the complex index of refraction n_c in terms of the other parameters. For convenience the symbol "n_c" will be used in all cases for the complex index of refraction so that it must be stressed that its form will in general change for each different mode.

Since the quantity β is given by (3-19) it is convenient to multiply through by $(n_c^2 - 1)$ yielding

$$\frac{\omega}{\Omega} n_c^4 + \left[\frac{\omega_p^2}{\Omega\omega} - i\left(1 \pm \frac{\omega_b}{\omega}\right) - \frac{\omega}{\Omega}\right] n_c^2 + i\left(1 \pm \frac{\omega_b}{\omega}\right) - i\frac{\omega_p^2}{\omega^2} = 0, \tag{4-59}$$

with the possible solutions

$$2\frac{\omega}{\Omega}n_c^2 = \left[\frac{\omega_p^2}{\Omega\omega} - i\left(1 \pm \frac{\omega_b}{\omega}\right) - \frac{\omega}{\Omega}\right]\left\{-1 \pm \right.$$

$$\left.\sqrt{1 - \frac{4i\frac{\omega}{\Omega}\left(1 \pm \frac{\omega_b}{\omega} - \frac{\omega_p^2}{\omega^2}\right)}{\left[\frac{\omega_p^2}{\Omega\omega} - i\left(1 \pm \frac{\omega_b}{\omega}\right) - \frac{\omega}{\Omega}\right]^2}}\right\}. \tag{4-60}$$

In view of the complexity of (4-60) and since

$$\frac{\omega}{\Omega}|n_c^2| < 1 \tag{4-61}$$

must hold in accordance with (4-50), several special cases involving different values of the parameters of the plasma will be discussed.

Before beginning the discussion of special cases it seems best to study solutions of the general quadritic equation when a condition such as (4-61) holds.

The general quadratic equation is

$$ax^2 + bx + c = 0, \tag{4-62}$$

with the possible solutions

$$ax = \frac{1}{2} b\left[-1 \pm \sqrt{1 - \frac{4ac}{b^2}}\right]. \tag{4-63}$$

Imposing the condition

$$|ax| < 1 \tag{4-64}$$

limits the solutions in a way depending on the quadratic coefficients. Several possibilities are as follows:

(1) $|b| < 1, \ \left|\frac{4ac}{b^2}\right| \leqslant 1.$

Under these conditions, both roots satisfy (4-64).

(2) $|b| < 1, \ \left|\frac{4ac}{b^2}\right| > 1.$

Here the roots become

$$ax \cong -\frac{1}{2} b \pm i\sqrt{ac}\left(1 - \frac{b^2}{8ac}\right) \tag{4-65}$$

with the condition that

$$|ac|^{1/2} \leq 1. \tag{4-66}$$

If it happens that

$$\left| \frac{4ac}{b^2} \right| \gg 1 , \tag{4-67}$$

then

$$ax = \pm i\sqrt{ac} . \tag{4-68}$$

(3) $|b| \cong 1.$

If

$$\left| 1 - \frac{4ac}{b^2} \right|^{1/2} < 1 , \tag{4-69}$$

then both roots (4-63) can give valid results.

(4) $|b| > 1.$

Only the positive sign in front of the radical in (4-63) may be used with the additional requirement that

$$\left| 1 - \frac{4ac}{b^2} \right| \cong 1 , \tag{4-70}$$

implying that

$$\left| \frac{4ac}{b^2} \right| < 1, \tag{4-71}$$

so that the root may be approximated by

$$ax = -\frac{ac}{b}\left(1 - \frac{ac}{b^2}\right). \tag{4-72}$$

Having completed this preliminary discussion, it is now possible to begin an analysis of the transverse modes under various conditions. The parameters to be varied are the characteristic turbulent frequency Ω , the electron cyclotron frequency ω_b, and the Fourier frequency ω .

4.2.2.1 Limiting Case of No Turbulence. In passing to the case of no turbulence ($\Omega \to \infty$), (4-59) becomes

$$-i\left(1 \pm \frac{\omega_b}{\omega}\right) n_c^2 + i\left[\left(1 \pm \frac{\omega_b}{\omega}\right) - \frac{\omega_p^2}{\omega^2}\right] = 0 \tag{4-73}$$

and

$$n_c^2 = \frac{1 \pm \frac{\omega_b}{\omega} - \frac{\omega_p^2}{\omega^2}}{1 \pm \frac{\omega_b}{\omega}} . \tag{4-74}$$

In order to calculate the mobility for this situation as well as for the others in the remainder of this chapter, it is easiest to use the relation

$$\gamma + \delta = \alpha \pm i\omega_b \tag{4-75}$$

which follows immediately from (4-57).

Substituting in (4-35) there results

$$\mu_{xx} = \frac{e}{m}\,\frac{\alpha \pm i\omega_b}{\alpha^2 \pm 2i\alpha\omega_b} \tag{4-76}$$

with similar expressions for μ_{xy}, μ_{yx}, and μ_{yy}.

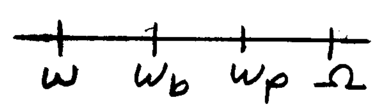
According to (4-20)

$$\alpha = i\,\frac{\omega_p^2}{\omega}\,\beta, \tag{4-77}$$

so that in view of (4-41) the mobility may be expressed in terms of ω alone for any particular mode.

4.2.2.2 Moderate Turbulence and Moderate Magnetic Field.
($\omega_p/\Omega < 1$, $\omega_b/\omega_p < 1$)

(1) $\omega/\omega_b < 1$.



Under these conditions (4-59) becomes

$$\frac{\omega}{\Omega} n_c^4 + \left[\frac{\omega_p^2}{\Omega\omega} \mp i \frac{\omega_b}{\omega} \right] n_c^2 - i \frac{\omega_p^2}{\omega^2} = 0 . \tag{4-78}$$

Since $|b| > 1$ only one root may be used. This means essentially that the term $\frac{\omega}{\Omega} n_c^4$ may be dropped so that

$$n_c^2 = \frac{i \frac{\omega_p^2}{\omega^2}}{\frac{\omega_p^2}{\Omega\omega} \mp i \frac{\omega_b}{\omega}} \tag{4-79}$$

or

$$n_c^2 = \frac{\omega_p^2}{\omega^2} \left[\left(\frac{\omega_p^2}{\Omega\omega} \right)^2 + \frac{\omega_b^2}{\omega^2} \right]^{-1/2} e^{i\left(\frac{\pi}{2} \pm \theta\right)} , \tag{4-80}$$

where

$$\tan \theta = \frac{\omega_b \Omega}{\omega_p^2} . \tag{4-81}$$

Since equation (4-71) must also be satisfied, it must be that

$$4\,\frac{\omega\Omega}{\omega_p^2} < 1. \tag{4-82}$$

In the case of particularly low magnetic fields and stronger turbulence, it might be that

$$\frac{\omega_b\Omega}{\omega_p^2} << 1, \tag{4-83}$$

in which case writing (4-80) as

$$n_c^2 = \frac{\Omega}{\omega}\Big[1 + \frac{\omega_b^2\Omega^2}{\omega_p^4}\Big]^{-1/2} e^{i\,(\frac{\pi}{2} \pm \theta)}, \tag{4-84}$$

there results

$$n_R \cong \left(\frac{\Omega}{\omega}\right)^{1/2} \cos\frac{\pi}{4}. \tag{4-85}$$

To calculate the mobility we use the quantity β. Since $\Omega >> \omega$ and from (4-83), $|n_c^2| > 1$ so that

$$\beta \cong \frac{1}{n_c^2} \tag{4-86}$$

or

$$\beta = -i\,\frac{\omega}{\Omega}\left[1 + \frac{\omega_b^2\Omega^2}{2\omega_p^4}\right]. \qquad (4\text{-}87)$$

Substituting this value for β in (4-76), the mobility to second order is

$$\mu_{xx} = \frac{e}{m}\frac{\Omega}{\omega_p^2}\Bigg(1 - \frac{\omega_b^2\Omega^2}{2\omega_p^4} \mp i\,\frac{\omega_b\Omega}{\omega_p^2}\Bigg). \qquad (4\text{-}88)$$

For high magnetic fields and relatively weak turbulence, it may be that

$$\frac{\omega_b\Omega}{\omega_p^2} > 1\,, \qquad (4\text{-}89)$$

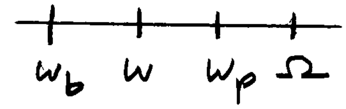
in which case (4-84) yields

$$n_R \cong 0\ ; \qquad (+\omega_b) \qquad (4\text{-}90)$$

and

$$n_R \cong \left(\frac{\omega_p^2}{\omega\omega_b}\right)^{1/2}; \qquad (-\omega_b). \qquad (4\text{-}91)$$

(2) $\frac{\omega}{\omega_b} \gtrsim 1, \; \frac{\omega}{\omega_p} < 1.$



In this situation (4-59) is

$$\frac{\omega}{\Omega} n_c^4 + \left[-i\left(1 \pm \frac{\omega_b}{\omega}\right)\right] n_c^2 - i\,\frac{\omega_p^2}{\omega^2} = 0 . \tag{4-92}$$

Taking the positive sign first, $|b| > 1$ so that the first term may be neglected leading to

$$n_c^2 = -\frac{\omega_p^2}{\omega^2}\left(1 + \frac{\omega_b}{\omega}\right)^{-1}, \tag{4-93}$$

which is simply the appropriate form of (4-74), the non-turbulent case.

Similar results are obtained with the negative sign unless $\omega_b/\omega \cong 1$ in which case $|b| < 1$. For this situation, the results depend on the quantity

$$\left|\frac{4ac}{b^2}\right| = 4\,\frac{\omega_p}{\Omega}\,\frac{\omega_p}{\omega}\left(1 - \frac{\omega_b}{\omega}\right)^{-2}. \tag{4-94}$$

If $\left|\frac{4ac}{b^2}\right| \leq 1$ there results

$$n_c^2 = -i\frac{\Omega}{\omega}\left(1-\frac{\omega_b}{\omega}\right)\Big[-1 + i\frac{\omega}{\Omega}\frac{\omega_p^2}{\omega^2}\left(1-\frac{\omega_b}{\omega}\right)^{-2}\Big] \tag{4-95}$$

and

$$n_c^2 = i\left(1-\frac{\omega_b}{\omega}\right)^{-1}\frac{\omega_p^2}{\omega^2} \tag{4-96}$$

as the two roots of (4-92).

If $\left|\frac{4ac}{b^2}\right| > 1$ the roots are according to (4-65)

$$n_c^2 = \frac{1}{2}i\frac{\Omega}{\omega}\left(1-\frac{\omega_b}{\omega}\right) \pm \frac{\omega_p}{\omega}\sqrt{\frac{\Omega}{\omega}}\Big[1 + \frac{\Omega\omega}{8\omega_p^2}\left(1-\frac{\omega_b}{\omega}\right)^2\Big]e^{i\frac{\pi}{4}}, \tag{4-97}$$

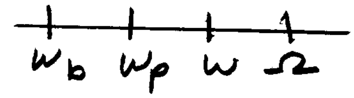
or if $\left|\frac{4ac}{b^2}\right| \gg 1$ the roots become

$$n_c^2 = \pm\frac{\omega_p}{\omega}\sqrt{\frac{\Omega}{\omega}}\,e^{i\pi/4} \tag{4-98}$$

in accordance with (4-68).

Expressions for the index of refraction and the mobilities may be obtained from these results in the usual manner.

(3) $\frac{\omega}{\omega_p} \geq 1, \; \frac{\omega}{\Omega} < 1.$



For the higher Fourier frequencies considered here, the first term in (4-59) may be neglected and the results are similar to (4-74) with the appropriate approximations.

(4) $\frac{\omega}{\Omega} > 1.$

In this case, (4-59) becomes

$$\frac{\omega}{\Omega} n_c^4 - \frac{\omega}{\Omega} n_c^2 + i = 0 \tag{4-99}$$

and $|b| > 1$ leading to the possible root

$$n_c^2 = \frac{\Omega}{\omega} i , \tag{4-100}$$

which; however, violates the condition (4-50) so that the turbulence is too strong for the iterative procedure and $|\rho^{(2)}|/|\rho^{(1)}| > 1$.

