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RADIATIVE TRANSFER IN A CLOUDY ATMOSPHERE

by R. E. Samuelson

Goddard Space Flight Center
Greenbelt, Md.

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R. E. Samuelson

April, 1965

1. Page 28, last paragraph. The second sentence should be changed to read "An application of the flux integral reveals further that M_0 in Equation 94 becomes zero in this case, as is demonstrated in Appendix C."

2. Page 29, Equation 96 should read

$$I^{(0)}\left(\tau,\mu_i\right) = \frac{1}{4} F_0 \left\{ \sum_{a=1}^{n-1} \frac{M_a \, e^{-k_a \tau}}{1+\mu_i \, k_a} \left[\sum_{l=0}^{N} \tilde{\omega}_l \, \xi_l \left(+k_a\right) P_l\left(\mu_i\right) \right] + \frac{\gamma_0 \, e^{-\tau/\mu_0}}{1+\mu_i/\mu_0} \left[\sum_{l=0}^{N} \tilde{\omega}_l \, \xi_l \left(\frac{1}{\mu_0}\right) P_l\left(\mu_i\right) \right] + 3\mu_0 \, M_n \right\} \quad \left(\tilde{\omega}_0 = 1;\ i = \pm 1,\ \cdots,\ \pm n\right) . \tag{96}$$

The foregoing changes have become necessary because of an incorrect derivation in Appendix C. The following is a corrected version of the relevant derivation.

3. Page 70 et seq. The section from Equation C9 through Equation C21 should read:

Upon substituting $\lambda = 1$ in Equations C8 and C9, Equation C6 reduces to

$$F(\tau) = F_0 \left\{ \sum_{\alpha=1}^{n} M_\alpha \, e^{-k_\alpha \tau} \, \xi_1 (+k_\alpha) + \sum_{\alpha=1}^{n} M_{-\alpha} \, e^{+k_\alpha \tau} \, \xi_1 (-k_\alpha) \right.$$

$$\left. + \, e^{-\tau/\mu_0} \sum_{l=0}^{N} \tilde{\omega}_l \, \gamma_l \left(\frac{1}{\mu_0}\right) D_{1,l} \left(\frac{1}{\mu_0}\right) \right\}, \tag{C10a}$$

where

$$\gamma_l \left(\frac{1}{\mu_0}\right) = \gamma_0 \, \xi_l \left(\frac{1}{\mu_0}\right) \qquad (l = 0, \cdots, N) . \tag{C10b}$$

It can be shown (Reference 3) that γ_l obeys the relation

$$\gamma_l \left(\frac{1}{\mu_0}\right) = P_l (-\mu_0) + \sum_{\lambda=0}^{N} \tilde{\omega}_\lambda \, \gamma_\lambda \left(\frac{1}{\mu_0}\right) D_{l,\lambda} \left(\frac{1}{\mu_0}\right) . \tag{C10c}$$

By virtue of Equations C10b - C10c, Equation C10a reduces to

$$F(\tau) = F_0 \left\{ \sum_{\alpha=1}^{n} M_\alpha \, e^{-k_\alpha \tau} \, \xi_1 (+k_\alpha) + \sum_{\alpha=1}^{n} M_{-\alpha} \, e^{+k_\alpha \tau} \, \xi_1 (-k_\alpha) \right.$$

$$\left. + \left[\gamma_0 \, \xi_1 \left(\frac{1}{\mu_0}\right) + \mu_0 \right] e^{-\tau/\mu_0} \right\}. \tag{C10d}$$

Upon setting $\lambda = 0$ in Equation C7, and remembering that $\xi_{-1} = 0$, Equation C10d further reduces to

$$F(\tau) = F_0 \left\{ \mu_0 \, e^{-\tau/\mu_0} + \left(1 - \tilde{\omega}_0\right) \left[\sum_{\alpha=1}^{n} \frac{1}{k_\alpha} \left(M_{-\alpha} \, e^{+k_\alpha \tau} - M_\alpha \, e^{-k_\alpha \tau} \right) - \mu_0 \, \gamma_0 \, e^{-\tau/\mu_0} \right] \right\} . \tag{C11}$$

The net flux crossing the top of the cloud is found, upon setting $\tau = 0$, from

$$F(0) = F_0 \left\{ \mu_0 + \left(1 - \tilde{\omega}_0\right) \left[\sum_{\alpha=1}^{n} \frac{1}{k_\alpha} \left(M_{-\alpha} - M_\alpha \right) - \mu_0 \, \gamma_0 \right] \right\}, \tag{C12}$$

and crossing the bottom of the cloud, upon setting $\tau = \tau_1$, from

$$F(\tau_1) = F_0\left\{\mu_0\, e^{-\tau_1/\mu_0} + \left(1 - \tilde{\omega}_0\right)\left[\sum_{\alpha=1}^{n} \frac{1}{k_\alpha}\left(M_{-\alpha}\, e^{+k_\alpha \tau_1} - M_\alpha\, e^{-k_\alpha \tau_1}\right) - \mu_0\, \gamma_0\, e^{-\tau_1/\mu_0}\right]\right\}. \tag{C13}$$

If the cloud is semi-infinite, all $M_{-\alpha} = 0 \quad (\alpha = 1, \cdots, n)$, and Equation C12 becomes

$$F(0) = F_0\left\{\mu_0 - \left(1 - \tilde{\omega}_0\right)\left[\sum_{\alpha=1}^{n}\left(\frac{M_\alpha}{k_\alpha}\right) + \mu_0\, \gamma_0\right]\right\}. \tag{C14}$$

In the special case of conservative scattering ($\tilde{\omega}_0 = 1$), Equations C12-C14 are no longer necessarily valid relations. In this case the azimuth-independent term of the intensity in the case of a finitely thick cloud is given by (Equation 94)

$$\begin{aligned} I^{(0)}(\tau, \mu_i) = \frac{1}{4} F_0 \Bigg\{ & \sum_{\alpha=1}^{n-1} \frac{M_\alpha\, e^{-k_\alpha \tau}}{1 + \mu_i\, k_\alpha}\left[\sum_{l=0}^{N} \tilde{\omega}_l\, \xi_l\, (+k_\alpha)\, P_l(\mu_i)\right] \\ & + \sum_{\alpha=1}^{n-1} \frac{M_{-\alpha}\, e^{+k_\alpha \tau}}{1 - \mu_i\, k_\alpha}\left[\sum_{l=0}^{N} \tilde{\omega}_l\, \xi_l\, (-k_\alpha)\, P_l(\mu_i)\right] + \frac{\gamma_0\, e^{-\tau/\mu_0}}{1 + \mu_i/\mu_0}\left[\sum_{l=0}^{N} \tilde{\omega}_l\, \xi_l\left(\frac{1}{\mu_0}\right) P_l(\mu_i)\right] \\ & + 3\mu_0\left[\left\{\left(1 - \frac{1}{3}\tilde{\omega}_1\right)\tau + \mu_i\right\} M_0 + M_n\right] \Bigg\}, \end{aligned} \tag{C15}$$

where $M_{\pm\alpha}$ ($\alpha = 1, \cdots, n - 1$), M_0, and M_n are the $2n$ constants of integration. It is clear from Equation C11 and the preceding discussion that in this case the flux integral must reduce to

$$F(\tau) = F_0\left\{\mu_0\, e^{-\tau/\mu_0} + \frac{3}{2}\sum_i a_i\, \mu_i\, \mu_0\left[\left\{\left(1 - \frac{1}{3}\tilde{\omega}_1\right)\tau + \mu_i\right\} M_0 + M_n\right]\right\}. \tag{C16}$$

Since

$$\sum_i a_i\, \mu_i = 0; \qquad \sum_i a_i\, \mu_i^2 = \frac{2}{3}, \tag{C17}$$

Equation C16 reduces after some algebra to

$$F(\tau) = \mu_0\, F_0\left[M_0 + e^{-\tau/\mu_0}\right]. \tag{C18a}$$

This is the net flux of the *diffuse* radiation field. Since the net flux of the *direct* field (from the point source) is given by

$$F_D(\tau) = -\mu_0\, F_0\, e^{-\tau/\mu_0}, \tag{C18b}$$

the net flux of the *total* radiation field is given by

$$F_T(\tau) = \mu_0 F_0 M_0 . \tag{C18c}$$

The flux integral $F_T(\tau)$ is seen to be independent of optical depth, as it must be for conservative scattering.

If the cloud is semi-infinite and $\tilde{\omega}_0 = 1$, then, as $\tau \to 0$, Equation C18a must reduce to

$$F(0) = \mu_0 F_0 ; \tag{C19}$$

i.e., the "albedo" of the cloud must be unity. Thus M_0 is zero, and the expression for $I^{(0)}(\tau,\mu_i)$ in this case is seen by inspection to be

$$I^{(0)}(\tau,\mu_i) = \frac{1}{4} F_0 \left\{ \sum_{\alpha=1}^{n-1} \frac{M_\alpha e^{-k_\alpha \tau}}{1 + \mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l (+k_\alpha) P_l (\mu_i) \right] + \frac{\gamma_0 e^{-\tau/\mu_0}}{1 + \mu_i/\mu_0} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l \left(\frac{1}{\mu_0}\right) P_l (\mu_i) \right] + 3\mu_0 M_n \right\} , \tag{C20}$$

where M_α $(\alpha = 1, \cdots, n-1)$ and M_n are the n constants of integration to be determined from the n boundary conditions

$$I^{(0)}(0,-\mu_i) = 0 \qquad (i = 1, \cdots, n) . \tag{C21}$$

Continue with the section on *The Law of Darkening,* page 72.

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R. E. Samuelson

Goddard Space Flight Center

SUMMARY

The equation of radiative transfer in the context of a partially thermally emitting, partially anisotropically scattering plane-parallel cloudy atmosphere is derived. The derivation allows an exact interpretation of the auxiliary quantities in the equation of transfer in terms of the Mie scattering parameters. Explicit solutions are given by the method of discrete ordinates in accordance with Chandrasekhar's procedure, and extended to include thermal emission at infrared wavelengths. Solutions of this type, restricted to plane-parallel layers bounded on both sides by a vacuum and characteristic of a phase function for single scattering independent of optical depth, are referred to as solutions to the *restricted* problem. To extend the treatment to the *general* problem, a procedure wherein layers with different scattering properties are combined is developed in terms of the restricted solutions; two explicit examples are worked out. Such quantities as the angular distribution of outgoing radiation and the net outgoing flux are consequences of these solutions.

CONTENTS

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R. E. Samuelson
Goddard Space Flight Center

INTRODUCTION

The study of radiative transfer in a cloudy atmosphere is extremely involved due to the fact that clouds are composed primarily of particles having physical dimensions comparable to the wavelengths of radiation of interest. This realm of particle sizes requires a much more elaborate treatment of the theory of single scattering (scattering of radiation by one particle) than do the theories for single scattering for particles of quite different sizes.

Even where the theory for scattering by single particles is well understood, it still remains to describe the process of the transfer of radiation through a field of these particles, and, in particular, to describe the angular distribution of radiation from such a field. It is due to the order of difficulty in describing this compound process quantitatively that many investigators in the past have been led to treat various simplifications of the theory. An example of one such simplification involves adding a diffuse (isotropic) component to the intensities derived from a regular single scattering problem. Another is the two stream method used by various authors, whereby the essential assumption made is that the radiation field may be divided into two streams normal to a stratified cloud layer. Havard (Reference 1) has improved upon this method by replacing the specific intensity with the flux density as the dependent variable. Both methods suffer from the difficulty that nothing can be said about the angular distribution of radiation emitted from the cloud.

On the other hand, Churchill *et al.* (Reference 2) have obtained exact solutions for the angular distribution of outgoing diffusely reflected and transmitted radiation which are valid for three-term single scattering phase functions, and have developed a program which will solve transfer problems involving phase functions of much greater complexity exactly, provided among other assumptions that the phase function is independent of optical depth. The major limiting factors are the number of terms carried and the time involved on the electronic computer. A further limitation, which in practice is relevant only at wavelengths of a few microns and beyond, is that no account of thermal emission can be made.

It will be the purpose of this paper to formulate in a logical manner the appropriate equation of radiative transfer consistent with restricted physical situations, indicate how numerical solutions may be obtained for these situations, and finally, show how these restricted solutions lead in a natural way to more general solutions for physically realistic model cloudy atmospheres. In order to interpret the various single scattering parameters contained in the equation of transfer, it is convenient to derive this equation in the context of some single scattering theory. We have chosen to adopt the Mie theory implicitly for this purpose.

Beginning with the fundamental concept of individual photon-particle interactions, we are led in a natural way to the exact interpretation of such physically significant quantities as the phase function for single scattering, the albedo for single scattering, and the optical depth, all of which are explicitly contained in the basic equation of transfer. This development further illuminates the nature of the various approximations which are employed in order to make the scalar equation of transfer, in the context of polydispersed plane-parallel media, amenable to solution in some finite order of approximation.

Once the equation of transfer is formulated we proceed to the results of an analysis by Chandrasekhar. In particular, the solutions for the angular distribution of diffusely reflected and transmitted radiation at the surfaces of a plane-parallel cloud are given. In addition, an extension of this analysis to include radiation thermally emitted by a cloud having an arbitrary (but known) temperature profile is made, and analogous solutions to the more extended problem are derived. The solutions in all cases are obtained by the method of discrete ordinates compatible with the relevant equation of transfer. These solutions are referred to as solutions to the "restricted" problems; i.e., the relevant strata are assumed to be bounded on both sides by a vacuum, and are irradiated at most by a single outside point source.

The problem is then extended to include the effects of the surrounding atmosphere and ground. The two explicit problems considered involve the solutions for the net flux and angular distribution of: 1) radiation which is diffusely reflected from the top of an optically thin atmosphere overlying an optically thick cloud, and 2) radiation which is thermally emitted from the top of an atmosphere containing an optically thin cloud. In the latter case only the cloud is assumed to contribute to the scattering of radiation.

The formulation of the first of these more general problems consists of a linear integral equation containing as free parameters solutions to the more restricted problems previously discussed. This integral equation is shown to reduce to a system of independent linear integral equations of the second kind containing one independent variable only. Once these equations are solved by standard methods the remainder of the solution becomes quite straightforward.

The formulation of the second of these general problems consists only of a combination of the solutions, and integrals of these solutions, of the relevant restricted problems previously alluded to. By virtue of formulating the problem in the context of only one scattering layer, no integrals containing the dependent variable appear in the formulation. The other layers (including the ground) are assumed to contribute only purely emitted radiation.

Many extensions to the basic theory are possible. The solutions obtained here are primarily restricted to those which are not excessively difficult to solve numerically with the aid of an electronic computer. Inherent in the final solutions are any defects in the formalism due to the following nine major assumptions:

1. plane-parallel atmospheres,
2. unpolarized radiation,
3. spherical cloud particles,
4. well-separated cloud particles,
5. phase function for single scattering independent of optical depth,
6. local thermodynamic equilibrium,
7. blackbody emission from the ground at long wavelengths,
8. finite Legendre polynomial expansion of the phase function for single scattering,
9. finite approximation solution to the equation of transfer.

Besides these approximations, there is the additional restriction of having to approximate functions, which vary continuously with certain physical parameters, by step functions. Examples are the variation of the complex index of refraction with wavelength and the variation of particle distribution with particle size. It is true that these steps may be taken rather fine, but practical considerations of time and efficiency are going to limit the accuracy.

In the above approximations, it is felt that only the concept of plane-parallel atmospheres and the independence of the phase function for single scattering upon optical depth are subject to severe criticism. A partial compensation can be made for polarization in the sense that those intensities which are incorporated in the general solutions which result from a Rayleigh scattering atmosphere will be considered to be those intensities which result from the correct matrix formulation of the equation of transfer for the relevant restricted problem. The other approximations are not felt to be particularly severe. Perhaps the concept of spherical cloud particles should be modified; cirrus clouds are an example of one physical situation which might require this. If the cloud were divided into more than one layer, and each layer treated separately, the approximation of requiring the phase function for single scattering to the independent of optical depth could also be modified. However, this would make the *general problem*, that of combining the separate *restricted solutions* into a general solution through the use of certain integral relations to be derived in the last section of this paper, extremely complex, and hence will not be considered.

Five appendices are included in this paper.

Appendix A shows that the phase function for single scattering is normalized to the albedo for single scattering when this phase function is approximated by a finite series expansion of Legendre polynominals.

Appendix B gives the solution for a particular integral satisfying the equation of transfer which includes effects of thermal emission.

Appendix C gives solutions for the various flux integrals which arise from solutions to the equation of transfer.

Appendix D suggests methods of numerically solving certain integrals and integral equations. The integrals are replaced by Gaussian sums, and each integral equation is correspondingly replaced by a system of n equations in n unknowns.

Appendix E contains a list of symbols used in this paper.

FUNDAMENTAL QUANTITIES

Introduction

In a gross way we may think of observables, namely the radiative specific intensity, outgoing flux density, and so on, as being field quantities, and we will want to inquire how clouds affect the radiation field as a whole. But in order to do this it will be necessary to investigate individual photon-particle interactions, for these individual processes are responsible for variations of the field.

Regardless of the physical processes involved, the study must be undertaken in some frame of reference. The kinds of clouds which will be adopted as models will be plane-parallel, i.e., stratified with no horizontal inhomogeneities. It can be anticipated that the optical path length of radiation in clouds is very short compared to the radius of curvature of any planetary atmosphere, and that vertical inhomogeneities will far outweigh those of horizontal extent. Only "corrugated tops" of clouds are expected to cause serious difficulty; in fact this deviation from plane-parallelism is in practice probably the most serious defect of the models.

The Coordinate System

Since it will be necessary to consider such processes as absorption, scattering, and emission of radiant energy on a microscopic scale, we shall need the concept of an element of mass dm containing many particles, restricted to a volume dV, and characterized by a mean density ρ of matter in the element. The position of this mass element will be specified by the vertical distance z measured positively from the ground to dm. Directions at dm will be specified by the cosine of the zenith angle θ, denoted by μ, and the azimuthal angle ϕ (Figure 1). The zenith angle θ is measured positively from zero (the zenith) to π (the nadir). The azimuthal angle ϕ is measured through 2π radians in the plane of stratification from some arbitrary angle ϕ_0. Both the values of ϕ_0 and the direction of measurement will be specified at a later time. In general directions will be indicated by the symbol (μ, ϕ). For example, $I(z, \mu, \phi)$ is the intensity of radiation at dm (at a level z) in the direction (μ, ϕ). This direction is inclined to the normal to the plane of stratification by an angle θ, and is contained in the plane defined by the normal and the azimuthal angle ϕ.

The Specific Intensity

The amount of radiant energy which is transmitted across an elemental surface area $d\sigma$ in a time dt and in a direction inclined at an angle θ to the normal to $d\sigma$ will be given by δE_ν. δE_ν is further restricted to the frequency range $(\nu, \nu + d\nu)$ and to a solid angle $d\omega$. The dependent variable of principal interest is the "monochromatic" specific intensity defined by the limiting ratio

$$I_\nu = \frac{\delta E_\nu}{\cos\theta \, d\sigma \, d\omega \, d\nu \, dt} \tag{1}$$

where $d\sigma$, $d\omega$, $d\nu$, $dt \to 0$ in any manner. Thus I_ν is the rate at which radiant energy confined to a unit solid angle and unit frequency interval crosses a unit surface area which is normal to the direction of radiation.

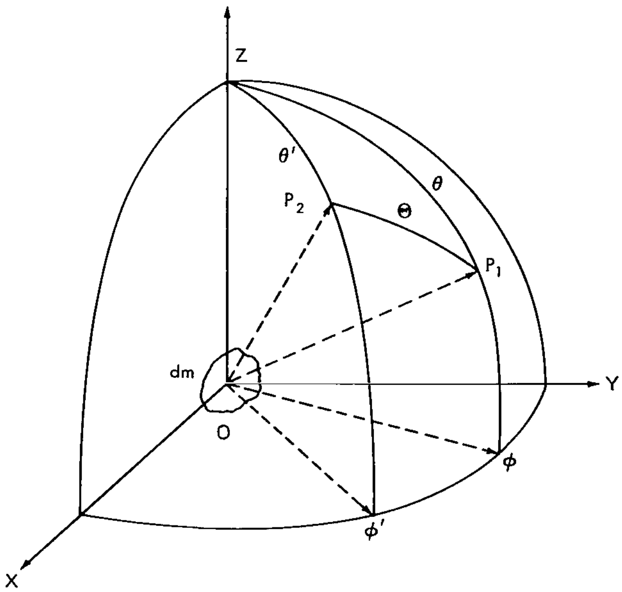

Figure 1 — The relation between the scattering angle Θ and the coordinate angles ϕ, ϕ', θ, and θ'. Radiation is assumed to be incident on the mass element dm at O in the direction $\vec{OP_1} = (\mu, \phi)$ and scattered by dm through the angle Θ into the direction $\vec{OP_2} = (\mu', \phi')$.

The Phase Function for Single Scattering

We shall need the concept of a phase function for single scattering, $\mathcal{P}(\cos\Theta)$, which describes the angular distribution of radiation scattered once. If ΔE_ν represents the fractional amount of energy E_ν incident on dm in unit time in the direction (μ, ϕ), which is either absorbed or scattered in all directions in unit time, and $d(\Delta E_\nu)$ is that fraction of ΔE_ν which is scattered into the direction (μ', ϕ') and contained in the solid angle $d\omega'$, we may formally represent this fraction of scattered radiation by

$$d\left[\Delta E_\nu (z, \mu', \phi')\right] = \mathcal{P}(\cos\Theta)\, \Delta E_\nu (z, \mu, \phi)\, \frac{d\omega'}{4\pi}, \tag{2}$$

where Θ is the angle through which the radiation is scattered.*

If the albedo for single scattering $\tilde{\omega}_0$ is defined to be the ratio of radiation scattered in all directions in unit time to the radiation extinguished (absorbed plus scattered) in unit time, we clearly have

$$\frac{1}{\Delta E_\nu (z, \mu, \phi)} \int_{\omega'} d\left[\Delta E_\nu (z, \mu', \phi')\right] = \tilde{\omega}_0 = \int_{\omega'} \mathcal{P}(\cos\Theta)\, \frac{d\omega'}{4\pi}, \tag{3}$$

*The tacit assumption has been made that multiple scattering does not take place in dm. The consequences of this assumption will be made apparent in the next section.

where the integration is performed over all solid angles. The function $\mathcal{P}(\cos\Theta)$ may formally be replaced by

$$\mathcal{P}(\cos\Theta) = p(\mu', \phi'; \mu, \phi) \tag{4}$$

since obviously the scattering angle can be expressed in terms of the coordinate system; $p(\mu', \phi'; \mu, \phi)$ then refers to radiation originally in the direction (μ, ϕ) which has been scattered into the direction (μ', ϕ'). If $\tilde{\omega}_0 = 1$ we have conservative scattering (no absorption), and $p(\mu', \phi'; \mu, \phi)$ is normalized to unity.

The relation between $\cos\Theta$ and μ, ϕ, μ', and ϕ' can be easily obtained from Figure 1. Let us suppose that radiation is incident on a mass element dm at O in the direction $\vec{OP}_1$, and is scattered at O through an angle Θ into the direction $\vec{OP}_2$. From the spherical triangle ZP_1P_2 we obtain the following relation:

$$\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi'-\phi) = \mu\mu' + \left(1-\mu^2\right)^{1/2}\left(1-\mu'^2\right)^{1/2}\cos(\phi'-\phi) \tag{5}$$

where $\mu = \cos\theta$ and $\mu' = \cos\theta'$. The angles θ and θ' are respectively the zenith angles of incidence and scattering, and ϕ and ϕ' are the corresponding azimuthal angles. It should further be noted from Equation 5 that $p(\mu', \phi'; \mu, \phi)$ obeys Helmholtz's principle of reciprocity; i.e.,

$$p(\mu', \phi'; \mu, \phi) = p(\mu, \phi; \mu', \phi') . \tag{6}$$

The Net Flux

Equation 1 gives the rate at which energy confined to a unit solid angle and unit frequency interval flows across a unit area which is normal to the direction of radiation. The "monochromatic" net flux πF_ν is the net rate at which energy per unit frequency interval flows across a unit surface in all directions, and this is given by

$$\pi F_\nu = \int_\omega I_\nu \cos\theta \, d\omega , \tag{7}$$

where the integration is to be performed over all solid angles. It should be noted that Equation 7 gives, in effect, the difference between the upward and downward fluxes across a unit surface area; therefore, deviations from a null net flux reflect deviations from conditions equivalent to those inside a perfectly insulated isothermal cavity.

In terms of the coordinate system previously specified the net flux becomes

$$\pi F_\nu = \int_0^{2\pi}\int_{-1}^{+1} \mu \, I_\nu(z, \mu, \phi)\, d\mu\, d\phi . \tag{8}$$

It is clear from Equation 8 that F_ν is in general a function of z. An extended account of the explicit integrations which will be required in the context of the specific problems encountered in this paper is given in Appendix C.

BASIC CONCEPTS AND THE EQUATION OF RADIATIVE TRANSFER

Introduction

It will be easiest to consider the interaction of radiation with matter from the "Lagrangian" point of view; i.e., the movement of individual particles (photons in our present context) will be followed. However, the observable parameter of interest, namely the monochromatic specific intensity I_ν, is a field of these photons, and we shall be interested in following variations of this field as it interacts with matter in some specified way. For this purpose it will be more practical to study local variations of I_ν without regard to the individual history of each photon. This is the "Eulerian" point of view (Reference 4).

An abrupt transition from the Lagrangian to the Eulerian point of view is somewhat unappealing in that it seems rather artificial. In order to appease this feeling of discontinuity of logical consequence we shall endeavor to smooth out the transition by retaining something of the Lagrangian point of view while introducing the Eulerian point of view, and in doing so give some kind of picture illustrating the essential identity of these two concepts.

Classification of Photon-Particle Interactions

Consider the photons of frequency ν interacting with dm to be classified by "type", each type depending upon the result of interaction as well as upon the intrinsic characteristics of the photon itself. Each class of photons is therefore a field of these photons which behaves exactly as every member of the field individually. We will want to restrict our attention to one field at a time, which in essence is the same as restricting our attention to one photon of this field at a time. Thus, the field from the Lagrangian point of view (in a less strict sense of the phrase) will be followed.

Since each photon behaves (by definition) like every other photon in the field, it is clear that not only must each photon be identical with every other photon of this class, but also the system of particles which this class as a whole interacts with must be composed of exactly the same kind (dimensions, refractive index, etc.) of individual particles in order that the separate interactions be identical; and if more than one interaction per photon takes place, the order, number, and character of these interactions must be the same for each photon. Since the last restriction creates an insurmountable strain on the imagination, it is seen immediately that we must restrict the volume dV of the mass element dm to dimensions considerably less than the mean free path of an individual photon, so that only single interactions are possible in dm. The additional problem of correctly specifying the orientation of each particle can be circumvented only by limiting ourselves to homogeneous spherical particles.

We define four classes of photons, each photon in the frequency (energy) range $(\nu, \nu + d\nu)$, and each interacting with one (spherical) particle in the radius range $(r_i, r_i + dr_i)$ which is of homogeneous composition of complex refractive index $n - i\kappa$. The four classes are distinguished by four different types of interactions, and are defined as follows (cf. Figure 2)*:

1. That system of photons which is incident upon dm in a time dt and in the direction (μ, ϕ) contained in the solid angle $d\omega$, and is (singly) scattered into the solid angle $d\omega'$ in the direction (μ', ϕ') by interactions in dm with particles in the radius range $(r_i, r_i + dr_i)$. This scattering process may either be considered as a scattering of a certain fraction of the number of incident photons into $d\omega'$, or as a re-direction of a fraction $d[\delta E_\nu (z, \mu', \phi')]$ of the incident energy $\delta E_\nu (z, \mu, \phi)$ into $d\omega'$.

2. That system of photons which is incident upon dm in time dt and in the direction (μ, ϕ) contained in the solid angle $d\omega$, and is absorbed by the particles under consideration in dm. This absorption process may be considered as an absorption of a certain fraction of incident photons or as a diminution of the incident energy $\delta E_\nu (z, \mu, \phi)$ by an amount $d[\delta E_\nu (z, \mu, \phi)]$.

3. That system of photons which is incident on dm in time dt and in the direction (μ', ϕ') contained in the solid angle $d\omega'$, and is (singly) scattered into the solid angle $d\omega$ in the direction (μ, ϕ). Again, this scattering process may be considered either as a scattering of certain fraction of incident individual photons into $d\omega$, or as a re-direction of a fraction $d[\delta E_\nu(z, \mu, \phi)]$ of the incident energy $\delta E_\nu (z, \mu', \phi')$ into $d\omega$.