4.2.2.3 Moderate Turbulence and High Magnetic Field.

$\left(\omega_p/\omega_b < 1 \ , \ \omega_b/\Omega < 1 \right)$ $\qquad \omega_p \quad \omega_b \quad \Omega$

With one exception, to first order, these conditions lead to the non-turbulent case (4-74). The one exception is when $\omega_b/\omega \cong 1$ so that $|b| < 1$. In this instance, (4-59) is

$$\frac{\omega}{\Omega} n_c^4 - \frac{\omega}{\Omega} n_c^2 - i \frac{\omega_p^2}{\omega^2} = 0 , \qquad (4\text{-}101)$$

with the roots

$$n_c^2 = \frac{1}{2} \left[1 \pm \sqrt{1 + 4i \frac{\omega_p^2}{\omega^2} \frac{\Omega}{\omega}} \right]. \qquad (4\text{-}102)$$

4.3 Propagation along the x-axis

The other degenerate case for which the equations in section 4.1 become manageable is for propagation perpendicular to the magnetic field. The axes are chosen such that

$$\underline{k} = k \hat{a}_x . \qquad (4\text{-}103)$$

Again neglecting the collision and pressure terms in (3-85), the momentum equations become

$$\gamma v_x \quad - \omega_b v_y \qquad = \frac{e}{m} E_x , \tag{4-104}$$

$$\omega_b v_x + (\gamma + \delta) v_y \qquad = \frac{e}{m} E_y \tag{4-105}$$

and

$$(\gamma + \delta) v_z = \frac{e}{m} E_z , \tag{4-106}$$

where again

$$\gamma = -i\omega + h^2 \sigma \tag{4-107}$$

and

$$\delta = \frac{\omega_p^2 \sigma}{c^2} (1 + \beta) , \tag{4-108}$$

which yield the mobility in the form

$$\underline{\underline{\mu}} = \frac{e}{m} \begin{pmatrix} \frac{\gamma+\delta}{D} & \frac{\omega_b}{D} & 0 \\ -\frac{\omega_b}{D} & \frac{\gamma}{D} & 0 \\ 0 & 0 & \frac{1}{\gamma+\delta} \end{pmatrix} , \tag{4-109}$$

where

$$D = \gamma(\gamma+\delta) + \omega_b^2 . \tag{4-110}$$

Making use of (3-39) in the momentum equations, the dispersion relations are obtained and are

$$\left(\gamma + i\frac{\omega_p^2}{\omega}\right)v_x - \omega_b v_y = 0, \tag{4-111}$$

$$\omega_b v_x + (\gamma+\delta-\alpha)v_y = 0 \tag{4-112}$$

and

$$(\gamma+\delta-\alpha)v_z = 0, \tag{4-113}$$

so that the dispersion determinant is

$$\begin{vmatrix} \gamma + i\frac{\omega_p^2}{\omega} & -\omega_b & 0 \\ \omega_b & \gamma+\delta-\alpha & 0 \\ 0 & 0 & \gamma+\delta-\alpha \end{vmatrix} = 0, \tag{4-114}$$

where again

$$\alpha = i\frac{\omega_p^2}{\omega}\beta . \tag{4-115}$$

4.3.1 Longitudinal Modes

Longitudinal modes are those whose motion is confined to the x-axis which is the direction of propagation so that $v_x \neq 0$ and no velocity component exists in the yz-plane. Under these conditions the mobility is easily obtained from

$$\gamma + \delta = \alpha, \tag{4-116}$$

which comes from the dispersion determinant. Using this relation, the mobility is given by

$$\mu_{zz} = -\frac{e}{m}\frac{\omega}{\omega_p^2}(n_c^2 - 1). \tag{4-117}$$

The complex index of refraction may also be obtained from (4-116) in the form

$$\begin{aligned} -i\omega + \frac{\omega^2}{\Omega} n_c^2 + \frac{\omega_p^2}{\Omega}(1+\beta) \\ - i\frac{\omega_p^2}{\omega}\beta = 0, \end{aligned} \tag{4-118}$$

which may be rewritten

$$\frac{\omega}{\Omega} n_c^4 + \left[\frac{\omega}{\Omega}\left(\frac{\omega_p^2}{\omega^2} - 1\right) - i \right] n_c^2$$
$$+ \, i\left(1 - \frac{\omega_p^2}{\omega^2}\right) = 0 .$$

(4-119)

4.3.1.1 Limiting Case of No Turbulence. In the case of no turbulence ($\Omega \to \infty$) there results

$$n_c^2 = 1 - \frac{\omega_p^2}{\omega^2} , \qquad (4\text{-}120)$$

so that the mode is non-propagating if $\omega < \omega_p$.

4.3.1.2 Turbulence with Moderate Density. Here the case $\omega_p/\Omega < 1$ will be considered. Since $|b| > 1$, it must be that $\left| \frac{4ac}{b^2} \right| < 1$ so that

$$\left| \frac{4 \frac{\omega}{\Omega}\left(1 - \frac{\omega_p^2}{\omega^2}\right)}{\left[\frac{\omega}{\Omega}\left(\frac{\omega_p^2}{\omega^2} - 1\right) - i \right]^2} \right| < 1 .$$

(4-121)

There are two general situations to be considered.

(1) $\left|\frac{\omega}{\Omega}\left(1-\frac{\omega_p^2}{\omega^2}\right)\right| << 1 .$

Using equation (4-119) in the form

$$n_c^2 = \frac{1}{2}\frac{\Omega}{\omega}\left[\frac{\omega}{\Omega}\left(\frac{\omega_p^2}{\omega^2}-1\right)-i\right]\left\{-1 \pm \sqrt{1-\frac{4i\frac{\omega}{\Omega}\left(1-\frac{\omega_p^2}{\omega^2}\right)}{\left[\frac{\omega}{\Omega}\left(\frac{\omega_p^2}{\omega^2}-1\right)-i\right]^2}}\right\}, \tag{4-122}$$

along with the above approximation, the results are

$$n_c^2 = 1-\frac{\omega_p^2}{\omega^2} \tag{4-123}$$

and

$$n_c^2 = i\frac{\Omega}{\omega} . \tag{4-124}$$

The other possibility to be considered is

(2) $\left|\frac{\omega}{\Omega}\left(1-\frac{\omega_p^2}{\omega^2}\right)\right| > 1 .$

For this case, (4-112) becomes

$$\frac{\omega}{\Omega} n_c^4 + \frac{\omega}{\Omega}\left(\frac{\omega_p^2}{\omega^2} - 1\right) n_c^2 + i\left(1 - \frac{\omega_p^2}{\omega^2}\right) = 0, \tag{4-125}$$

so that

$$n_c^2 = i\frac{\Omega}{\omega}\left[1 - i\frac{\Omega}{\omega}\left(\frac{\omega_p^2}{\omega^2} - 1\right)^{-1}\right] \tag{4-126}$$

is the only root.

4.3.2 Transverse Modes

For these modes, motion is restricted to the yz-plane which is transverse to the direction of propagation so that the dispersion determinant yields

$$(\gamma + \delta - \alpha)\left(\gamma + i\frac{\omega_p^2}{\omega}\right) + \omega_b^2 = 0, \tag{4-127}$$

which rewritten is

$$\left[-i + \frac{\omega}{\Omega} n_c^2 + \frac{\omega_p^2}{\omega\Omega}(1+\beta) - i\frac{\omega_p^2}{\omega^2}\beta\right]\left[-i + \frac{\omega}{\Omega} n_c^2 + i\frac{\omega_p^2}{\omega^2}\right] + \frac{\omega_b^2}{\omega^2} = 0 . \tag{4-128}$$

Employing the condition (4-50) in the form

$$\left| \frac{\omega}{\Omega} n_c^2 \right| < 1 , \tag{4-129}$$

equation (4-128) becomes

$$\left[-i + \frac{\omega_p^2}{\omega \Omega}(1+\beta) - i \frac{\omega_p^2}{\omega^2} \beta \right] \left[-i + i \frac{\omega_p^2}{\omega^2} \right] + \frac{\omega_b^2}{\omega^2} = 0 , \tag{4-130}$$

so that

$$n_c^2 = i \left[\frac{\omega_b^2}{\omega^2} - \left(1 - \frac{\omega_p^2}{\omega^2}\right)^2 \right] \left[\left(1 - \frac{\omega_p^2}{\omega^2}\right) \left(\frac{\omega_p^2}{\omega \Omega} - i \right) + i \frac{\omega_b^2}{\omega^2} \right]^{-1} . \tag{4-131}$$

This is the root corresponding to the classical result slightly modified by the turbulent term. In making the approximation (4-129) the strictly turbulent mode; i.e., the mode which would not appear in the absence of turbulence was discarded.

4.4 Summary

Before going on to a numerical calculation of various quantities connected with the degenerate cases discussed in this chapter it may be well to summarize some of the main results concerning the modes present in a turbulent plasma.

By examining equation (4-25) and keeping in mind that the quantity β involves a k^2 term in the denominator, it is clear that each of the dispersion equations is of second degree in k^2. The dispersion determinant formed from the coefficients of the velocities in the three equations of type (4-25) leads to a sixth order algebraic equation in k^2 so that in general six classes of modes are expected. It might be mentioned that although twelve solutions for k result, each of the k^2 values represents two solutions travelling in opposite directions.

In previous sections, two degenerate cases of propagation along and propagation perpendicular to the applied magnetic field were examined. General expressions for the complex index of refraction were obtained for each mode from which the mobility may be calculated. Some of the salient aspects of the modes studied are summarized in figures 4.1 and 4.2. The various types of mode are indicated with a reference to the section in which they were studied. The turbulent modes are numbered with Roman numerals for future reference. Since any motion of a physical plasma may be described by the three classical modes as discussed in reference 28 taking the ions to immobile, it may be concluded that the turbulent modes are representative of the mixing brought about by the non-linear terms of the equation of motion. The term "turbulent mode" is employed in this work with the understanding that these modes are actually pseudowaves compounded from the classical modes. The relative importance of these turbulent modes is dependent on the value of the characteristic turbulent frequency Ω which was defined in section 4.2.1. It may be well to discuss briefly the physical meaning of this important quantity.

The quantity Ω appears in the various momentum equations studied in two possible ways;

$$C_1 = \frac{\omega}{\Omega} n_c^2 , \tag{4-132}$$

and

$$C_2 = \frac{\omega_p^2}{\Omega \omega} (1+\beta) , \tag{4-133}$$

as can be seen from the typical momentum equation (4-58) upon division by ω .

The characteristic turbulent frequency may be written as

$$\Omega = \frac{2\pi c}{\Lambda} , \tag{4-134}$$

where Λ is a characteristic interaction length between modes.

Now since

$$n_c = \frac{kc}{\omega}, \tag{4-135}$$

a complex Fourier wave length may be defined by

$$\lambda_c = \frac{2\pi}{k}, \tag{4-136}$$

so that (4-132) may be written as

$$C_1 = \frac{\Lambda}{\lambda_c} n_c , \tag{4-137}$$

indicating that the contribution from this term depends on the ratio of the complex wavelength of the Fourier component to the characteristic interaction length which approaches zero as the turbulence decreases ($\Omega \to \infty$).

Now writing (4-133) as

$$C_2 = \frac{\Lambda}{2\pi c} \frac{\lambda_c}{2\pi \nu_\phi} \cdot \frac{4\pi^2 \nu_\phi^2}{\lambda_p^2}(1+\beta) , \tag{4-138}$$

or more simply as

$$C_2 = \frac{\Lambda}{\lambda_p} \frac{\lambda_c}{\lambda_p} \frac{v_\phi}{c} (1+\beta), \tag{4-139}$$

where

$$\lambda_p = \frac{2\pi v_\phi}{\omega}; \tag{4-140}$$

it is seen that the contribution C_2 depends on the ratio of the characteristic interaction length to the plasma wavelength λ_p.

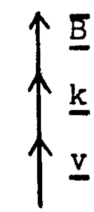
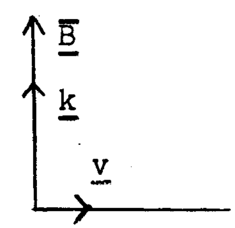
CHARACTER OF WAVE	DAMPING MECHANISM	VECTOR DIAGRAM
<u>Longitudinal Modes</u> (4.2.1)		
$v_x = v_y = 0.$	Momentum transfer through	
$v_z \neq 0.$	plasma oscillations.	
$\underline{k} = ka_z.$		
<u>Transverse Modes</u> (4.2.2)		
$v_x \neq 0,\ v_y \neq 0.$	Momentum transfer through	
$v_z = 0.$	ordinary, extraordinary and	
$\underline{k} = ka_z.$	turbulent modes (II,III).	

FIGURE 4.1 DEGENERATE MODES PRESENT IN TURBULENCE PROPAGATION ALONG Z-AXIS

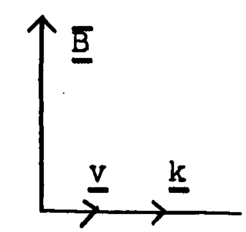
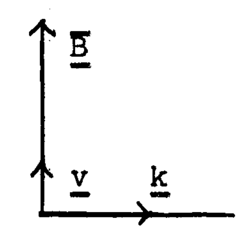
CHARACTER OF WAVE	DAMPING MECHANISM	VECTOR DIAGRAM
Longitudinal Modes (4.3.1)		
$v_x \neq 0,\ v_y = 0.$ $v_z = 0.$ $\underline{k} = ka_x.$	Momentum transfer through singular longitudinal waves and turbulent modes (I).	
Transverse Modes (4.3.2)		
$v_x = 0,\ v_y \neq 0.$ $v_z \neq 0.$ $\underline{k} = ka_x.$	Momentum transfer through elliptically polarized waves and turbulent modes (IV,V).	

FIGURE 4.2 DEGENERATE MODES PRESENT IN TURBULENCE PROPAGATION ALONG X-AXIS

CHAPTER V

COMPUTATION

5.1 Introduction

In this chapter the mobilities and indices of refraction will be computed for the degenerate cases considered in the previous chapter. The results will be displayed in graphical form in the final section.

In the momentum equations (4-27,28,29), use was made of the following quantities:

$$n_c = \frac{kc}{\omega}, \tag{5-1}$$

$$\Omega = c^2/\sigma, \tag{5-2}$$

$$\beta = [n_c^2 - 1]^{-1}, \tag{5-3}$$

$$\gamma = -i\omega + n_c^2 \frac{\omega^2}{\Omega} \tag{5-4}$$

and

$$\delta = \frac{\omega_p^2}{\Omega}(1+\beta). \tag{5-5}$$

These variables are used in the present chapter also in addition to the quantities

$$Y = \omega/\omega_p , \tag{5-6}$$

$$R = \omega_p/\Omega \tag{5-7}$$

and

$$S = \omega_b/\omega_p , \tag{5-8}$$

where R represents the ratio between the characteristic turbulent interaction length and the plasma wavelength. The quantity S is directly proportional to the average magnetic field. In the graphical representation of the results obtained in this chapter, Y will generally be employed as the running variable.

With these new definitions, there result

$$\gamma = \omega\left[-i + n_c^2 R Y\right], \tag{5-9}$$

$$\delta = \omega_p R(1+\beta), \tag{5-10}$$

and

$$\alpha = \omega_p i \frac{1}{Y} \beta . \tag{5-11}$$

5.2 Determination of the Complex Refractive Index

5.2.1 Propagation along $\bar{B}$

Making the above substitutions in the dispersion equations (4-17,18,19) there result

$$C_3 v_x - \frac{S}{Y} v_y = 0, \tag{5-12}$$

$$\frac{S}{Y} v_x + C_3 v_x = 0 \tag{5-13}$$

and

$$\left[-i + n_c^2 RY + i\frac{1}{Y^2}\right] v_z = 0, \tag{5-14}$$

where

$$C_3 = \left[-i + n_c^2 RY + \frac{R}{Y}(1+\beta) - i\frac{1}{Y^2}\beta\right]. \tag{5-15}$$

5.2.1.1 Longitudinal Modes. Since $v_z \neq 0$,

$$n_c^2 = \frac{i}{RY}\left[1 - \frac{1}{Y^2}\right] \tag{5-16}$$

and the index of refraction is

$$n_R = \left[\frac{1}{RY}\left(1-\frac{1}{Y^2}\right)\right]^{1/2} \cos\frac{\pi}{4};$$
$$[Y^2 \geq 1], \tag{5-17}$$

$$n_R = \left[\frac{1}{RY}\left(\frac{1}{Y^2}-1\right)\right]^{1/2} \sin\frac{\pi}{4};$$
$$[Y^2 \leq 1], \tag{5-18}$$

or simply

$$n_R = \left[\frac{1}{RY}\left|1-\frac{1}{Y^2}\right|\right]^{1/2} \cos\frac{\pi}{4}. \tag{5-19}$$

The results for $k = k_z$ and $v_z \neq 0$ are then

$$\mathrm{Re}\,(n_c^2) = 0, \tag{5-20}$$

$$\mathrm{Im}\,(n_c^2) = \frac{1}{RY}\left[1-\frac{1}{Y^2}\right], \tag{5-21}$$

$$n_R = \left[\frac{1}{RY}\left|\left(1-\frac{1}{Y^2}\right)\right]^{1/2} \cos\frac{\pi}{4} \tag{5-22}$$

and

$$n_I = n_R. \tag{5-23}$$

It is to be noted that these results are independent of the magnetic field, S.

5.2.1.2 Transverse Modes. This is the case for which $v_x \neq 0$, $v_y \neq 0$, $v_z = 0$, and $k_z = k$; it follows then, from (5-12,13) that

$$C_3^2 + \left(\frac{S}{y}\right)^2 = 0, \tag{5-24}$$

which may be factored and written as

$$C_3 = i \frac{S}{y} . \tag{5-25}$$

By convention the sign is included in S so that S may be positive or negative.

Since

$$C_3 = \left[-i + n_c^2 R y + \frac{R}{y}(1+\beta) - i \frac{1}{y^2}\beta\right], \tag{5-26}$$

a quadratic equation in n_c^2 results which is

$$R y\, n_c^4 + \left[\frac{R}{y} - R y - i\left(1 + \frac{S}{y}\right)\right] n_c^2 + i\left[1 - \frac{1}{y^2} + \frac{S}{y}\right] = 0. \tag{5-27}$$

This equation is of the form

$$A_c n_c^4 + B_c n_c^2 + C_c = 0, \tag{5-28}$$

where

$$A_c = A_R, \tag{5-29}$$

$$B_c = B_R + i B_I, \tag{5-30}$$

$$C_c = C_I, \tag{5-31}$$

$$A_R = RY, \tag{5-32}$$

$$B_R = \frac{R}{Y} - RY, \tag{5-33}$$

$$B_I = -(1 + S/Y) \tag{5-34}$$

and

$$C_I = 1 - \frac{1}{Y^2} + \frac{S}{Y} . \tag{5-35}$$

The final results for the transverse modes are then

$$n_c^2 = \frac{-B_c \pm |B_c^2 - 4A_cC_c|^{1/2} e^{i\theta_1/2}}{2A_c},$$

(5-33)

$$\tan\theta_1 = \mathrm{Im}(B_c^2 - 4A_cC_c)/\mathrm{Re}(B_c^2 - 4A_cC_c),$$

(5-34)

$$n_R = |n_c^2|^{1/2} \cos\psi_1/2,$$ (5-35)

$$n_I = |n_c^2|^{1/2} \sin\psi_1/2$$ (5-36)

and

$$\tan\psi_1 = \mathrm{Im}(n_c^2)/\mathrm{Re}(n_c^2),$$ (5-37)

which are convenient forms for numerical computation.

5.2.2 Propagation Perpendicular to $\underline{B}$

5.2.2.1 Longitudinal Modes. The relation for the complex index of refraction comes from (4-118) and from the fact that $v_z = 0$ so that

$$RYn_c^4 + \left(\frac{R}{Y} - RY - i\right)n_c^2 + i\left(1 - \frac{1}{Y^2}\right) = 0 \,. \tag{5-38}$$

This equation is quadratic in n_c^2 and is of the form

$$A_R n_c^4 + B_c n_c^2 + C_I = 0, \tag{5-39}$$

where in this section

$$A_R = RY, \tag{5-40}$$

$$B_R = \frac{R}{Y} - RY, \tag{5-41}$$

$$B_I = -1 \tag{5-42}$$

and

$$C_I = 1 - \frac{1}{Y^2}, \tag{5-43}$$

leading to the following results

$$n_c^2 = \frac{-B_c \pm \left|B_c^2 - 4A_R C_I\right|^{1/2} e^{i\theta_2/2}}{2A_R}, \tag{5-44}$$

$$\tan\theta_2$$
$$= \mathrm{Im}(B_c^2 - 4A_R C_I)/\mathrm{Re}(B_c^2 - 4A_R C_I), \qquad (5\text{-}45)$$

$$n_R = |n_c^2|^{1/2} \cos \psi_2/2 , \qquad (5\text{-}46)$$

$$n_I = |n_c^2|^{1/2} \sin \psi_2/2 \qquad (5\text{-}47)$$

and

$$\tan \psi_2 = \mathrm{Im}(n_c^2)/\mathrm{Re}(n_c^2) . \qquad (5\text{-}48)$$

5.2.2.2 Transverse Modes. Again the propagation here is along the x-axis, but the particle motion is in the yz-plane.

From (4-111,112), the dispersion relations are

$$\left(\frac{\gamma}{\omega} + i\frac{1}{y^2}\right) v_x - \frac{S}{y} v_y = 0 \qquad (5\text{-}49)$$

and

$$\frac{S}{y} v_x + \left(\frac{\gamma}{\omega} + \frac{\delta}{\omega} - \frac{\alpha}{\omega}\right) v_y = 0 . \qquad (5\text{-}50)$$

Unfortunately these equations do not show the symmetry obtained before in similar cases and there results a cubic in n_c^2 which is

$$R^2Y^2n_c^6 + (R^2 - R^2Y^2 - 2iRY + i\frac{R}{Y})n_c^4 + (\frac{1}{Y^2} - 1 + \frac{S^2}{Y^2} + 2iRY - 3i\frac{R}{Y} + i\frac{R}{Y^3})n_c^2 + (1 - \frac{2}{Y^2} + \frac{1}{Y^4} - \frac{S^2}{Y^2}) = 0. \tag{5-51}$$

This equation is of the form

$$n_c^6 + P_c n_c^4 + Q_c n_c^2 + R_c = 0, \tag{5-52}$$

where

$$P_c = P_R + iP_I, \tag{5-53}$$

$$Q_c = Q_R + iQ_I \tag{5-54}$$

and

$$R_c = R_R + iR_I. \tag{5-55}$$

The quantities on the right hand side of the above equations have the values

$$P_R = \frac{1}{y^2}(1-y^2), \tag{5-56}$$

$$P_I = \frac{1}{Ry^2}\left(\frac{1}{y} - 2y\right), \tag{5-57}$$

$$Q_R = \frac{1}{R^2y^2}\left(\frac{1}{y^2} - 1 + \frac{S^2}{y^2}\right), \tag{5-58}$$

$$Q_I = \frac{1}{Ry^2}\left(2y - \frac{3}{y} + \frac{1}{y^3}\right), \tag{5-59}$$

$$R_R = \left(1 - \frac{2}{y^2} + \frac{1}{y^4} - \frac{S^2}{y^2}\right)\frac{1}{R^2y^2}, \tag{5-60}$$

and

$$R_I = 0. \tag{5-61}$$

To obtain the reduced cubic, the substitution

$$h_c^2 = z - P_c/3 \tag{5-62}$$

made leading to

$$z^3 + A_c z + B_c = 0, \tag{5-63}$$

where

$$A_c = \frac{1}{3}(3Q_c - P_c^2) \qquad (5\text{-}64)$$

and

$$B_c = \frac{1}{27}(2P_c^3 - 9P_cQ_c + 27R_c). \qquad (5\text{-}65)$$

There are essentially two methods of solving the reduced cubic. One of these involves the taking of cube roots and leads to some difficulty in combining the contributions to Z. The method used here is that first published by Cardan (1545).

First making the substitution

$$Z = T - \frac{A_c}{3T}, \qquad (5\text{-}66)$$

there results

$$T^3 - \frac{A_c^3}{27T^3} + B_c = 0. \qquad (5\text{-}67)$$

It is easily seen that this is a quadratic in T^3 of the form

$$T^6 + B_cT^3 - \frac{A_c^3}{27} = 0. \qquad (5\text{-}68)$$

In solving (5-68) methods similar to those used in solving the quadratic equations obtained previously are used. One value for T^3 is then selected. It turns out that the other root of the quadratic will lead to the same results obtained below, but in a different order.

Now the following quantities are formed

$$T_1 = T, \tag{5-69}$$

$$T_2 = \omega T, \tag{5-70}$$

$$T_3 = \omega^2 T, \tag{5-71}$$

where

$$\omega = -0.5 + i\frac{\sqrt{3}}{2} \tag{5-72}$$

is one of the cube roots of unity.

The results for the square of the complex index of refraction for the transverse modes follow and are

$$n_{c1}^2 = T_1 - \frac{A_c}{3T_1} - \frac{P_c}{3}, \tag{5-73}$$

$$n_{c2}^2 = T_2 - \frac{A_c}{3T_2} - \frac{P_c}{3} \tag{5-74}$$

and

$$n_{c3}^2 = T_3 - \frac{A_c}{3T_3} - \frac{P_c}{3} \, . \qquad (5\text{-}75)$$

The refractive index and the extinction index may be obtained by using (5-46,47,48).

5.3 Determination of Mobilities

5.3.1 Propagation along $\underline{B}$

For convenience, the quantity $1/\omega_p$ is factored from the mobility tensor (4-35) so that the mobility components are of the form

$$\mu_{xx} = \frac{e}{m}\frac{1}{\omega_p}\left[\frac{\omega_p(\gamma+\delta)}{(\gamma+\delta)^2+\omega_b^2}\right]. \qquad (5\text{-}76)$$

In order to obtain generalized mobilities the above components are written in terms of R, S, and Y defined above. The results of these substitutions are summarized below.

5.3.1.1 Longitudinal Modes. For modes for which $\underline{v} = v_z$,

$$\mu_{zz} = \frac{e}{m}\frac{1}{\omega_p}(iY). \qquad (5\text{-}77)$$

5.3.1.2 Transverse Modes. For these modes the motion is in the xy-plane. The mobilities for this case are

$$\mu_{xx} = \frac{e}{m}\frac{1}{\omega_p}(-i)\,\frac{1 + YS\frac{1}{\beta}}{\frac{1}{Y}\beta + 2S}\,, \tag{5-78}$$

$$\mu_{xy} = \frac{e}{m}\frac{1}{\omega_p}\left[\frac{-SY(n_c^2-1)}{\frac{1}{Y}\beta + 2S}\right], \tag{5-79}$$

$$\mu_{xx} = \mu_{yy} \tag{5-80}$$

and

$$\mu_{xy} = -\mu_{yx}\,. \tag{5-81}$$

Here again S can be less than zero. The sign of S indicates the polarization of the mode.

5.3.2 Propagation along the x-axis

Expressions for the mobilities are obtained from (4-109) and (4-110). These are summarized below.

5.3.2.1 Longitudinal Modes. From (5-9) and the general expressions for the mobilities

$$\mu_{zz} = \frac{e}{m}\frac{1}{\omega_p}\left[-iY(n_c^2-1)\right]. \tag{5-82}$$

5.3.2.2 Transverse Modes. From (4-110)

$$\frac{D}{\omega^2} = (-i + n_c^2 RY)\left[-i + n_c^2 RY + \frac{R}{Y}(1+\beta)\right] + \left(\frac{S}{Y}\right)^2 ; \tag{5-83}$$

then from (4-109)

$$\mu_{xx} = \frac{e}{m}\frac{1}{\omega_p}\frac{1}{Y}\left[-i + n_c^2 RY + \frac{R}{Y}(1+\beta)\right]\left[\frac{D}{\omega^2}\right]^{-1}, \tag{5-84}$$

$$\mu_{yy} = \frac{e}{m}\frac{1}{\omega_p}\frac{1}{Y}(-i + n_c^2 RY) \times \left[\frac{D}{\omega^2}\right]^{-1} \tag{5-85}$$

and

$$\mu_{xy} = \frac{e}{m}\frac{1}{\omega_p}\frac{S}{Y^2}\left[\frac{D}{\omega^2}\right]^{-1}. \tag{5-86}$$

It is to be noted that

$$\mu_{xy} = -\mu_{yx} . \tag{5-87}$$

5.4 Graphical Results

Using the IBM 1107 computer at Case Institute of Technology, the more important expressions of this chapter were evaluated for various values of the parameters

$$Y = \omega/\omega_p , \tag{5-88}$$

$$R = \omega_p/\Omega \tag{5-89}$$

and

$$S = \omega_b/\omega_p . \tag{5-90}$$

On the graphs the quantity

$$M_{ij} = \left[\frac{e}{m}\frac{1}{\omega_p}\right]^{-1} |\mu_{ij}| \tag{5-91}$$

represents the mobility and the quantities n_R and n_I are represented by their absolute values. In all calculations the parameter

$$T = \frac{\omega}{\Omega} \, |n_c^2| \, , \tag{5-92}$$

which is realted to the assumption of semi-compressibility is less than 2.0.