4. That system of photons which is (thermally) emitted from the particles under consideration in dm into the solid angle $d\omega$ and in the direction (μ, ϕ) in a time dt. This thermal emission process may be considered either as an emission of many individual photons, each in the energy range $(h\nu, h\nu + dh\nu)$, or as a source of energy $d[\delta E_\nu (z, \mu, \phi)]$ which is emitted into $d\omega$ in the direction (μ, ϕ). In principle we need not have considered only thermal emission; however, in planetary atmospheres we would not expect other types of emission to play a very significant role.

The first two classes of photons are those which are lost from the radiation field in the direction (μ, ϕ) by scattering and absorption respectively. The last two classes consist of photons which are gained by the radiation field in the direction (μ, ϕ) respectively by scattering and emission. These are obviously not the total losses and gains to the radiation field, however, since only interactions with particles in the radius range $(r_i, r_i + dr_i)$ have been considered. Later on an integration over all particle sizes will have to be made.

The four classes of photons have the following characteristics in common:

a. frequency range $(\nu, \nu + d\nu)$.

b. interaction with one (spherical) particle in the radius range $(r_i, r_i + dr_i)$.

c. interaction with particles of homogeneous composition of complex refractive index $(n - i\kappa)$.

*Note that $d\omega$ and $d\omega'$ are elements of solid angle which are *both* subtended at dm.

The classes are also characterized by the differences tabulated below:

Class Characteristic	Incident Direction	Solid Angle Subtended at dm	Emergent Direction	Solid Angle Subtended at dm
1. Singly Scattered (Loss)	(μ, ϕ)	$d\omega$	(μ', ϕ')	$d\omega'$
2. Absorbed (Loss)	(μ, ϕ)	$d\omega$	None	None
3. Singly Scattered (Gain)	(μ', ϕ')	$d\omega'$	(μ, ϕ)	$d\omega$
4. Thermally Emitted (Gain)	None	None	(μ, ϕ)	$d\omega$

The classes may be characterized in terms of energy as follows:

1. Re-direction of fraction $d[\delta E_\nu (z, \mu', \phi')]$ of incident energy $\delta E_\nu (z, \mu, \phi)$ into $d\omega'$.
2. Absorption of fraction $d[\delta E_\nu (z, \mu, \phi)]$ of incident energy $\delta E_\nu (z, \mu, \phi)$ by dm.
3. Re-direction of fraction $d[\delta E_\nu (z, \mu, \phi)]$ of incident energy $\delta E_\nu (z, \mu', \phi')$ into $d\omega$.
4. Emission of energy $d[\delta E_\nu (z, \mu, \phi)]$ into $d\omega$.

Single Scattering

In order to describe the interaction of one photon with one particle quantitatively, an appeal will have to be made to some single scattering theory. The Mie Theory (Reference 1) will suit our purpose if we restrict ourselves to homogeneous spherical particles in a radiation field of plane waves.* Conceptually the Mie theory describes the outgoing "scattered" radiation in terms of a spherical wave front which depends upon the wavelength of the incoming plane wave and the size and complex index of refraction of the spherical particle. The radial component of the outgoing spherical wave dies out as the square of the distance, while the tangential component dies out as the first power of the distance. Thus, the outgoing spherical wave tends to take on more and more the character of a plane wave as the radial distance from the scattering center is steadily increased. Since it has been implicitly assumed that many of the incident plane wave "photons" have originated from scattering and emission processes in other mass elements, it is clear that the dimensions of the mass elements must be sufficiently large such that the outgoing spherical wave of scattered radiation from a particle in element dm′, say, has traveled a sufficient distance to become essentially a plane wave by the time it suffers another interaction with a particle in dm. In a crude way this means that the mean free path of a photon from mass element to adjacent mass element must be large compared to the wavelength of radiation, which is going to place an upper limit on the size of the particles compared with their mean separation distance. We shall always assume that physical conditions prevail such that the Mie theory gives a valid picture of single scattering in clouds.

It will not be the purpose of this paper to describe the Mie theory. Van de Hulst (Reference 6) and Born and Wolf (Reference 5) give a good account of it, and Havard (Reference 1) gives rather

*Photons are now regarded as plane waves, with the realization that this rather abrupt transition of our "picture" of a photon leaves something to be desired.

complete details on how to compute the phase function for single scattering $[\mathcal{P}(\cos\Theta) = p(\mu, \phi; \mu', \phi')]$ and the efficiency factors* for extinction, $Q_E^{(i)}$, absorption, $Q_A^{(i)}$, and scattering, $Q_S^{(i)}$. The relation

$$Q_E^{(i)} = Q_S^{(i)} + Q_A^{(i)} \tag{9}$$

always holds. With this brief digression we return to a more detailed discussion of the first class of photons.

The Radiation Field; Losses and Gains

Construct a convex closed surface δS around $dm = \rho dV$ such that the volume contained by δS is large compared with dV but small otherwise (Figure 2). Let dA and dA' be elements of δS such that the direction from dA to dm is (μ, ϕ) and the direction from dm to dA' is (μ', ϕ'). Further let $d\omega$ and $d\omega'$ be respectively the elements of solid angle containing dA and dA' as seen from dm, and $d\alpha$ and

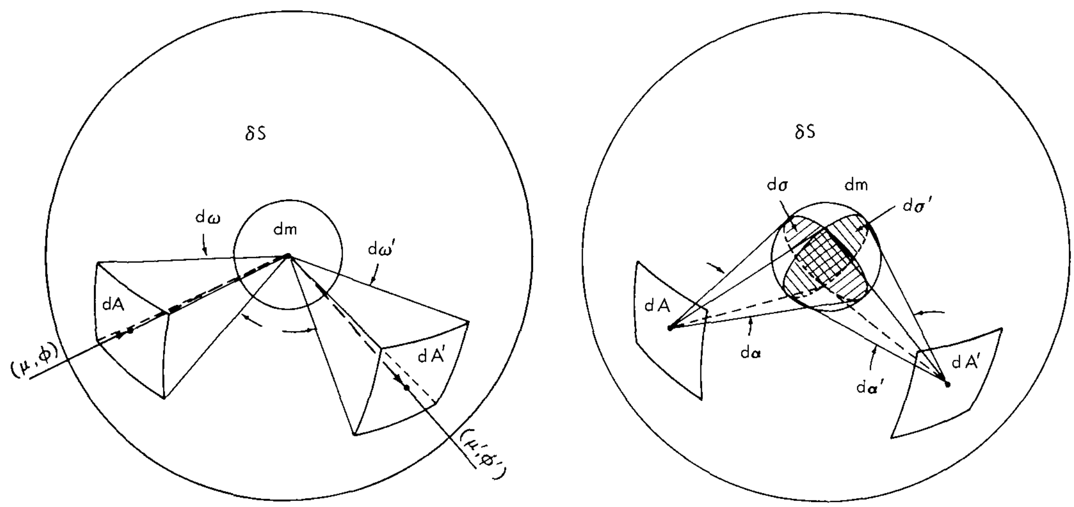

Figure 2 — Illustration of the elements of solid angle $d\omega$ and $d\omega'$ subtended at the mass element dm respectively by dA and dA', elements of the convex bounding surface δS. Also shown are the elements of solid angle $d\alpha$ and $d\alpha'$, subtended respectively at dA and dA' by dm.

*Each efficiency factor is defined as the ratio between the effective cross-section of the particle $\chi^{(i)}$ to the geometric cross-section πr_i^2, where r_i is the radius of the i^{th} particle. Thus the efficiency factors for extinction, absorption and scattering are

$$\left.\begin{aligned} Q_E^{(i)} &= \chi_E^{(i)}/\pi r_i^2 \\ Q_A^{(i)} &= \chi_A^{(i)}/\pi r_i^2 \\ Q_S^{(i)} &= \chi_S^{(i)}/\pi r_i^2 \end{aligned}\right\} .$$

$d\alpha'$ the elements of solid angle containing dm as seen respectively from dA and dA'. Referring to Equation 1, it is seen that the amount of energy $\delta E_\nu(z, \mu, \phi)$ crossing dm in a time dt and in the frequency range $(\nu, \nu + d\nu)$, which has originated outside δS and has also crossed dA, is

$$\delta E_\nu (z, \mu, \phi) = I_\nu (z, \mu, \phi) \mu_1 \, dA \, d\alpha \, d\nu \, dt \ , \tag{10}$$

where μ_1 is the cosine of the angle between the direction (μ, ϕ) and the normal to dA.

Of all the energy crossing dA which is contained in the solid angle $d\alpha$, a certain fraction will be singly scattered by dm into the solid angle $d\omega'$. The fraction of energy which is thus scattered in a time dt is seen to be (Equation 2)

$$\frac{d\left[\delta E_\nu (z, \mu', \phi')\right]}{\delta E_\nu (z, \mu, \phi)} = P_S^{(i)} \, p_i (\mu', \phi'; \mu, \phi) \frac{d\omega'}{4\pi} \ , \tag{11}$$

where $P_S^{(i)}$ is the probability that any one photon which is incident on dm is scattered in any direction whatsoever; $p_i(\mu', \phi'; \mu, \phi)$ in this context is the phase function for single scattering through the angle between the directions (μ, ϕ) and (μ', ϕ') normalized to unity. Now the probability of scattering $P_S^{(i)}$ of one photon is just the ratio of the total available effective scattering cross-section of all the particles in the radius range $(r_i, r_i + dr_i)$ contained in dm to the geometrical cross-section $d\sigma$ of dm as seen in the direction (μ, ϕ), and this is given by

$$P_S^{(i)} = \frac{Q_S^{(i)} \pi r_i^2 N_i \, dr_i \, dV}{d\sigma} \ , \tag{12}$$

where dV is the volume of dm, N_i is the number of particles per unit volume per unit radius range centered about r_i, and $Q_S^{(i)}$ is the efficiency factor for scattering from the Mie theory as described previously. Equation 12 is valid so long as there is no "shadow" effect, i.e., the probability that any one particle is screened off from any other particle is negligible; this requires that only single scattering prevails, or, put another way, that $P_S^{(i)}$ is small compared to unity. Collecting Equations 10, 11, and 12 we have:

$$d\left[\delta E_\nu (z, \mu', \phi')\right] = \frac{Q_S^{(i)} \pi r_i^2 N_i \, dr_i \, dV}{d\sigma} I_\nu (z, \mu, \phi) \mu_1 \, dA \, d\alpha \, d\nu \, dt \, p_i (\mu', \phi'; \mu, \phi) \frac{d\omega'}{4\pi} \ . \tag{13}$$

There must clearly be a loss of intensity $d\left[\delta_- I_S (z, \mu, \phi)\right]$ in the direction (μ, ϕ) which is associated with the loss of energy $d\left[\delta E_\nu (z, \mu', \phi')\right]$ from the direction (μ, ϕ), and this is given by (cf. Equation 10)

$$d\left[\delta E_\nu (z, \mu', \phi')\right] = d\left[\delta_- I_S (z, \mu, \phi)\right] \mu_1 \, dA \, d\alpha \, d\nu \, dt \ . \tag{14}$$

Comparing Equations 13 and 14, we obtain

$$d\left[\delta_- I_S(z,\mu,\phi)\right] = \frac{Q_S^{(i)}\,\pi r_i^2 N_i\, dr_i\, dV}{d\sigma} I_\nu(z,\mu,\phi)\, p_i(\mu',\phi';\mu,\phi)\frac{d\omega'}{4\pi} . \tag{15}$$

Equation 15 is valid for scattering of incident photons in the frequency range $(\nu, \nu + d\nu)$ into the solid angle $d\omega'$ by particles in the radius range $(r_i, r_i + dr_i)$. In order to obtain the effect of scattering from particles of all sizes in all directions, Equation 15 must be integrated over all r_i and ω'. Thus, the total intensity in the frequency range $(\nu, \nu + d\nu)$ and in the direction (μ, ϕ) which is lost from this direction by scattering from dm is

$$\delta_- I_S(z,\mu,\phi) = \frac{N_0\, dV}{d\sigma} I_\nu(z,\mu,\phi)\int_0^\infty \int_{\omega'} Q_S^{(i)}\,\pi r_i^2 D_i\, p_i(\mu',\phi';\mu,\phi)\frac{d\omega'}{4\pi}\, dr_i , \tag{16}$$

where N_0 is the total number of particles per unit volume and D_i (the relative number of particles per unit radius range centered about r_i) is the normalized distribution function of particle sizes; i.e., $N_i = N_0 D_i$.

By definition $(1/4\pi)\; p_i(\mu', \phi'; \mu, \phi)$ integrated over all solid angles is unity regardless of the particle size involved; however we shall delay this intergration until later.

We turn our attention now to the second class of photons defined previously. From Equations 10 and 11 and the corresponding discussion it should be clear that the energy in the frequency range $(\nu, \nu + d\nu)$ crossing dA (Figure 2) in a time dt, which is absorbed by the particles in the radius range $(r_i, r_i + dr_i)$, is given by

$$d\left[\delta E_\nu(z,\mu,\phi)\right] = I_\nu(z,\mu,\phi)\,\mu_1\, dA\, d\alpha\, d\nu\, dt\, P_A^{(i)} , \tag{17}$$

where $P_A^{(i)}$, the probability of absorption of a single photon in the energy range $(h\nu, h\nu + dh\nu)$, is given by (cf. Equation 12)

$$P_A^{(i)} = \frac{Q_A^{(i)}\,\pi r_i^2 N_i\, dr_i\, dV}{d\sigma} . \tag{18}$$

By analogy with Equation 14, the loss $d\left[\delta_- I_A(z, \mu, \phi)\right]$ of the intensity $I_\nu(z, \mu, \phi)$ which is incident on dm in the direction (μ, ϕ) is given by

$$d\left[\delta E_\nu(z,\mu,\phi)\right] = d\left[\delta_- I_A(z,\mu,\phi)\right]\mu_1\, dA\, d\alpha\, d\nu\, dt . \tag{19}$$

Comparing Equations 19 and 17, we obtain, with the aid of Equation 18,

$$d\left[\delta_- I_A(z,\mu,\phi)\right] = \frac{Q_A^{(i)}\,\pi r_i^2 N_i\, dr_i\, dV}{d\sigma} I_\nu(z,\mu,\phi) , \tag{20}$$

and this integrated over all particle sizes becomes

$$\delta_- I_A(z, \mu, \phi) = \frac{N_0\, dV}{d\sigma} I_\nu(z, \mu, \phi) \int_0^\infty Q_A^{(i)} \pi r_i^2 D_i\, dr_i \quad . \tag{21}$$

Equation 20 is the total intensity in the frequency range $(\nu, \nu + d\nu)$ and in the direction (μ, ϕ) which is lost from this direction by absorption in dm.

In order to establish the gain to the radiation field by the scattering of the third class of photons into the direction (μ, ϕ), it is necessary only to reverse the sense of direction of the radiation field in Figure 2. By once again tracing out the consequences of the scattering process it becomes clear that Equations 10 to 13 remain valid upon an interchange of the primed and unprimed quantities. Thus, the gain of energy $d[\delta E_\nu(z, \mu, \phi)]$ in a time dt and in the direction (μ, ϕ) contained in the solid angle $d\omega$, which has resulted from a scattering of the energy by particles in the radius range $(r_i, r_i + dr_i)$ in dm, and which was originally in the direction (μ', ϕ') and contained in the solid angle $d\omega'$, is

$$d[\delta E_\nu(z, \mu, \phi)] = \frac{Q_S^{(i)} \pi r_i^2 N_i\, dr_i\, dV}{d\sigma'} I_\nu(z, \mu', \phi') \mu_1'\, dA'\, d\alpha'\, d\nu\, dt\, p_i(\mu, \phi;\ \mu', \phi') \frac{d\omega}{4\pi} \quad . \tag{22}$$

Here $d\sigma'$ is the geometrical cross-section of the mass element dm as seen in the direction (μ', ϕ'), $d\alpha'$ is the solid angle subtended by $d\sigma'$ at dA', and μ_1' is the cosine of the angle between the direction (μ', ϕ') and the normal to dA'.

The corresponding gain in intensity at dm in the direction (μ, ϕ) must be given by

$$d[\delta E_\nu(z, \mu, \phi)] = d[\delta_+ I_S(z, \mu, \phi)]\, d\sigma\, d\omega\, d\nu\, dt \quad , \tag{23}$$

since $d[\delta E_\nu(z, \mu, \phi)]$ is just the energy in a time dt which crosses normal to the surface element $d\sigma$ at dm and is contained in the solid angle $d\omega$. Comparing Equations 23 and 22, we obtain

$$d[\delta_+ I_S(z, \mu, \phi)] = \frac{Q_S^{(i)} \pi r_i^2 N_i\, dr_i\, dV}{d\sigma} I_\nu(z, \mu', \phi')\, p_i(\mu, \phi;\ \mu', \phi') \frac{\mu_1'\, dA'\, d\alpha'}{4\pi\, d\sigma'} \quad . \tag{24}$$

If the distance between dm and dA' is denoted by r, then, from the geometry,

$$\mu_1'\, dA' = r^2\, d\omega' \tag{25}$$

and

$$d\sigma' = r^2\, d\alpha' \quad . \tag{26}$$

With the aid of Equations 25 and 26, Equation 24 becomes

$$d[\delta_+ I_S(z, \mu, \phi)] = \frac{Q_S^{(i)} \pi r_i^2 N_i\, dr_i\, dV}{d\sigma} p_i(\mu, \phi;\ \mu', \phi')\, I_\nu(z, \mu', \phi') \frac{d\omega'}{4\pi} \quad . \tag{27}$$

In order to obtain the total contribution to the intensity in the direction (μ, ϕ) by scattering from dm, Equation 27 must be integrated over all solid angles ω' and all particle sizes r_i; i.e. (cf. Equation 16),

$$\delta_+ I_S (z, \mu, \phi) = \frac{N_0 \, dV}{d\sigma} \int_0^\infty \int_{\omega'} Q_S^{(i)} \pi r_i^2 D_i \, p_i (\mu, \phi; \mu', \phi') \, I_\nu (z, \mu', \phi') \frac{d\omega'}{4\pi} \, dr_i \; . \tag{28}$$

The fourth class of photons describes the contribution to the radiation field by thermal emission from dm. In order to calculate this contribution, consider the surface of Figure 2 to be a perfectly insulated isothermal enclosure in which dm is maintained at a constant equilibrium temperature T. Consider $N_i \, dr_i$ particles in the radius range $(r_i, r_i + dr_i)$ contained in dm, each of internal temperature T. Since the radiation field within the enclosure is in equilibrium with its surroundings and isotropic, the amount of energy in the frequency interval $(\nu, \nu + d\nu)$ and in the direction (μ, ϕ) contained in the solid angle $d\omega$, which would be emitted from dm in a time dt upon an instantaneous removal of the cavity walls, is given by (cf. Equation 23)

$$d\left[\delta E_\nu (z, \mu, \phi)\right] = B_\nu (T) \, N_i \, dr_i \, \chi_A^{(i)} \, dV \, d\omega \, d\nu \, dt \; , \tag{29}$$

and this must be equivalent to

$$d\left[\delta E_\nu (z, \mu, \phi)\right] = d\left[\delta_+ I_E (z, \mu, \phi)\right] d\sigma \, d\omega \, d\nu \, dt \; , \tag{30}$$

where $d\left[\delta_+ I_E (z, \mu, \phi)\right]$ refers to the gain in intensity in the direction (μ, ϕ) due to thermal emission from dm, and where $B_\nu(T)$ is the Planck function in intensity units. If the total number of particles of all sizes per unit volume is N_0, and the size distribution in dm is given as before by D_i, then Equation 29 may be re-written as

$$\int_{\delta E_\nu} d\left[\delta E_\nu (z, \mu, \phi)\right] = \int_0^\infty N_0 \, dV \, Q_A^{(i)} \pi r_i^2 D_i \, B_\nu (T) \, d\omega \, d\nu \, dt \, dr_i = N_0 \, dV \, B_\nu (T) \, d\omega \, d\nu \, dt \int_0^\infty Q_A^{(i)} \pi r_i^2 D_i \, dr_i \; , \tag{31}$$

where $\chi_A^{(i)} = Q_A^{(i)} \pi r_i^2$. Relating Equation 30 to Equation 31 we have

$$\delta_+ I_E (z, \mu, \phi) = \int_{\delta_+ I_E} d\left[\delta_+ I_E (z, \mu, \phi)\right] = \frac{N_0 \, dV}{d\sigma} B_\nu (T) \int_0^\infty Q_A^{(i)} \pi r_i^2 D_i \, dr_i \; . \tag{32}$$

If the enclosure is not replaced, dm will be subjected to the anisotropic radiation field of arbitrary energy density. What happens now is largely a function of the relative importance of: (1) collisions between molecules contained in the particles, compared with (2) interaction between these molecules and the radiation field, as a cause of molecular absorptions and emissions. If interactions with the radiation field predominates, the emission will essentially consist of the spontaneous emission of photons from excited molecules, and induced emission of photons from excited molecules through perturbations due to the radiation field; the latter type of emission is proportional to the incident intensity and is therefore anisotropic in the same sense as the radiation field. The molecules will be excited through the absorption of incident radiation, and the local temperature will be strictly dependent upon the photon density. Thus the radiation emitted from the particles cannot be isotropic because of the contribution from induced emission, unless the radiation field itself is strictly isotropic.

At the other extreme, collisional battering of molecules predominates, and the thermal (isotropic) emission far outweighs emission induced by the radiation field. This will occur where the density of molecules is great enough so that the frequency of encounters among molecules is much larger than the frequency of encounters between molecules and incident photons. *We shall suppose this latter condition always to prevail in clouds.* Since the density outside the particles is much less than the density inside, we shall have to be more careful when considering emitted radiation from the surrounding gaseous medium.

Under the assumption made, Equation 32 is acceptable as it stands upon a removal of the cavity walls. The temperature T will depend only upon the energy available through collisions, and the radiation field does not have to be in equilibrium with the surrounding medium.

Averaging Techniques and the Equation of Transfer

We are now in a position to evaluate the net change in intensity in the direction (μ, ϕ) due to the presence of the mass element dm. Adding up all the net gains and losses from Equations 16, 21, 28, and 32 we obtain

$$
\begin{aligned}
\delta I_\nu(z, \mu, \phi) &= -\delta_- I_S(z, \mu, \phi) - \delta_- I_A(z, \mu, \phi) + \delta_+ I_S(z, \mu, \phi) + \delta_+ I_E(z, \mu, \phi) \\
&= -\frac{N_0\, dV}{d\sigma} I_\nu(z, \mu, \phi) \int_0^\infty \int_{\omega'} Q_S^{(i)} \pi r_i^2 D_i\, p_i(\mu', \phi'; \mu, \phi) \frac{d\omega'}{4\pi}\, dr_i \\
&\quad - \frac{N_0\, dV}{d\sigma} I_\nu(z, \mu, \phi) \int_0^\infty Q_A^{(i)} \pi r_i^2 D_i\, dr_i \\
&\quad + \frac{N_0\, dV}{d\sigma} \int_0^\infty \int_{\omega'} Q_S^{(i)} \pi r_i^2 D_i\, p_i(\mu, \phi; \mu', \phi') I_\nu(z, \mu', \phi') \frac{d\omega'}{4\pi}\, dr_i \\
&\quad + \frac{N_0\, dV}{d\sigma} B_\nu(T) \int_0^\infty Q_A^{(i)} \pi r_i^2 D_i\, dr_i \quad . \qquad (33)
\end{aligned}
$$

The terms on the right hand side of Equation 33 are negative or positive depending on whether they are respectively losses or gains.

After an examination of Equation 33, it becomes clear that a judicious choice of an effective phase function for single scattering $p_0(\mu, \phi; \mu', \phi')$ given by

$$p_0(\mu, \phi; \mu', \phi') \int_0^\infty Q_S^{(i)} \pi r_i^2 D_i \, dr_i = \int_0^\infty Q_S^{(i)} \pi r_i^2 D_i \, p_i(\mu, \phi; \mu', \phi') \, dr_i \tag{34}$$

will greatly simplify the equation. Upon multiplying both sides of Equation 34 by $d\omega'/4\pi$, integrating over all solid angles, and remembering that $\int_{\omega'} p_i(\mu, \phi; \mu', \phi') \, d\omega'/4\pi = 1$, we obtain the result

$$\int_{\omega'} p_0(\mu, \phi; \mu', \phi') \frac{d\omega'}{4\pi} = 1 . \tag{35}$$

Thus, $p_0(\mu, \phi; \mu', \phi')$ is normalized to unity just as $p_i(\mu, \phi; \mu', \phi')$. Upon inserting Equation 34 into Equation 33 and using the results of Equation 35 we obtain

$$\begin{aligned} \delta I_\nu(z, \mu, \phi) = & - \frac{N_0 \, dV}{d\sigma} I_\nu(z, \mu, \phi) \int_0^\infty \left[Q_S^{(i)} + Q_A^{(i)}\right] \pi r_i^2 D_i \, dr_i \\ & + \frac{N_0 \, dV}{d\sigma} \left[\int_0^\infty Q_S^{(i)} \pi r_i^2 D_i \, dr_i\right] \int_{\omega'} p_0(\mu, \phi; \mu', \phi') I_\nu(z, \mu', \phi') \frac{d\omega'}{4\pi} \\ & + \frac{N_0 \, dV}{d\sigma} B_\nu(T) \int_0^\infty Q_A^{(i)} \pi r_i^2 D_i \, dr_i . \end{aligned} \tag{36}$$

Upon using the relations $Q_E^{(i)} = Q_S^{(i)} + Q_A^{(i)}$, $\mu = \cos\theta$, and $d\omega = \sin\theta \, d\theta \, d\phi$, Equation 36 becomes

$$\begin{aligned} \frac{\delta I_\nu(z, \mu, \phi)}{-\frac{N_0 \, dV}{d\sigma} \int_0^\infty Q_E^{(i)} \pi r_i^2 D_i \, dr_i} = & \; I_\nu(z, \mu, \phi) \\ & - \frac{1}{4\pi} \left[\frac{\int_0^\infty Q_S^{(i)} \pi r_i^2 D_i \, dr_i}{\int_0^\infty Q_E^{(i)} \pi r_i^2 D_i \, dr_i}\right] \int_0^{2\pi} \int_{-1}^{+1} p_0(\mu, \phi; \mu', \phi') I_\nu(z, \mu', \phi') \, d\mu' \, d\phi' \\ & - \left[1 - \frac{\int_0^\infty Q_S^{(i)} \pi r_i^2 D_i \, dr_i}{\int_0^\infty Q_E^{(i)} \pi r_i^2 D_i \, dr_i}\right] B_\nu(T) . \end{aligned} \tag{37}$$

Equation 37 may be made less bulky by defining an effective extinction cross-section χ_E, an effective scattering cross-section χ_S, and incidentally an effective absorption cross-section χ_A, such that

$$
\left.\begin{aligned}
\chi_E &= \int_0^\infty Q_E^{(i)}\, \pi r_i^2\, D_i\, dr_i \\
\chi_S &= \int_0^\infty Q_S^{(i)}\, \pi r_i^2\, D_i\, dr_i \\
\chi_A &= \int_0^\infty Q_A^{(i)}\, \pi r_i^2\, D_i\, dr_i
\end{aligned}\right\} . \tag{38}
$$

The effective albedo for single scattering is then given by

$$\tilde{\omega}_0 = \frac{\chi_S}{\chi_E} . \tag{39}$$

In the context of plane-parallel atmospheres it is clear that, for any layer of thickness dz, a volume element cylinder dV of height dz and base area $d\sigma_z$ is defined by

$$dV = d\sigma_z\, dz . \tag{40}$$

If now the linear dimensions of $d\sigma_z$ are made arbitrarily much larger than dz, but still small enough so that $d\sigma_z$ remains an element of surface area, then the sides of the cylinder may be neglected relative to $d\sigma_z$ in determining the effective cross-section of dV when seen from a slant-path in the direction (μ, ϕ). The cross-section $d\sigma$ of dV then becomes

$$d\sigma = \mu\, d\sigma_z . \tag{41}$$

Upon utilizing Equations 38 through 41, Equation 37 may be rewritten as

$$\mu \frac{\delta I_\nu (z, \mu, \phi)}{-N_0 \chi_E\, dz} = I_\nu (z, \mu, \phi)$$

$$- \frac{\tilde{\omega}_0}{4\pi} \int_0^{2\pi} \int_{-1}^{+1} p_0 (\mu, \phi;\ \mu', \phi')\, I_\nu (z, \mu', \phi')\, d\mu'\, d\phi' - \left(1 - \tilde{\omega}_0\right) B_\nu (T) . \tag{42}$$

It will be convenient to define a normal optical depth τ_ν measured from the top of the cloud inward such that

$$d\tau_\nu = - N_0 \chi_E\, dz , \tag{43}$$

and a phase function for single scattering $p(\mu, \phi; \mu', \phi')$ normalized to $\tilde{\omega}_0$ such that

$$p(\mu, \phi; \mu', \phi') = \tilde{\omega}_0 p_0 (\mu, \phi; \mu', \phi') . \tag{44}$$

Upon letting the ratio $\delta I_\nu (z, \mu, \phi)/d\tau_\nu$ approach its limit as $d\tau_\nu \to 0$, and upon replacing $I_\nu (z, \mu, \phi)$ everywhere with $I(\tau, \mu, \phi)$ and dropping the subscript ν, Equation 42 becomes

$$\mu \frac{dI(\tau, \mu, \phi)}{d\tau} = I(\tau, \mu, \phi) - \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{+1} p(\mu, \phi; \mu', \phi') I(\tau, \mu', \phi') d\mu' d\phi' - \left(1 - \tilde{\omega}_0\right) B(T) . \tag{45}$$

It remains understood that Equation 45 is implicitly a function of the frequency of radiation ν.