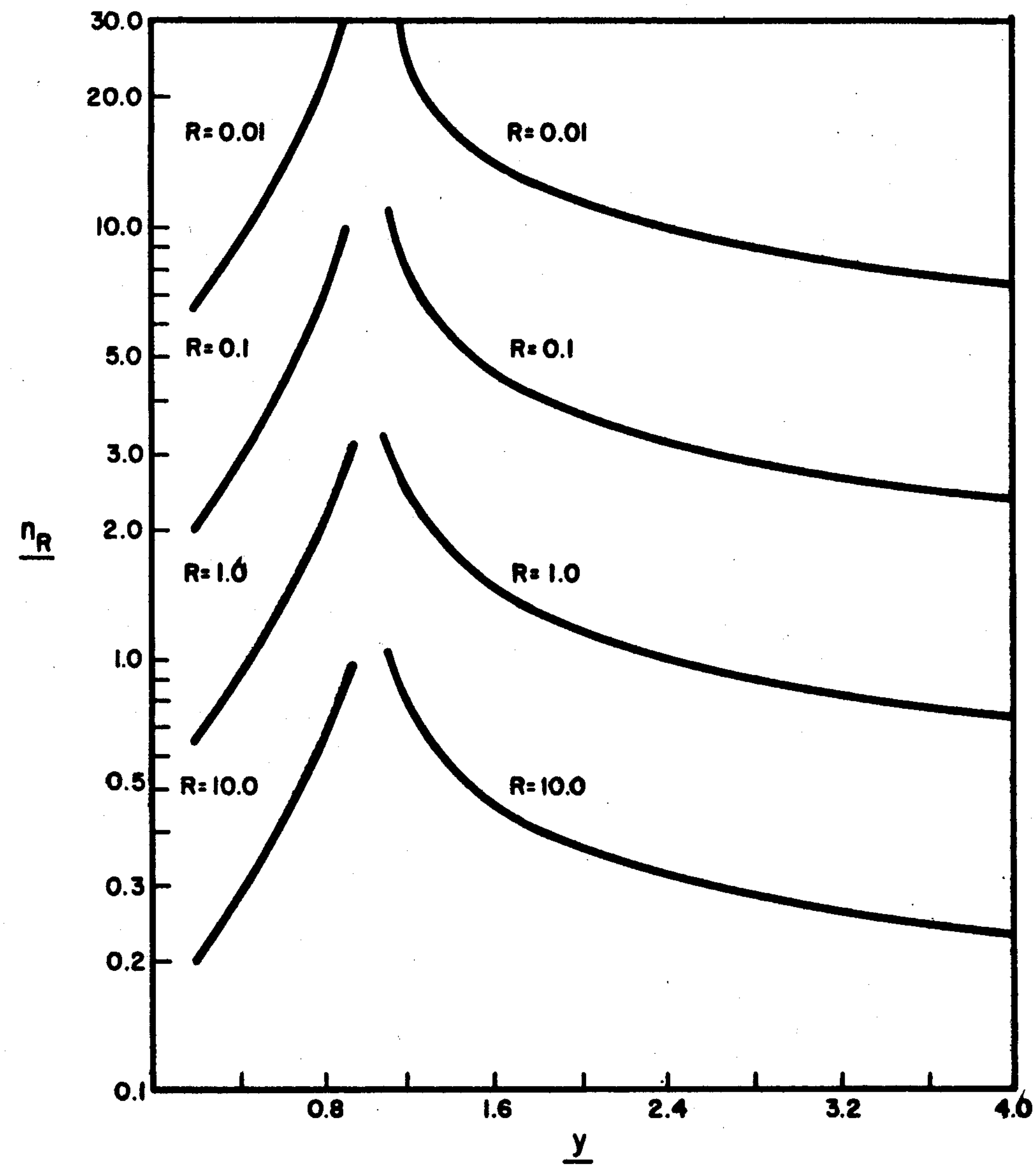


FIGURE 5.1 INDEX OF REFRACTION FOR PLASMA OSCILLATIONS

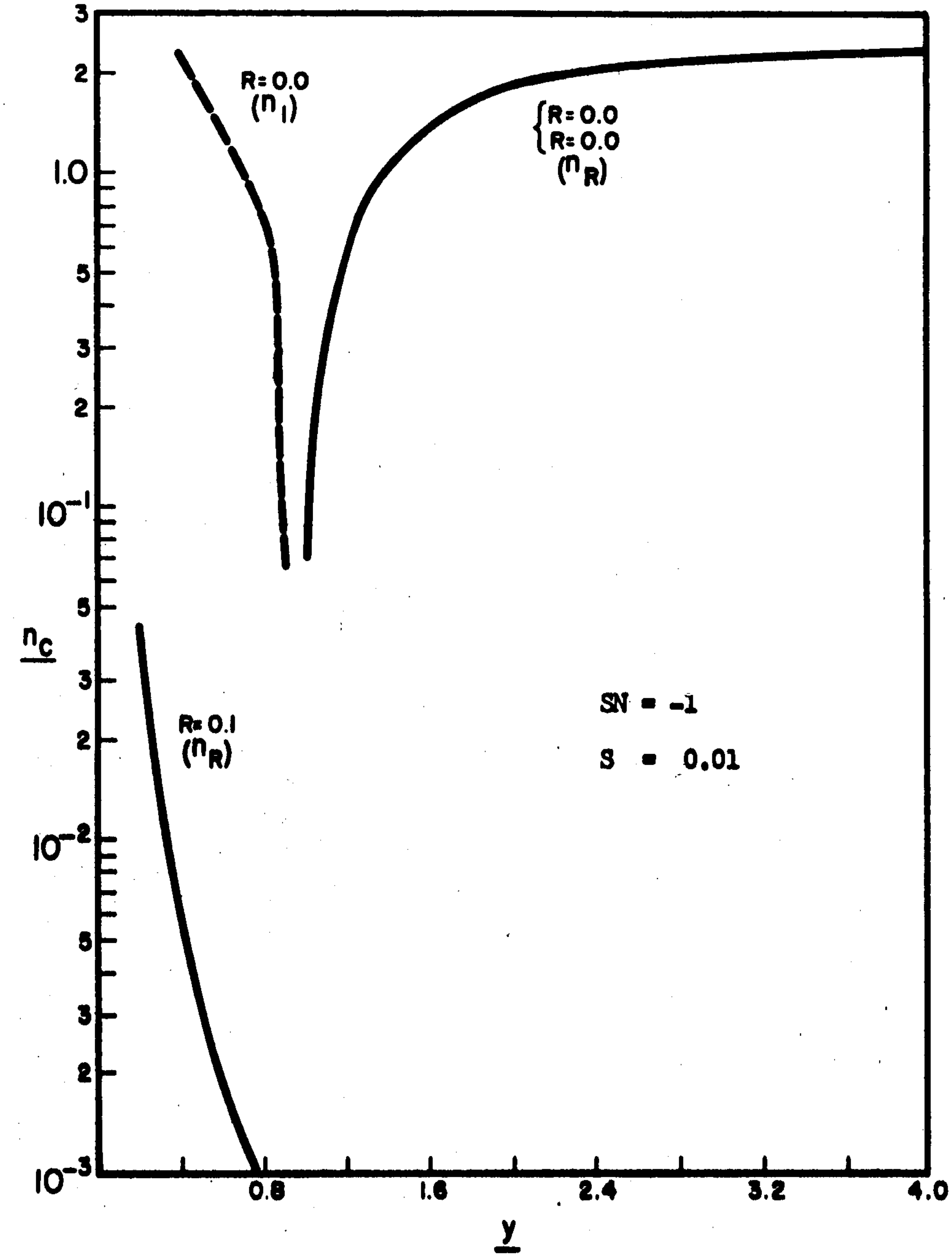


FIGURE 5.2 COMPLEX INDEX OF REFRACTION FOR ORDINARY MODES; MODERATE TURBULENCE, WEAK MAGNETIC FIELD

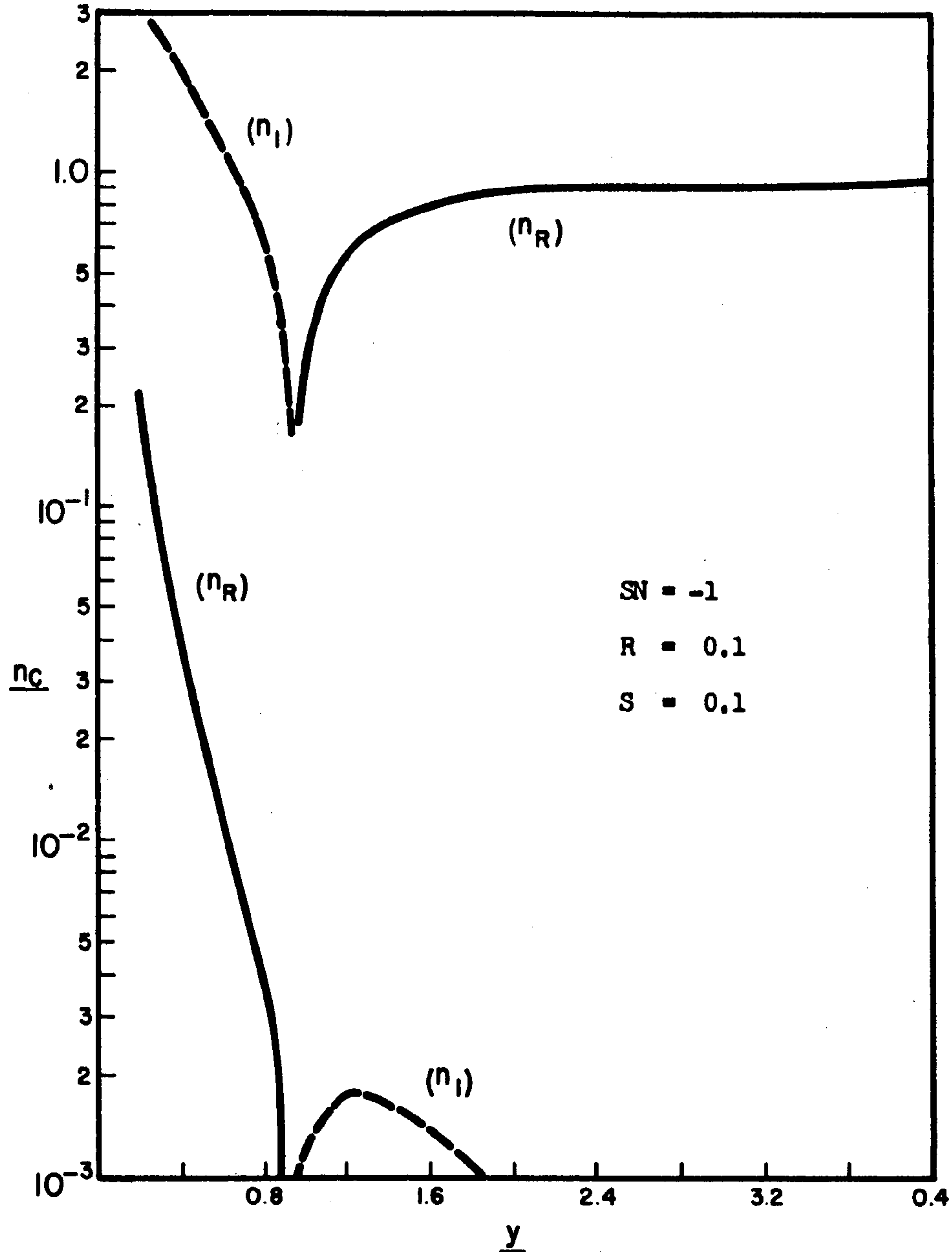


FIGURE 5.3 COMPLEX INDEX OF REFRACTION FOR ORDINARY MODES: MODERATE TURBULENCE, MODERATE MAGNETIC FIELD

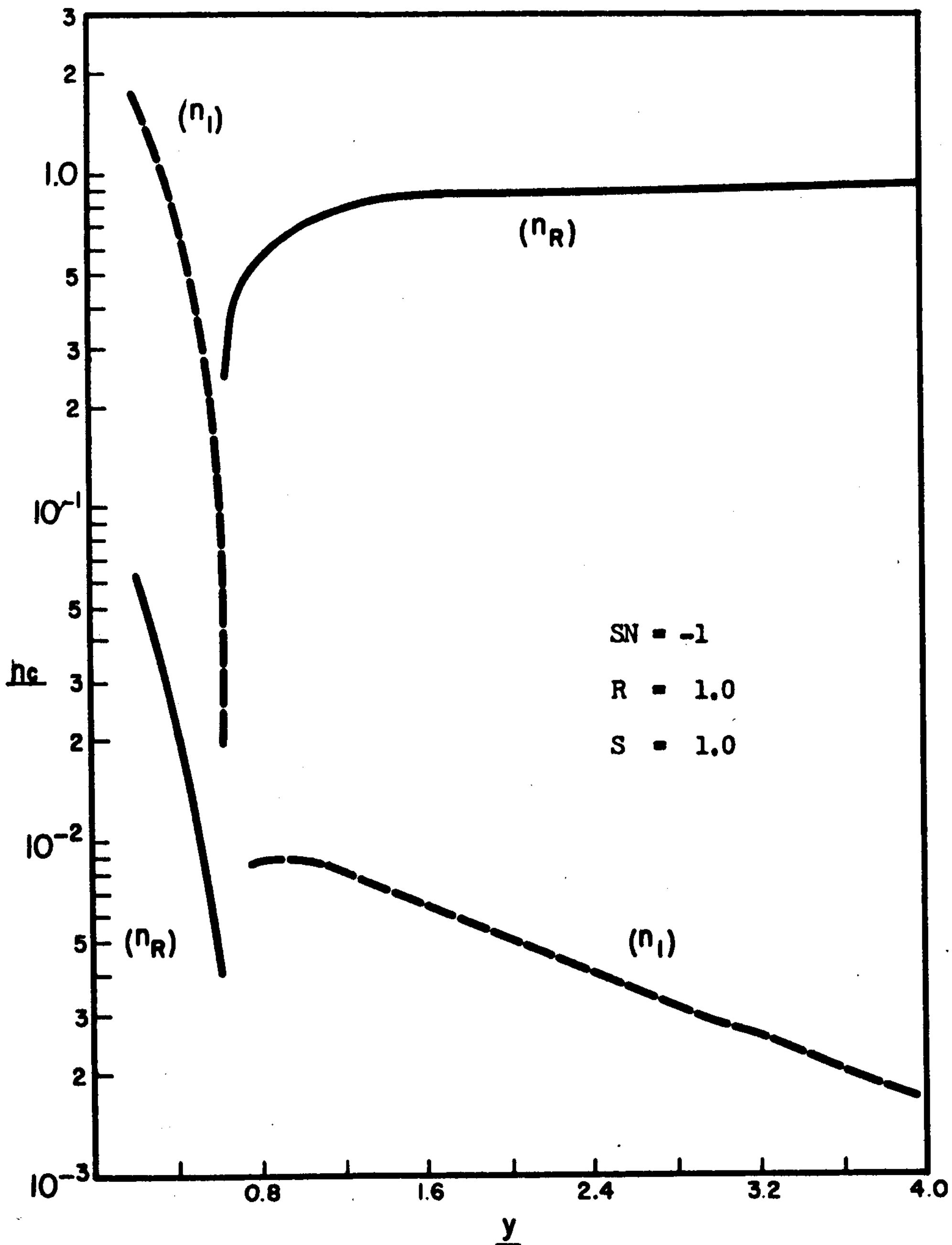


FIGURE 5.4 COMPLEX INDEX OF REFRACTION FOR ORDINARY MODES; STRONG TURBULENCE

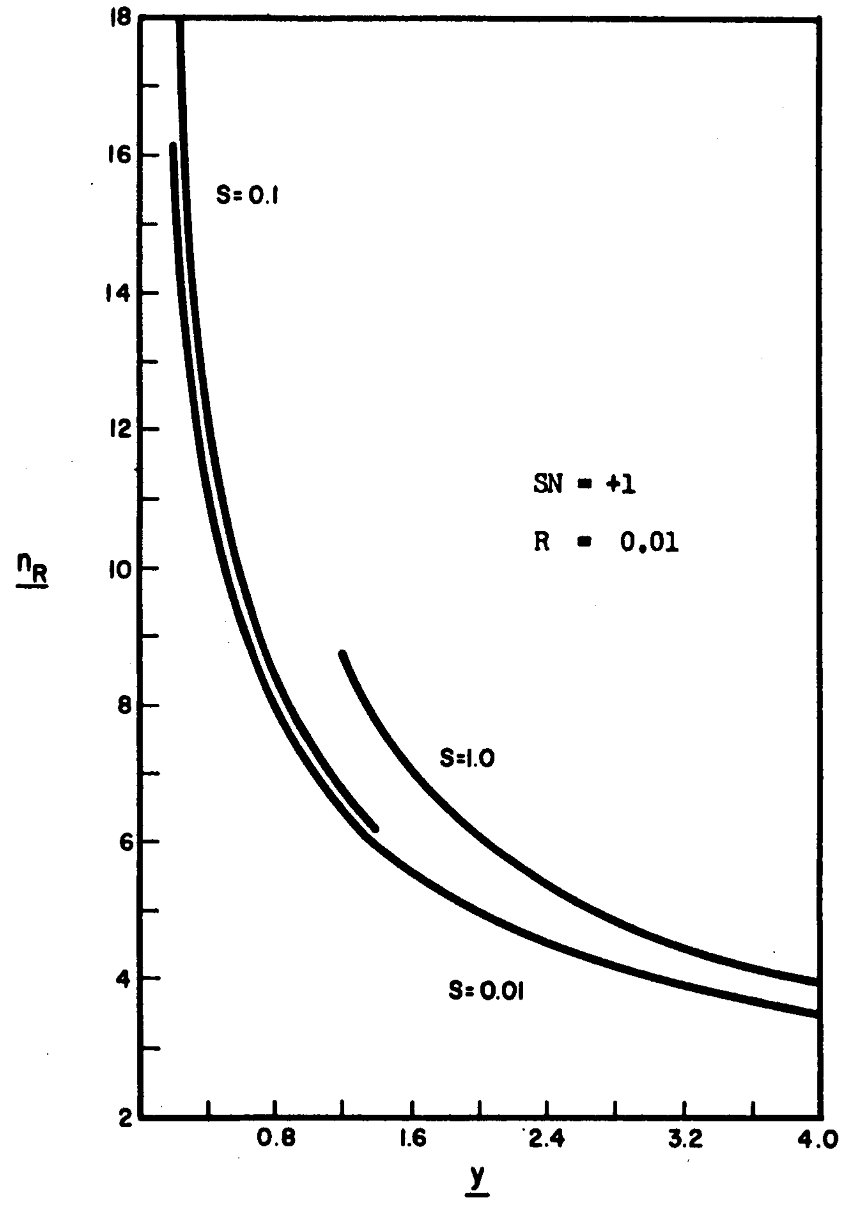


FIGURE 5.5 INDEX OF REFRACTION FOR TURBULENT MODE (II)

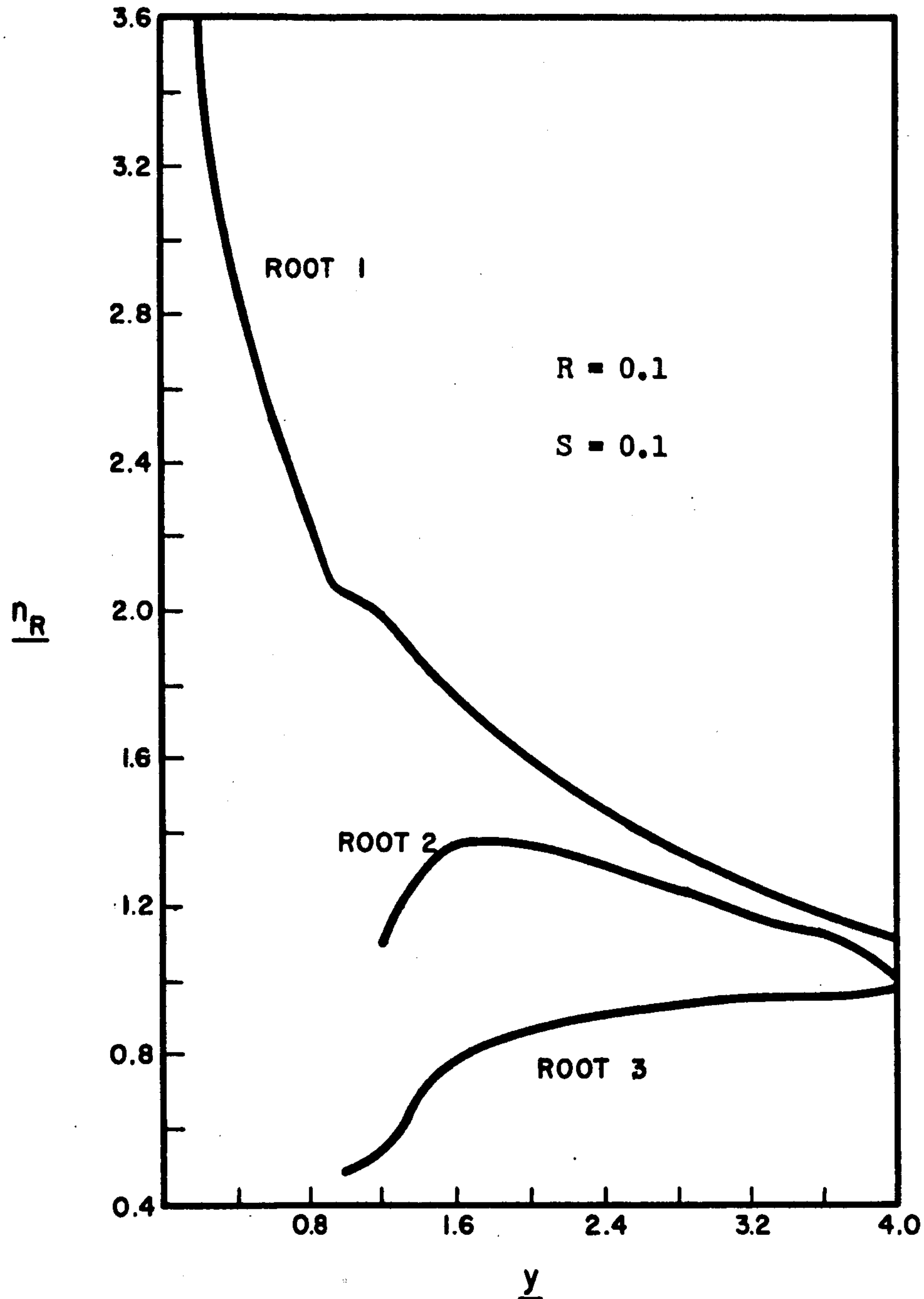


FIGURE 5.6 INDEX OF REFRACTION FOR ELLIPTICALLY POLARIZED MODE AND FOR TURBULENT MODES (IV, V)