Equation 45 is the equation of transfer for an arbitrary field of radiation. In general this radiation field is the diffuse radiation field which has originated from thermal emission from the cloud, surrounding atmosphere, and the ground. There will also be a contribution at relevant wavelengths from the sun, which in fact is the sole source of radiation at visible and shorter wavelengths. It will be convenient to distinguish between the reduced incident radiation $\pi F_0 e^{-\tau/\mu_0}$ from the sun, which penetrates to the level τ without suffering any scattering or absorption processes, and the diffuse radiation field which has arisen through one or more scattering and/or emission processes; here πF_0 is the total flux in the frequency interval $(\nu, \nu + d\nu)$ of a beam of radiation crossing in unit time a surface of unit area normal to the beam, and μ_0 is the cosine of the zenith angle of the sun. The sun is assumed to be approximately a point source.

It will turn out that the intensity of the diffuse radiation field is the quantity most easily handled in the equation of transfer, because the boundary conditions are much simpler to impose correctly in this case than for the case in which the intensity of the total radiation field is the dependent variable. In order to separate the diffuse field from the directly transmitted radiation from the sun, we proceed in the manner given below.

The Diffuse Radiation Field

Consider a parallel beam of radiation from a point source at infinity of flux πF_0 crossing a unit surface normal to the beam. The magnitude of the flux crossing a unit surface which is in the plane of the top of the cloud is $\pi \mu_0 F_0$, where the cosine of the zenith angle of the point source is μ_0, and this is (Reference 3)

$$\pi \mu_0 F_0 = \int_0^{2\pi} \int_0^{1} \mu I(0, -\mu, \phi) d\mu d\phi , \tag{46}$$

where $I(0, -\mu, \phi)$ is the downward intensity of radiation in the direction $(-\mu, \phi)$. Since the only contribution is in the direction $(-\mu_0, \phi_0)$ from the point source, the intensity $I(0, -\mu, \phi)$ should be of the form

$$I(0, -\mu, \phi) = f(\mu)\, \delta(\mu - \mu_0)\, g(\phi)\, \delta(\phi - \phi_0) \ , \tag{47}$$

where $\delta(\mu - \mu_0)$ and $\delta(\phi - \phi_0)$ are Dirac delta functions, and $f(\mu)$ are $g(\phi)$ are respectively functions yet to be determined of (μ) and (ϕ). Inserting Equation 47 into Equation 46 we obtain

$$\pi\mu_0 F_0 = \int_0^1 \mu\, f(\mu)\, \delta(\mu - \mu_0)\, d\mu \int_0^{2\pi} g(\phi)\, \delta(\phi - \phi_0)\, d\phi = \mu_0\, f(\mu_0)\, g(\phi_0) \tag{48}$$

From Equations 47 and 48 the downward intensities at the top of the cloud become

$$I(0, -\mu, \phi) = \pi F_0\, \delta(\mu - \mu_0)\, \delta(\phi - \phi_0) \ . \tag{49}$$

The total intensity $I(\tau, \mu, \phi)$ associated with Equation 45 is the sum of the intensity $I_D(\tau, \mu, \phi)$ arising from the diffuse radiation field and the intensity directly transmitted from the point source to the level τ. By analogy with Equation 49 this latter intensity may be expressed in the form

$$I_T(\tau, \mu, \phi) = \pi F_0\, \delta(\mu + \mu_0)\, \delta(\phi - \phi_0)\, h(\tau) \ , \tag{50}$$

where $h(\tau)$ is a function of τ alone yet to be determined. The total intensity then becomes

$$I(\tau, \mu, \phi) = I_D(\tau, \mu, \phi) + \pi F_0\, \delta(\mu + \mu_0)\, \delta(\phi - \phi_0)\, h(\tau) \ , \tag{51}$$

and the equation of transfer (Equation 45) after some reduction becomes

$$\begin{aligned} \mu \frac{dI_D(\tau, \mu, \phi)}{d\tau} + \pi\mu F_0\, \delta(\mu + \mu_0)\, \delta(\phi - \phi_0) \frac{dh(\tau)}{d\tau} &= I_D(\tau, \mu, \phi) + \pi F_0\, \delta(\mu + \mu_0)\, \delta(\phi - \phi_0)\, h(\tau) \\ &- \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{+1} p(\mu, \phi;\ \mu', \phi')\, I_D(\tau, \mu', \phi')\, d\mu'\, d\phi' - \frac{1}{4} F_0\, h(\tau)\, p(\mu, \phi;\ -\mu_0 \phi_0) - (1 - \tilde{\omega}_0)\, B(T) \ . \end{aligned} \tag{52}$$

We suppose that $dh(\tau)/d\tau$ and $h(\tau)$ are in general non-zero. Then in the limit as $\mu \to -\mu_0$ and $\phi \to \phi_0$, Equation 52 becomes

$$\lim_{\substack{\mu \to -\mu_0 \\ \phi \to \phi_0}} \left[\pi F_0\, \mu\, \delta(\mu + \mu_0)\, \delta(\phi - \phi_0) \frac{dh(\tau)}{d\tau}\right] = \lim_{\substack{\mu \to -\mu_0 \\ \phi \to \phi_0}} \left[\pi F_0\, \delta(\mu + \mu_0)\, \delta(\phi - \phi_0)\, h(\tau)\right] \tag{53}$$

yielding

$$h(\tau) = c_0 e^{-\tau/\mu_0} , \tag{54}$$

where c_0 is the constant of integration. Upon replacing $h(\tau)$ in Equation 50 with its equivalent from Equation 54 and letting $\tau \to 0$, it is observed that (cf. Equation 49)

$$c_0 \equiv 1 . \tag{55}$$

In general then the equation of transfer Equation 52 for the *diffuse* radiation field becomes

$$\mu \frac{dI_D(\tau, \mu, \phi)}{d\tau} = I_D(\tau, \mu, \phi) - \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{+1} p(\mu, \phi; \mu', \phi') I_D(\tau, \mu', \phi') d\mu' d\phi'$$

$$- \frac{1}{4} F_0 e^{-\tau/\mu_0} p(\mu, \phi; -\mu_0, \phi_0) - (1 - \tilde{\omega}_0) B(T) . \tag{56}$$

The contribution to the total intensity due to the *directly* transmitted flux may be obtained from (cf. Equations 46 and 50)

$$F(\tau) = F_0 e^{-\tau/\mu_0} . \tag{57}$$

It should be clear that Equation 52 could be modified to include n point sources by including n terms in the source function of the form

$$- \frac{1}{4} F_0^{(i)} e^{-\tau/\mu_0^{(i)}} p(\mu, \phi; -\mu_0^{(i)}, \phi_0^{(i)}) , \tag{58}$$

where (i) refers to the i^{th} point source. However, this shall be considered to be beyond the scope of this paper, even though in the far infrared terms of this form could be included to account for isolated emitting sources, such as discrete cumulus-type clouds. In any event Equation 56 will be the basic equation of transfer considered in this paper. Solutions to Equation 56 will be sought which describe the radiation field at the surfaces of plane-parallel clouds of arbitrary optical thickness which are bounded on both sides by a vacuum. These solutions will hereafter be referred to as solutions to the *restricted problem.* The solutions to the restricted problem will then be incorporated as functions in the formulation of certain integral equations which will describe the *general problem;* i.e., each vacuum will be replaced by a realistic atmospheric model which will include any ground or outside cloud effects, where relevant. This concludes the formulation of the basic equation of radiative transfer.

THE RESTRICTED PROBLEM

Introduction

The restricted problem deals with the evaluation of the angular distribution of radiation and net flux at the surfaces of a plane-parallel slab of particulate medium of arbitrary optical thickness which is bounded on both sides by a vacuum. There are two major tasks involved:

1. the determination of the angular distribution of radiation diffusely reflected from and transmitted through the slab which is irradiated by an outside point source, and

2. the determination of the angular distribution of radiation emitted by the slab from sources of radiation internal to the slab.

It will be assumed that these two fields of radiation do not interfere and thus may be calculated independently of each other.

The basic equation of radiative transfer which the total diffuse radiation field must obey is (Equation 56)

$$\mu \frac{dI(\tau, \mu, \phi)}{d\tau} = I(\tau, \mu, \phi) - \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} p(\mu, \phi; \mu', \phi')\, I(\tau, \mu', \phi')\, d\mu'\, d\phi'$$

$$- \frac{1}{4} F_0\, e^{-\tau/\mu_0}\, p\left(\mu, \phi; -\mu_0, \phi_0\right) - \left(1 - \tilde{\omega}_0\right) B(\tau) \quad . \qquad (59)$$

It is understood in Equation 59 that $I(\tau, \mu, \phi)$ refers to the *diffuse* radiation field; hence the subscript D is dropped, as it will be throughout the rest of this paper. It should also be realized that all parameters in Equation 59 except the angular coordinates are dependent upon the frequency of radiation; therefore Equation 59 is valid only over a frequency interval in which all these parameters are sensibly constant. And finally, the Planck function $B(T)$ is replaced by $B(\tau)$ (ν understood), which implies that the temperature T may be uniquely expressed as a function of τ alone.

The Equation of Transfer in the n^{th} Approximation

The method of solution that will be followed throughout this section is the method of discrete ordinates in the n^{th} approximation. Most of the work has already been well described by Chandrasekhar (References 3 and 7), and we shall for the most part simply list the results and indicate the necessary equations which the reader would need in order to obtain numerical solutions. Some derivations are needed, and these are contained for the most part in Appendices A, B, and C. In particular, a derivation for the particular integral needed to complete the solution to the equation of transfer containing the Planck function is new, and this is given in Appendix B.

In the method of discrete ordinates the continuous radiation field is approximated by $2n$ linearly independent beams of radiation. Each of these beams is characterized by a unique direction

μ_i, and there are a total of $2n$ of these directions, n each in the upward and downward directions. The discrete values of μ_i $(i = \pm 1, \cdots, \pm n)$ are picked such that the integral over μ in the source function of the equation of transfer may "most accurately" be evaluated by a quadrature formula containing $2n$ terms which is chosen for that purpose. The formula which will be considered in this paper is of the form

$$\sum_i a_i \, f(\mu_i) \approx \int_{-1}^{+1} f(\mu) \, d\mu , \tag{60}$$

where the a_i's are the weights appropriate to a quadrature formula based on the division μ_i of the interval (-1, +1). This specific formula picked by Chandrasekhar is due to Gauss, where the values of μ_i are the roots of the even-degree Legendre polynomial $P_{2n}(\mu)$, and the values of a_i are determined from the expression

$$a_i = \left[\left(\frac{dP_{2n}(\mu)}{d\mu}\right)_{\mu=\mu_i}\right]^{-1} \int_{-1}^{+1} \frac{P_{2n}(\mu)}{\mu - \mu_i} \, d\mu \qquad (i = \pm 1, \cdots, \pm n) . \tag{61}$$

The identities

$$a_i = a_{-i} \quad \text{and} \quad \mu_i = -\mu_{-i} \tag{62}$$

are valid for all values of n.

This use of the Gaussian formula was criticized by Kourganoff (Reference 4) because it approximates an integration over the whole interval (-1, +1), and the integral of interest (Equation 59) is discontinuous at the boundaries of the cloud at $\mu = 0$. King and Florence (Reference 8) have recommended the Sykes double-Gauss method of fitting the Gaussian formula separately over the ranges (-1, 0) and (0, +1). The merits of this method appear to be quite sound and should be investigated more thoroughly.

Another approximation which will be incorporated consists of replacing the phase function $p(\mu, \phi; \mu', \phi')$ determined from the Mie theory with a finite Legendre polynomial series expansion of the argument, $\cos\Theta$, where Θ is the scattering angle; i.e., it is required that (Equation 4)

$$p(\mu, \phi; \mu', \phi') = \mathcal{P}(\cos\Theta) \approx \sum_{l=0}^{N} \tilde{\omega}_l \, P_l(\cos\Theta) , \tag{63}$$

where $\tilde{\omega}_0$ is the albedo for single scattering and the values of $\tilde{\omega}_l$ $(l = 1, \cdots, N)$ are constants independent of Θ which provide a "best" fit to Equation 63 in the sense of least squares. A proof is given in Appendix A showing that the series approximation in Equation 63 is normalized to $\tilde{\omega}_0$.

From Equation 5 it is seen that

$$P_l(\cos\Theta) = P_l\left[\mu\mu' + \left(1-\mu^2\right)^{1/2}\left(1-\mu'^2\right)^{1/2}\cos(\phi'-\phi)\right] . \tag{64}$$

Expanding Equation 64 in accordance with the addition theorem of spherical harmonics we obtain

$$P_l(\cos\Theta) = P_l(\mu)P_l(\mu') + 2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\mu)P_l^m(\mu')\cos m(\phi'-\phi) . \tag{65}$$

If it is now assumed that the intensity in Equation 59 may be written in the form

$$I(\tau,\mu,\phi) = I^{(0)}(\tau,\mu) + \sum_{m=1}^{N} I^{(m)}(\tau,\mu)\cos m\left(\phi_0-\phi\right) , \tag{66}$$

it can be verified (Reference 7) by direct substitution that Equation 59 splits up into the $(N+1)$ independent equations

$$\mu\frac{dI^{(o)}(\tau,\mu)}{d\tau} = I^{(0)}(\tau,\mu) - \frac{1}{2}\sum_{l=0}^{N}\tilde{\omega}_l P_l(\mu)\int_{-1}^{+1}P_l(\mu')I^{(0)}(\tau,\mu')d\mu'$$

$$-\frac{1}{4}F_0 e^{-\tau/\mu_0}\sum_{l=0}^{N}(-1)^l\tilde{\omega}_l P_l(\mu)P_l\left(\mu_0\right) - \left(1-\tilde{\omega}_0\right)B(\tau) \tag{67}$$

and

$$\mu\frac{dI^{(m)}(\tau,\mu)}{d\tau} = I^{(m)}(\tau,\mu) - \frac{1}{2}\sum_{l=m}^{N}\frac{(l-m)!}{(l+m)!}\tilde{\omega}_l P_l^m(\mu)\int_{-1}^{+1}P_l^m(\mu')I^{(m)}(\tau,\mu')d\mu'$$

$$-\frac{1}{2}F_0 e^{-\tau/\mu_0}\sum_{l=m}^{N}(-1)^{l+m}\frac{(l-m)!}{(l+m)!}\tilde{\omega}_l P_l^m(\mu)P_l^m\left(\mu_0\right) \qquad (m = 1, \cdots, N) . \tag{68}$$

At this stage it will be expedient to define the form of the Planck function, $B(\tau)$. It is not unreasonable to suppose that in any realistic physical situation $B(\tau)$ is sufficiently smooth and well-behaved so that it may be expressed exactly by an explicit infinite power series in τ. A term-by-term differentiation of this power series would be expected to correspond exactly to the derivative of the original function.

In keeping with the spirit of approximating the phase function for single scattering by a series expansion containing a finite number of terms, N, we shall correspondingly approximate the Planck function by a power series also containing N terms. The function $B(\tau)$ in Equation 67 is then replaced by

$$B(\tau) \approx \sum_{r=0}^{N} b_r \tau^r , \tag{69}$$

where the values of b_r ($r = 0, \cdots, N$) are constants independent of τ which provide the "best" fit to $B(\tau)$ in the sense of least squares.

In the method of discrete ordinates the integrals in Equations 67 and 68 are replaced by sums according to Gauss's quadrature formula, and each of the $(N + 1)$ equations is replaced by an equivalent system of linear equations of order $2n$. Solutions must be sought in approximations n such that

$$4n - 1 > 2N . \tag{70}$$

The $2n(N+1)$ linear inhomogeneous differential equations which replace the $(N+1)$ linear inhomogeneous integro-differential equations given by Equations 67 and 68 are

$$\mu_i \frac{dI^{(0)}(\tau, \mu_i)}{d\tau} = I^{(0)}(\tau, \mu_i) - \frac{1}{2}\sum_{l=0}^{N} \tilde{\omega}_l P_l(\mu_i)\left[\sum_j a_j P_l(\mu_j) I^{(0)}(\tau, \mu_j)\right]$$

$$- \frac{1}{4} F_0 e^{-\tau/\mu_0} \sum_{l=0}^{N} (-1)^l \tilde{\omega}_l P_l(\mu_i) P_l(\mu_0) - \left(1 - \tilde{\omega}_0\right) \sum_{r=0}^{N} b_r \tau^r$$

$$(i = \pm 1, \cdots, \pm n;\ j = \pm 1, \cdots, \pm n) \tag{71}$$

and

$$\mu_i \frac{dI^{(m)}(\tau, \mu_i)}{d\tau} = I^{(m)}(\tau, \mu_i) - \frac{1}{2}\sum_{l=m}^{N} \frac{(l-m)!}{(l+m)!} \tilde{\omega}_l P_l^m(\mu_i)\left[\sum_j a_j P_l^m(\mu_j) I^{(m)}(\tau, \mu_j)\right]$$

$$- \frac{1}{2} F_0 e^{-\tau/\mu_0} \sum_{l=m}^{N} (-1)^{l+m} \frac{(l-m)!}{(l+m)!} \tilde{\omega}_l P_l^m(\mu_i) P_l^m(\mu_0)$$

$$(i = \pm 1, \cdots, \pm n;\ j = \pm 1, \cdots, \pm n;\ m = 1, \cdots, N). \tag{72}$$

The complete solution to the system of Equations 71 and 72 will involve $2n(N+1)$ constants of integration which can be determined from the $2n(N+1)$ boundary conditions

$$I^{(0)}(0, \mu_{-i}) = I^{(m)}(0, \mu_{-i}) = 0 \tag{73}$$

and

$$I^{(0)}\left(\tau_1, \mu_i\right) = I^{(m)}\left(\tau_1, \mu_i\right) = 0 \qquad (i = 1, \cdots, n;\ m = 1, \cdots, N) \ , \tag{74}$$

where τ_1 is the normal optical thickness of the cloud. If the cloud is semi-infinite in extent the $n(N + 1)$ boundary conditions*

$$e^{-\tau} I^{(0)}\left(\tau, \mu_i\right) \to e^{-\tau} I^{(m)}\left(\tau, \mu_i\right) \to 0 \quad \text{as} \quad \tau \to \infty \qquad (i = 1, \cdots, n;\ m = 1, \cdots, N) \tag{75}$$

replace those of Equation 74 in order to secure the boundedness of the solution.

The Complete Solution

The complete solution to Equations 71 and 72 is required where the frequency of radiation is such that neither radiation from the sun nor thermal emission from the cloud can be neglected. Since both Equations 71 and 72 are linear, this solution may be obtained by first solving the homogeneous parts of the equations, and then finding a particular integral, which, when added to the general solutions, satisfies the complete system of inhomogeneous equations.

The Azimuth-Independent Solution.

The solution to the homogeneous part of Equation 71 is (Reference 7)

$$I_G^{(0)}\left(\tau, \mu_i\right) = \sum_{\alpha=1}^{n} \frac{M_\alpha e^{-k_\alpha \tau}}{1 + \mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l \left(+ k_\alpha\right) P_l \left(\mu_i\right)\right] + \sum_{\alpha=1}^{n} \frac{M_{-\alpha} e^{+k_\alpha \tau}}{1 - \mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l \left(- k_\alpha\right) P_l \left(\mu_i\right)\right] \qquad (i = \pm 1, \cdots, \pm n) \ , \tag{76}$$

*Often it is not clear that these conditions are sufficient, although they are always necessary. The actual criteria involved are that the formal solutions to the equation of transfer (Reference 3)

$$I(0, \mu_i) = \int_0^\infty J(\tau, \mu_i)\, e^{-\tau/\mu_i} \frac{d\tau}{\mu_i} \qquad (i = 1, \cdots, n) \ ,$$

where $J(\tau, \mu_i)$ is the source function, do not become unbounded as $\tau \to \infty$. For solutions in the n^{th} approximation, $J(\tau, \mu_i)$ contains terms of the form (Equations 71, 72, 92 and 98) $M_{-\alpha} e^{+k_\alpha \tau}$ and $M_{-\alpha}^{m} e^{+k_\alpha \tau}$ which, when multiplied by $(1/\mu_i)\, e^{-\tau/\mu_i}$ and integrated over all τ, do *not* become unbounded for all values of k_α and k_α^m (Equations 79 and 104), as has been verified numerically by the author for several specific cases ($\tilde{\omega}_0 \neq 0$). Thus the requirements

$$M_{-\alpha} = M_{-\alpha}^{m} = 0 \qquad (\alpha = 1, \cdots, n)$$

needed to make all constants of integration determinate are not automatically insured by Equation 75. However, for all single scattering phase functions expressible as a finite series of Legendre polynomials, the form of the solutions are the same, and the requirements $M_{-\alpha} = M_{-\alpha}^{m} = 0$ ($\alpha = 1, \cdots, n$) shall be expected to hold for semi-infinite clouds in general.

where $M_{\pm\alpha}$ are the 2n constants of integration, and the quantities $\xi_l(x)$ are defined, apart from an arbitrary multiplying constant, by the relations

$$\xi_l(x) = \sum_{\lambda=0}^{N} \tilde{\omega}_\lambda \xi_\lambda(x) D_{l,\lambda}(x) \qquad (l = 0, \cdots, N) \tag{77}$$

where

$$D_{l,\lambda}(x) = \frac{1}{2} \sum_j \frac{a_j P_l(\mu_j) P_\lambda(\mu_j)}{1 + \mu_j x} \qquad (j = \pm 1, \cdots, \pm n) . \tag{78}$$

The arbitrariness may be removed by defining $\xi_0(x)$ to be unity. Upon combining Equations 77 and 78, and setting $l = 0$, the characteristic equation for k results:

$$1 = \frac{1}{2} \sum_j \frac{a_j \sum_{\lambda=0}^{N} \tilde{\omega}_\lambda \xi_\lambda(k) P_\lambda(\mu_i)}{1 + \mu_j k} . \tag{79}$$

This equation is of order n in k^2 and admits, in general, 2n distinct nonvanishing roots which must occur in pairs as $\pm k_\alpha (\alpha = 1, \cdots, n)$. It can be shown that the ξ_λ's obey the recursion relation

$$\xi_{l+1} = -\frac{2l + 1 - \tilde{\omega}_l}{k(l+1)} \xi_l - \frac{l}{l+1} \xi_{l-1} , \tag{80}$$

where

$$\xi_0 = 1 \quad ; \quad \xi_\lambda = 0 \quad (\lambda < 0) . \tag{81}$$

Equation 80 may be used to generate $\xi_l (l = 0, \cdots, N)$ as a function of k, and Equation 79 subsequently solved for the 2n roots $\pm k_\alpha (\alpha = 1, \cdots, n)$. In practice, for large N, it is probably more practical to solve Equations 79 and 80 interdependently on an electronic computer.

In order to complete the solution, a particular integral must be found which, when added to Equation 76, satisfies Equation 71. In practice it will be easiest to find two particular integrals by setting first one, and then the other term of the inhomogeneous part of Equation 71 equal to zero. The sum of these two integrals thus found will then comprise the particular integral sought.

If the term in Equation 79 containing $\sum_{r=0}^{N} b_r \tau^r$ is suppressed, the resultant particular integral (Reference 7) is

$$I_i(\mu_0) = \frac{1}{4} F_0 \frac{\gamma_0 e^{-\tau/\mu_0}}{1 + \mu_i/\mu_0} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l \left(\frac{1}{\mu_0} \right) P_l(\mu_i) \right] \qquad (i = \pm 1, \cdots, \pm n) , \tag{82}$$

where

$$\gamma_0 = H(\mu_0)\,H(-\mu_0) \tag{83}$$

and

$$H(x) = \frac{1}{\mu_1 \cdots \mu_n} \frac{\prod_{i=1}^{n}(x+\mu_i)}{\prod_{\alpha=1}^{n}(1+k_\alpha x)} . \tag{84}$$

Alternatively, if instead, the term containing $e^{-\tau/\mu_0}$ is suppressed, the resultant particular integral may be expressed as

$$I_i\left[B(\tau)\right] = \sum_{r=0}^{N} \tau^r \left[\sum_{s=r}^{N} C_{r,s}\, P_{s-r}(\mu_i)\right] , \tag{85}$$

where the values of $C_{r,s}$ are constants (independent of τ and μ_i) which may be written as explicit functions of b_r and $\tilde{\omega}_l$ alone. The complete set of relations required to generate the various values of $C_{r,s}$ ($r = 0, \cdots, N$; $s = r, \cdots, N$) are as follows:

$$C_{N,N} = b_N , \tag{86}$$

$$C_{N-1,N-1} = b_{N-1} , \tag{87}$$

$$C_{r,r} = b_r + \frac{(r+1)}{3\left(1-\tilde{\omega}_0\right)} C_{r+1,r+2} \qquad (r = 0, \cdots, N-2) , \tag{88}$$

$$C_{r,N} = \frac{\left[\dfrac{(r+1)(N-r)}{2(N-r)-1}\right]}{\left[1-\dfrac{\tilde{\omega}_{N-r}}{2(N-r)+1}\right]} C_{r+1,N} \qquad (r = 0, \cdots, N-1) , \tag{89}$$

$$C_{r,N-1} = \frac{\left[\dfrac{(r+1)(N-r-1)}{2(N-r)-3}\right]}{\left[1-\dfrac{\tilde{\omega}_{N-r-1}}{2(N-r)-1}\right]} C_{r+1,N-1} \qquad (r = 0, \cdots, N-2) , \tag{90}$$

and

$$C_{r,s} = \frac{(r+1)}{\left[1-\dfrac{\tilde{\omega}_{s-r}}{2(s-r)+1}\right]} \left[\frac{(s-r+1)}{2(s-r)+3} C_{r+1,s+2} + \frac{(s-r)}{2(s-r)-1} C_{r+1,s}\right] \quad (r = 0, \cdots, N-3;\ s = r+1, \cdots, N-2) . \tag{91}$$

A derivation of Equations 86 through 91 and a systematic method of generating the various values of $C_{r,s}$ are given in Appendix B.

The two integrals given by Equations 82 and 85, when added to the general solution in Equation 76, comprise the complete solution to Equation 71, and this is

$$I^{(0)}(\tau, \mu_i) = \sum_{\alpha=1}^{n} \frac{M_\alpha e^{-k_\alpha \tau}}{1 + \mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l(+k_\alpha) P_l(\mu_i) \right] + \sum_{\alpha=1}^{n} \frac{M_{-\alpha} e^{+k_\alpha \tau}}{1 - \mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l(-k_\alpha) P_l(\mu_i) \right]$$

$$+ \frac{1}{4} F_0 \frac{\gamma_0 e^{-\tau/\mu_0}}{1 + \mu_i/\mu_0} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l\left(\frac{1}{\mu_0}\right) P_l(\mu_i) \right] + \sum_{r=0}^{N} \tau^r \left[\sum_{s=r}^{N} C_{r,s} P_{s-r}(\mu_i) \right]$$

$$(i = \pm 1, \cdots, \pm n) . \qquad (92)$$

The $2n$ constants of integration M_α ($\alpha = \pm 1, \cdots, \pm n$) are to be determined in accordance with the $2n$ boundary conditions

$$I^{(0)}(0, \mu_{-i}) = I^{(0)}(\tau_1, \mu_i) = 0 \qquad (i = 1, \cdots, n) . \qquad (93)$$

In the case of conservative scattering $(\tilde{\omega}_0 = 1)$ it is shown in Reference 7 that the solution to Equation 71 can be written in the form

$$I^{(0)}(\tau, \mu_i) = \frac{1}{4} F_0 \left\{ \sum_{\alpha=1}^{n-1} \frac{M_\alpha e^{-k_\alpha \tau}}{1 + \mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l(+k_\alpha) P_l(\mu_i) \right] + \sum_{\alpha=1}^{n-1} \frac{M_{-\alpha} e^{+k_\alpha \tau}}{1 - \mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l(-k_\alpha) P_l(\mu_i) \right] \right.$$

$$\left. + \frac{\gamma_0 e^{-\tau/\mu_0}}{1 + \mu_i/\mu_0} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l\left(\frac{1}{\mu_0}\right) P_l(\mu_i) \right] + 3\mu_0 \left[\left\{ \left(1 - \frac{1}{3}\tilde{\omega}_1\right) \tau + \mu_i \right\} M_0 + M_n \right] \right\}$$

$$(i = \pm 1, \cdots, \pm n) , \qquad (94)$$

where M_α ($\alpha = \pm 1, \cdots, \pm[n-1]$), M_0, and M_n are the $2n$ constants of integration to be determined in accordance with Equation 93. The summations over α are restricted to $(n-1)$ terms, since Equation 79 in this case admits only $(2n-2)$ distinct and nonvanishing roots.

If the cloud is semi-infinite in extent, the boundedness of the solution requires that the terms containing the constants $M_{-\alpha}$ in Equations 92 and 94 all be suppressed (cf. Equation 75). An application of the flux integral reveals further that M_0 in Equation 94 becomes zero ~~unity~~ in this case, as is demonstrated in Appendix C. The solutions contained in Equations 92 and 94 in the semi-infinite case then become respectively

$$I^{(0)}(\tau, \mu_i) = \sum_{\alpha=1}^{n} \frac{M_\alpha e^{-k_\alpha \tau}}{1 + \mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l(+k_\alpha) P_l(\mu_i) \right] + \frac{1}{4} F_0 \frac{\gamma_0 e^{-\tau/\mu_0}}{1 + \mu_i/\mu_0} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l\left(\frac{1}{\mu_0}\right) P_l(\mu_i) \right]$$

$$+ \sum_{r=0}^{N} \tau^r \left[\sum_{s=r}^{N} C_{r,s} P_{s-r}(\mu_i) \right] \qquad (\tilde{\omega}_0 \neq 1;\ i = \pm 1, \cdots, \pm n) \qquad (95)$$

and

$$I^{(0)}(\tau, \mu_i) = \frac{1}{4} F_0 \left\{ \sum_{\alpha=1}^{n-1} \frac{M_\alpha e^{-k_\alpha \tau}}{1 + \mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l (+k_\alpha) P_l(\mu_i) \right] + \frac{\gamma_0 e^{-\tau \mu_0}}{1 + \mu_i/\mu_0} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l \left(\frac{1}{\mu_0}\right) P_l(\mu_i) \right] \right.$$

see errata

$$\left. + 3\mu_0 \left[\left(1 - \frac{1}{3} \tilde{\omega}_1\right) \tau + \mu_i + M_n \right] \right\} \quad (\tilde{\omega}_0 = 1;\; i = \pm 1, \cdots, \pm n) \ . \qquad (96)$$

The constants M_α ($\alpha = 1, \cdots, n$) in Equation 95, and M_α ($\alpha = 1, \cdots, n-1$) and M_n in Equation 96, are to be determined in accordance with the n boundary conditions

$$I^{(0)}(0, \mu_{-i}) = 0 \qquad (i = 1, \cdots, n) \ . \qquad (97)$$

The Azimuth-Dependent Solutions.

In a fashion entirely analogous to that of obtaining the solution of Equation 71 it can be shown (Reference 7) that the complete solutions to the azimuth-dependent equations (Equations 72) can be written in the form

$$I^{(m)}(\tau, \mu_i) = \frac{1}{2} F_0 P_m^m(\mu_0) \left\{ \sum_{\alpha=1}^{n} \frac{M_\alpha^m e^{-k_\alpha^m \tau}}{1 + \mu_i k_\alpha^m} \left[\sum_{l=m}^{N} \tilde{\omega}_l \frac{(l-m)!}{(l+m)!} \xi_l^m (+k_\alpha^m) P_l^m(\mu_i) \right] \right.$$

$$+ \sum_{\alpha=1}^{n} \frac{M_{-\alpha}^m e^{+k_\alpha^m \tau}}{1 - \mu_i k_\alpha^m} \left[\sum_{l=m}^{N} \tilde{\omega}_l \frac{(l-m)!}{(l+m)!} \xi_l^m (-k_\alpha^m) P_l^m(\mu_i) \right]$$

$$\left. + \frac{\gamma_0^m e^{-\tau/\mu_0}}{1 + \mu_i/\mu_0} \left[\sum_{l=m}^{N} \tilde{\omega}_l \frac{(l-m)!}{(l+m)!} \xi_l^m \left(\frac{1}{\mu_0}\right) P_l^m(\mu_i) \right] \right\} \quad (i = \pm 1, \cdots, \pm n;\; m = 1, \cdots, N), \qquad (98)$$

where the $2nN$ constants $M_{\pm\alpha}^m$ are to be found from the conditions

$$I^{(m)}(0, \mu_{-i}) = I^{(m)}(\tau_1, \mu_i) = 0 \qquad (i = 1, \cdots, n;\; m = 1, \cdots, N) \ , \qquad (99)$$

and τ_1 is again the normal optical thickness of the cloud. If the cloud is semi-infinite in extent, the boundedness of the solutions requires that the terms in Equation 98 containing $M_{-\alpha}^m$ all be suppressed. In this case the solutions become

$$I^{(m)}(\tau, \mu_i) = \frac{1}{2} F_0 P_m^m(\mu_0) \left\{ \sum_{\alpha=1}^{n} \frac{M_\alpha^m e^{-k_\alpha^m \tau}}{1 + \mu_i k_\alpha^m} \left[\sum_{l=m}^{N} \tilde{\omega}_l \frac{(l-m)!}{(l+m)!} \xi_l^m (+k_\alpha^m) P_l^m(\mu_i) \right] \right.$$

$$\left. + \frac{\gamma_0^m e^{-\tau/\mu_0}}{1 + \mu_i/\mu_0} \left[\sum_{l=m}^{N} \tilde{\omega}_l \frac{(l-m)!}{(l+m)!} \xi_l^m \left(\frac{1}{\mu_0}\right) P_l^m(\mu_i) \right] \right\} \quad (i = \pm 1, \cdots, \pm n;\; m = 1, \cdots, N) \ , \qquad (100)$$

where the nN constants M_α^m are found from the conditions

$$I^{(m)}\left(0, \mu_{-i}\right) = 0 \qquad (i = 1, \cdots, n;\ m = 1, \cdots, N)\ .. \tag{101}$$

The quantities $\xi_l^m(x)$ are defined, apart from an arbitrary multiplying constant, by the relations

$$\xi_l^m(x) = \sum_{\lambda=m}^{N} \tilde{\omega}_\lambda \frac{(\lambda - m)!}{(\lambda + m)!} \xi_\lambda^m(x) D_{l,\lambda}^m(x) \qquad (l = m, \cdots, N;\ m = 1, \cdots, N)\ , \tag{102}$$

where

$$D_{l,\lambda}^m(x) = \frac{1}{2} \sum_j \frac{a_j P_l^m(\mu_j) P_\lambda^m(\mu_j)}{1 + \mu_j x} \qquad (j = \pm 1, \cdots, \pm n)\ . \tag{103}$$

The arbitrariness is removed upon defining $\xi_m^m(x)$ to be unity. The resulting characteristic equation, upon setting $l = m$, becomes

$$1 = \frac{1}{2} \sum_j \left[\frac{a_j \sum_{\lambda=m}^{N} \tilde{\omega}_\lambda \frac{(\lambda - m)!}{(\lambda + m)!} \xi_\lambda^m(k^m) P_m^m(\mu_j) P_\lambda^m(\mu_j)}{1 + \mu_j k^m} \right] \qquad (j = \pm 1, \cdots, \pm n)\ . \tag{104}$$

Equation 104 is of order n in $(k^m)^2$ and admits, in general, $2n$ distinct nonvanishing roots which must occur in pairs as

$$\pm k_\alpha^m \qquad (\alpha = 1, \cdots, n;\ m = 1, \cdots, N)\ .$$

It may also be verified that the ξ_λ^m's obey the recursion relation

$$\xi_{l+1}^m = -\frac{2l + 1 - \tilde{\omega}_l}{k^m(l - m + 1)} \xi_l^m - \frac{l + m}{l - m + 1} \xi_{l-1}^m\ , \tag{105}$$

where

$$\xi_m^m = 1\ ; \qquad \xi_\lambda^m = 0 \quad (\lambda < m)\ . \tag{106}$$

As before, Equation 105 may be used to generate the functions of $\xi_\lambda^m(k^m)$ required for the evaluation of the roots of Equation 104. Once these roots have been obtained, the values of $\xi_\lambda^m(k^m)$ follow from Equation 105.

The values of γ_0^m in Equation 98 are made determinate from the relations

$$\gamma_0^m = H^{(m)}(\mu_0)\, H^{(m)}(-\mu_0) \qquad (m = 1, \cdots, N) , \tag{107}$$

where

$$H^{(m)}(x) = \frac{1}{\mu_1 \cdots \mu_n} \frac{\prod_{i=1}^{n} (x + \mu_i)}{\prod_{a=1}^{n} (1 + k_a^m x)} . \tag{108}$$

The Angular Distribution of Outgoing Radiation

It is required to evaluate Equation 66 for the angular distribution of outgoing radiation at the surfaces ($\tau = 0$ and $\tau = \tau_1$) of the cloud, and this may be accomplished through nothing more than simple substitution. The form of the solution will depend upon the number and type of sources of radiation involved. It will be convenient in the present context to treat each source separately and add the separate resultant solutions to obtain the final solution. This will entail no loss in generality and will circumvent any ambiguities which might arise later in the treatment of the general problem (in the next section), where the bounding vacuums in the restricted problem are replaced by more realistic physical conditions. The only assumption required is that the separate radiation fields do not interfere with each other.

In order to reduce the space required to write the various solutions, the following artifices will be employed:

1. The azimuth-independent and dependent terms will be combined by extending the summations over m to include ($m = 0, \cdots, N$); e.g., in this abbreviated form Equation 66 becomes, in the scheme of the n^{th} approximation,

$$I(\tau, \mu_i, \phi) = \sum_{m=0}^{N} I^{(m)}(\tau, \mu_i) \cos m(\phi_0 - \phi) \qquad (i = \pm 1, \cdots, \pm n) . \tag{109}$$

2. The symbols $\delta_{\beta,\lambda}$ and $\tilde{\omega}_l^m$ will be respectively defined by

$$\delta_{\beta,\lambda} = \begin{cases} 1 & (\lambda = \beta) \\ 0 & (\lambda \neq \beta) \end{cases} \tag{110}$$

and

$$\tilde{\omega}_l^m = \tilde{\omega}_l \frac{(l - m)!}{(l + m)!} . \tag{111}$$

3. The definition

$$k_{-\alpha}^{m} = -k_{\alpha}^{m} \qquad (\alpha = 1, \cdots, n) \tag{112}$$

will be employed, and the symbol $\left(\sum_{\alpha}\right)$ will be understood to mean the summation over α, where the range will be specified in accordance with the relevant solution.

With these definitions in mind we now turn to a description of the angular distribution of radiation at the boundaries of a plane-parallel cloud within the context of the restricted problem.

Diffuse Reflection and Transmission.

The solutions relevant to radiation from an outside point source which is diffusely reflected from a cloud of normal optical thickness τ_1 are (cf. Equations 92, 94, 98, and 109 through 112), for $\tilde{\omega}_0 \neq 1$:

$$I(0, \mu_i, \phi) = \frac{1}{4} F_0 \sum_{m=0}^{N} \left(2 - \delta_{0,m}\right) P_m^m(\mu_0) \cos m(\phi_0 - \phi) \left\{ \sum_{\alpha} \frac{M_{\alpha}^{m}}{1 + \mu_i k_{\alpha}^{m}} \left[\sum_{l=m}^{N} \tilde{\omega}_l^m \xi_l^m (k_{\alpha}^{m}) P_l^m(\mu_i) \right] \right.$$

$$\left. + \frac{\gamma_0^m}{1 + \mu_i/\mu_0} \left[\sum_{l=m}^{N} \tilde{\omega}_l^m \xi_l^m \left(\frac{1}{\mu_0}\right) P_l^m(\mu_i) \right] \right\} \qquad (\alpha = \pm 1, \cdots, \pm n;\ i = 1, \cdots, n)\ , \tag{113}$$

and, for $\tilde{\omega}_0 = 1$:

$$I(0, \mu_i, \phi) = \frac{1}{4} F_0 \sum_{m=0}^{N} \left(2 - \delta_{0,m}\right) P_m^m(\mu_0) \cos m(\phi_0 - \phi) \left\{ \sum_{\alpha} \left(1 - \delta_{n,|\alpha|-m}\right) \frac{M_{\alpha}^{m}}{1 + \mu_i k_{\alpha}^{m}} \left[\sum_{l=m}^{N} \tilde{\omega}_l^m \xi_l^m (k_{\alpha}^{m}) P_l^m(\mu_i) \right] \right.$$

$$\left. + \frac{\gamma_0^m}{1 + \mu_i/\mu_0} \left[\sum_{l=m}^{N} \tilde{\omega}_l^m \xi_l^m \left(\frac{1}{\mu_0}\right) P_l^m(\mu_i) \right] + 3\, \delta_{0,m} \mu_0 \left[M_0 \mu_i + M_n\right] \right\} \qquad (\alpha = \pm 1, \cdots, \pm n;\ i = 1, \cdots, n)\ . \tag{114}$$

If the cloud is semi-infinite in extent, the range of summation over α in Equations 113 and 114 must be restricted to ($\alpha = 1, \cdots, n$), and the constant M_0 in Equation 114 identified with unity (cf. Equations 95 and 96). In all cases the usual relevant boundary conditions (Equations 99 and 101) have been imposed in determining the various constants of integration.

The corresponding solutions for the radiation diffusely transmitted through a cloud of normal optical thickness τ_1 through analogous considerations become respectively, for $\tilde{\omega}_0 \neq 1$:

$$I(\tau_1, -\mu_i, \phi) = \frac{1}{4} F_0 \sum_{m=0}^{N} \left(2 - \delta_{0,m}\right) P_m^m(\mu_0) \cos m(\phi_0 - \phi) \times \left\{ \sum_{\alpha} \frac{M_{\alpha}^{m} e^{-k_{\alpha}^{m}\tau_1}}{1 - \mu_i k_{\alpha}^{m}} \left[\sum_{l=m}^{N} \tilde{\omega}_l^m \xi_l^m (k_{\alpha}^{m}) P_l^m(-\mu_i) \right] \right.$$

$$\left. + \frac{\gamma_0^m e^{-\tau_1/\mu_0}}{1 - \mu_i/\mu_0} \left[\sum_{l=m}^{N} \tilde{\omega}_l^m \xi_l^m \left(\frac{1}{\mu_0}\right) P_l^m(-\mu_i) \right] \right\} \qquad (\alpha = \pm 1, \cdots, \pm n;\ i = 1, \cdots, n)\ , \tag{115}$$

and, for $\tilde{\omega}_0 = 1$:

$$I(\tau_1, -\mu_i, \phi) = \frac{1}{4} F_0 \sum_{m=0}^{N} (2-\delta_{0,m}) P_m^m(\mu_0) \cos m(\phi_0 - \phi) \times \left\{ \sum_{\alpha} (1-\delta_{n,|\alpha|-m}) \frac{M_\alpha e^{-k_\alpha^m \tau_1}}{1-\mu_i k_\alpha^m} \left[\sum_{l=m}^{N} \tilde{\omega}_l^m \xi_l^m (k_\alpha^m) P_l^m(-\mu_i) \right] \right.$$

$$\left. + \frac{\gamma_0^m e^{-\tau_1/\mu_0}}{1-\mu_i/\mu_0} \left[\sum_{l=m}^{N} \tilde{\omega}_l^m \xi_l^m \left(\frac{1}{\mu_0}\right) P_l^m(-\mu_i) \right] + 3\delta_{0,m}\mu_0 \left[\left\{ \left(1 - \frac{1}{3}\tilde{\omega}_1\right)\tau_1 - \mu_i \right\} M_0 + M_n \right] \right\} \quad (\alpha = \pm 1, \cdots, \pm n;\ i = 1, \cdots, n) \,. \tag{116}$$

Thermal Emission and The Law of Darkening.

In view of the fact that no outside sources contribute to the diffuse radiation field in our present context, the solution for the angular distribution of outgoing radiation from a thermally emitting cloud reduces to an axially symmetric one, and the azimuth-dependent terms in Equation 109 are all suppressed. From Equation 92 it is seen that the solutions at the top and bottom of a cloud of normal optical thickness τ_1 are, respectively, for $\tau = 0$:

$$I(0, \mu_i) = \sum_{\alpha} \frac{M_\alpha}{1+\mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l (k_\alpha) P_l(\mu_i) \right] + \sum_{s=0}^{N} C_{0,s} P_s(\mu_i)$$

$$(\alpha = \pm 1, \cdots, \pm n;\ i = 1, \cdots, n) \,, \tag{117}$$

and, for $\tau = \tau_1$:

$$I(\tau_1, -\mu_i) = \sum_{\alpha} \frac{M_\alpha e^{-k_\alpha \tau_1}}{1-\mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l (k_\alpha) P_l(-\mu_i) \right] + \sum_{r=0}^{N} \tau_1^r \left[\sum_{s=r}^{N} C_{r,s} P_{s-r}(-\mu_i) \right]$$

$$(\alpha = \pm 1, \cdots, \pm n;\ i = 1, \cdots, n) \,. \tag{118}$$

If the cloud is semi-infinite in extent, it can be seen from Equation 95 that the range of integration over α in Equation 117 must be restricted to ($\alpha = 1, \cdots, n$).

It is clear from Equation 67 that in the special case of conservative scattering ($\tilde{\omega}_0 = 1$) in a semi-infinite cloud the law of darkening is independent of the temperature. Since no sources or sinks are available to the radiation field the net flux is constant (independent of τ). In terms of this constant net flux πF the solution to Equation 71, upon suppressing the inhomogeneous terms, can be shown to be (Reference 3)

$$I(\tau, \mu_i) = \frac{3}{4} F \left\{ \sum_{\alpha=1}^{n-1} \frac{M_\alpha e^{-k_\alpha \tau}}{1+\mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l (+k_\alpha) P_l(\mu_i) \right] + \left(1 - \frac{1}{3}\tilde{\omega}_1\right)\tau + \mu_i + M_n \right\} \quad (i = \pm 1, \cdots, \pm n) \,, \tag{119}$$

from which it follows that the law of darkening is given by

$$I(0, \mu_i) = \frac{3}{4} F \left\{ \sum_{\alpha=1}^{n-1} \frac{M_\alpha}{1+\mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l (+k_\alpha) P_l(\mu_i) \right] + \mu_i + M_n \right\} \quad (i = 1, \cdots, n) \,. \tag{120}$$

The values of ξ_l and k_α may be determined from Equations 79 and 80 and the constants M_α ($\alpha = 1, \cdots, n-1$) and M_n from the n boundary conditions

$$I(0, -\mu_i) = 0 \quad (i = 1, \cdots, n) . \tag{121}$$

It should be noted that any time-independent solution for the angular distribution of conservatively scattered radiation in a cloud of finite optical thickness is meaningless.

Molecular Effects

In general it may be supposed that the atmosphere within the cloud contributes to the absorption and thermal emission of radiation, but not significantly to the scattering. Any mass element dm in the cloud then absorbs radiation through both particle absorption and absorption by the free molecules contained in dm.

Let E be the radiant energy incident on dm from all directions in unit time which is extinguished (absorbed plus scattered) by dm. If g is that fraction of E which is extinguished by the particles contained in dm, then (1 - g) must be that fraction of E which is absorbed by the free molecules in dm. If E_P is the amount of energy extinguished by the particles in unit time, then

$$E_P^S = \tilde{\omega}_0 E_P = \tilde{\omega}_0 g E \tag{122}$$

is the amount of energy scattered by these particles in unit time, and

$$E_P^A = (1 - \tilde{\omega}_0) E_P = (1 - \tilde{\omega}_0) g E \tag{123}$$

is the amount of energy absorbed by the particles in unit time. It is further clear that

$$E_M^A = (1 - g) E \tag{124}$$

must be the energy which is absorbed by the free molecules in dm in unit time. Upon adding Equations 123 and 124 it is seen that the total amount of energy absorbed by dm in unit time must be

$$E_P^A + E_M^A = (1 - \tilde{\omega}_0 g) E . \tag{125}$$

The ratio of the energy scattered in unit time to the energy extinguished in unit time, i.e., the effective albedo for single scattering of dm, is given by (Equation 122)

$$\frac{E_P^S}{E} = \tilde{\omega}_0 g . \tag{126}$$

If further, local thermodynamic equilibrium is assumed to prevail in dm, the rate ($\mathcal{J}$) of emitted energy in accordance with Kirchhoff's law is seen from Equation 125 to be

$$\mathcal{J} \;=\; \left(1 - \tilde{\omega}_0\, g\right) B(\tau) \quad . \tag{127}$$

And finally, since the effective extinction cross-section of dm has been increased by a factor of $(1/g)$, the differential optical thickness $d\tau$ becomes (Equation 43)

$$d\tau \;=\; -\frac{1}{g}\, N_0\, \chi_E\, dz \quad . \tag{128}$$

If g is independent of τ, it should be clear from Equations 42 and 126 through 128 that, upon including the effects of free molecules upon the radiation field, a system of solutions results which is equivalent in every respect to the system of solutions discussed previously, merely by replacing χ_E and $\tilde{\omega}_l$ ($l = 0, \ldots, N$) everywhere with $(1/g)\chi_E$ and $\tilde{\omega}_l\, g$ ($l = 0, \ldots, N$) respectively.

THE GENERAL PROBLEM

Introduction

In order to assess properly the radiation leaving a planetary atmosphere, the interplay of radiation with the gaseous atmosphere and aerosols, as well as with clouds, must be taken into account except perhaps for limited regions of the electromagnetic spectrum. To further complicate the picture ground effects must be considered where clouds are optically thin.

Several points of view are possible in any treatment of a physical problem. Ours shall be one directed toward (though not restricted to) practical numerical solutions connected with certain physically realistic problems associated with the transfer of radiation through the Earth's cloudy atmosphere. Since our interest is primarily utilitarian in nature, it will be necessary to be somewhat restrictive in the formulation and solution of the problems considered. In particular the nine approximations enumerated in the Introduction will be maintained.

Two basic problems are considered in this section. The first problem considered is the determination of the intensity distribution of short wavelength radiation diffusely reflected from an optically thin Rayleigh scattering atmosphere overlying an optically thick cloud. The second problem is the determination of the intensity distribution of long wavelength radiation at the top of an atmosphere containing an optically thin cloud. In the first problem an outside point source (e.g., the sun) is assumed to illuminate the atmosphere-cloud composite from above. In the second problem the radiation is assumed to arise from the atmosphere-cloud-ground composite by thermal emission associated with a characteristic temperature T at an optical depth τ. In the latter problem the ground is assumed to emit radiation in accordance with its characteristic blackbody temperature T. In neither case is a contribution due to aerosols considered, although an obvious extension of the problem could be carried out to include them. In view of the utilitarian purpose of this paper it is felt that further extensions of this sort (including optically thin clouds at short wavelengths) would lead us too far from the main theme, since the numerical solutions would rapidly become overwhelmingly complex.

Diffuse Reflection and Transmission

Consider a plane-parallel slab of a gaseous or particulate medium of normal optical thickness τ_1 bounded on both sides by a vacuum. We shall want to define a scattering function

$$S(\tau_1;\ \mu, \phi;\ \mu_0, \phi_0)$$

and a transmission function

$$T(\tau_1;\ \mu, \phi;\ \mu_0, \phi_0)$$

such that the diffusely reflected intensity of radiation at the top of the slab is given by

$$I(0, \mu, \phi) = \frac{F_0}{4\mu} S(\tau_1;\ \mu, \phi;\ \mu_0, \phi_0) , \tag{129}$$

and the intensity of radiation diffusely transmitted through the slab at the bottom of the slab is given by

$$I(\tau_1, -\mu, \phi) = \frac{F_0}{4\mu} T(\tau_1;\ \mu, \phi;\ \mu_0, \phi_0) . \tag{130}$$

These intensities are solutions to the so-called restricted problem. The radiation in Equations 129 and 130 is assumed to be originally in the direction $(-\mu_0, \phi_0)$ and thence redirected respectively into the directions (μ, ϕ) and $(-\mu, \phi)$, where the cosine of the zenith angle μ is in the range $(0 \leq \mu \leq 1)$ and the azimuthal angle ϕ is in the range $(0 \leq \phi \leq 2\pi)$. The quantity F_0 is related to the net flux πF_0 normal to the incident beam of radiation.

It should be noted that the reflected and transmitted intensities refer only to the radiation which has suffered one or more scattering processes. For example, if we denote the direction of incident radiation by $(-\mu_0, \phi_0)$, $I(\tau_1, -\mu, \phi)$ does not include the directly transmitted flux $\pi F_0 e^{-\tau_1/\mu_0}$.

The particular forms of the scattering and transmission functions in Equations 129 and 130 were chosen from the following considerations:

If the intensity of an outside point source is given by Equation 49, it follows from the flux integral (Equation 8) that the net flux diffusely reflected by the slab is

$$\pi F_s = \pi F_0 \int_\omega S(\tau_1;\ \mu, \phi;\ \mu_0, \phi_0) \frac{d\omega}{4\pi} , \tag{131}$$

and the net flux diffusely transmitted through the slab is

$$\pi F_T = \pi F_0 \int_\omega T\left(\tau_1;\ \mu,\ \phi;\ \mu_0,\ \phi_0\right) \frac{d\omega}{4\pi} \ , \tag{132}$$

where, from the boundary conditions,

$$S\left(\tau_1;\ -\mu,\ \phi;\ \mu_0,\ \phi_0\right) = T\left(\tau_1;\ -\mu,\ \phi;\ \mu_0,\ \phi_0\right) = 0 \quad (0 < \mu \leq 1) \ , \tag{133}$$

and the integrations are to be performed over all solid angles. Thus $S\left(\tau_1;\ \mu,\ \phi;\ \mu_0,\ \phi_0\right)$ and $T\left(\tau_1;\ \mu,\ \phi;\ \mu_0,\ \phi_0\right)$ are normalized so that their integrals are respectively a measure of the fluxes diffusely reflected from and transmitted through the slab relative to the flux of incoming radiation crossing a unit area normal to the incoming beam.

The scattering and transmission functions as defined obey Helmholtz's principle of reciprocity; i.e.,

$$S\left(\tau_1;\ \mu,\ \phi;\ \mu',\ \phi'\right) = S\left(\tau_1;\ \mu',\ \phi';\ \mu,\ \phi\right) \tag{134}$$

and

$$T\left(\tau_1;\ \mu,\ \phi;\ \mu',\ \phi'\right) = T\left(\tau_1;\ \mu',\ \phi';\ \mu,\ \phi\right) \ , \tag{135}$$

where the more general primed quantities replace the explicit values μ_0 and ϕ_0. With these definitions in mind we turn now to the formulation and solution of the integral relations which express the intensity of diffusely reflected radiation at the top of the atmosphere in terms of the restricted problems.

Formulation of the General Problem.

The solutions to the restricted problems suggested by Equations 129 and 130 will be presumed to be known. For example, Chandrasekhar (Reference 3) has given the restricted solutions to the problem of diffuse reflection and transmission for a plane-parallel slab of gaseous atmosphere, and the comparable solutions for a plane-parallel cloud layer have been indicated in the previous section of this paper.

The problem now is to unite the solutions of the restricted problems with the scattering and transmission functions which arise as formal solutions to the "composite" problem in such a way as to allow a tractable solution for these functions. It will turn out that two interdependent linear integral equations plus one linear algebraic equation suffice to formulate the problem. Figure 3 illustrates the role of each intensity component which is considered in the following discussion.

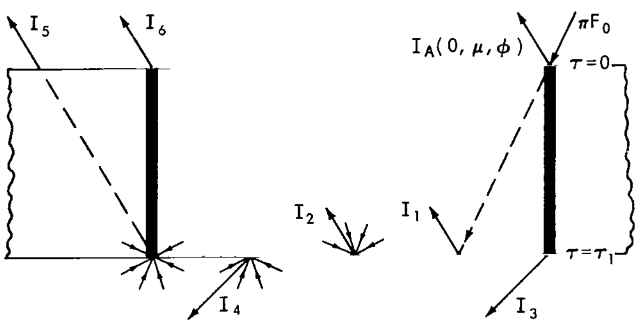

Figure 3 — The intensity components arising through the interaction between an incident beam of parallel radiation of flux πF_0 and a model of the atmosphere-cloud composite. The tops of the cloud and atmosphere are situated respectively at $\tau = \tau_1$ and $\tau = 0$.

Let

$$I_A(0, \mu, \phi) = \frac{F_0}{4\mu} S_A(\tau_1; \mu, \phi; \mu_0, \phi_0) \tag{136}$$

and

$$I_A(\tau_1, -\mu, \phi) = \frac{F_0}{4\mu} T_A(\tau_1; \mu, \phi; \mu_0, \phi_0) \tag{137}$$

be the restricted solutions for the intensities at the top and bottom of a slab of gas respectively as derived by Chandrasekhar (Reference 3).