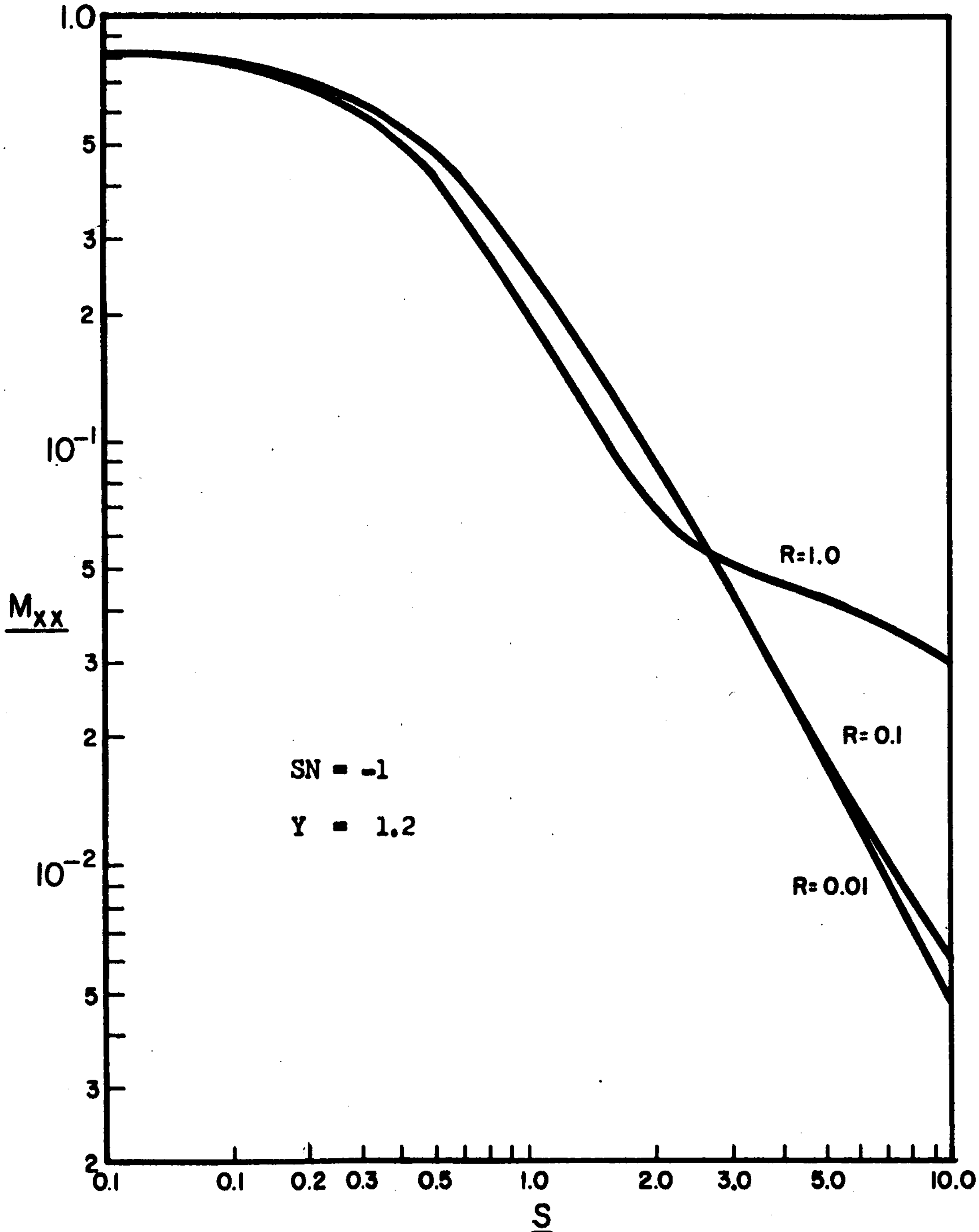


FIGURE 5.7 MOBILITY (XX) FOR ORDINARY MODE

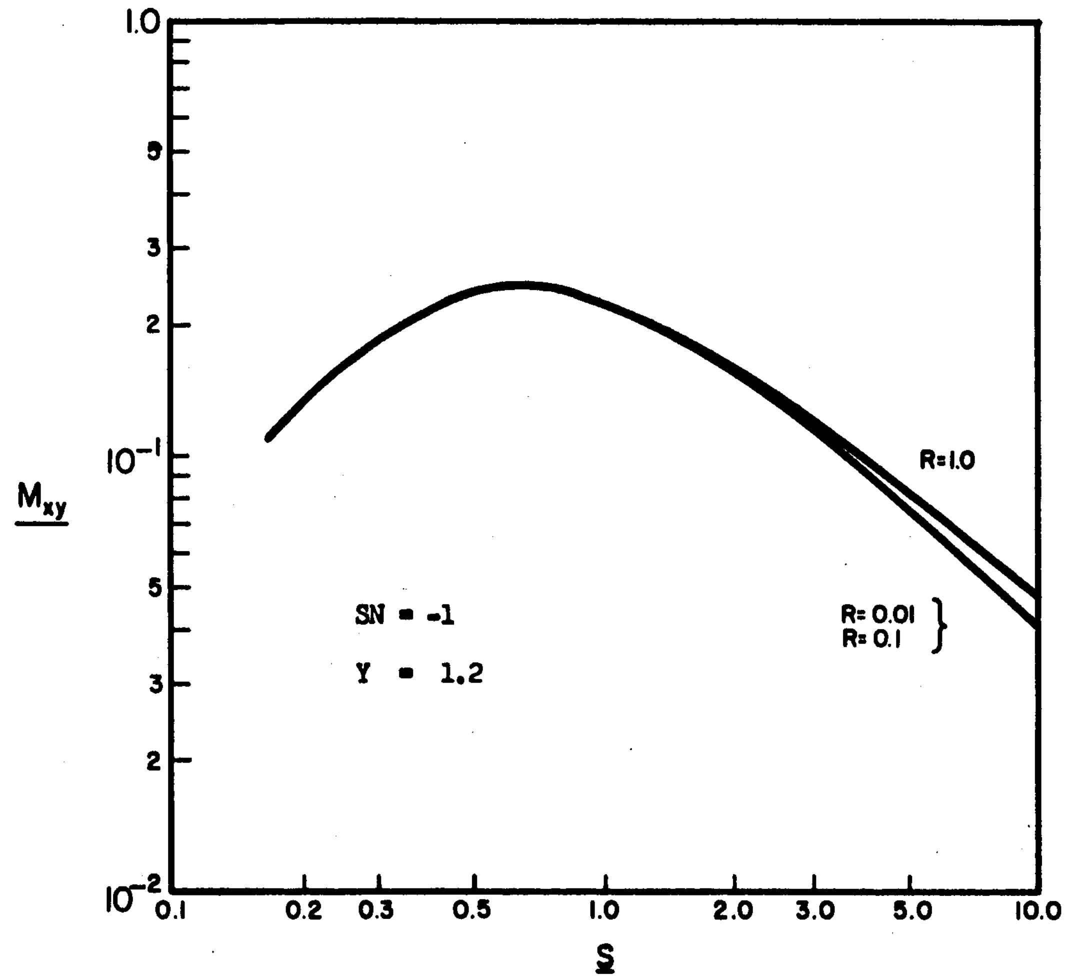


FIGURE 5.8 MOBILITY (XY) FOR ORDINARY MODE

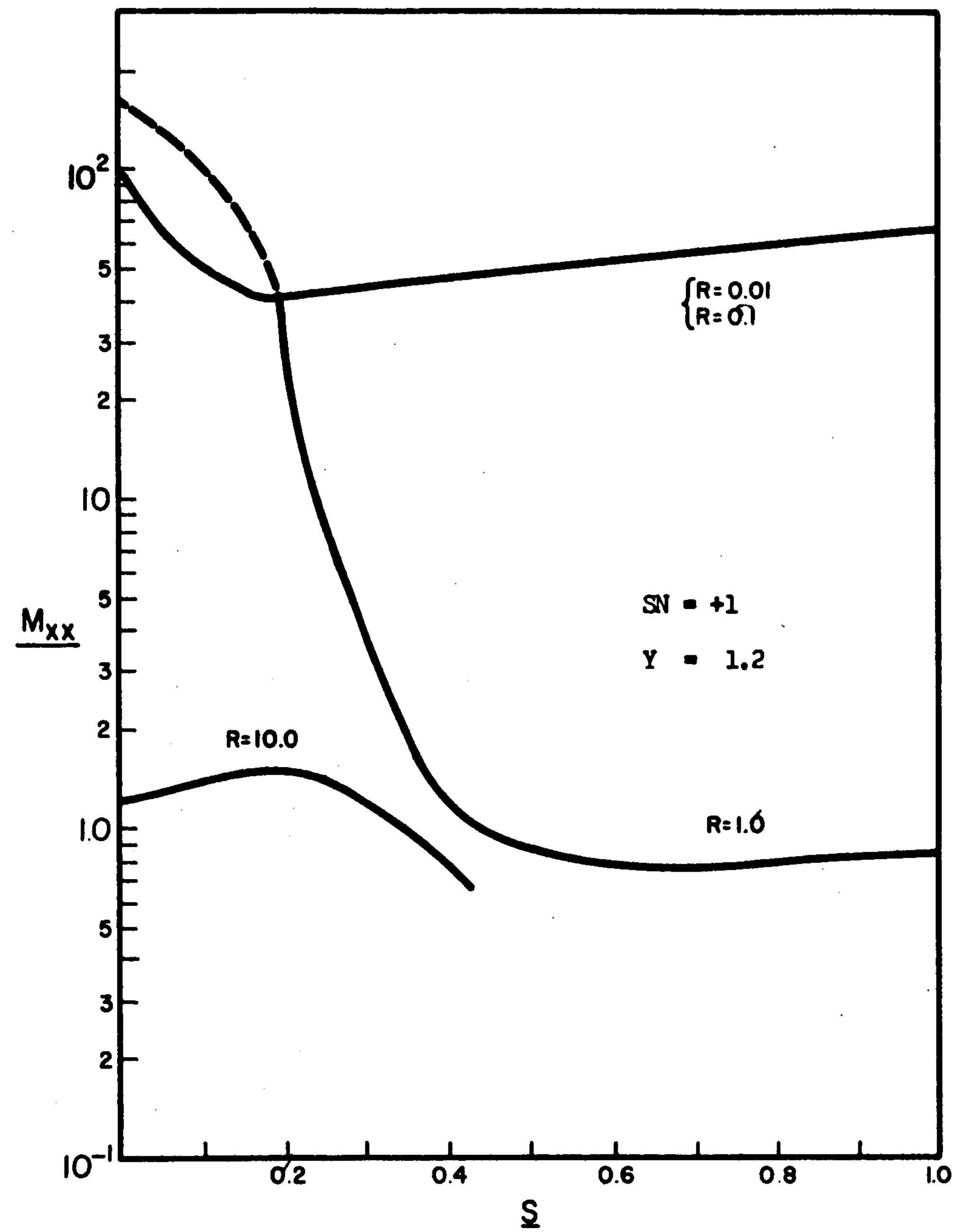


FIGURE 5.9 MOBILITY (XX) FOR TURBULENT MODE (II)