Further let

$$I_C(0, \mu, \phi) = \frac{F_0}{4\mu} S_C(\mu, \phi; \mu_0, \phi_0) \tag{138}$$

be the restricted solution for the intensity at the top of a semi-infinitely thick cloud due to diffuse reflection from the cloud.

In what follows the subscript "A" shall always refer to the atmosphere above the cloud and the subscript "C" for the cloud itself. All quantities with an asterisk will refer to unknown intensities arising from the interplay of radiation between the cloud and atmosphere. All quantities without an asterisk will refer to known intensities arising as solutions to the "restricted" problems. All quantities with (~) above them indicate a reversal of direction, or a so-called change of parity; for example, if T denotes the transmission function through a slab in one direction, $\tilde{T}$ denotes the transmission function in the opposite direction.

Let $I_C^*(\tau_1, \mu, \phi)$ be the intensity in the direction (μ, ϕ) at the top of the cloud arising from the interaction of radiation with both the atmosphere and cloud, and $I_A^*(\tau_1, -\mu, \phi)$ be the corresponding intensity at the top of the cloud in the direction $(-\mu, \phi)$. The normal optical thickness τ_1 of the atmosphere is the optical depth at which these intensities are located.

The following equations are somewhat more general than those of Chandrasekhar, where he considers the more restrictive case of an isotropically scattering ground in his discussion of the "planetary problem" (Reference 3, page 269). The derivations of the formulas parallel his.

$I_C^*(\tau_1, \mu, \phi)$ will be composed of two parts. One part is due to the radiation directly transmitted through the atmosphere before being diffusely reflected by the cloud. Symbolically we may indicate this component as

$$I_1 = e^{-\tau_1/\mu_0} I_C(0, \mu, \phi) \quad , \tag{139}$$

where the multiplying factor $e^{-\tau_1/\mu_0}$ is due to the attenuation the directly transmitted radiation undergoes in traversing the atmosphere of normal optical thickness τ_1.

The second component is due to the intensity of radiation from any direction in the atmosphere incident on the cloud top which is diffusely reflected by the cloud into the direction (μ, ϕ). Symbolically

$$I_2 = \frac{1}{4\pi\mu} \int_0^{2\pi}\int_0^1 S_C(\mu, \phi; \mu', \phi') I_A^*(\tau_1, -\mu', \phi') \, d\mu' \, d\phi' \ . \tag{140}$$

The integration is performed over all solid angles in the upward hemisphere in order to account for the radiation from the sky incident on the cloud from all directions. Note that $I_A^*(\tau_1, -\mu', \phi')$ is the "unknown" intensity of sky radiation at the cloud top in the direction $(-\mu', \phi')$.

The intensity $I_A^*(\tau_1, -\mu, \phi)$ is similarly composed of two parts. One component is due to the intensity of radiation diffusely transmitted through the atmosphere in the direction $(-\mu, \phi)$ which originates with the main source (e.g., the sun). Symbolically we have

$$I_3 = I_A(\tau_1, -\mu, \phi) = \frac{F_0}{4\mu} T_A(\tau_1; \mu, \phi; \mu_0, \phi_0) \quad , \tag{141}$$

the transmitted intensity derived from the restricted problem.

The second component is due to the intensity of radiation $I_C^*(\tau_1, \mu', \phi')$ in the direction (μ', ϕ') arising from the cloud top, which is diffusely reflected by the atmosphere into the direction $(-\mu, \phi)$. Symbolically we have

$$I_4 = \frac{1}{4\pi\mu} \int_0^{2\pi}\int_0^1 \tilde{S}_A(\tau_1; -\mu, \phi; -\mu', \phi') \, I_C^*(\tau_1, \mu', \phi') \, d\mu' \, d\phi' \quad , \tag{142}$$

where the integration is carried out over all solid angles in the upward hemisphere.

Upon collecting Equations 139 through 142, we obtain two independent integral equations relating the intensities $I_C^*(\tau_1, \mu, \phi)$ and $I_A^*(\tau_1, -\mu, \phi)$ at the top of the cloud, namely

$$I_C^*(\tau_1, \mu, \phi) = \frac{F_0}{4\mu} e^{-\tau_1/\mu_0} S_C(\mu, \phi; \mu_0, \phi_0) + \frac{1}{4\pi\mu}\int_0^{2\pi}\int_0^1 S_C(\mu, \phi; \mu', \phi') I_A^*(\tau_1, -\mu', \phi') \, d\mu' \, d\phi' \quad (143)$$

and

$$I_A^*(\tau_1, -\mu, \phi) = \frac{F_0}{4\mu} T_A(\tau_1; \mu, \phi; \mu_0, \phi_0) + \frac{1}{4\pi\mu}\int_0^{2\pi}\int_0^1 \tilde{S}_A(\tau_1; -\mu, \phi; -\mu', \phi') I_C^*(\tau_1, \mu', \phi') \, d\mu' \, d\phi' \quad . (144)$$

Upon inserting Equation 144 into 143 we obtain

$$I_C^*(\tau_1, \mu, \phi) = \frac{F_0}{4\mu} e^{-\tau_1/\mu_0} S_C(\mu, \phi; \mu_0, \phi_0) + \frac{1}{4\pi\mu}\int_0^{2\pi}\int_0^1 S_C(\mu, \phi; \mu', \phi') \left\{ \frac{F_0}{4\mu'} T_A(\tau_1; \mu', \phi'; \mu_0, \phi_0) \right.$$

$$\left. + \frac{1}{4\pi\mu'}\int_0^{2\pi}\int_0^1 \tilde{S}_A(\tau_1; -\mu', \phi'; -\mu'', \phi'') I_C^*(\tau_1, \mu'', \phi'') \, d\mu'' \, d\phi'' \right\} d\mu' \, d\phi' \quad . \quad (145)$$

Equation 145 is an integral equation for $I_C^*(\tau_1, \mu, \phi)$ expressed in terms of the known scattering and transmission functions of the restricted solutions.

Finally, we must obtain the intensity of radiation in the direction (μ, ϕ) at the top of the atmosphere due to the presence of both the atmosphere and cloud. This intensity $I^*(0, \mu, \phi)$ is composed of three parts. The first part is the radiation from the source diffusely reflected by the atmosphere, $I_A(0, \mu, \phi)$, as given by the restricted solution. The second part is the intensity arising from the cloud top in the direction (μ, ϕ) directly transmitted through the atmosphere and attenuated by the amount $e^{-\tau_1/\mu}$. Symbolically this intensity is

$$I_5 = I_C^*(\tau_1, \mu, \phi) \, e^{-\tau_1/\mu} \quad . \quad (146)$$

The third component is the intensity of radiation arising from the cloud top in the direction (μ', ϕ') and diffusely transmitted through the atmosphere into the direction (μ, ϕ). Symbolically this intensity is

$$I_6 = \frac{1}{4\pi\mu}\int_0^{2\pi}\int_0^1 \tilde{T}_A\left(\tau_1;\ -\mu,\phi;\ -\mu',\phi'\right) I_C^*\left(\tau_1,\mu',\phi'\right)\, d\mu'\, d\phi' \quad . \tag{147}$$

Once Equation 145 has been solved for $I_C^*\left(\tau_1, \mu, \phi\right)$, the outgoing intensity $I^*(0, \mu, \phi)$ at the top of the atmosphere may be found with the aid of Equations 146 and 147 and the related discussion. Thus

$$I^*(0,\mu,\phi) = I_A(0,\mu,\phi) + I_C^*\left(\tau_1,\mu,\phi\right) e^{-\tau_1/\mu} + \frac{1}{4\pi\mu}\int_0^{2\pi}\int_0^1 \tilde{T}_A\left(\tau_1;\ -\mu,\phi;\ -\mu',\phi'\right) I_C^*\left(\tau_1,\mu',\phi'\right)\, d\mu'\, d\phi' \quad . \tag{148}$$

Equations 145 and 148 represent the formal solution to the outgoing intensity of diffusely reflected visible radiation in the direction (μ, ϕ) at the top of an optically thin atmosphere overlying an optically thick cloud.

Solution to the Problem.

The formulation of the problem is contained in Equations 145 and 148. In keeping with the definition of scattering functions we may define a scattering function such that (Equation 129)

$$I_C^*\left(\tau_1,\mu,\phi\right) = \frac{F_0}{4\mu} S_C^*\left(\tau_1;\ \mu,\phi;\ \mu_0,\phi_0\right) \quad . \tag{149}$$

Upon inserting Equation 149 into 145 we obtain

$$S_C^*\left(\tau_1;\ \mu,\phi;\ \mu_0\phi_0\right) = e^{-\tau_1/\mu_0} S_C\left(\mu,\phi;\ \mu_0,\phi_0\right) + \frac{1}{4\pi}\int_0^{2\pi}\int_0^1 S_C\left(\mu,\phi;\ \mu',\phi'\right)\frac{1}{\mu'} T_A\left(\tau_1;\mu',\phi';\ \mu_0,\phi_0\right) d\mu'\, d\phi'$$

$$+ \frac{1}{16\pi^2}\int_0^{2\pi}\int_0^1\int_0^{2\pi}\int_0^1 S_C\left(\mu,\phi;\ \mu',\phi'\right)\frac{1}{\mu'}\tilde{S}_A\left(\tau_1;-\mu',\phi';\ -\mu'',\phi''\right)\frac{1}{\mu''} S_C^*\left(\tau_1;\ \mu'',\phi'';\ \mu_0,\phi_0\right) d\mu''\, d\phi''\, d\mu'\, d\phi' \quad . \tag{150}$$

We shall require the solution for $S_C^*\left(\tau_1;\ \mu,\phi;\ \mu_0,\ \phi_0\right)$ from Equation 150.

The restricted solutions in Equation 150 have been solved in general in the so-called n^{th} approximation in the form (Equation 109)

$$I\left(\tau,\mu_i,\phi\right) = \sum_{m=0}^{N} I^{(m)}\left(\tau,\mu_i\right)\cos m\left(\phi_0-\phi\right) \qquad (i = \pm 1, \cdots, \pm n) \quad , \tag{151}$$

where $\mu_{\pm i}$ are the intervals appropriate to an n-point quadrature formula picked for the purpose of evaluating the integrals which appear in the source function of the equation of transfer.

In keeping with Equation 151 we re-write the relevant scattering and transmission functions in the form

$$S_C(\mu, \phi; \mu', \phi') = \sum_{k=0}^{N} S_C^{(k)}(\mu, \mu') \cos k(\phi' - \phi) \quad , \tag{152}$$

$$\tilde{S}_A\left(\tau_1; -\mu', \phi'; -\mu'', \phi''\right) = \sum_{l=0}^{N} \tilde{S}_A^{(l)}(-\mu', -\mu'') \cos l(\phi'' - \phi') \quad , \tag{153}$$

$$S_C^*\left(\tau_1; \mu, \phi; \mu_0, \phi_0\right) = \sum_{m=0}^{N} S_C^{*(m)}\left(\mu, \mu_0\right) \cos m\left(\phi_0 - \phi\right) \quad , \tag{154}$$

and

$$T_A\left(\tau_1; \mu', \phi'; \mu_0, \phi_0\right) = \sum_{n=0}^{N} T_A^{(n)}\left(\mu', \mu_0\right) \cos n\left(\phi_0 - \phi'\right) \quad , \tag{155}$$

where the dependence on τ_1 is not indicated merely for the sake of convenience.

Upon replacing the second term on the right-hand side of Equation 150 with $\mathcal{J}_1$, we have successively with the aid of Equations 152 and 155 the relations

$$\begin{aligned}
\mathcal{J}_1 &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^1 S_C(\mu, \phi; \mu', \phi') \frac{1}{\mu'} T_A\left(\tau_1; \mu', \phi'; \mu_0, \phi_0\right) d\mu'\, d\phi' \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^1 \left[\sum_{k=0}^{N} S_C^{(k)}(\mu, \mu') \cos k(\phi' - \phi)\right] \frac{1}{\mu'} \left[\sum_{n=0}^{N} T_A^{(n)}\left(\mu', \mu_0\right) \cos n\left(\phi_0 - \phi'\right)\right] d\mu'\, d\phi' \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^1 \frac{1}{\mu'}\left[\sum_{k=0}^{N}\sum_{n=0}^{N} S_C^{(k)}(\mu, \mu')\, T_A^{(n)}\left(\mu', \mu_0\right) \cos k(\phi' - \phi) \cos n\left(\phi_0 - \phi'\right)\right] d\mu'\, d\phi' \quad .
\end{aligned} \tag{156}$$

With the aid of the relation

$$\int_0^{2\pi} \cos k(\phi' - \phi) \cos n\left(\phi_0 - \phi'\right) d\phi' = \left\{\begin{matrix} 0 & (k \neq n) \\ \pi \cos n\left(\phi_0 - \phi\right) & (k = n \neq 0) \\ 2\pi & (k = n = 0) \end{matrix}\right\} \tag{157}$$

Equation 156 becomes successively

$$\mathcal{I}_1 = \frac{1}{4\pi}\int_0^1 \frac{1}{\mu'}\left[\sum_{k=0}^{N}\sum_{n=0}^{N} S_C^{(k)}(\mu,\mu')\, T_A^{(n)}(\mu',\mu_0)\int_0^{2\pi}\cos k(\phi'-\phi)\cos n(\phi_0-\phi')\, d\phi'\right] d\mu'$$

$$= \frac{1}{4}\sum_{n=0}^{N}\left[\int_0^1 (1+\delta_{0,n})\, S_C^{(n)}(\mu,\mu')\, T_A^{(n)}(\mu',\mu_0)\frac{d\mu'}{\mu'}\right]\cos n(\phi_0-\phi) \quad , \tag{158}$$

where

$$\delta_{0,\lambda} = \begin{Bmatrix} 1 & (\lambda = 0) \\ 0 & (\lambda \neq 0) \end{Bmatrix} \quad . \tag{159}$$

In like manner, for the third term on the right-hand side of Equation 150, denoted by $\mathcal{I}_2$, we obtain with the aid of Equations 152 through 154 successively the relations

$$\mathcal{I}_2 = \frac{1}{16\pi^2}\int_0^{2\pi}\int_0^1\int_0^{2\pi}\int_0^1 S_C(\mu,\phi;\mu',\phi')\frac{1}{\mu'}\tilde{S}_A(\tau_1;-\mu',\phi';-\mu'',\phi'')\frac{1}{\mu''} S_C^*(\tau_1;\mu'',\phi'';\mu_0,\phi_0)\, d\mu''\, d\phi''\, d\mu'\, d\phi'$$

$$= \frac{1}{16\pi^2}\int_0^{2\pi}\int_0^1\int_0^{2\pi}\int_0^1 \left[\sum_{k=0}^{N} S_C^{(k)}(\mu,\mu')\cos k(\phi'-\phi)\right]\frac{1}{\mu'}\left[\sum_{l=0}^{N}\tilde{S}_A^{(l)}(-\mu',-\mu'')\cos l(\phi''-\phi')\right]$$

$$\times \frac{1}{\mu''}\left[\sum_{m=0}^{N} S_C^{*(m)}(\mu'',\mu_0)\cos m(\phi_0-\phi'')\right] d\mu''\, d\phi''\, d\mu'\, d\phi'$$

$$= \frac{1}{16\pi^2}\int_0^{2\pi}\int_0^1\left\{\int_0^{2\pi}\int_0^1 \frac{1}{\mu'}\left[\sum_{k=0}^{N}\sum_{l=0}^{N} S_C^{(k)}(\mu,\mu')\,\tilde{S}_A^{(l)}(-\mu',-\mu'')\cos k(\phi'-\phi)\cos l(\phi''-\phi')\right] d\mu'\, d\phi'\right\}$$

$$\times \frac{1}{\mu''}\left[\sum_{m=0}^{N} S_C^{*(m)}(\mu'',\mu_0)\cos m(\phi_0-\phi'')\right] d\mu''\, d\phi'' \quad . \tag{160}$$

By analogy with the derivation of Equation 158, Equation 160 becomes successively

$$\mathcal{J}_2 = \frac{1}{16\pi}\int_0^{2\pi}\int_0^1 \left\{ \sum_{l=0}^{N} \left[\int_0^1 \left(1 + \delta_{0,l}\right) S_C^{(l)}(\mu, \mu')\, \tilde{S}_A^{(l)}(-\mu', -\mu'')\, \frac{d\mu'}{\mu'} \right] \cos l(\phi'' - \phi) \right\}$$

$$\times \frac{1}{\mu''}\left[\sum_{m=0}^{N} S_C^{*(m)}\left(\mu'', \mu_0\right) \cos m\left(\phi_0 - \phi''\right)\right] d\mu''\, d\phi''$$

$$= \frac{1}{16\pi}\int_0^1 \left\{ \int_0^{2\pi}\int_0^1 \frac{1}{\mu''}\left[\sum_{l=0}^{N} \left(1 + \delta_{0,l}\right) \sum_{m=0}^{N} S_C^{(l)}(\mu, \mu')\, \tilde{S}_A^{(l)}(-\mu', -\mu'')\, S_C^{*(m)}\left(\mu'', \mu_0\right) \right.\right.$$

$$\left.\left. \times \cos l(\phi'' - \phi) \cos m\left(\phi_0 - \phi''\right) \right] d\mu''\, d\phi'' \right\} \frac{d\mu'}{\mu'}$$

$$= \frac{1}{16}\sum_{l=0}^{N} \left\{ \left(1 + \delta_{0,l}\right)^2 \cos l\left(\phi_0 - \phi\right) \left[\int_0^1 S_C^{*(l)}\left(\mu'', \mu_0\right) \left(\int_0^1 S_C^{(l)}(\mu, \mu')\, \tilde{S}_A^{(l)}(-\mu', -\mu'') \frac{d\mu'}{\mu'} \right) \frac{d\mu''}{\mu''} \right] \right\} . \tag{161}$$

Upon replacing k, l, and m by n and utilizing Equations 152, 154, 158, and 161, Equation 150 becomes

$$\sum_{n=0}^{N} S_C^{*(n)}\left(\mu, \mu_0\right) \cos n\left(\phi_0 - \phi\right) = e^{-\tau_1/\mu_0} \sum_{n=0}^{N} S_C^{(n)}\left(\mu, \mu_0\right) \cos n\left(\phi_0 - \phi\right)$$

$$+ \frac{1}{4}\sum_{n=0}^{N} \left\{ \left(1 + \delta_{0,n}\right) \cos n\left(\phi_0 - \phi\right) \left[\int_0^1 S_C^{(n)}(\mu, \mu')\, T_A^{(n)}\left(\mu', \mu_0\right) \frac{d\mu'}{\mu'} \right] \right\}$$

$$+ \frac{1}{16}\sum_{n=0}^{N} \left\{ \left(1 + \delta_{0,n}\right)^2 \cos n\left(\phi_0 - \phi\right) \left[\int_0^1 S_C^{*(n)}\left(\mu'', \mu_0\right) \left(\int_0^1 S_C^{(n)}(\mu, \mu')\, \tilde{S}_A^{(n)}(-\mu', -\mu'') \frac{d\mu'}{\mu'} \right) \frac{d\mu''}{\mu''} \right] \right\} . \tag{162}$$

The exponential in the first term on the right-hand side of Equation 162 serves to remind us that each scattering and transmission function which is subscripted with an "A" or superscripted with an asterisk is an implicit function of the optical thickness τ_1 of the overlying atmosphere. Let us define

$$\int_0^1 S_C^{(n)}(\mu, \mu')\, T_A^{(n)}\left(\mu', \mu_0\right) \frac{d\mu'}{\mu'} = h_1^{(n)}\left(\mu, \mu_0\right) , \tag{163}$$

a "known" function of (μ, μ_0) for a given τ_1, and

$$\int_0^1 S_C^{(n)}(\mu, \mu')\, \widetilde{S}_A^{(n)}(-\mu', -\mu'')\, \frac{d\mu'}{\mu'} = h_2^{(n)}(\mu, \mu'') \quad , \tag{164}$$

a "known" function of (μ, μ'') for a given τ_1.

With the aid of Equations 163 and 164, Equation 162 may be re-written as

$$\sum_{n=0}^{N} S_C^{*(n)}(\mu, \mu_0) \cos n(\phi_0 - \phi) = e^{-\tau_1/\mu_0} \sum_{n=0}^{N} S_C^{(n)}(\mu, \mu_0) \cos n(\phi_0 - \phi)$$

$$+ \frac{1}{4} \sum_{n=0}^{N} (1 + \delta_{0,n})\, h_1^{(n)}(\mu, \mu_0) \cos n(\phi_0 - \phi)$$

$$+ \frac{1}{16} \sum_{n=0}^{N} (1 + \delta_{0,n})^2 \left[\int_0^1 h_2^{(n)}(\mu, \mu'')\, S_C^{*(n)}(\mu'', \mu_0)\, \frac{d\mu''}{\mu''}\right] \cos n(\phi_0 - \phi) \quad . \tag{165}$$

Equation 165 must be valid for all ϕ. We therefore have a set of $(N + 1)$ equations of the form

$$S_C^{*(n)}(\tau_1;\ \mu, \mu_0) = e^{-\tau_1/\mu_0} S_C^{(n)}(\mu, \mu_0) + \frac{1}{4}(1 + \delta_{0,n})\, h_1^{(n)}(\tau_1;\ \mu, \mu_0)$$

$$+ \frac{1}{16}(1 + \delta_{0,n})^2 \int_0^1 h_2^{(n)}(\tau_1;\ \mu, \mu'')\, S_C^{*(n)}(\tau_1;\ \mu'', \mu_0)\, \frac{d\mu''}{\mu''} \quad (n = 0, \cdots, N) \quad , \tag{166}$$

where the dependence of the various terms upon the parameter τ_1 has been re-indicated.

Equations 166 may be written in the more concise notation

$$S_C^{*(n)}(\tau_1;\ \mu, \mu_0) = f^{(n)}(\tau_1;\ \mu, \mu_0) + \int_0^1 g^{(n)}(\tau_1;\ \mu, \mu')\, S_C^{*(n)}(\tau_1;\ \mu', \mu_0)\, d\mu' \quad (n = 0, \cdots, N) \quad , \tag{167}$$

where

$$f^{(n)}(\tau_1;\ \mu, \mu_0) = e^{-\tau_1/\mu_0} S_C^{(n)}(\mu, \mu_0) + \frac{1}{4}(1 + \delta_{0,n})\, h_1^{(n)}(\tau_1;\ \mu, \mu_0) \quad , \tag{168}$$

$$g^{(n)}(\tau_1;\ \mu, \mu') = \frac{1}{16\mu'}(1 + \delta_{0,n})^2\, h_2^{(n)}(\tau_1;\ \mu, \mu') \quad , \tag{169}$$

and τ_1 and μ_0 are given parameters, constant for a given problem.

Equations 167 are formally examples of Fredholm's linear integral equation of the second kind and may be solved by standard methods. One method which lends itself well to the particular problem at hand is presented in Appendix D.

The final quantity which we wish to solve for is the outgoing intensity in the direction (μ, ϕ) at the top of the atmosphere given by (Equation 148)

$$I^*(0, \mu, \phi) = I_A(0, \mu, \phi) + e^{-\tau_1/\mu} I_C^*(\tau_1, \mu, \phi)$$

$$+ \frac{1}{4\pi\mu} \int_0^{2\pi}\int_0^1 \tilde{T}_A(\tau_1; -\mu, \phi; -\mu', \phi') \, I_C^*(\tau_1, \mu', \phi') \, d\mu' \, d\phi' \quad ,$$

or, in terms of the relevant scattering and transmission functions,

$$S^*(\tau_1; \mu, \phi; \mu_0, \phi_0) = S_A(\tau_1; \mu, \phi; \mu_0, \phi_0) + e^{-\tau_1/\mu} S_C^*(\tau_1; \mu, \phi; \mu_0, \phi_0)$$

$$+ \frac{1}{4\pi} \int_0^{2\pi}\int_0^1 \tilde{T}_A(\tau_1; -\mu, \phi; -\mu', \phi') \, \frac{1}{\mu'} \, S_C^*(\tau_1; \mu', \phi'; \mu_0, \phi_0) \, d\mu' \, d\phi' \quad , \tag{170}$$

where

$$I^*(0, \mu, \phi) = \frac{F_0}{4\mu} S^*(\tau_1; \mu, \phi; \mu_0, \phi_0) \; . \tag{171}$$

Upon denoting the third term on the right-hand side of Equation 170 by $\mathcal{I}_3$, we find with the aid of Equation 154 and the relation

$$\tilde{T}_A(\tau_1; -\mu, \phi; -\mu', \phi') = \sum_{n=0}^{N} \tilde{T}_A^{(n)}(-\mu, -\mu') \cos n(\phi' - \phi) \tag{172}$$

that $\mathcal{I}_3$ may be successively written as

$$\mathcal{I}_3 = \frac{1}{4\pi} \int_0^{2\pi}\int_0^1 \tilde{T}_A(\tau_1; -\mu, \phi; -\mu', \phi') \, \frac{1}{\mu'} \, S_C^*(\tau_1; \mu', \phi'; \mu_0, \phi_0) \, d\mu' \, d\phi'$$

$$= \frac{1}{4\pi} \int_0^{2\pi}\int_0^1 \left[\sum_{n=0}^{N} \tilde{T}_A^{(n)}(-\mu, -\mu') \cos n(\phi' - \phi)\right] \frac{1}{\mu'} \left[\sum_{m=0}^{N} S_C^{*(m)}(\mu', \mu_0) \cos m(\phi_0 - \phi')\right] d\mu' \, d\phi'$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^1 \frac{1}{\mu'}\left[\sum_{n=0}^{N}\sum_{m=0}^{N} \tilde{T}_A^{(n)}(-\mu, -\mu')\, S_C^{*(m)}(\mu', \mu_0)\cos n(\phi' - \phi)\cos m(\phi_0 - \phi')\right] d\mu'\, d\phi' \quad . \tag{173}$$

By analogy with the comparison of Equations 156 and 158, we obtain successively

$$\mathcal{I}_3 = \frac{1}{4}\int_0^1 \left[\sum_{n=0}^{N} (1 + \delta_{0,n})\, \tilde{T}_A^{(n)}(-\mu, -\mu')\, S_C^{*(n)}(\mu', \mu_0)\cos n(\phi_0 - \phi)\right]\frac{d\mu'}{\mu'} \tag{174}$$

or

$$\mathcal{I}_3 = \sum_{n=0}^{N} \mathcal{O}^{(n)}(\mu, \mu_0)\cos n(\phi_0 - \phi) \quad , \tag{175}$$

where

$$\mathcal{O}^{(n)}(\mu, \mu_0) = \frac{1}{4}(1 + \delta_{0,n})\int_0^1 \tilde{T}_A^{(n)}(-\mu, -\mu')\, S_C^{*(n)}(\mu', \mu_0)\frac{d\mu'}{\mu'} \quad . \tag{176}$$

From the manner in which it enters Equation 170, we are led to assume that the scattering function $S^*(\tau_1; \mu, \phi; \mu_0, \phi_0)$ may be written in the form

$$S^*(\tau_1; \mu, \phi; \mu_0, \phi_0) = \sum_{n=0}^{N} S^{*(n)}(\mu, \mu_0)\cos n(\phi_0 - \phi) \quad . \tag{177}$$

Upon gathering terms from Equations 170 and 175 we find with the aid of Equations 154 and 177 and the relation

$$S_A(\tau_1; \mu, \phi; \mu_0, \phi_0) = \sum_{n=0}^{N} S_A^{(n)}(\mu, \mu_0)\cos n(\phi_0 - \phi) \tag{178}$$

that Equation 170 may be written in the form

$$\sum_{n=0}^{N} S^{*(n)}(\mu, \mu_0)\cos n(\phi_0 - \phi) = \sum_{n=0}^{N} S_A^{(n)}(\mu, \mu_0)\cos n(\phi_0 - \phi)$$

$$+ e^{-\tau_1/\mu}\sum_{n=0}^{N} S_C^{*(n)}(\mu, \mu_0)\cos n(\phi_0 - \phi) + \sum_{n=0}^{N} \mathcal{O}^{(n)}(\mu, \mu_0)\cos n(\phi_0 - \phi) \quad . \tag{179}$$

Since Equation 179 must be valid for all ϕ, it separates into $(N+1)$ equations of the form

$$S^{*(n)}\left(\tau_1;\ \mu,\mu_0\right) = S_A^{(n)}\left(\tau_1;\ \mu,\mu_0\right) + e^{-\tau_1/\mu} S_C^{*(n)}\left(\tau_1;\ \mu,\mu_0\right) + \Theta^{(n)}\left(\tau_1;\ \mu,\mu_0\right) \quad (n = 0, \cdots, N), \tag{180}$$

where the dependence of the various terms on τ_1 has been re-indicated. A method in keeping with the context of the problem for evaluating the integral $\Theta^{(n)}(\tau_1;\mu,\mu_0)$ in Equation 176 is given in Appendix D.