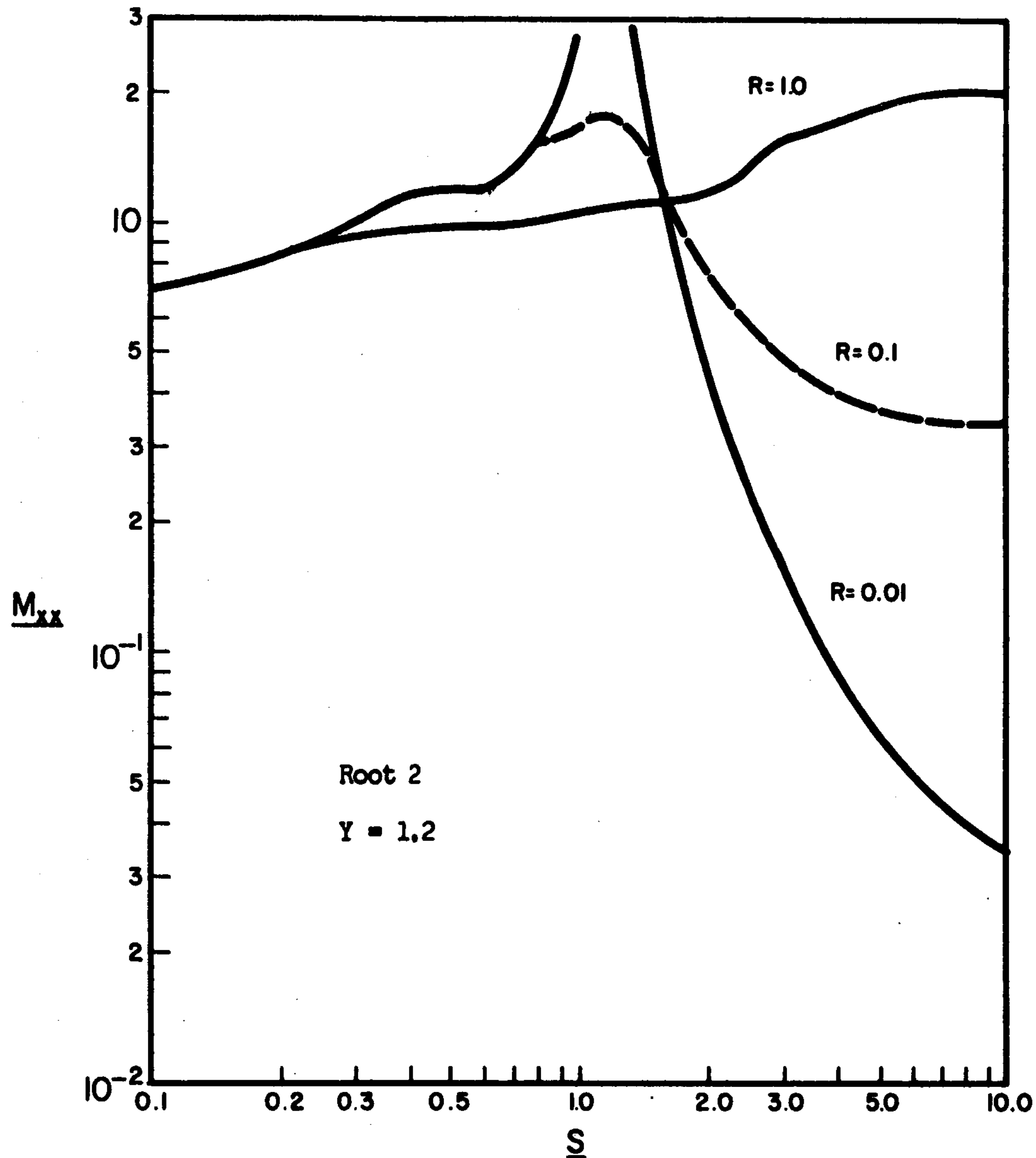


FIGURE 5.10 MOBILITY (XX) FOR ELLIPTICALLY POLARIZED MODE

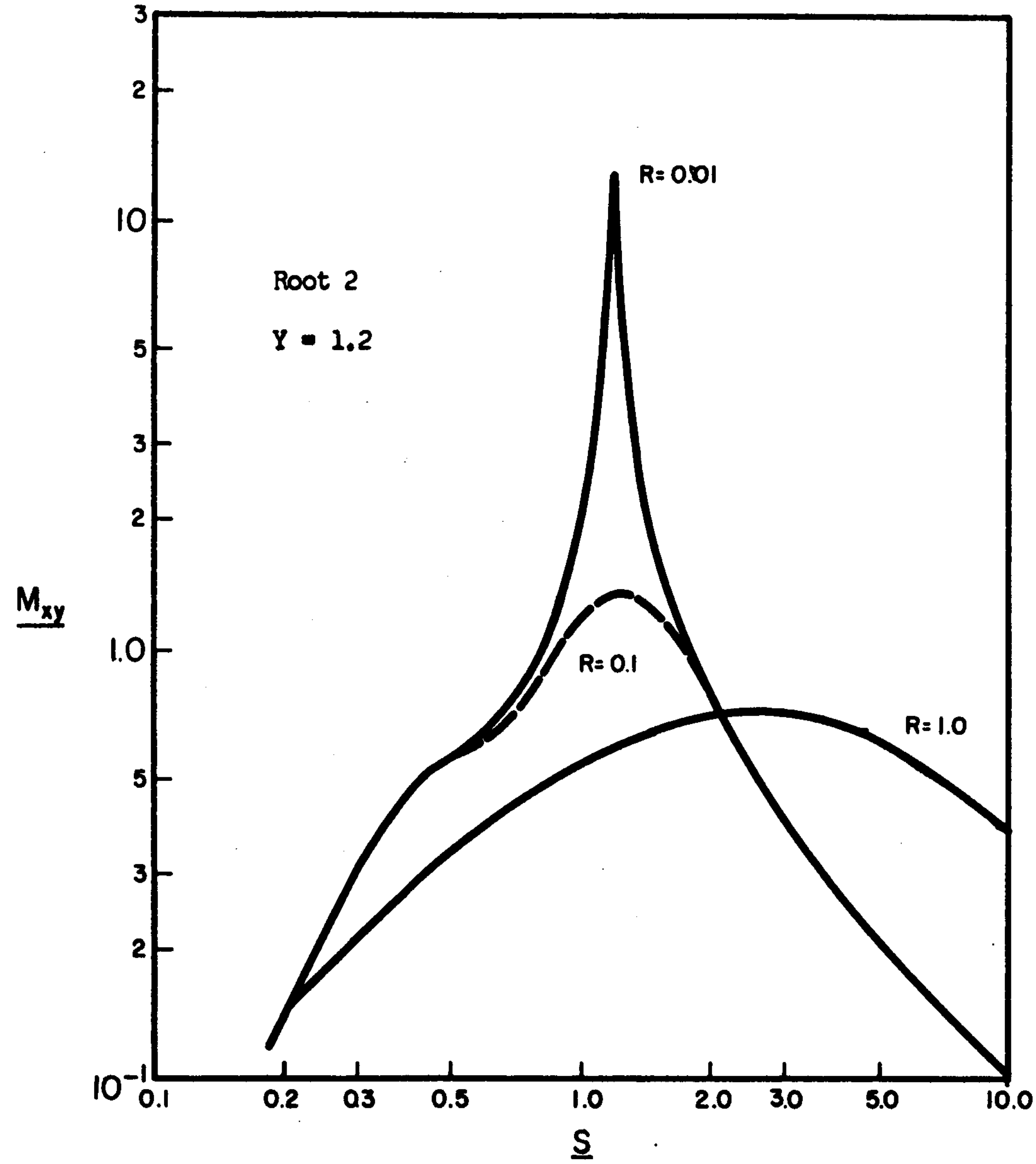


FIGURE 5.11 MOBILITY (XY) FOR ELLIPTICALLY POLARIZED MODE

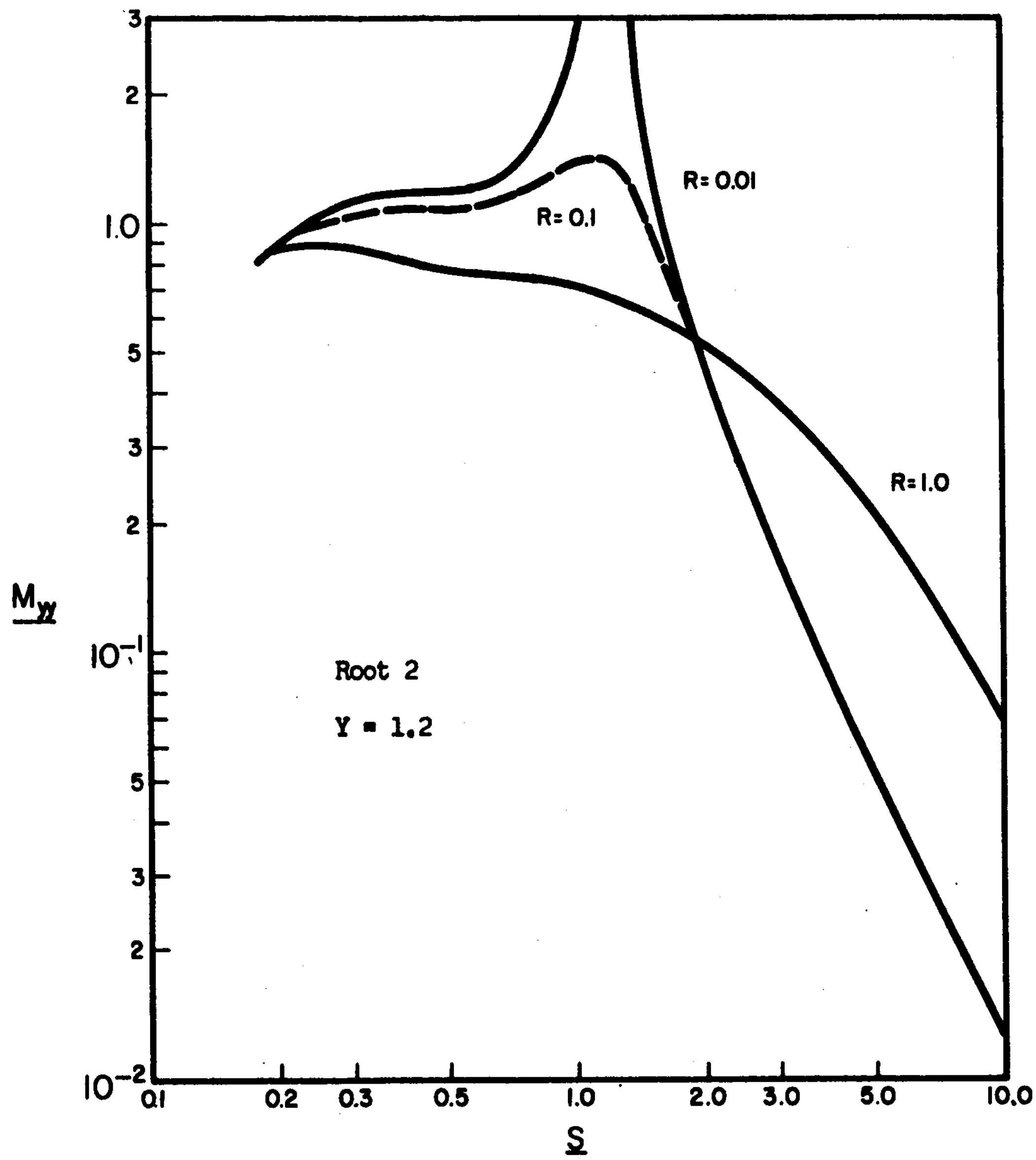


FIGURE 5.12 MOBILITY (YY) FOR ELLIPTICALLY POLARIZED MODE

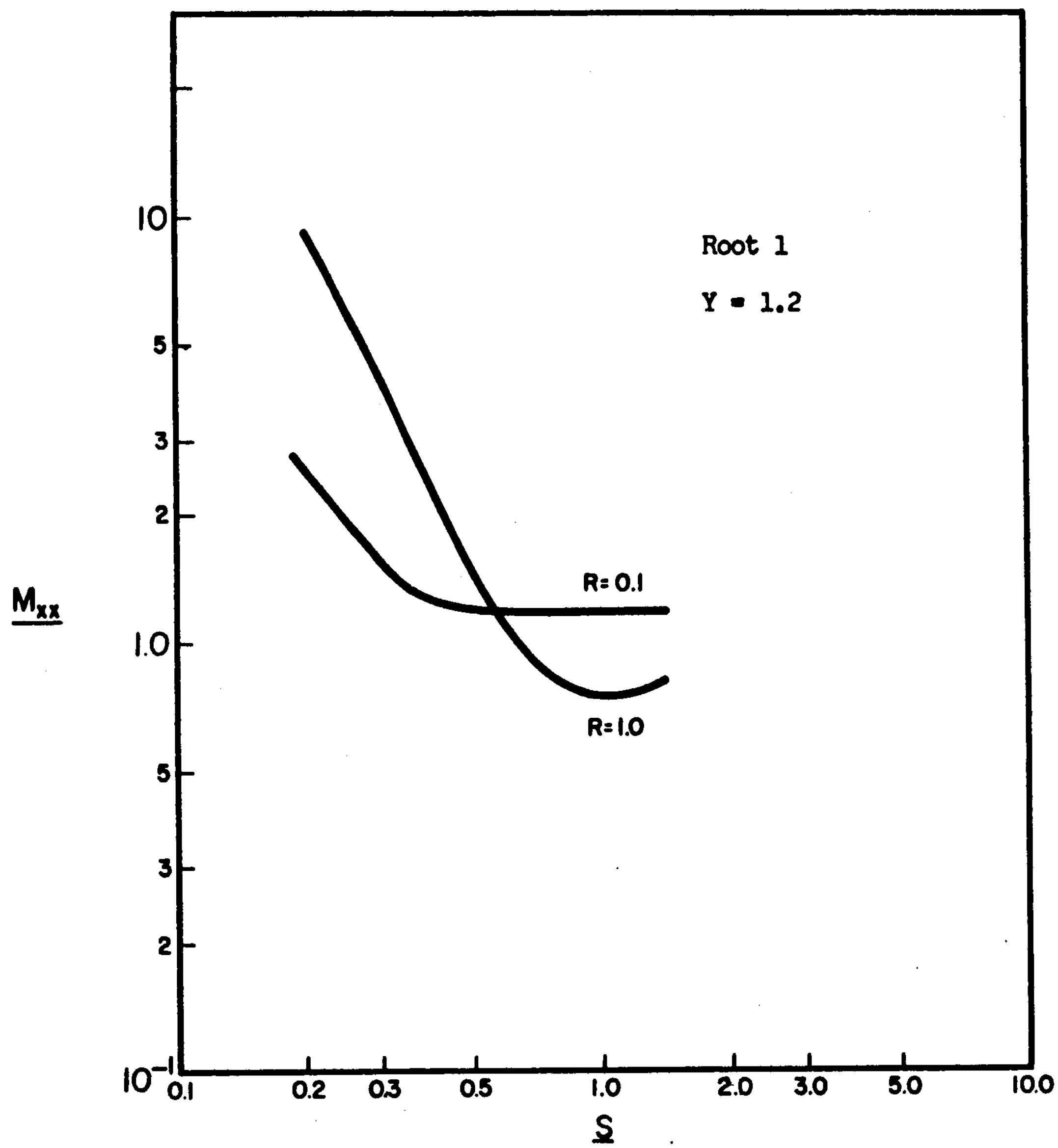


FIGURE 5.13 MOBILITY (XX) FOR TURBULENT MODE (IV)

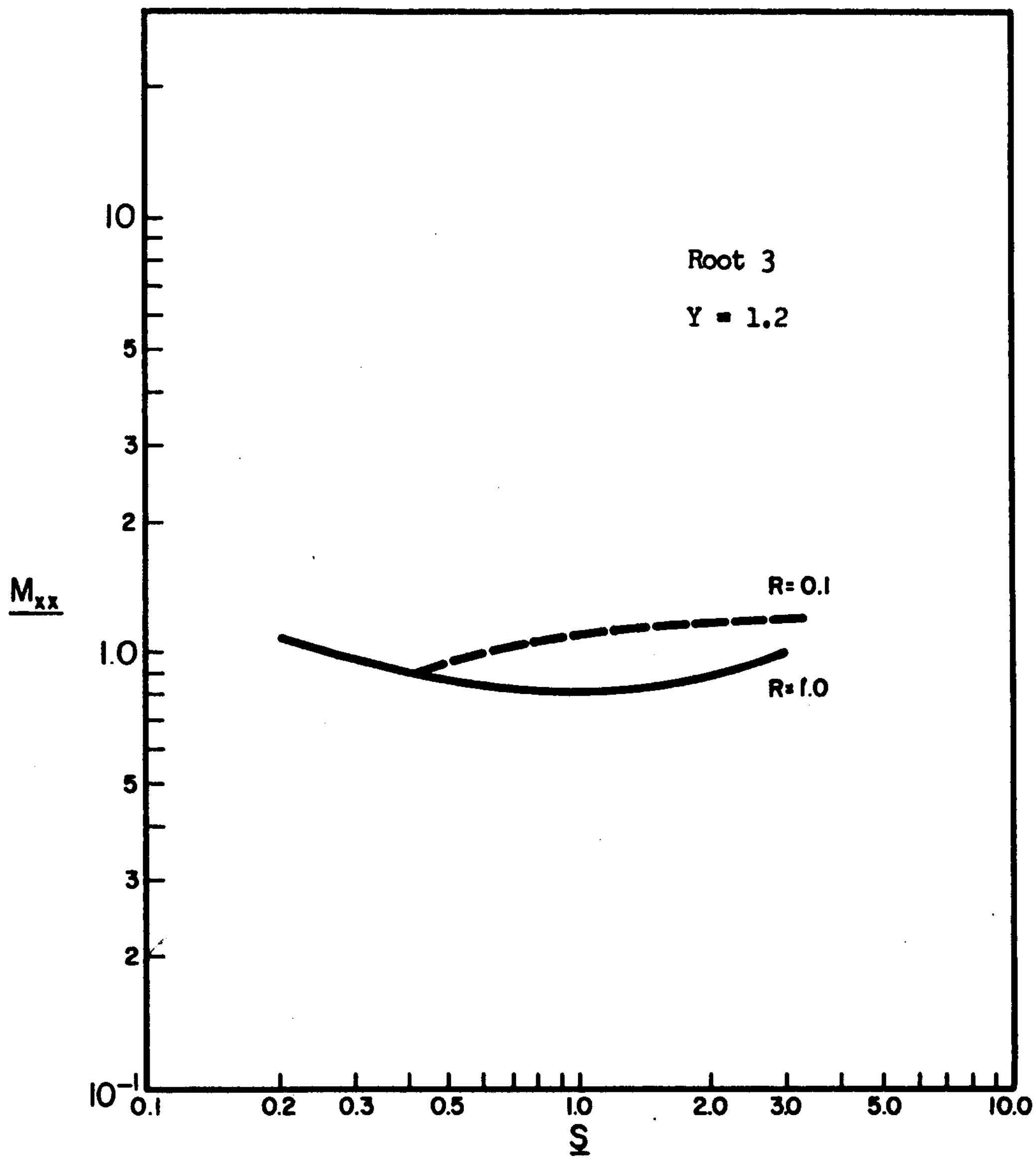


FIGURE 5.14 MOBILITY (XX) FOR TURBULENT MODE (V)

CHAPTER VI

CONCLUSION

Before turning to a discussion of the graphical results of the preceeding chapter, it may be well to summarize briefly some of the results obtained earlier.

In accordance with condition (3-12), a parameter T was defined in section 4.2.1 which describes limits for which a real plasma may be considered semi-compressible. In order to use the results of this work in any real situation, the quantity T must be kept lower than a certain maximum value.

Another parameter introduced in section 4.2.1 which has been frequently used is the characteristic turbulent frequency Ω. The quantity Ω was shown to be related to a characteristic interaction length for correlation between the modes of turbulence.

The character of the modes present in turbulence is indicated by inspecting the general dispersion equations of the type (4-25) which show that in general six modes of wave motion are to be expected. Three of these modes turn out to be modified classical modes as obtained for a non-turbulent Lorentz gas. The other three modes are mixed and express the coupling introduced by the non-linear terms in the equation of motion; these modes are designated by the term "turbulent modes". In the degenerate cases of propagation along and perpendicular to the average magnetic field the nature of these six modes is as outlined in figures 4.1 and 4.2.

Turning now to the graphical results, figure 5.1 indicates the behavior of the index of refraction for plasma oscillations in the presence of turbulence. In a non-turbulent Lorentz gas these oscillations do not propagate when the pressure is neglected. In the presence of turbulence propagation takes place although the damping is very strong, the extinction index being equal to the index of refraction. As the turbulence is increased, the phase velocity given by

$$v_\phi = c/n_R \tag{6-1}$$

increases.

Figures 5.2,3,4 indicate the behavior of the complex index of refraction for the ordinary modes. To the left of the singular point near the plasma frequency these waves are strongly damped as is indicated by the fact that the magnitude of the index of refraction (n_R) is much less than the magnitude of the extinction index (n_I). To the right of the critical point propagation takes place with some damping which increases with increase in turbulence. As the turbulence is increased it is also noticed that the critical point is displaced to the left as is to be expected from the presence of a viscosity-like term.

Figure 5.5 portrays the behavior of the index of refraction of turbulent modes (II) with varying magnetic field strengths. It is to be noted that for this type of turbulent mode, as with all of the others, the index of refraction is approximately equal in magnitude to the index of extinction so that the turbulent modes are all strongly damped.

Figure 5.6 shows the behavior of the index of refraction for all three mode types found for propagation perpendicular to the average magnetic field. The mode designated as "root 2" is identified as the elliptically polarized mode since the index of extinction is much less than the index of refraction indicating propagation with slight damping. The other two modes are turbulent or mixed since the damping is severe. One noteworthy feature of the curves as shown is the similarity between results for "root 2" and for "root 3" indicating that these two mode types may be related. The same similarity is to be found between the curve for "root 1" and the behavior of the index of refraction in figure 5.5 for turbulent mode (II). The inference to be drawn from these similarities is that the turbulent modes do indeed represent a mixing of the three possible types of modes found in a non-turbulent Lorentz gas. Another way of stating this is that excitation of one particular mode will result in the excitation of others including those with propagation vectors in other directions. The impossibility of a one-dimensional turbulence is rather clear in view of these observations.

The results shown in figure 5.7 are most significant in that in the presence of moderate turbulence ($R = 1.0$) there is clearly a critical magnetic field above which the magnetic field dependence of the mobility is of an entirely different character. For sufficiently high magnetic fields and for $R = 1.0$ the mobility shown is proportional to $B^{0.5}$ in contrast to the B^2 dependence of the non-turbulent case. This type of behavior has indeed been observed in the D.C. mobility[17] so that the results presented here are highly suggestive.

Figure 5.8 indicates that the cross-diffusion, M_{xy} is but little affected by the presence of turbulence.

Figure 5.9 is a study of the behavior of the mobility for turbulent modes (II) associated with propagation along the magnetic field. The curves are incomplete due to the fact that the parameter T was to be less than 2.0.

Figures 5.10,11,12 show the behavior of M_{xx}, M_{xy}, and M_{yy} for the elliptically polarized modes propagating in the x-direction. Of particular interest is the fact that M_{yy} and M_{xx} behave quite differently with increase in turbulence above a critical magnetic field.

The graphical results are concluded with figures 5.13 and 5.14 which indicate the behavior of the mobility of the two turbulent modes (IV,V) associated with a propagation vector in the x-direction.

To conclude it may be well to discuss possible ways of relating the results of the present study to experimentally verifiable quantities.

At several places in the preceeding pages, expressions and values are obtained for the indices of refraction and extinction. It may be possible to excite a particular mode above its normal level in an already turbulent plasma, and by means of measuring the decay of the mode at that frequency a check could be made on the values predicted. The modes present in the turbulence could be inferred from a study of the behavior of the index of refraction, the dynamic viscosity being an experimental parameter.