One remark should be made at this point. If the solutions to the restricted problems have been determined in the n^{th} approximation, the solutions to the composite problem as indicated by Equations 167 and 180, for example, should also be solved in the n^{th} approximation (cf. Appendix D). This completes our discussion of the problem of diffuse reflection of visible radiation in an optically thick cloud plus overlying atmosphere composite.

The Law of Darkening

The second problem which we shall investigate is the problem of limb darkening in the infrared region of the spectrum at wavelengths of a few microns where an optically thin cloud is present. There is nothing in principle that makes the solution for the outgoing intensity more difficult for a finitely thick cloud as far as the restricted problem is concerned. One simply replaces the n boundary conditions (Equation 75)

$$I\left(\tau,\mu_i,\phi\right) e^{-\tau} \rightarrow 0 \text{ as } \tau \rightarrow \infty \qquad (i = 1, \cdots, n)$$

with the n boundary conditions

$$I\left(\tau_1,\mu_i,\phi\right) = 0 \qquad (i = 1, \cdots, n) ,$$

where τ_1 is the normal optical thickness of the cloud. It is now necessary to solve $2n$ linear algebraic equations for $2n$ unknowns in this case.

The problem becomes considerably more bulky when ground and atmospheric effects are incorporated into the results. However, if one is willing to admit the approximation that the ground emits as a blackbody at its characteristic temperature and that aerosols and dust in the atmosphere contribute a negligible amount of scattering, the problem reduces to a rather simple situation.

Let us suppose in a gross fashion that there are four contributions to the intensity $I^*(0,\mu;\tau_3)$ in the direction μ at the top of the atmosphere ($\tau = 0$). One contribution will be from the atmosphere above the cloud. The second will be from the cloud itself. The third will be from the atmosphere beneath the cloud. The fourth will be from the ground.

Before formulating the problem a slight change in notation should be indicated. We shall in general indicate an intensity as

$$I\left(\tau_\alpha, \mu; \tau_\gamma - \tau_\beta\right) ,$$

where τ_α refers to the optical depth at which the intensity is considered, μ refers to the cosine of the zenith angle of interest, and $(\tau_\gamma - \tau_\beta)$ refers to the optical thickness of the layer in question. Thus $I^*(0, \mu; \tau_3)$, the intensity of radiation in the direction μ at the top of the atmosphere ($\tau = 0$) as indicated above, refers to the total optical thickness τ_3 of the atmosphere-cloud composite. We shall specify that the normal optical depths ($\tau = 0, \tau_1, \tau_2, \tau_3$) refer respectively to the top of the atmosphere, top of the cloud, bottom of the cloud, and the ground level. The azimuth-independent nature of the radiation field as implied by the lack of specification of the azimuthal angle ϕ in our notation should be evident in view of the fact that the problem has been stated in the context of plane-parallel strata with no horizontal inhomogeneities.

One point should be clearly emphasized; as the problem has been stated there is no scattering of radiation except by the cloud itself. Thus aerosols, dust, and the ground are assumed to contribute no scattered energy to the radiation field. If this approximation is admitted in any particular case a considerable simplification results. We turn now to a formulation and solution of the problem of limb darkening in a cloudy atmosphere. Figure 4 illustrates the various intensity components which will arise in the course of our investigation.

The nine assumptions specified in the Introduction will be maintained in the following discussion. In what follows we shall first want to formulate the expression for the intensity $I_c^*(\tau_1, \mu; \tau_3 - \tau_1)$ at the top of the cloud in the direction $+\mu$. Once the expression for $I_c^*(\tau_1, \mu; \tau_3 - \tau_1)$ has been obtained, we shall want to use this expression to formulate properly the expression for the intensity $I^*(0, \mu; \tau_3)$ in the direction μ at the top of the atmosphere.

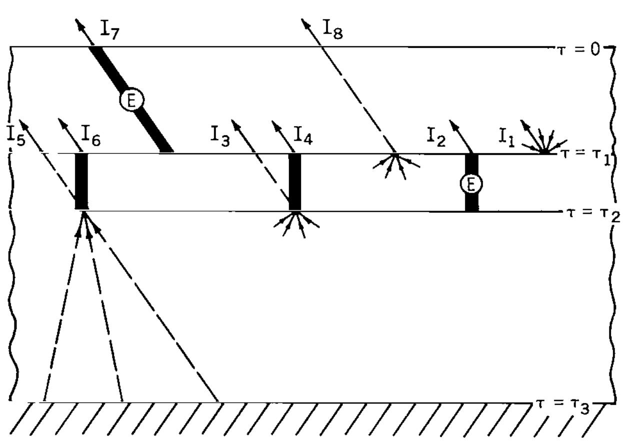

Figure 4 — The various intensity components arising through the interaction between an axially symmetric thermal radiation field and a model atmosphere-cloud-ground composite. The symbol Ⓔ refers to radiation originating in the relevant stratum. The tops of the upper atmosphere, cloud, lower atmosphere, and ground are situated respectively at $\tau = 0, \tau_1, \tau_2,$ and τ_3.

We proceed by considering first the contribution to the intensity $I_c^*(\tau_1, \mu; \tau_3 - \tau_1)$ from the atmosphere of optical thickness τ_1 above the cloud. This component will be the intensity integrated over all solid angles in the upward hemisphere which originates in the atmosphere above the cloud in the direction $(-\mu', \phi')$ and is diffusely reflected by the cloud into the direction (μ, ϕ). Upon integrating over ϕ' all the terms of the Fourier expansion in $\cos n(\phi' - \phi)$ (cf. Equations 152 and 157) drop out except for $n = 0$, leaving us with the expression

$$I_1 = \frac{1}{2\mu}\int_0^1 S_C^{(0)}\left(\mu, \mu'; \tau_2-\tau_1\right) I_A\left(\tau_1, -\mu'; \tau_1\right) d\mu' \quad , \tag{181}$$

where $I_A\left(\tau_1, -\mu'; \tau_1\right)$ is the axially symmetric solution of the restricted atmosphere of optical thickness τ_1 for the intensity at τ_1 in the direction $-\mu'$.

Note that the intensity $I_A\left(\tau_1, -\mu'; \tau_1\right)$ from the atmosphere above the cloud has no component due to the scattering of radiation in the atmosphere from a source (e.g., the cloud layer) outside this atmospheric layer. Of course radiation originating from the cloud, for example, is absorbed and re-emitted by the atmosphere above the cloud, and this will contribute to a change in the temperature profile in the atmosphere above the cloud. However, this temperature profile is presumed to be known (e.g., determined from radiosonde data) and thus $I_A\left(\tau_1, -\mu'; \tau_1\right)$ known (for a given mixing ratio of absorbing constituents, etc.).

The next contribution to the intensity $I_C^*\left(\tau_1, \mu; \tau_3-\tau_1\right)$ to be considered originates from within the cloud itself. But this component of the intensity is simply

$$I_2 = I_C\left(\tau_1, \mu; \tau_2-\tau_1\right) \quad , \tag{182}$$

the solution to the restricted problem (cf. Equation 117 and Appendix B).

The third contribution originates in the atmosphere below the cloud, and will consist of two components. The first component is the intensity directly transmitted through the cloud and thus reduced by the factor $e^{-(\tau_2-\tau_1)/\mu}$. The second component is the intensity integrated over all solid angles in the lower hemisphere originating in the direction (μ', ϕ') and diffusely transmitted through the cloud into the direction (μ, ϕ).

These components are respectively given by

$$I_3 = e^{-(\tau_2-\tau_1)/\mu} I_A\left(\tau_2, \mu; \tau_3-\tau_2\right) \tag{183}$$

and

$$I_4 = \frac{1}{2\mu}\int_0^1 \tilde{T}_C^{(0)}\left(-\mu, -\mu'; \tau_2-\tau_1\right) I_A\left(\tau_2, \mu'; \tau_3-\tau_2\right) d\mu' \quad . \tag{184}$$

Here the intensity $I_A\left(\tau_2, \mu; \tau_3-\tau_2\right)$ is the restricted solution for the pertinent layer of atmosphere below the cloud. The transmission function $\tilde{T}_C^{(0)}\left(-\mu, -\mu'; \tau_2-\tau_1\right)$ is the azimuth-independent solution of the restricted problem for the "inverted" cloud. In other words, if we were to turn the cloud upside down and irradiate the cloud from what is now "above" in the direction $-\mu'$, the appropriate transmission function becomes

$$I_C\left(\tau_1, -\mu;\ \tau_2-\tau_1\right) = \frac{F_0}{4\mu} T_C^{(0)}\left(\mu, \mu';\ \tau_2-\tau_1\right) = \frac{F_0}{4\mu} \tilde{T}_C^{(0)}\left(-\mu, -\mu';\ \tau_2-\tau_1\right) \quad . \tag{185}$$

The only reason for writing the transmission function as $\tilde{T}_C^{(0)}\left(-\mu, -\mu';\ \tau_2-\tau_1\right)$ is to remind ourselves that the parity, or sense of direction, is reversed from what it is normally.

The fourth and last contribution to the intensity $I_C^*\left(\tau_1, \mu;\ \tau_3-\tau_1\right)$ is due to the isotropically radiating ground, here considered to be a blackbody radiating in accordance with Kirchhoff's law. This contribution also consists of two components. The first component is the intensity directly transmitted through the atmosphere below the cloud and thence through the cloud itself in the direction (μ, ϕ). The second component is the intensity directly transmitted through the atmosphere below the cloud in the direction (μ', ϕ'), and thence diffusely transmitted through the "inverted" cloud into the direction (μ, ϕ).

Denoting the intensity of the ground by the constant I_G we have respectively for these two components:

$$I_5 = I_G\, e^{-(\tau_3-\tau_1)/\mu} \tag{186}$$

and

$$I_6 = \frac{1}{2\mu}\int_0^1 \tilde{T}_C^{(0)}\left(-\mu, -\mu';\ \tau_2-\tau_1\right) I_G\, e^{-(\tau_3-\tau_2)/\mu'}\, d\mu' \ , \tag{187}$$

where the integration in Equation 187 is once again implicitly performed over the lower hemisphere, thus accounting for the notation of the transmission function.

Upon adding up all the components I_1 through I_6 indicated in Equations 181 through 187 we obtain for the outgoing intensity at the top of the cloud the expression

$$\begin{aligned} I_C^*\left(\tau_1, \mu;\ \tau_3-\tau_1\right) &= I_C\left(\tau_1, \mu;\ \tau_2-\tau_1\right) + e^{-(\tau_2-\tau_1)/\mu} I_A\left(\tau_2, \mu;\ \tau_3-\tau_2\right) \\ &+ I_G\, e^{-(\tau_3-\tau_1)/\mu} + \frac{1}{2\mu}\int_0^1 \tilde{T}_C^{(0)}\left(-\mu, -\mu';\ \tau_2-\tau_1\right) I_A\left(\tau_2, \mu';\ \tau_3-\tau_2\right) d\mu' \\ &+ \frac{1}{2\mu}\int_0^1 \tilde{T}_C^{(0)}\left(-\mu, -\mu';\ \tau_2-\tau_1\right) I_G\, e^{-(\tau_3-\tau_2)/\mu'}\, d\mu' + \frac{1}{2\mu}\int_0^1 S_C^{(0)}\left(\mu, \mu';\ \tau_2-\tau_1\right) I_A\left(\tau_1, -\mu';\ \tau_1\right) d\mu' \ . \end{aligned} \tag{188}$$

The expression for the outgoing intensity $I^*\left(0, \mu;\ \tau_3\right)$ at the top of the atmosphere may now be derived. This intensity may be thought to consist of two components. The first component is due to thermal emission in the atmosphere above the cloud alone and is nothing more than the intensity corresponding to the restricted solution for such a layer of atmosphere. The second component is the intensity at the top of the cloud in the direction μ as given by Equation 188 reduced by the factor $e^{-\tau_1/\mu}$. These components are respectively

$$I_7 = I_A(0, \mu; \tau_1) \tag{189}$$

and

$$I_8 = I_C^*(\tau_1, \mu; \tau_3 - \tau_1)\, e^{-\tau_1/\mu} \quad . \tag{190}$$

The exponential in Equation 190 is required to account for the reduction in intensity suffered by the radiation originating at the cloud top in traversing an optical path length (τ_1/μ) through the atmosphere above the cloud.

Upon collecting terms from Equations 188 to 190 we finally obtain for the outgoing intensity in the direction μ at the top of the atmosphere the expression

$$\begin{aligned} I^*(0, \mu; \tau_3) = {} & I_A(0, \mu; \tau_1) + e^{-\tau_1/\mu} I_C(\tau_1, \mu; \tau_2 - \tau_1) + e^{-\tau_2/\mu} I_A(\tau_2, \mu; \tau_3 - \tau_2) \\ & + I_G\, e^{-\tau_3/\mu} + \frac{e^{-\tau_1/\mu}}{2\mu} \int_0^1 \Big\{ \tilde{T}_C^{(0)}(-\mu, -\mu'; \tau_2 - \tau_1) \left[I_A(\tau_2, \mu'; \tau_3 - \tau_2) \cdot I_G\, e^{-(\tau_3 - \tau_2)/\mu'} \right] \\ & + S_C^{(0)}(\mu, \mu'; \tau_2 - \tau_1)\, I_A(\tau_1, -\mu'; \tau_1) \Big\}\, d\mu' \quad . \end{aligned} \tag{191}$$

For a semi-infinite cloud Equation 191 reduces to

$$\begin{aligned} I^*(0, \mu) = {} & I_A(0, \mu; \tau_1) + e^{-\tau_1/\mu} I_C(\tau_1, \mu) \\ & + \frac{e^{-\tau_1/\mu}}{2\mu} \int_0^1 S_C^{(0)}(\mu, \mu')\, I_A(\tau_1, -\mu'; \tau_1)\, d\mu' \quad . \end{aligned} \tag{192}$$

Appendix D gives a convenient method for solving the integrals in Equations 191 and 192 by mechanical quadratures compatible with solutions to the restricted problems in the scheme of the n^{th} approximation. In this case, we would seek solutions of the form

$$I^*(0, \mu_i; \tau_3) \approx I^*(0, \mu; \tau_3) \qquad (\mu = \mu_i;\ i = 1, \cdots, n) \quad , \tag{193}$$

where μ_i are the intervals appropriate to n-point quadrature formula chosen for the purpose.

If we were to consider more than one scattering layer, such as two cloud decks for example, we would encounter a problem of somewhat more complexity than that of the first problem considered in this section—the problem of describing the outgoing visible radiation field diffusely reflected from a cloud-atmosphere composite. Since this would take us too far astray from our main theme, which is to discuss problems which are not excessively difficult to solve numerically on an electronic computer, we shall conclude our investigation at this point.

The Albedo and Net Outgoing Flux

Two quantities which follow immediately from the results of this section, where relevant, are the monochromatic albedo A^* and net outgoing monochromatic flux. The net outgoing monochromatic flux πF follows from the relation

$$F = 2\int_0^1 \mu I^*(0, \mu; \tau_3)\, d\mu \, . \tag{194}$$

The monochromatic albedo A^* of visible radiation follows from the relation

$$A^* = \frac{F}{\mu_0 F_0} \, , \tag{195}$$

where πF_0 is the net monochromatic incident flux crossing a unit area normal to the direction of incident radiation.

CONCLUDING REMARKS

The analysis leading from the concept of individual photon-particle interactions to the final solutions for the angular distribution of outgoing radiation from the top of a cloudy atmosphere has required at various times the need for certain assumptions and approximations which are, to a greater or lesser extent, at variance with physical reality. At this point it would be well worth while to review briefly these various restrictions and, where possible, make a partial assessment of the limitations they will logically be expected to impose on the accuracy and validity of the solutions.

In the first place, the polarization of the radiation field has been completely ignored throughout the formulation and solution of the equation of transfer. A proper account of polarization through the correct matrix formulation of the equation of transfer (References 3 and 9) is particularly difficult and consequently has not been considered. However, most of the radiation scattered once is in general confined to a rather small range of scattering angles near zero degrees (Reference 10) and *at* $\Theta = 0°$ the degree of polarization must go to zero from considerations of symmetry. We would therefore expect the degree of polarization of the net radiation field always to be small, and the corresponding effect on the resultant intensity quite weak. For a clearer picture of how a proper account of polarization affects the total intensity in the case of Rayleigh scattering, where forward scattering does *not* dominate and where the degree of polarization in general is considerable, the reader is referred to Tables XV and XXIV and Figures 24 and 25 of Reference 3.

The assumption that multiple scattering does not take place in a mass element dm which is large enough to contain a representative particle size distribution would appear to be quite sound for the wavelengths of radiation of interest. For a volume element of one cubic centimeter containing 300 particles, each having an efficiency factor for extinction $Q_E^{(i)} = 3$ and a radius $r_i = 20\mu$ (microns) (References 6 and 11), it is seen from Equations 38 and 43 that, upon setting $dz = 1$,

$$d\tau \approx (300)(3)(\pi)\left(20 \times 10^{-4}\right)^2 \mathrm{cm}^{-1}\, dz \approx 0.01 \; . \tag{196}$$

Since the numbers in Equation 196 are on the large side, it would be reasonable to assume that in general no more than a small fraction of a percent of any radiation scattered once in dm would be scattered again before leaving dm.

In order to determine how valid the plane wave approximation of radiation incident on each particle is, we must obtain an estimate of how far an outgoing scattered wave must travel before becoming essentially a plane wave. It can be shown (Reference 5) that the magnitude of the radial components $E_r^{(s)}$ and $H_r^{(s)}$ of the scattered wave are proportional to $(\lambda/r)^2$, where r is the distance from the scattering center and λ is the wavelength of radiation, whereas the tangential components $E_T^{(s)}$ and $H_T^{(s)}$ of the scattered wave are proportional to (λ/r). If we impose the requirement that that magnitude of the radial components be about one percent of the magnitude of the tangential components before the outgoing scattered wave may be considered plane, we obtain the relation

$$r \geq 100\,\lambda \; , \tag{197}$$

and from Equation 196 it may be inferred that more than 99% of all scattered radiation of wavelengths less than $\lambda = 0.1$ mm may be considered plane before suffering another scattering process. It should also be noted that, for particle densities of $N_0 \approx 300$ particles/cm^3, the mean separation distance of particles is on the order of a millimeter, and this in turn implies that for wavelengths less than this order of magnitude all interactions of radiation with matter may be considered as "particle" phenomena; i.e., it is proper to consider clouds to be comprised of individual particles as opposed to a continuous fluid. No attempt will be made to discuss how deviations from spherical particles and/or particles containing impurities will affect the effective cross-sections or angular scattering patterns and the degree of polarization.

The concept of local thermodynamic equilibrium in the pressure and temperature ranges of interest would appear to rest on an equally sound foundation for the atmosphere in general (Reference 12). Since densities in the interior of particles are considerably greater than densities in the surrounding atmosphere, Kirchhoff's law should be an even better approximation with regard to particle absorption and emission than for the surrounding atmosphere.

Treating the ground as an ideal blackbody in the infrared region of the spectrum is open to question. Variations of the emissivity of the ground from unity by several percent are quite possible, although oceans would be expected to be quite black.

The approximation of replacing an outside source with a point source removed to infinity was incorporated in the derivation of the equation of transfer which describes the diffuse radiation field. This approximation is also implicity extended to include any part of the radiation field wherever numerical solutions by quadratures in the n^{th} approximation are sought. The general validity of this method would appear to be born out from an inspection of results obtained by the author (unpublished) where the equation of transfer is solved in varying degrees of approximation for certain special problems and the results compared with one another. Little change was noticed in the solutions beyond the fifth approximation. By analogy with these results, it would also appear that any source extended over several degrees may be approximated rather accurately by a point source, so long as the extended source does not vary in intensity from point to point in too discontinuous a manner.

The replacement of the phase function for single scattering with an infinite series expansion of Legendre polynomials would be expected to yield quite rigorous results for any reasonably well-behaved phase function. The validity of approximating this function with a finite series of Legendre polynomials would, then, appear to depend only on the number of terms retained. In principle this is of no concern; in practice, however, the number of terms carried must be quite large, since the forward scattering nature of the phase function in general is quite extreme and the form of the function is not very amenable to reproduction by means of periodic functions. Unpublished results by the author would indicate that the scattering and transmission functions of the previous section reflect the oscillatory nature of poorly fitted phase functions (low values of N), although mean curves fitted through the results appear fairly accurate within 2% or so of the exact solution, at least for orders of N such that the magnitude of the oscillations in the approximate phase function do not exceed 10% of the magnitude of the forward peak. For ratios of forward to backward scattering of $10^3 - 10^4$, the number of terms retained should correspond to $N \approx 15 - 20$ or more. The solutions for the law of darkening do not tend to reflect these oscillations significantly, since multiple scattering and thermal emission play a much more dominant role, and the effect of single scattering is greatly reduced.

A report by Neiburger (Reference 13) indicates that the particle size distribution in stratus clouds is fairly independent of optical depth except near the base of the cloud even though the liquid water content is quite dependent upon τ. Whether this is true of clouds in general is doubtful, as is indicated from data obtained for a cumulus congestus cloud (Figure 48 of Reference 11). What is really required for thick clouds, however, is a knowledge of how constant the particle size distribution is near the boundaries of the cloud, since it is here that the contribution to the outgoing diffuse radiation field is largest. More data, both observational and theoretical, are needed before a proper evaluation can be made of this problem. An inclusion of molecular effects from the pervading atmosphere compounds the difficulty.

The effect of scattering from a cubic centimeter of air at STP can be calculated from the expression for Rayleigh scattering:

$$N_0 \chi_s = \frac{8\pi^3}{3} \frac{(\tilde{n}^2 - 1)^2}{N_0 \lambda^4} \quad . \tag{198}$$

Using the value of $\tilde{n} = 1.00029$, valid for $\lambda \approx 3000\text{Å}$, it can be verified that the optical thickness of a one centimeter column of atmosphere (assuming no absorption) is (Equation 43)

$$d\tau = N_0 \chi_s \, dz \approx \left(\frac{10^{-6}\ \text{cm}}{\lambda}\right)^4 \quad , \tag{199}$$

where dz is set equal to one centimeter. If $\lambda \approx 3000\text{Å}$, the optical thickness $d\tau$ becomes about

$$d\tau \approx 1.2 \times 10^{-6} \quad . \tag{200}$$

Again, if we assume a particle density of $N_0 = 100$ particles/cm^3, an efficiency factor for scattering of $Q_s^{(i)} = 2$, and a (rather low) average particle radius of $r_i = 5 \times 10^{-4}$ cm, it is found from Equation 199 that the optical thickness of a one centimeter column due to the particles alone is

$$d\tau = (100)(2)(\pi)\left(5 \times 10^{-4}\right)^2 \approx 1.57 \times 10^{-4} \quad , \tag{201}$$

or a factor of about 100 greater than the optical thickness resulting from molecular scattering. We conclude that scattering due to molecules contributes no more than one percent to the total scattered radiation at $\lambda = 3000$ Å, and this contribution becomes rapidly less as λ increases.

The final approximation to be considered is probably the most serious one—that of considering the whole problem in the context of plane-parallel layers containing no horizontal inhomogeneities. The really serious part of the difficulty with regard to real clouds is in connection with the inherently "rough" surfaces of clouds being replaced in the theory by perfectly smooth and level surfaces.

It is felt that no intuitive approach is going to be completely satisfactory. Intuitively one is led to believe that, in the first approximation at least, an inclusion of a rough surface in the problem would lead to substantially the same results, since the roughness is only of a higher degree than that already incorporated into the problem implicitly through the concept of discrete particles as opposed to the concept of a continuous fluid. Just as intuitively it might be argued that the outgoing radiation field will tend to become more isotropic so long as the roughness is shallow, because different orientations of the surfaces of each irregularity, and internal reflections among these irregularities, will tend to be random. If the roughness is great enough there will also be numerous shadow effects caused by each irregularity (although internal reflections among the different individual irregularities will tend to reduce this effect), and the argument breaks down.

In this case one can even conceive of predominantly backward scattering, since the shadow effect is minimum in this direction.

The reason the first intuitive approach is invalid is because the roughness of the cloud surface is not comparable to the "roughness" implicitly introduced into the equation of transfer through the concept of discrete scattering centers. Each mass element dm must now be considered large enough to contain a representative distribution of "irregularity" sizes, and, apart from considerations of the shapes, associated phase functions, and dependence upon optical depth of these irregularities, there is also clearly the impossibility of neglecting multiple scattering within dm itself. On the other hand, if dm is restricted to a size such that only single scattering prevails, then clearly dm does not contain a representative particle size distribution, and is furthermore quite dependent upon real horizontal inhomogeneities.

The other intuitive approach is invalid for the very reason that the whole problem was undertaken in the first place. In other words, the extreme anisotropic nature of the phase function for single scattering makes it impossible to discuss with any degree of confidence any tendency the radiation field might have toward isotropy at the surface of the cloud due to random orientations of the surfaces of the irregularities and internal reflections among them.

If the task were not too prohibitive, a possible direction of investigation of these effects would be to consider the effect of different types of mass elements distributed in accordance with some distribution function, random or otherwise. For higher degrees of roughness it might be advantageous to pursue the problem from the point of view of considering each irregularity as an entity within itself, solve for the outgoing radiation field from each irregularity within the framework of an idealized geometry, and obtain the solution to the composite problem in a manner analogous to that adopted in the previous section. It is anticipated, however, that either approach is likely to be subject to a great deal more labor than the problem warrants.

Three problems of quite general significance which have not been considered in this paper include:

1. The dependence of the phase function for single scattering upon optical depth.

2. The polarization of the radiation field.

3. The solution to the equation of transfer within the framework of geometries other than plane-parallel.

The first and third problems are of particular interest in the context of radiative transfer through aerosols, where the optical depth remains small for considerable linear distances. Furthermore, a correct account of polarization is of significant interest in connection with an interpretation of the characteristics of clouds in other planetary atmospheres, notably those of Jupiter and Venus. And finally, it is stressed that these problems should be attacked with a certain degree of sophistication. It is very difficult to deal intuitively with them, and the advent of high speed electronic computers has made the necessity for crude approximations less requisite than ever before.

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Appendix A

Normalization of the Phase Function for Single Scattering

It is assumed that the phase function for single scattering, $\mathcal{P}(\cos\Theta)$, can be expressed as a finite series expansion of Legendre polynomials in the form

$$\mathcal{P}(\cos\Theta) = \sum_{l=0}^{N} \tilde{\omega}_l P_l(\cos\Theta) \quad . \tag{A1}$$

It is required to show that $\mathcal{P}(\cos\Theta)$ expressed in this way is normalized to $\tilde{\omega}_0$, the albedo for single scattering. This requirement is equivalent to the expression

$$\int_{\omega} \mathcal{P}(\cos\Theta) \frac{d\omega}{4\pi} = \tilde{\omega}_0 \quad , \tag{A2}$$

where the integration is performed over all solid angles. Thus

$$\tilde{\omega}_0 = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left[\sum_{l=0}^{N} \tilde{\omega}_l P_l(\cos\Theta) \right] \sin\Theta \, d\Theta \, d\phi \quad , \tag{A3}$$

where spherical coordinates are chosen so that the polar angle is picked to be the scattering angle Θ, and the azimuthal angle ϕ is the angle of rotation about the axis of symmetry defined by $\Theta = 0$.

Upon reversing the order of integration and summation, and replacing $\cos\Theta$ by α, we obtain

$$\tilde{\omega}_0 = \frac{1}{4\pi} \sum_{l=0}^{N} \tilde{\omega}_l \int_{-1}^{+1} P_l(\alpha) \, d\alpha \int_0^{2\pi} d\phi = \frac{1}{2} \sum_{l=0}^{N} \tilde{\omega}_l \int_{-1}^{+1} P_l(\alpha) \, d\alpha \quad . \tag{A4}$$

From the relations

$$1 = P_0(\alpha) \tag{A5}$$

and

$$\int_{-1}^{+1} P_m(\alpha) P_l(\alpha) \, d\alpha = \frac{2\delta_{m,l}}{2l + 1} = \begin{Bmatrix} 0 & (m \neq l) \\ \dfrac{2}{2l+1} & (m = l) \end{Bmatrix} \tag{A6}$$

we obtain

$$\tilde{\omega}_0 = \frac{1}{2}\sum_{l=0}^{N} \tilde{\omega}_l \int_{-1}^{+1} P_0(\alpha)\, P_l(\alpha)\, d\alpha = \tilde{\omega}_0 , \tag{A7}$$

and the required identity is established.

Appendix B

A Particular Integral to the Equation of Transfer Appropriate for an Emitting Cloud

In context with the correct solution to the intensity distribution at the top of an emitting cloud, the equation of transfer appropriate to the restricted problem is

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2}\int_{-1}^{+1} p(\mu, \mu') I(\tau, \mu') d\mu' - \left(1 - \tilde{\omega}_0\right) B(\tau) , \tag{B1}$$

where $p(\mu, \mu')$ has the form

$$p(\mu, \mu') = \sum_{l=0}^{N} \tilde{\omega}_l P_l(\mu) P_l(\mu') \tag{B2}$$

and $B(\tau)$ has the form

$$B(\tau) = \sum_{r=0}^{N} b_r \tau^r . \tag{B3}$$

A particular integral $I[B(\tau)]$ satisfying Equation B1 will be required which, when added to the general solution to the homogeneous part of Equation B1, gives the required solution for the angular distribution of emitted radiation.

Since we are concerned with solutions to the equation of transfer in the n^{th} approximation, Equation B1 can be replaced with $2n$ equations of the form

$$\begin{aligned}\mu_i \frac{dI(\tau, \mu_i)}{d\tau} &= I(\tau, \mu_i) - \frac{1}{2}\sum_j a_j I(\tau, \mu_j) \sum_{l=0}^{N} \tilde{\omega}_l P_l(\mu_i) P_l(\mu_j) - \left(1 - \tilde{\omega}_0\right) \sum_{r=0}^{N} b_r \tau^r \\ &= I(\tau, \mu_i) - \frac{1}{2}\sum_{l=0}^{N} \tilde{\omega}_l P_l(\mu_i) \sum_j a_j I(\tau, \mu_j) P_l(\mu_j) - \left(1 - \tilde{\omega}_0\right) \sum_{r=0}^{N} b_r \tau^r\end{aligned}$$

$$\left(i = \pm 1, \cdots, \pm n ; \quad j = \pm 1, \cdots, \pm n ; \quad n > \frac{2N+1}{4}\right) . \tag{B4}$$

From the manner in which it enters into the equation of transfer, and from the nature of the Planck function, we are led to assume that the required particular integral may be expressed in the form

$$I_i[B(\tau)] = \sum_{r=0}^{N} \tau^r \sum_{s=r}^{N} C_{r,s} P_{s-r}(\mu_i) , \tag{B5}$$

where the $C_{r,s}$'s are functions of $\tilde{\omega}_l$ and b_r only, and obey certain recursion relations to be derived below.

From Equation B5 and the relation

$$(2l+1)\mu P_l(\mu) = (l+1) P_{l+1}(\mu) + l P_{l-1}(\mu) , \tag{B6}$$

Equation B4 becomes successively

$$\begin{aligned}
\mu_i \frac{dI_i[B(\tau)]}{d\tau} &= \mu_i \sum_{r=1}^{N} r\tau^{r-1} \sum_{s=r}^{N} C_{r,s} P_{s-r}(\mu_i) \\
&= \sum_{r=1}^{N} r\tau^{r-1} \sum_{s=r}^{N} \frac{C_{r,s}}{2(s-r)+1} \left[(s-r+1) P_{s-r+1}(\mu_i) + (s-r) P_{s-r-1}(\mu_i)\right] \\
&= \sum_{r=0}^{N} \tau^r \sum_{s=r}^{N} C_{r,s} P_{s-r}(\mu_i) - \frac{1}{2} \sum_{l=0}^{N} \tilde{\omega}_l P_l(\mu_i) \sum_j a_j P_l(\mu_j) \sum_{r=0}^{N} \tau^r \sum_{s=r}^{N} C_{r,s} P_{s-r}(\mu_j) \\
&\quad - (1-\tilde{\omega}_0) \sum_{r=0}^{N} b_r \tau^r .
\end{aligned} \tag{B7}$$

Upon using the identity

$$\sum_j a_j P_l(\mu_j) P_{s-r}(\mu_j) = \int_{-1}^{+1} P_l(\mu') P_{s-r}(\mu') d\mu' = \frac{2\delta_{l,s-r}}{2(s-r)+1} , \tag{B8}$$

where

$$\delta_{l,s-r} = \begin{Bmatrix} 1 & (l = s-r) \\ 0 & (l \neq s-r) \end{Bmatrix} , \tag{B9}$$

the second term of the right-hand side of Equation B7 becomes successively

$$\frac{1}{2}\sum_{l=0}^{N}\tilde{\omega}_l P_l(\mu_i)\sum_j a_j P_l(\mu_j)\sum_{r=0}^{N}\tau^r\sum_{s=r}^{N}C_{r,s}P_{s-r}(\mu_j) = \frac{1}{2}\sum_{l=0}^{N}\tilde{\omega}_l P_l(\mu_i)\sum_{r=0}^{N}\tau^r\sum_{s=r}^{N}C_{r,s}\sum_j a_j P_l(\mu_j)P_{s-r}(\mu_j)$$

$$= \sum_{r=0}^{N}\tau^r\sum_{s=r}^{N}\frac{\tilde{\omega}_{s-r}C_{r,s}}{2(s-r)+1}P_{s-r}(\mu_i) \quad . \tag{B10}$$

Therefore Equation B7 becomes, upon replacing r by (r + 1) on the left-hand side,

$$\sum_{r=0}^{N-1}(r+1)\tau^r\sum_{s=r+1}^{N}\frac{C_{r+1,s}}{2(s-r)-1}\left[(s-r)P_{s-r}(\mu_i)+(s-r-1)P_{s-r-2}(\mu_i)\right]$$

$$= \sum_{r=0}^{N}\tau^r\sum_{s=r}^{N}C_{r,s}P_{s-r}(\mu_i)-\sum_{r=0}^{N}\tau^r\sum_{s=r}^{N}\frac{\tilde{\omega}_{s-r}C_{r,s}}{2(s-r)+1}P_{s-r}(\mu_i)-(1-\tilde{\omega}_0)\sum_{r=0}^{N}b_r\tau^r \, . \tag{B11}$$

The left-hand side of Equation B11 is zero for r = s = N. Therefore

$$C_{N,N} = b_N \quad . \tag{B12}$$

This reduces Equation B11 to

$$\sum_{r=0}^{N-1}(r+1)\tau^r\sum_{s=r+1}^{N}\frac{C_{r+1,s}}{2(s-r)-1}\left[(s-r)P_{s-r}(\mu_i)+(s-r-1)P_{s-r-2}(\mu_i)\right]$$

$$= \sum_{r=0}^{N-1}\tau^r\sum_{s=r}^{N}C_{r,s}\left[1-\frac{\tilde{\omega}_{s-r}}{2(s-r)+1}\right]P_{s-r}(\mu_i)-(1-\tilde{\omega}_0)\sum_{r=0}^{N-1}b_r\tau^r \quad . \tag{B13}$$

Since Equation B13 is valid for all τ we require that

$$(r+1)\sum_{s=r+1}^{N}\frac{C_{r+1,s}}{2(s-r)-1}\left[(s-r)P_{s-r}(\mu_i)+(s-r-1)P_{s-r-2}(\mu_i)\right]$$

$$= \sum_{s=r}^{N}C_{r,s}\left[1-\frac{\tilde{\omega}_{s-r}}{2(s-r)+1}\right]P_{s-r}(\mu_i)-(1-\tilde{\omega}_0)b_r \quad (0 \le r < N-1) \; . \tag{B14}$$

Equation B14 may be re-written as

$$(r+1)\sum_{s=r+1}^{N}\frac{C_{r+1,s}}{2(s-r)-1}(s-r)P_{s-r}(\mu_i) + (r+1)\sum_{s=r}^{N-2}\frac{C_{r+1,s+2}}{2(s-r)+3}(s-r+1)P_{s-r}(\mu_i)$$

$$= \sum_{s=r}^{N}C_{r,s}\left[1-\frac{\tilde{\omega}_{s-r}}{2(s-r)+1}\right]P_{s-r}(\mu_i)-(1-\tilde{\omega}_0)b_r \tag{B15}$$

for $(0 \leq r \leq N-1)$. Note that the second term on the left-hand side of Equation B15 is zero for $s = r - 1$; the summation is therefore started from $s = r$.

Since Equation B15 must be valid for all μ_i, we require that

$$\frac{(r+1)(N-r)}{2(N-r)-1} C_{r+1,N} = \left[1 - \frac{\tilde{\omega}_{N-r}}{2(N-r)+1}\right] C_{r,N} \qquad (0 \leq r \leq N-1) \tag{B16}$$

and

$$\frac{(r+1)(N-r-1)}{2(N-r)-3} C_{r+1,N-1} = \left[1 - \frac{\tilde{\omega}_{N-r-1}}{2(N-r)-1}\right] C_{r,N-1} \qquad (0 \leq r \leq N-2) . \tag{B17}$$

Equations B16 and B17 were obtained by setting $s = N$ and $s = N - 1$ respectively in Equation B15.

From the identity $P_0(\mu_i) \equiv 1$, we further obtain from Equation B15, upon setting $s = r$, the relation

$$\frac{1}{3}(r+1) C_{r+1,r+2} = \left(1 - \tilde{\omega}_0\right)\left(C_{r,r} - b_r\right) \tag{B18}$$

for $s = r\ (0 \leq r \leq N\text{-}2)$.

In the case $s = r = N - 1$ it follows from Equation B15 that

$$C_{N-1,N-1} = b_{N-1} . \tag{B19}$$

With the aid of Equations B12 and B16 through B19, Equation B15 reduces to

$$(r+1)\sum_{s=r+1}^{N-2} \frac{C_{r+1,s}}{2(s-r)-1}(s-r)P_{s-r}(\mu_i) + (r+1)\sum_{s=r+1}^{N-2} \frac{C_{r+1,s+2}}{2(s-r)+3}(s-r+1)P_{s-r}(\mu_i)$$

$$= \sum_{s=r+1}^{N-2} C_{r,s}\left[1 - \frac{\tilde{\omega}_{s-r}}{2(s-r)+1}\right] P_{s-r}(\mu_i) . \tag{B20}$$

Since Equation B20 must be valid for all μ_i, the general recursion relation

$$\left[1 - \frac{\tilde{\omega}_{s-r}}{2(s-r)+1}\right] C_{r,s} = (r+1)\left[\frac{(s-r+1)}{2(s-r)+3} C_{r+1,s+2} + \frac{(s-r)}{2(s-r)-1} C_{r+1,s}\right] \tag{B21}$$

for $(0 \leq r \leq N-3;\ r+1 \leq s \leq N-2)$ is obtained.

In view of the fact that Equations B12, B16 through B19, and B21 are just sufficient to determine all the required values of $C_{r,s}$ $(0 \leq r \leq N;\ r \leq s \leq N)$, we conclude that Equation B5 is a valid representation for the required particular integral.

Two sets of values of $c_{r,s}$ may be generated independently of each other, namely the set for which $(N - s)$ is even and the set for which $(N - s)$ is odd. Let us denote these sets by Equations B-I and B-II respectively. In what follows we shall assume N to be arbitrary.

For Set B-I...

From Equation B12 it is noted that

$$c_{N,N} = b_N \quad . \tag{B-I-1}$$

From Equation B16 are generated successively

$$\left.\begin{aligned}
c_{N-1,N} &= \frac{N}{\left(1 - \frac{1}{3}\tilde{\omega}_1\right)} c_{N,N} \quad , \\
c_{N-2,N} &= \frac{2(N-1)}{3\left(1 - \frac{1}{5}\tilde{\omega}_2\right)} c_{N-1,N} \quad , \\
&\;\;\vdots \\
c_{0,N} &= \frac{N}{(2N-1)\left(1 - \frac{\tilde{\omega}_N}{2N+1}\right)} c_{1,N} \quad .
\end{aligned}\right\} \tag{B-I-2}$$

To proceed, we obtain from Equation B18

$$c_{N-2,N-2} = b_{N-2} + \frac{(N-1)}{3\left(1 - \tilde{\omega}_0\right)} c_{N-1,N} \quad . \tag{B-I-3}$$

From Equation B21 are generated successively

$$\left.\begin{aligned}
c_{N-3,N-2} &= \frac{(N-2)}{\left(1 - \frac{1}{3}\tilde{\omega}_1\right)} \left[\frac{2}{5} c_{N-2,N} + c_{N-2,N-2}\right] \quad , \\
c_{N-4,N-2} &= \frac{(N-3)}{\left(1 - \frac{1}{5}\tilde{\omega}_2\right)} \left[\frac{3}{7} c_{N-3,N} + \frac{2}{3} c_{N-3,N-2}\right] \quad , \\
&\;\;\vdots \\
c_{0,N-2} &= \frac{1}{\left(1 - \frac{\tilde{\omega}_{N-2}}{2N-3}\right)} \left[\frac{N-1}{2N-1} c_{1,N} + \frac{N-2}{2N-5} c_{1,N-2}\right] \quad .
\end{aligned}\right\} \tag{B-I-4}$$

Upon returning to Equation B18 we obtain

$$c_{N-4,N-4} = b_{N-4} + \frac{N-3}{3\left(1 - \tilde{\omega}_0\right)} c_{N-3,N-2} \quad . \tag{B-I-5}$$

Again, returning to Equation B21, we generate successively $C_{N-5,N-4}$, $C_{N-6,N-4}$, $\cdots$, $C_{0,N-4}$. Thus Equations B18 and B21 are alternately employed as required until all the $C_{r,s}$'s for $(N-s)$ even are generated.

The $C_{r,s}$'s in Set (B-II), i.e., for $(N-s)$ odd, are then obtained as follows.

For Set B-II...

From Equation B19 it is noted that

$$C_{N-1,N-1} = b_{N-1} \quad . \tag{B-II-1}$$

From Equation B17 are generated successively

$$\left.\begin{aligned}
C_{N-2,N-1} &= \frac{(N-1)}{\left(1-\frac{1}{3}\tilde{\omega}_1\right)} C_{N-1,N-1} \quad , \\
C_{N-3,N-1} &= \frac{2(N-2)}{3\left(1-\frac{1}{5}\tilde{\omega}_2\right)} C_{N-2,N-1} \quad , \\
&\vdots \\
C_{0,N-1} &= \frac{(N-1)}{(2N-3)\left(1-\frac{\tilde{\omega}_{N-1}}{2N-1}\right)} C_{1,N-1} \quad .
\end{aligned}\right\} \tag{B-II-2}$$

To proceed, we obtain from Equation B18

$$C_{N-3,N-3} = b_{N-3} + \frac{(N-2)}{3\left(1-\tilde{\omega}_0\right)} C_{N-2,N-1} \quad . \tag{B-II-3}$$

From Equation B21 are generated successively

$$\left.\begin{aligned}
C_{N-4,N-3} &= \frac{(N-3)}{\left(1-\frac{1}{3}\tilde{\omega}_1\right)} \left[\frac{2}{5} C_{N-3,N-1} + C_{N-3,N-3}\right] \quad , \\
C_{N-5,N-3} &= \frac{(N-4)}{\left(1-\frac{1}{5}\tilde{\omega}_2\right)} \left[\frac{3}{7} C_{N-4,N-1} + \frac{2}{3} C_{N-4,N-3}\right] \quad , \\
&\vdots \\
C_{0,N-3} &= \frac{1}{\left(1-\frac{\tilde{\omega}_{N-3}}{2N-5}\right)} \left[\frac{N-2}{2N-3} C_{1,N-1} + \frac{N-3}{2N-7} C_{1,N-3}\right] \quad .
\end{aligned}\right\} \tag{B-II-4}$$

Upon returning to Equation B18 we obtain

$$C_{N-5,N-5} = b_{N-5} + \frac{N-4}{3\left(1-\tilde{\omega}_0\right)} C_{N-4,N-3} \quad . \tag{B-II-5}$$

Again, returning to Equation B21 we generate successively $C_{N-6,N-5}$, $C_{N-7,N-5}$, $\cdots$, $C_{0,N-5}$. Thus Equations B18 and B21 are alternately employed as required until all the $C_{r,s}$'s for $(N-s)$ odd are generated. The procedure is exactly the same regardless of whether N is even or odd.

Since the $C_{r,s}$'s are uniquely determined, and there are no internal inconsistencies in their derivation, we conclude that (Equation B5)

$$I_i\left[B(\tau)\right] = \sum_{r=0}^{N} \tau^r \sum_{s=r}^{N} C_{r,s} P_{s-r}\left(\mu_i\right)$$

is the required particular integral, where the $C_{r,s}$'s are determined in accordance with the preceding discussion.

In particular, for $\tau = 0$,

$$I_i\left[B(0)\right] = \sum_{s=0}^{N} C_{0,s} P_s\left(\mu_i\right) \qquad (i = \pm 1, \cdots, \pm n) \ , \tag{B22}$$

and for $\tau = \tau_1$,

$$I_i\left[B\left(\tau_1\right)\right] = \sum_{r=0}^{N} \tau_1{}^r \sum_{s=r}^{N} C_{r,s} P_{s-r}\left(\mu_i\right) \quad (i = \pm 1, \cdots, \pm n) \ . \tag{B23}$$

Equations B22 and B23, along with the 2n boundary conditions $I\left(0,-\mu_i\right) = I\left(\tau_1, \mu_i\right) = 0$ $(i = 1, \cdots, n)$ and the general solution to the homogeneous part of the equation of transfer, suffice to determine the appropriate law of darkening at the top of the cloud for the restricted problem.

Appendix C

The Flux Integral

The net flux πF_ν in the frequency range $(\nu, \nu + d\nu)$ is defined by

$$\pi F_\nu = \int_0^{2\pi} \int_{-1}^{+1} \mu I_\nu (\tau_\nu, \mu, \phi)\, d\mu\, d\phi \tag{C1}$$

where the integration is performed over all solid angles. If $I_\nu(\tau_\nu, \mu, \phi)$ can be written as

$$I_\nu(\tau_\nu, \mu, \phi) = \sum_{n=0}^{N} I_\nu^{(n)}(\tau_\nu, \mu) \cos n(\phi_0 - \phi) \ , \tag{C2}$$

the flux integral may be reduced to

$$F_\nu = 2\int_{-1}^{+1} \mu I_\nu^{(0)}(\tau_\nu, \mu)\, d\mu \ , \tag{C3}$$

where the integration over the azimuth-dependent terms all go to zero.

Diffuse Reflection and Transmission.

If a single outside point source is considered to be the sole source of radiation, the relevant equation of transfer describing the intensity of radiation at any point in the cloud in any direction is, upon dropping the subscript ν,

$$\begin{aligned} \mu \frac{dI(\tau, \mu, \phi)}{d\tau} = {} & I(\tau, \mu, \phi) \\ & - \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} p(\mu, \phi; \mu', \phi')\, I(\tau, \mu', \phi')\, d\mu'\, d\phi' \\ & - \frac{1}{4} F_0\, e^{-\tau/\mu_0}\, p(\mu, \phi; -\mu_0, \phi_0) \ . \end{aligned} \tag{C4}$$

The solution for $I^{(0)}(\tau, \mu)$ for this equation of transfer, using the method of discrete ordinates, is

$$I^{(0)}(\tau, \mu_i) = \frac{1}{4} F_0 \left\{ \sum_{\alpha=1}^{n} \frac{M_\alpha e^{-k_\alpha \tau}}{1 + \mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l(+k_\alpha) P_l(\mu_i) \right] + \sum_{\alpha=1}^{n} \frac{M_{-\alpha} e^{+k_\alpha \tau}}{1 - \mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l(-k_\alpha) P_l(\mu_i) \right] \right.$$

$$\left. + \frac{\gamma_0 e^{-\tau/\mu_0}}{1 + \mu_i/\mu_0} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l\left(\frac{1}{\mu_0}\right) P_l(\mu_i) \right] \right\} \quad (i = \pm 1, \cdots, \pm n) \tag{C5}$$

if $\tilde{\omega}_0 \neq 1$ and $\tau_1 \neq \infty$. πF_0 is the flux of incoming radiation crossing a unit surface area and normal to it. The net outgoing flux is found from Equations C3 and C5, after interchanging the order of summation, from

$$F(\tau) = 2 \sum_i a_i \mu_i I^{(0)}(\tau, \mu_i) = \frac{1}{2} F_0 \sum_{\alpha=1}^{n} M_\alpha e^{-k_\alpha \tau} \sum_{l=0}^{N} \tilde{\omega}_l \xi_l(+k_\alpha) \sum_i \frac{a_i \mu_i P_l(\mu_i)}{1 + \mu_i k_\alpha}$$

$$+ \frac{1}{2} F_0 \sum_{\alpha=1}^{n} M_{-\alpha} e^{+k_\alpha \tau} \sum_{l=0}^{N} \tilde{\omega}_l \xi_l(-k_\alpha) \sum_i \frac{a_i \mu_i P_l(\mu_i)}{1 - \mu_i k_\alpha} + \frac{1}{2} F_0 \gamma_0 e^{-\tau/\mu_0} \sum_{l=0}^{N} \tilde{\omega}_l \xi_l\left(\frac{1}{\mu_0}\right) \sum_i \frac{a_i \mu_i P_l(\mu_i)}{1 + \mu_i/\mu_0} . \tag{C6}$$

In deriving the recursion relation (cf. Equations 77-80)

$$\xi_{\lambda+1} = - \frac{2\lambda + 1 - \tilde{\omega}_\lambda}{k(\lambda + 1)} \xi_\lambda - \frac{\lambda}{\lambda + 1} \xi_{\lambda-1} , \tag{C7}$$

use is made of the identity

$$\xi_\lambda(x) = \sum_{l=0}^{N} \tilde{\omega}_l \xi_l(x) D_{\lambda, l}(x) \quad (\lambda = 0, \cdots, N) \tag{C8}$$

where $D_{\lambda, l}(x)$ is defined by

see errata

$$D_{\lambda, l}(x) = \frac{1}{2} \sum_i \frac{a_i P_\lambda(\mu_i) P_l(\mu_i)}{1 + \mu_i x} . \tag{C9}$$

Upon substituting $\lambda = 1$ in Equations C8 and C9, Equation C6 reduces to

see errata.

$$F(\tau) = F_0 \left\{ \sum_{\alpha=1}^{n} M_\alpha e^{-k_\alpha \tau} \xi_1(+k_\alpha) + \sum_{\alpha=1}^{n} M_{-\alpha} e^{+k_\alpha \tau} \xi_1(-k_\alpha) + \gamma_0 e^{-\tau/\mu_0} \xi_1\left(\frac{1}{\mu_0}\right) \right\} . \tag{C10}$$

Upon setting $\lambda = 0$ in Equation C7, and remembering that $\xi_{-1} = 0$, Equation C10 further reduces to

$$F(\tau) = \left(1-\tilde{\omega}_0\right) F_0 \left\{ \sum_{\alpha=1}^{n} \frac{1}{k_\alpha} \left(M_{-\alpha}\, e^{+k_\alpha \tau} - M_\alpha\, e^{-k_\alpha \tau} \right) - \mu_0 \gamma_0\, e^{-\tau/\mu_0} \right\} . \tag{C11}$$

The net flux crossing the top of the cloud is found, upon setting $\tau = 0$, from

$$F(0) = \left(1-\tilde{\omega}_0\right) F_0 \left\{ \left[\sum_{\alpha=1}^{n} \frac{1}{k_\alpha} \left(M_{-\alpha} - M_\alpha \right) \right] - \mu_0 \gamma_0 \right\} , \tag{C12}$$

and crossing the bottom of the cloud, upon setting $\tau = \tau_1$, from

$$F(\tau_1) = \left(1-\tilde{\omega}_0\right) F_0 \left\{ \left[\sum_{\alpha=1}^{n} \frac{1}{k_\alpha} \left(M_{-\alpha}\, e^{+k_\alpha \tau_1} - M_\alpha\, e^{-k_\alpha \tau_1} \right) \right] - \mu_0 \gamma_0\, e^{-\tau_1/\mu_0} \right\} . \tag{C13}$$

If the cloud is semi-infinite, all $M_{-\alpha} = 0$ ($\alpha = 1, \cdots, n$), and Equation C12 becomes

$$F(0) = -\left(1-\tilde{\omega}_0\right) F_0 \left\{ \sum_{\alpha=1}^{n} \left(\frac{M_\alpha}{k_\alpha} \right) + \mu_0 \gamma_0 \right\} . \tag{C14}$$

In the special case of conservative scattering $\left(\tilde{\omega}_0 = 1\right)$ it is clear that Equations C13 and C14 are no longer valid relations. In this case, as can be verified by a direct substitution into the equation of transfer, the azimuth-independent term of the intensity in the case of a finitely thick cloud is given by (Equation 94)

$$\begin{aligned} I^{(0)}\left(\tau, \mu_i\right) = \frac{1}{4} F_0 \Bigg\{ & \sum_{\alpha=1}^{n-1} \frac{M_\alpha\, e^{-k_\alpha \tau}}{1+\mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l\, \xi_l\left(+k_\alpha\right) P_l\left(\mu_i\right) \right] + \sum_{\alpha=1}^{n-1} \frac{M_{-\alpha}\, e^{+k_\alpha \tau}}{1-\mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l\, \xi_l\left(-k_\alpha\right) P_l\left(\mu_i\right) \right] \\ & + \frac{\gamma_0\, e^{-\tau/\mu_0}}{1+\mu_i/\mu_0} \left[\sum_{l=0}^{N} \tilde{\omega}_l\, \xi_l\left(\frac{1}{\mu_0}\right) P_l\left(\mu_i\right) \right] + 3\mu_0 \left[\left\{ \left(1 - \frac{1}{3}\tilde{\omega}_1\right) \tau + \mu_i \right\} M_0 + M_n \right] \Bigg\} , \end{aligned} \tag{C15}$$

where $M_{\pm\alpha}$ ($\alpha = 1, \cdots, n-1$), M_0, and M_n are the $2n$ constants of integration. It is clear from Equation C11 and the preceding discussion that in this case the flux integral must reduce to

$$F(\tau) = 2 \sum_{i} a_i\, \mu_i\, \frac{1}{4} F_0 \left(3\mu_0 \left\{ \left[\left(1 - \frac{1}{3}\tilde{\omega}_1\right) \tau + \mu_i \right] M_0 + M_n \right\} \right) . \tag{C16}$$

Since

$$\sum_i a_i \mu_i = 0 \; ; \qquad \sum_i a_i \mu_i^2 = \frac{2}{3} \; , \tag{C17}$$

Equation C16 reduces after some algebra to

$$F(\tau) = \mu_0 F_0 M_0 \; . \tag{C18}$$

The flux integral $F(\tau)$ is seen to be independent of optical depth, as it must be for conservative scattering.

If the cloud is semi-infinite and $\tilde{\omega}_0 = 1$, Equation C18 must reduce to

$$F(\tau) = \mu_0 F_0 \; ; \tag{C19}$$

i.e., the "albedo" of the cloud must be unity. The expression for $I^{(0)}(\tau, \mu_i)$ in this case is seen by inspection to be

$$I^{(0)}(\tau, \mu_i) = \frac{1}{4} F_0 \left\{ \sum_{\alpha=1}^{n-1} \frac{M_\alpha e^{-k_\alpha \tau}}{1 + \mu_i k_\alpha} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l (+k_\alpha) P_l(\mu_i) \right] + \frac{\gamma_0 e^{-\tau/\mu_0}}{1 + \mu_i/\mu_0} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l \left(\frac{1}{\mu_0}\right) P_l(\mu_i) \right] + 3\mu_0 \left[\left(1 - \frac{1}{3}\tilde{\omega}_1\right) \tau + \mu_i + M_n \right] \right\} , \tag{C20}$$

where M_α ($\alpha = 1, \cdots, n-1$) and M_n are the n constants of integration to be determined from the n boundary conditions

$$I^{(0)}(0, -\mu_i) = 0 \qquad (i = 1, \cdots, n) \; . \tag{C21}$$

The Law of Darkening.