In certain special cases it may be possible to determine the relations governing velocity correlations and electric field correlations or correlations of other types. Indications of how this may be done are discussed in the introduction of chapter four.

It is also possible to study the phenomenon of diffusion by defining a diffusion coefficient such that

$$v_i = D_{ij} k_j \frac{\rho}{\bar{\rho}} . \qquad (6\text{-}2)$$

Now from (3-85) leaving out the electric field term since diffusion is being considered, and neglecting ν there results

$$\begin{aligned}[-i\omega + \frac{n_i^2\omega}{\Omega}] v_x = &-ia^2 \frac{\partial \rho}{\partial x} \\ &+ \omega_b [v_y \hat{a}_x - v_x \hat{a}_y] \cdot \hat{a}_x \\ &- \frac{\omega_p^2}{k^2\Omega} [-\tfrac{1}{2} k_x k_y v_y \\ &+ (k_y^2 + k_z^2) v_x - \tfrac{1}{2} k_x k_z v_z] . \end{aligned} \qquad (6\text{-}3)$$

Since the dispersion relations are identical for the calculation of the diffusion and the mobility, the only change in the analysis is that where $\frac{e}{m} E_x$ occurs for the mobility, the term $-ia^2 \frac{\partial \rho}{\partial x}$ appears so that the diffusion coefficient is given by

$$D_{ij} = -i \frac{m}{e} a^2 \mu_{ij} . \qquad (6\text{-}4)$$

Note that v_1 here is that part of the contribution to the velocity due to the turbulent density gradient, and is in general much smaller than the v_1 contribution of the turbulent electric field.

If D_{ij} is independent of $\underline{k}$ then the average velocity correlation due to the average A.C. density gradient may be found in a manner similar to that used in developing equation (4-15).

APPENDIX A

SPECIAL SYMBOLS

A_1 For a semi-compressible plasma = 1 otherwise less than 1.

A_1' For a semi-compressible plasma = 0 otherwise greater than 0.

$G = 4 \frac{\omega_p^2 \Omega}{\omega^3}$.

$G_B = \frac{\Omega\omega_b}{\omega^2}$.

$H = \frac{\omega_p^2}{\omega\omega_b}$.

$\underline{k}$ Complex wave vector.

$\underline{k}_R$ Wave vector; real part of $\underline{k}$.

$\underline{k}_I$ Attenuation vector - imaginary part of $\underline{k}$.

$M_{ij} = |[\frac{e}{m} \frac{1}{\omega_p}]^{-1} \mu_{ij}|$.

n_c Complex index of refraction.

n_R Index of Refraction; real part of n_c.

n_I Extinction Index; imaginary part of n_c.

$R = \omega_p/\Omega$.

$S = \omega_b/\omega_p$.

SN Indicates sign used in quadratic formula; may be +1 or -1. SN = +1 indicates a turbulent mode; SN = -1 indicates a classical mode.

$T = \frac{\omega}{\Omega} \left| n_c^2 \right|$.

$T_{1,2,3,4} (\underline{k}, \omega)$ see section 3.2.

v_ϕ Phase velocity = ω/k_R .

$\underline{v}' = \underline{v}(\underline{k}', \omega')$

$\underline{v}'' = \underline{v}(\underline{k} - \underline{k}', \omega - \omega')$.

$Y = \omega/\omega_p.$

$\alpha = i\,\dfrac{\omega_p^2}{\omega}\,\beta.$

$\beta = \dfrac{1}{n_c^2 - 1}.$

$\gamma = -i\omega + k^2\sigma.$

$\delta = \dfrac{\omega_p^2\sigma}{c^2}(1 + \beta).$

$\rho' = \rho(\underline{k}',\omega')$

$\rho'' = \rho(\underline{k} - \underline{k}', \omega - \omega').$

σ^* Dynamic viscosity.

σ Dynamic viscosity (Heisenberg).

$\omega_p^2 = \dfrac{e^2\rho}{m^2\varepsilon_o}.$

$\Omega = c^2/\sigma.$

APPENDIX B

FOURIER TRANSFORM OF A PRODUCT

Let $A(\underline{r})$ and $B(\underline{r})$ be two space dependent quantities such that

$$A(\underline{r}) = \int A(\underline{k}') e^{i\underline{k}'\underline{r}} d\underline{k}' \tag{B-1}$$

and

$$B(\underline{r}) = \int B(\underline{k}) e^{i\underline{k}\underline{r}} d\underline{k} . \tag{B-2}$$

It is desired to find an expression for the quantity $AB(\underline{k})$ where

$$A(\underline{r})B(\underline{r}) = \int AB(\underline{k}) e^{i\underline{k}\underline{r}} d\underline{k} . \tag{B-3}$$

Now from the Fourier integral theorem

$$AB(\underline{k}) = \frac{1}{8\pi^3} \int A(\underline{r}) B(\underline{r}) e^{-i\underline{k}\underline{r}} d\underline{r} , \tag{B-4}$$

and substituting for $A(\underline{r})$ according to (B-1), there results

$$AB(\underline{k}) = \frac{1}{8\pi^3}\iint A(\underline{k}')B(\underline{r})e^{-i(\underline{k}'-\underline{k})\underline{r}}\,d\underline{k}'\,d\underline{r}\,. \tag{B-5}$$

Now the Fourier transform of (B-2) is

$$B(\underline{k}) = \frac{1}{8\pi^3}\int B(\underline{r})e^{-i\underline{k}\underline{r}}\,d\underline{r}\,, \tag{B-6}$$

so that integrating (B-5) over $\underline{r}$ first, it follows that

$$AB(\underline{k}) = \int A(\underline{k}')B(\underline{k}-\underline{k}')\,d\underline{k}' \tag{B-7}$$

which is the convolution theorem.

The above demonstration still holds if $\underline{k}$ is complex in which case the imaginary part of $\underline{k}$ strictly belongs to the amplitude so that

$$A(\underline{r}) = \int A(\underline{k}')e^{-\underline{k}_I'\underline{r}}\,e^{i\underline{k}_R\underline{r}}\,d\underline{k}_R'\,. \tag{B-8}$$

where

$$\underline{k} = \underline{k}_R + i\,\underline{k}_I\,. \tag{B-9}$$

Note that this extra amplitude factor cancels in the proof.

APPENDIX C

TRANSPORT IN THE ABSENCE OF TURBULENCE

C.1 Mobilities

In the absence of turbulence, the mobility does not depend on the refractive index nor on the type of mode involved and is given by[24]

$$\underline{\underline{\mu}} = \frac{e}{m} \begin{pmatrix} \frac{-i\omega}{D_1} & \frac{\omega_b}{D_1} & 0 \\ \frac{-\omega_b}{D_1} & \frac{-i\omega}{D_1} & 0 \\ 0 & 0 & \frac{i}{\omega} \end{pmatrix}, \qquad \text{(C-1)}$$

where collisions are neglected and

$$D_1 = \omega_b^2 - \omega^2. \qquad \text{(C-2)}$$

In terms of the general parameters of chapter five equation (C-1) becomes

$$\underline{\underline{\mu}} = \frac{e}{m}\frac{1}{\omega_p}\begin{pmatrix} -\frac{i}{D_2} & \frac{S}{YD_2} & 0 \\ -\frac{S}{YD_2} & -\frac{i}{D_2} & 0 \\ 0 & 0 & \frac{i}{Y} \end{pmatrix}, \tag{C-3}$$

where

$$D_2 = Y\left(\frac{S^2}{Y^2} - 1\right). \tag{C-4}$$

C.2 Refractive Indices

C.2.1 Propagation along $\underline{B}$

For a wave in a non-turbulent Lorentz gas propagating along the average magnetic field, the index of refraction is determined from the dispersion relations which follow from (4-37,38,39) neglecting turbulent terms so that

$$-i\omega v_x = i\frac{\omega_p^2}{\omega^2}\beta\omega v_x + \omega_b v_y, \tag{C-5}$$

$$-i\omega v_y = i\frac{\omega_p^2}{\omega^2}\beta\omega v_y - \omega_b v_x \tag{C-6}$$

and

$$-i\omega v_z = -i\frac{\omega_p^2}{\omega^2}\omega v_z . \tag{C-7}$$

From (C-7) it is seen that

$$\omega = \omega_p \tag{C-8}$$

and

$$n_c^2 \text{ indeterminate}, \tag{C-9}$$

which is the result for the usual non-propagating electron plasma oscillations in the absence of the pressure term.

The other two dispersion relations yield

$$n_c^2 = 1 - \frac{1}{SY+Y^2}, \tag{C-10}$$

where S may be positive or negative corresponding to ordinary and extraordinary waves.

C.2.2 Propagation Perpendicular to $\underline{B}$

In this case the dispersion equations are

$$\left(-i\omega+i\frac{\omega_p^2}{\omega}\right)v_x-\omega_b v_y=0\,, \tag{C-11}$$

$$\omega_b v_x+\left(-i\omega-i\frac{\omega_p^2}{\omega}\beta\right)v_y=0 \tag{C-12}$$

and

$$\left(-i\omega-i\frac{\omega_p^2}{\omega}\beta\right)v_z=0\,, \tag{C-13}$$

leading to

$$n_c^2=1-\frac{1}{\gamma^2} \tag{C-14}$$

for the singular longitudinal waves and to

$$n_c^2=1+\frac{1-1/\gamma^2}{S^2-(\gamma^2-1)} \tag{C-15}$$

for the elliptically polarized transverse waves.

APPENDIX D

TWO DIMENSIONAL TURBULENCE

A considerable simplification takes place when restriction is made to two dimensional turbulence. Assuming that $v_z = 0$ and using the momentum equation (3-85) there result

$$\left[-i\omega + \frac{n_c^2\omega^2}{\Omega} + \frac{\omega_p^2}{k^2\Omega}(1+\beta)(k_y^2 + k_z^2)\right]v_x + \left[-\omega_b - \frac{1}{2}\frac{\omega_p^2}{k^2\Omega}(1+\beta)k_x k_y\right]v_y = \frac{e}{m}E_x , \tag{D-1}$$

$$\left[\omega_b - \frac{1}{2}\frac{\omega_p^2}{k^2\Omega}(1+\beta)k_x k_y\right]v_x + \left[-i\omega + \frac{n_c^2\omega^2}{\Omega} + \frac{\omega_p^2}{k^2\Omega}(1+\beta)(k_x^2 + k_z^2)\right]v_y = \frac{e}{m}E_y \tag{D-2}$$

and

$$-\frac{1}{2}\frac{\omega_p^2}{k^2\Omega}(1+\beta)k_z(k_x v_x + k_y v_y) = \frac{e}{m}E_z . \tag{D-3}$$

If the z-component of the electric field,

$$E_z = -i\frac{m}{e}\frac{\omega_p^2}{\omega^3}\beta c^2 k_z (k_x v_x + k_y v_y) \qquad \text{(D-4)}$$

is examined in the light of (D-3) it is seen that there are two cases.

If $k_z \neq 0$,

$$\left[i \frac{c^2}{\omega^3}\beta - \frac{1}{2}\frac{1}{k^2\Omega}(1+\beta)\right](k_x v_x + k_y v_y) = 0 ; \qquad k_z \neq 0, \qquad \text{(D-5)}$$

which since $v_z = 0$ implies that

$$\rho^{(1)} = 0 ; \qquad k_z \neq 0. \qquad \text{(D-6)}$$

The other alternative is that $k_z = 0$ in which case (D-6) does not hold. The latter case will be considered first.

D.1 Propagation perpendicular to $\overline{B}$

Since $\underline{k}$ is in the xy-plane

$$k_x = k \cos\theta, \qquad \text{(D-7)}$$

$$k_y = k \sin\theta, \tag{D-8}$$

and the momentum equations are

$$\left[-i\omega + \frac{n_c^2\omega^2}{\Omega} + \frac{\omega_p^2}{\Omega}(1+\beta)\sin^2\theta\right] v_x + \left[-\omega_b - \frac{1}{2}\frac{\omega_p^2}{\Omega}(1+\beta)\sin\theta\cos\theta\right] v_y = \frac{e}{m}E_x \tag{D-9}$$

and

$$\left[\omega_b - \frac{1}{2}\frac{\omega_p^2}{\Omega}(1+\beta)\sin\theta\cos\theta\right] v_x + \left[-i\omega + \frac{n_c^2\omega^2}{\Omega} + \frac{\omega_p^2}{\Omega}(1+\beta)\cos^2\theta\right] v_y = \frac{e}{m}E_y, \tag{D-10}$$

from which the mobilities may be obtained.

Using (3-39), the dispersion equations are

$$
\begin{aligned}
&\Big[-i\omega + \frac{n_c^2\omega^2}{\Omega} + \frac{\omega_p^2}{\Omega}(1+\beta)\sin^2\theta \\
&\quad - i\omega\frac{\beta}{y^2}\Big(1-\frac{k^2c^2}{\omega^2}\cos^2\theta\Big)\Big]v_x \\
&\quad + \Big[-\omega_b - \frac{1}{2}\frac{\omega_p^2}{\Omega}(1+\beta)\sin\theta\cos\theta \\
&\quad + i\omega\frac{\theta}{y^2}n_c^2\sin\theta\cos\theta\Big]v_y = 0
\end{aligned}
\tag{D-11}
$$

and

$$
\begin{aligned}
&\Big[\omega_b - \frac{1}{2}\frac{\omega_p^2}{\Omega}(1+\beta)\sin\theta\cos\theta \\
&\quad + i\omega\frac{\beta}{y^2}n_c^2\sin\theta\cos\theta\Big]v_x \\
&\quad + \Big[-i\omega + \frac{n_c^2\omega^2}{\Omega} + \frac{\omega_p^2}{\Omega}(1+\beta)\cos^2\theta \\
&\quad - i\omega\frac{\beta}{y^2}(1-n_c^2\sin^2\theta)\Big]v_y = 0
\end{aligned}
\tag{D-12}
$$

which leads to a quartic equation in n_c^2 which may be solved numerically.

D.2 Propagation Parallel to $\bar{B}$

If the propagation vector has a z-component, it must have components in no other direction so that $v_z = 0$. These waves are identical to those studied in section 4.2.2.

By using the relations obtained above, the transport properties of two-dimensional turbulence may be studied in detail since the important quantities may be obtained numerically for propagation vectors of arbitrary direction.

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