The equation of transfer appropriate to the case of axial symmetry (no outside sources) where thermal emission is taken into account is (Equation B1)

$$\mu \frac{dI(\tau_\nu, \mu)}{d\tau} = I(\tau_\nu, \mu) - \frac{1}{2} \int_{-1}^{+1} p_\nu(\mu, \mu') \, I(\tau_\nu, \mu') \, d\mu' - (1 - \tilde{\omega}_0) B_\nu(T) \; . \tag{C22}$$

Upon dropping the subscript ν, the solutions for $I(\tau, \mu)$ for this equation of transfer, using the method of discrete ordinates, are

$$I(\tau, \mu_i) = \sum_{a=1}^{n} \frac{M_a e^{-k_a \tau}}{1 + \mu_i k_a} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l (+k_a) P_l(\mu_i) \right]$$

$$+ \sum_{a=1}^{n} \frac{M_{-a} e^{+k_a \tau}}{1 - \mu_i k_a} \left[\sum_{l=0}^{N} \tilde{\omega}_l \xi_l (-k_a) P_l(\mu_i) \right]$$

$$+ \sum_{r=0}^{N} \tau^r \left[\sum_{s=r}^{N} C_{r,s} P_{s-r}(\mu_i) \right] \qquad (i = \pm 1, \cdots, \pm n) . \qquad \text{(C23)}$$

The relation for the net flux is therefore (cf. Equations C5, C6, and C11)

$$F(\tau) = 4(1 - \tilde{\omega}_0) \sum_{a=1}^{n} \frac{1}{k_a} \left(M_{-a} e^{+k_a \tau} - M_a e^{-k_a \tau} \right) + 2 \sum_i a_i \mu_i \left\{ \sum_{r=0}^{N} \tau^r \left[\sum_{s=r}^{N} C_{r,s} P_{s-r}(\mu_i) \right] \right\} . \qquad \text{(C24)}$$

Upon rearranging the order of summation, the second term on the right-hand side of Equation C24 becomes successively (cf. Equation C17)

$$2 \sum_i a_i \mu_i \left\{ \sum_{r=0}^{N} \tau^r \left[\sum_{s=r}^{N} C_{r,s} P_{s-r}(\mu_i) \right] \right\} = 2 \sum_{r=0}^{N} \tau^r \left\{ \sum_{s=r}^{N} C_{r,s} \left[\sum_i a_i P_1(\mu_i) P_{s-r}(\mu_i) \right] \right\} = \frac{4}{3} \sum_{r=0}^{N-1} C_{r,r+1} \tau^r . \qquad \text{(C25)}$$

The final expression also results from the identity

$$\sum_i a_i P_m(\mu_i) P_n(\mu_i) = \frac{2\delta_{n,m}}{2m+1} , \qquad \text{(C26)}$$

where

$$\delta_{n,m} = \left\{ \begin{matrix} 0 & (n \neq m) \\ 1 & (n = m) \end{matrix} \right\} . \qquad \text{(C27)}$$

Upon collecting the results from Equations C24 and C25, the expression for the net flux becomes

$$F(\tau) = 4 \left[(1 - \tilde{\omega}_0) \sum_{a=1}^{n} \frac{1}{k_a} \left(M_{-a} e^{+k_a \tau} - M_a e^{-k_a \tau} \right) + \frac{1}{3} \sum_{r=0}^{N-1} C_{r,r+1} \tau^r \right] . \qquad \text{(C28)}$$

The expressions for the net flux at the top and bottom of the cloud are respectively

$$F(0) = 4\left[\left(1-\tilde{\omega}_0\right)\sum_{\alpha=1}^{n}\frac{1}{k_\alpha}\left(M_{-\alpha}-M_\alpha\right)+\frac{1}{3}C_{0,1}\right] \tag{C29}$$

and

$$F(\tau_1) = 4\left[\left(1-\tilde{\omega}_0\right)\sum_{\alpha=1}^{n}\frac{1}{k_\alpha}\left(M_{-\alpha}e^{+k_\alpha\tau_1}-M_\alpha e^{-k_\alpha\tau_1}\right)+\frac{1}{3}\sum_{r=0}^{N-1}C_{r,r+1}\tau_1^{\,r}\right] . \tag{C30}$$

If the cloud is semi-infinitely thick, Equation C28 reduces to

$$F(\tau) = -4\left[\left(1-\tilde{\omega}_0\right)\sum_{\alpha=1}^{n}\left(\frac{M_\alpha}{k_\alpha}\right)e^{-k_\alpha\tau}-\frac{1}{3}\sum_{r=0}^{N-1}C_{r,r+1}\tau^{r}\right] , \tag{C31}$$

and Equation C29 reduces to

$$F(0) = -4\left[\left(1-\tilde{\omega}_0\right)\sum_{\alpha=1}^{n}\left(\frac{M_\alpha}{k_\alpha}\right)-\frac{1}{3}C_{0,1}\right] . \tag{C32}$$

Appendix D

A Note on the Solutions of Certain Integrals and Integral Equations

Consider the integral

$$\int_0^1 g(\mu, \mu')\, h(\mu', \mu'')\, d\mu' = G(\mu, \mu'') \quad . \tag{D1}$$

If $g(\mu, \mu')$ and $h(\mu', \mu'')$ correspond to so-called n^{th} approximation solutions to the restricted problem, the integral in Equation D1 may be solved by replacing the integral with the sum

$$G(\mu_i, \mu_k) = \sum_{j=1}^{n} a_j\, g(\mu_i, \mu_j)\, h(\mu_j, \mu_k) \quad \left\{\begin{array}{l} \mu = \mu_i \\ \mu' = \mu_j \\ \mu'' = \mu_k \end{array}\right\} \tag{D2}$$

for discrete values of μ_i and μ_k. The divisions μ_j are the positive roots of the Legendre polynomial $P_{2n}(\mu)$ $(0 \le \mu \le 1)$, and the weights a_j may be determined from the formula (Equation 61)

$$a_j = \frac{1}{\left(\dfrac{d[P_{2n}(\mu)]}{d\mu}\right)_{\mu=\mu_j}} \int_{-1}^{+1} \frac{P_{2n}(\mu)}{\mu - \mu_j}\, d\mu \quad . \tag{D3}$$

Now consider the integral equation

$$S(\mu, \mu_0) = f(\mu, \mu_0) + \int_0^1\!\!\int_0^1 g(\mu, \mu')\, h(\mu', \mu'')\, S(\mu'', \mu_0)\, d\mu''\, d\mu' \quad , \tag{D4}$$

where f, g, and h are known functions and S is to be solved for.

Let (Equation D2)

$$G(\mu, \mu'') = \int_0^1 g(\mu, \mu')\, h(\mu', \mu'')\, d\mu' \approx \sum_{j=1}^{n} a_j\, g(\mu, \mu_j)\, h(\mu_j, \mu'') \quad . \tag{D5}$$

Then Equation D4 becomes

$$S(\mu, \mu_0) = f(\mu, \mu_0) + \int_0^1 G(\mu, \mu'') \, S(\mu'', \mu_0) \, d\mu'' \quad , \tag{D6}$$

or,

$$S(\mu, \mu_0) \approx f(\mu, \mu_0) + \int_0^1 \left[\sum_{j=1}^{n} a_j \, g(\mu, \mu_j) \, h(\mu_j, \mu'') \right] S(\mu'', \mu_0) \, d\mu''$$

$$\approx f(\mu, \mu_0) + \sum_{k=1}^{n} \left\{ a_k \left[\sum_{j=1}^{n} a_j \, g(\mu, \mu_j) \, h(\mu_j, \mu_k) \right] S(\mu_k, \mu_0) \right\} \quad . \tag{D7}$$

Not let μ take on the discrete values

$$\mu = \mu_i \qquad (i = 1, \cdots, n) \quad . \tag{D8}$$

Equation D7 now becomes

$$S(\mu_i, \mu_0) = f(\mu_i, \mu_0) + \sum_{k=1}^{n} \left\{ a_k \left[\sum_{j=1}^{n} a_j \, g(\mu_i, \mu_j) \, h(\mu_j, \mu_k) \right] S(\mu_k, \mu_0) \right\} \quad . \tag{D9}$$

We have from Equation D9 a set of n equations in n unknowns of the form

$$\left. \begin{aligned} S(\mu_1, \mu_0) &= f(\mu_1, \mu_0) + C_{1,j,1} \, S(\mu_1, \mu_0) + \cdots + C_{1,j,n} \, S(\mu_n, \mu_0) \\ S(\mu_2, \mu_0) &= f(\mu_2, \mu_0) + C_{2,j,1} \, S(\mu_1, \mu_0) + \cdots + C_{2,j,n} \, S(\mu_n, \mu_0) \\ &\vdots \\ S(\mu_n, \mu_0) &= f(\mu_n, \mu_0) + C_{n,j,1} \, S(\mu_1, \mu_0) + \cdots + C_{n,j,n} \, S(\mu_n, \mu_0) \end{aligned} \right\} , \tag{D10}$$

where

$$C_{i,j,k} = a_k \left[\sum_{j=1}^{n} a_j \, g(\mu_i, \mu_j) \, h(\mu_j, \mu_k) \right] \qquad (i = 1, \cdots, n; \quad k = 1, \cdots, n) \quad . \tag{D11}$$

We thus may obtain from Equations D10 and D11 the n values $S(\mu_i, \mu_0)$ which correspond to the function $S(\mu, \mu_0)$ through the expression

$$S(\mu, \mu_0) \approx S(\mu_i, \mu_0) \qquad (\mu = \mu_i;\ i = 1, \cdots, n) \ . \tag{D12}$$

The reader should be reminded that the solutions of Equations D2 and D9 depend implicitly in the present context on the solutions to the restricted problems which in turn require the evaluation at times of certain functions which are undefined for the divisions $\mu_i = \mu_0$ $(i = 1, \cdots, n)$. More explicitly, it is required to evaluate the quantity

$$\gamma_0^m = H^{(m)}(\mu_0)\, H^{(m)}(-\mu_0) \ , \tag{D13}$$

where

$$H^{(m)}(x) = \frac{1}{\mu_1 \cdots \mu_n} \frac{\prod_{j=1}^{n} (x + \mu_j)}{\prod_{\alpha=1}^{n} (1 + k_\alpha^m x)} \tag{D14}$$

in the n^{th} approximation, and then evaluate the functions

$$\mathcal{F}^{(m)}(\mu_i, \mu_0) = \frac{\gamma_0^m}{1 - \mu_i/\mu_0} \ . \tag{D15}$$

We note that Equation D15 is undefined for $\mu_i = \mu_0$. Equation D15 may, however, be evaluated quite easily by writing it in the form

$$\mathcal{F}^{(m)}(\mu_i, \mu_0) = -\mu_0(\mu_0 + \mu_i)\, \mathcal{H}^{(m)}(\mu_i, \mu_0)\, \mathcal{H}^{(m)}(\mu_i, -\mu_0) \ , \tag{D16}$$

where $\mathcal{H}^{(m)}(\mu_i, x)$ is defined by the expression

$$\mathcal{H}^{(m)}(\mu_i, x) = \frac{1}{\mu_1 \cdots \mu_n} \frac{\left[\prod_{j=1}^{i-1} (x + \mu_j)\right] \left[\prod_{j=i+1}^{n} (x + \mu_j)\right]}{\prod_{\alpha=1}^{n} (1 + k_\alpha^m x)} \ . \tag{D17}$$

Equation D16 is defined for μ_0 in the range $(0 \le \mu_0 \le 1)$ including the values $\mu_0 = \mu_i$ $(i = 1, \cdots, n)$, but excluding the values $(\mu_0 = 1/k_\alpha^m)$ (cf. Equation D17). It is never necessary, however, to include the latter values of μ_0 in any theoretical discussions of outgoing intensity distributions since other more convenient values of μ_0 are just as useful.

It should be noted that, for low values of n, Equation D3 will in general not be the most advantageous one to use in determining the appropriate weights needed in evaluating Equations D2 and D11, and the roots (μ_i, μ_j, μ_k) of $P_{2n}(\mu)$ are correspondingly the incorrect discrete divisions of evaluation. The integrations in Equations D1 and D4 are over the interval $(0 \leq \mu \leq 1)$, and Equations D2 and D11 evaluate these integrals over the interval $(-1 \leq \mu \leq 1)$. In effect this is accomplished by implicitly setting the relevant functions in the integrands identically equal to zero in the interval $(-1 \leq \mu < 0)$; however this has the effect of causing the integrands to be discontinuous at $(\mu = 0)$ and therefore impossible to approximate accurately with a polynomial in μ of low degree (small number of terms).

A better method (Reference 8) of evaluating Equations D1 and D4 is through the use of the expressions

$$\bar{a}_i = \frac{1}{2} a_i \tag{D18}$$

and

$$\bar{\mu}_i = \frac{1}{2}(1 + \mu_i) \quad (i = \pm 1, \cdots, \pm n), \tag{D19}$$

yielding the 2n values $\bar{a}_i$ and $\bar{\mu}_i$ which are to replace the n values a_i and μ_i $(i = 1, \cdots, n)$ in Equations D2 and D11. This artifice effectively maps the interval $(-1 \leq \mu \leq 1)$ onto the interval $(0 \leq \mu \leq 1)$.

In order to reduce the number of terms involved in the relevant equations, it should suffice to solve now for the roots (μ_i) of $P_{2n}(\mu)$ for low values of n. It will also be required to develop some kind of interpolation scheme for obtaining the values of functions at the divisions μ_i and $\bar{\mu}_i$ which were previously determined numerically at other divisions for higher orders of n. The efficacy of the method will be most manifest where it is desired to restrict the number of computations to a minimum while still obtaining accurate results. For details of this method the reader is directed to the reference previously cited.

Appendix E

List of Symbols

The list of symbols is divided into the following seven categories: Coordinates; General Symbols; General Functions; Subscripts; Superscripts; Energy Functions; and Intensity, Scattering, and Transmission Functions.

For the most part this list is intended to be a guide and is in no sense complete. In many cases only the form of the symbol is given with the understanding that the explicit meaning of each symbol is to be obtained from the text; this requirement is expedient in view of the fact that to give a complete list, a great deal of repetition would be involved, and this would be both a waste of space and wearing upon the reader. The division of the list into various categories was performed with the aim of isolating certain symbols, such as scattering and transmission functions, which the reader will have recourse to most frequently.

Parentheses enclosing a number in the right column refer to an equation in which the symbol is defined or illustrated.

Coordinates:

θ, θ'	Zenith angle
μ, μ', μ''	Cosine of the zenith angle treated as a continuous variable
μ_i, μ_j, μ_k, $\bar{\mu}_i$	Cosine of the zenith angle treated as a discrete number
μ_0	Cosine of the zenith angle of an outside point source
ϕ, ϕ', ϕ''	Azimuthal angle
ϕ_0	π radians removed from the azimuthal angle of an outside point source
τ, τ_ν	Monochromatic optical depth
z	Geometric height

General Symbols:

A, A'	Area

A^*	Albedo (195)
a_i, a_j, a_k, $\bar{a}_i$	Gaussian weights (61)
b_r	Coefficients of the terms of a power series expansion for the Planck function (B3)
$C_{i,j,k}$	Coefficients of scattering functions involved in a system of linear algebraic equations (D11)
$C_{r,s}$	Coefficients of the terms in a series expansion for a particular integral (86-91)
c_0	Constant of integration (54)
D_i	Particle size distribution (16)
E	Energy (122); electric field vector
F	Related to net flux (C1)
F_0	Related to incident flux (46)
g	Probability of extinction by particles (122)
H	Magnetic field vector
I	Monochromatic specific intensity (1)
$\mathcal{J}$	Rate of emitted radiation (127)
k, k_α, k_α^m	Roots of characteristic equations (79, 104)
$M_{\pm\alpha}$, $M_{\pm\alpha}^m$	Constants of integration (76, 98)
m	Mass
N	Number of terms in series expansion (63, 69)
n	Degree of approximation (70); real part of refractive index
$\tilde{n}$	Complex refractive index
N_i	Number of particles per unit volume per unit radius range centered about r_i (12)
N_0	Total number of particles per unit volume (16)
$P_j^{(i)}$ (j = S, A)	Probability of scattering (S) or absorption (A) of one photon by a particle of radius r_i in dm

$Q_j^{(i)}$ (j = E, S, A)	Efficiency factor for extinction (E), scattering (S), or absorption (A) of a particle of radius r_i
r	Radius; distance
S	Surface (Figure 2)
T	Temperature
t	Time
V	Volume
x	Argument of function; in general indicates that the function is one of several arguments
α	Solid angle; variable of integration
β	General argument of function
Θ	Angle through which radiation is singly scattered (Figure 1)
κ	Imaginary part of complex index of refraction
λ	Wavelength; running number of summation
ν	Frequency of radiation
ρ	Density
σ	Area
τ_a ($a \neq \nu$)	Normal optical thickness
χ_j (j = E, S, A)	Effective cross-section for extinction (E), scattering (S), or absorption (A) of mass element dm (38)
ω	Solid angle
$\tilde{\omega}_0$	Effective albedo for single scattering (39)
$\tilde{\omega}_l$, $\tilde{\omega}_l^m$	Coefficients of terms in associated Legendre polynomial series expansion of the phase function for single scattering

General Functions:

$\left.\begin{array}{l} B_\nu(T) \\ B(\tau) \end{array}\right\}$	Planck function (29, 69)
$\left.\begin{array}{l} D_{l,\lambda}(x) \\ D_{l,\lambda}^m(x) \end{array}\right\}$	Characteristic functions (78, 103)

$\mathcal{F}(\mu_i, \mu_0)$	Function associated with H functions (D15)
$f(\mu)$	Undetermined function (47)
$f(\mu, \mu_0)$ $f^{(n)}(\tau_1;\ \mu, \mu_0)$	Inhomogeneous terms of integral equations (168, D4)
$G(\mu, \mu')$ $G(\mu_i, \mu_k)$	Integrals (D1, D2)
$g(\mu, \mu')$ $g^{(n)}(\tau_1;\ \mu, \mu')$	Factors in integral equations (169, D4)
$g(\phi)$	Undetermined function (47)
$H(x)$ $H^{(m)}(x)$	H functions (84, 108)
$\mathcal{H}^{(m)}(\mu_i, x)$	Modified H function (D17)
$h(\mu', \mu'')$	Factor in integral equation (D4)
$h_1^{(n)}(\mu, \mu_0)$ $h_2^{(n)}(\mu, \mu'')$	Integrals (163, 164)
$h(\tau)$	Undetermined function (50)
$J(\tau, \mu_i)$	Source function
$\mathcal{O}^{(n)}(\mu, \mu_0)$ $\mathcal{O}^{(n)}(\tau_1;\ \mu, \mu_0)$	Integrals (176, 180)
$P_l(x)$	Legendre polynomial
$P_l^m(x)$	Associated Legendre polynomial
$p(\mu, \phi;\ \mu', \phi')$	Phase function for single scattering (4, 6)
$\mathcal{P}(\cos\Theta)$	Phase function for single scattering (3)
γ_0 γ_0^m	Products of H functions (83, 107)
$\delta_{\alpha,\beta}$	Kronecker delta function (110, B9)
$\delta(\alpha - \beta)$	Dirac delta function (47)
$\xi_l(x)$ $\xi_l^m(x)$	Xi functions (80, 105)

*Subscripts:**

A	Absorption; atmosphere
C	Cloud

*All numerical subscripts except (0) refer in general to explicit values or functions.

D	Diffuse
E	Extinction; emission
G	General; ground
i	Discrete value or function; explicit particle size; running number of summation
j	Discrete value; running number of summation
k	Discrete value; running number of summation
l	Degree of function; discrete value; running number of summation
M	Molecule
m	Discrete value
n	Degree of function; discrete value
P	Particle
r	Degree of function; running number of summation; radial
S	Scattered
s	Degree of function; running number of summation
T	Transmitted; tangential
z	Height
α	Discrete value; running number of summation
β	Discrete value
λ	Degree of function; running number of summation
ν	Frequency
0	General subscript—meaning to be derived from text

Superscripts:

i	Explicit particle size; discrete value
k, l, m, n	Associated with or explicit order of function or argument; order of azimuth-independent terms in the Fourier series expansion of a function

0	Zero - order

Energy functions:

$E_{\nu}(z, \mu, \phi)$	Energy at a level (height) z which crosses an element of area in uni time and is restricted to the frequency range $(\nu, \nu + d\nu)$ and to an element of solid angle containing the direction (μ, ϕ). The level, frequency, and directional dependence in general may or may not be indicated
$\delta E_{\nu}(z, \mu, \phi)$	That fraction of $E_{\nu}(z, \mu, \phi)$ which is restricted to the time interval $(t, t + dt)$
$d[\delta E_{\nu}(z, \mu, \phi)]$	That fraction of $\delta E_{\nu}(z, \mu, \phi)$ which has arisen through an interaction with matter in the mass element dm
$\Delta E_{\nu}(z, \mu, \phi)$	That fraction of $E_{\nu}(z, \mu, \phi)$ which is extinguished by the matter in dm
$d[\Delta E_{\nu}(z, \mu', \phi')]$	That fraction of $\Delta E_{\nu}(z, \mu, \phi)$ which is scattered only into the solid angle $d\omega'$ containing the direction (μ', ϕ')

Intensity, Scattering, and Transmission Functions:

Because of the great number of intensity, scattering, and transmission functions found in the text, only the different forms will be indicated in the list of symbols. With the aid of the accompanying definitions this reduced list of symbols should suffice to allow a ready interpretation of any such function from the form of the function, even though the specific notation will vary.

Intensity functions of the form. . .

$\left.\begin{matrix} I(\tau, \mu, \phi) \\ I_{\nu}(z, \mu, \phi) \end{matrix}\right\}$	Intensity of monochromatic radiation at respectively an optical depth τ or a height z in the direction (μ, ϕ)
$I_A(\tau, \mu, \phi)$	Same as $I(\tau, \mu, \phi)$ above referred to a restricted solution for a stratum of gaseous atmosphere
$I_C(\tau, \mu, \phi)$	Same as $I(\tau, \mu, \phi)$ above referred to a restricted solution for a cloud
$I_A^*(\tau, \mu, \phi)$	Same as $I_A(\tau, \mu, \phi)$ above referred to a composite solution; e.g., a component due to a neighboring cloud is added to $I_A(\tau, \mu, \phi)$
$I_C^*(\tau, \mu, \phi)$	Same as $I_C(\tau, \mu, \phi)$ above referred to a composite solution; e.g., a component due to a neighboring stratum of gas is added to $I_C(\tau, \mu, \phi)$

$I^*(0, \mu, \phi)$	Resultant intensity at the top of the atmosphere ($\tau = 0$) for the total composite problem
$I(\tau_\alpha, \mu; \tau_\gamma - \tau_\beta)$	Intensity of monochromatic radiation at a normal optical depth τ_α in the direction μ referred to a stratum of normal optical thickness $(\tau_\gamma - \tau_\beta)$
$I_A(\tau_\alpha, \mu; \tau_\gamma - \tau_\beta)$	Same as $I(\tau_\alpha, \mu; \tau_\gamma - \tau_\beta)$ above referred to a restricted solution for a stratum of gaseous atmosphere of normal optical thickness $(\tau_\gamma - \tau_\beta)$
$I_C(\tau_\alpha, \mu; \tau_\gamma - \tau_\beta)$	Same as $I(\tau_\alpha, \mu; \tau_\gamma - \tau_\beta)$ above referred to a restricted solution for a stratum of cloud of normal optical thickness $(\tau_\gamma - \tau_\beta)$
$I_C^*(\tau_\alpha, \mu; \tau_\gamma - \tau_\beta)$	Same as $I_C(\tau_\alpha, \mu; \tau_\gamma - \tau_\beta)$ above referred to a composite solution; e.g., a component due to a neighboring stratum of gas is added to $I_C(\tau_\alpha, \mu; \tau_\gamma - \tau_\beta)$
$I^*(0, \mu; \tau_3)$	Same as $I^*(0, \mu, \phi)$ above in the direction μ instead of (μ, ϕ) referred to an atmosphere-cloud composite of total normal optical thickness τ_3
$I_C(\tau_\alpha, \mu)$	Same as $I_C(\tau_\alpha, \mu; \tau_\gamma - \tau_\beta)$ above for a cloud of infinite optical thickness
$I^*(0, \mu)$	Same as $I^*(0, \mu; \tau_3)$ above for τ_3 infinite
$I^{(n)}(\tau, \mu)$	The azimuth-independent factor of the n^{th} term in the Fourier series expansion of $I(\tau, \mu, \phi)$ above
I_G	Intensity of monochromatic thermal radiation isotropically emitted by the ground (186)
$I_i[B(\tau)]$	A particular integral of the equation of transfer appropriate to a restricted solution to the problem of limb darkening (B5)
$I_i(\mu_0)$	A particular integral of the equation of transfer appropriate to a restricted solution to the problem of diffuse reflection and transmission (82)
$I_G^{(0)}(\tau, \mu_i)$	The general solution to the azimuth-independent equation of transfer
$d[\delta_\pm I_j(z, \mu, \phi)]$ $(j = S, A, E)$	A loss or gain [respectively (-) or (+)] to the intensity of the radiation field at z in the direction (μ, ϕ) due to scattering (S), absorption (A), or emission (E) processes with particles in the radius range $(r_i, r_i + dr_i)$ contained in the mass element dm

$\delta_{\pm} I_j (z, \mu, \phi) (j = S, A, E)$	The same as $d[\delta_{\pm} I_j (z, \mu, \phi)]$ above integrated over all particle sizes

Scattering functions of the form. . .

$S(\tau_1; \mu, \phi; \mu', \phi')$	Scattering function (cf. Equation 129) appropriate to a stratum of normal optical thickness τ_1 for radiation scattered from the direction $(-\mu', \phi')$ into the direction (μ, ϕ)
$S_A(\tau_1; \mu, \phi; \mu', \phi')$	Same as $S(\tau_1; \mu, \phi; \mu', \phi')$ referred to the restricted solution appropriate to a stratum of gaseous atmosphere
$S_C(\mu, \phi; \mu', \phi')$	Same as $S(\tau_1; \mu, \phi; \mu', \phi')$ above referred to the restricted solution appropriate to a semi-infinitely thick cloud
$\tilde{S}_A(\tau_1; -\mu, \phi; -\mu', \phi')$	Same as $S_A(\tau_1; \mu, \phi; \mu', \phi')$ but in the reverse sense, i.e., the reverse side of the stratum
$S_C^*(\tau_1; \mu, \phi; \mu', \phi')$	Same as $S_C(\mu, \phi; \mu', \phi')$ above referred to the composite problem; e.g., a contribution is added to $S_C(\mu, \phi; \mu', \phi')$ due to an overlying atmospheric stratum of normal optical thickness τ_1
$S_C^{(0)}(\mu, \mu')$	The azimuth-independent analog to $S_C(\mu, \phi; \mu', \phi')$ above
$S_C^{(0)}(\mu, \mu'; \tau_2 - \tau_1)$	Same as $S_C^{(0)}(\mu, \mu')$ above for a cloud of optical thickness $(\tau_2 - \tau_1)$
$\left.\begin{array}{l} S_A^{(n)}(\mu, \mu') \\ S_A^{(n)}(\tau_1; \mu, \mu') \end{array}\right\}$	Azimuth-independent factor of the n^{th} term in the Fourier series expansion of $S_A(\tau_1; \mu, \phi; \mu', \phi')$ above
$S_C^{(n)}(\mu, \mu')$	Azimuth-independent factor of the n^{th} term in the Fourier series expansion of $S_C(\mu, \phi; \mu', \phi')$ above
$\left.\begin{array}{l} S_C^*(\mu, \mu') \\ S_C^*(\tau_1; \mu, \mu') \end{array}\right\}$	Azimuth-independent factor of the n^{th} term in the Fourier series expansion of $S_C^*(\tau_1; \mu, \phi; \mu', \phi')$ above
$\tilde{S}_A^{(n)}(-\mu, -\mu')$	Azimuth-independent factor of the n^{th} term in the Fourier series expansion of $\tilde{S}_A(\tau_1; -\mu, \phi; -\mu', \phi')$ above
$S^*(\tau_1; \mu, \phi; \mu_0, \phi_0)$	Scattering function appropriate to the solution of the total composite problem at the top of the atmosphere
$\left.\begin{array}{l} S^{*(n)}(\mu, \mu_0) \\ S^{*(n)}(\tau_1; \mu, \mu_0) \end{array}\right\}$	Azimuth-independent factor of the n^{th} term in the Fourier series expansion of $S^*(\tau_1; \mu, \phi; \mu_0, \phi_0)$ above

$T(\tau_1;\ \mu, \phi;\ \mu', \phi')$	Transmission function (cf. Equation 130) appropriate to a stratum of normal optical thickness τ_1 for radiation originally in the direction $(-\mu', \phi')$ which is transmitted diffusely through this stratum into the direction $(-\mu, \phi)$
$T_A(\tau_1;\ \mu, \phi;\ \mu', \phi')$	Same as $T(\tau_1;\ \mu, \phi;\ \mu', \phi')$ referred to the restricted solution appropriate to a stratum of gaseous atmosphere
$\tilde{T}_A(\tau_1;\ -\mu, \phi;\ -\mu', \phi')$	Same as $T_A(\tau_1;\ \mu, \phi;\ \mu', \phi')$ but in the reverse sense, i.e., the reverse side of the stratum
$T_A^*(\tau_1;\ \mu, \phi;\ \mu', \phi')$	Same as $T_A(\tau_1;\ \mu, \phi;\ \mu', \phi')$ above referred to a composite solution; e.g., a component due to a neighboring cloud is added to $T_A(\tau_1;\ \mu, \phi;\ \mu', \phi')$
$T_A^{(n)}(\mu, \mu')$	Azimuth-independent factor of the n^{th} term in the Fourier series expansion of $T_A(\tau_1;\ \mu, \phi;\ \mu', \phi')$
$\tilde{T}_A^{(n)}(-\mu, -\mu')$	Azimuth-independent factor of the n^{th} term in the Fourier series expansion of $\tilde{T}_A(\tau_1;-\mu, \phi;-\mu', \phi')$
$T_C^{(0)}(\mu, \mu';\ \tau_2 - \tau_1)$	The azimuth-independent transmission function appropriate to the restricted solution for a stratum of cloud of optical thickness $(\tau_2 - \tau_1)$
$\tilde{T}_C^{(0)}(-\mu, -\mu';\ \tau_2 - \tau_1)$	Same as $T_C^{(0)}(\mu, \mu';\ \tau_2 - \tau_1)$ above except in the reverse sense through the cloud

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

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