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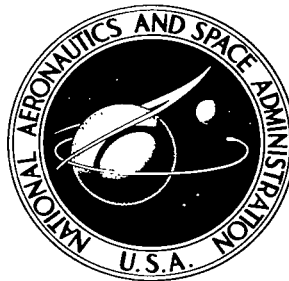
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TRANSIENT SURFACE TEMPERATURE DISTRIBUTION OF A THIN-WALLED SPHERE SUBJECTED TO RADIATION IN SPACE

by Glenn E. Schober

Lewis Research Center

Cleveland, Ohio

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SUMMARY

This report derives and investigates the partial differential equation governing the transient temperature distribution of a thin-walled sphere in the conditions of outer space. The sphere is initially at a uniform temperature and then subjected to radiant energy transfer. A source, such as the Sun, radiates energy to the sphere, and the sphere radiates energy to a sink, such as outer space, at some background temperature. Energy is transferred within the sphere wall by conduction.

Formulas are derived for cases when the differential equation can be approximated, and approximations of the complete solution are given. From the solutions for a special case it is possible to deduce the rapid growth of the temperature distribution on the sphere and the symmetry connected with it.

In addition, formulas are derived for the variation of the average temperature level of the sphere with time both in the case of a constant specific heat and in the case of a temperature dependent (Debye model) specific heat. These formulas also apply to a body cooling down in the absence of a radiating source.

INTRODUCTION

In the study of objects outside the Earth's atmosphere, the role of thermal radiation has become increasingly important. Since convective heat transfer is no longer a factor, a body's temperature level and temperature distribution are determined by radiative energy transfer both to and from the body and by conduction within the body itself. For a vehicle or satellite encircling the Earth, the temperature depends on whether the object is exposed to the Sun or is in the Earth's shadow. For this reason, the transient tem-

perature distribution of the object is of interest because of the effect it may have on the behavior of the vehicle and its content.

Various facilities have been developed to simulate the conditions in space. In one of these (described in ref. 1), the temperatures and other characteristics of models have been examined. For such tests as well as for space vehicles, it is important to have a transient theory in addition to the steady state to compare with experimental results. The time required to approximate steady-state conditions can then be estimated. Also, characteristics of the model, such as surface emissivity and absorptivity, can be measured by comparing the experimental data to the theoretical values.

This report examines the nonlinear partial differential equation that governs the transient temperature distribution of an object in space. Because the transient temperature may cover a wide range before steady-state conditions are achieved, the precise variation of specific heat with temperature can be quite significant in many cases. The Debye model for the specific heat was used to account for this variation.

The object is taken to be a sphere, although the results on the variation of average temperature with time are applicable to other objects as well.

Since an exact solution for the complete nonlinear equation is not known, various approximations are used to investigate cases that have physical significance; from them, an approximation of the complete solution is given.

Nondimensionalization of Energy Equation

In deriving the partial differential equation that governs the transient temperature distribution of a sphere, the following assumptions are made.

(1) The sphere has a uniform initial temperature T_0^*.

(2) The background temperature T_{sp}^* is constant.

(3) The walls of the sphere are so thin that the inside and outside temperatures at a point are identical.

(4) All energy transfer to the sphere is by parallel radiation from a single source.

(5) Lambert's cosine law governs absorption.

(6) The wall material has an absorptivity and an emissivity that are independent of temperature and wavelength.

(7) Internal radiation is not considered so that the temperature variation on the sphere will be affected only by conduction in the wall.

Consider the elemental volume in figure 1. The energy balance per unit time for the element may be expressed by the following partial differential equation (ref. 2):

$$\rho b c^* \frac{\partial T^*}{\partial t^*} = \frac{kb}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \, \frac{\partial T^*}{\partial \theta} \right) - \epsilon \sigma \left(T^{*4} - T_{sp}^{*4} \right) + \alpha \Phi \cos \theta \, \eta(\theta) \tag{1}$$
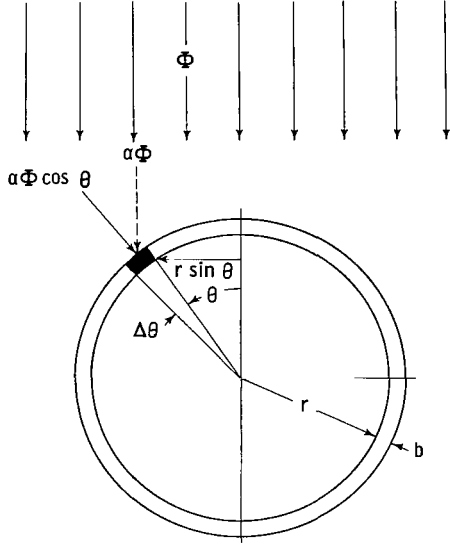
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Figure 1. - Notation for sphere.

$$\eta(\theta) = \begin{cases} 1 \text{ for } 0 \leq \theta \leq \dfrac{\pi}{2} \\[2ex] 0 \text{ for } \dfrac{\pi}{2} < \theta \leq \pi \end{cases}$$

(All symbols are defined in appendix A.) Equation (1) may be made nondimensional by selecting a reference temperature and time. A convenient normalizing temperature T_∞^* based on infinite thermal conductivity and steady-state conditions is defined as follows (see appendix B):

$$\alpha\Phi \equiv 4\epsilon\sigma\left(T_\infty^{*4} - T_{sp}^{*4}\right) \qquad (B4)$$

A reference time t_r^* may be defined as follows:

$$t_r^* = \frac{\rho b c_\infty^*}{\epsilon\sigma T_\infty^{*3}}$$

The nondimensional quantities then become

$$T = \frac{T^*}{T_\infty^*} \qquad\qquad t = \frac{t^*}{t_r^*} \qquad c = \frac{c^*}{c_\infty^*}$$

$$\mu = \frac{kb}{r^2\epsilon\sigma T_\infty^{*3}} \qquad T_{sp} = \frac{T_{sp}^*}{T_\infty^*} \qquad x = \cos\theta$$

Using these quantities results in

$$c\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\mu(1-x^2)\frac{\partial T(x,t)}{\partial x}\right] - \left[T^4(x,t) - T_{sp}^4\right] + 4\left(1 - T_{sp}^4\right)x\eta(x) \qquad (2)$$

where

$$\eta(x) = \begin{cases} 1 \text{ for } 0 \leq x \leq 1 \\[2ex] 0 \text{ for } -1 \leq x < 0 \end{cases}$$

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Analysis of Initial Temperature Development

Linearization of energy equation. - To linearize the T^4 term in equation (2), consider the initial departure from a uniform temperature $T_0 = T_0^*/T_\infty^*$. In the range where this departure is small, assume that

$$T = T_0(1 + \delta)$$

where $\delta \ll 1$. Since

$$T^4 = T_0^4 + 4T_0^4\delta + O(\delta^2) = 4T_0^3 T - 3T_0^4 + O(\delta^2)$$

the linearized equation (neglecting terms of order δ^2) is

$$c\,\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\mu(1 - x^2)\,\frac{\partial T(x,t)}{\partial x}\right] - \left[4T_0^3 T(x,t) - 3T_0^4 - T_{sp}^4\right] + 4(1 - T_{sp}^4)x\eta(x) \qquad (3)$$

In general, the conductivity k and, therefore, the conduction parameter μ will depend on temperature. For the present small departure from uniform temperature, however, μ can be assumed to be constant.

Equation (3) is solved in appendix C for the case of constant specific heat. (The case $T_0 = 0$ is treated in appendix D.) The solution is

$$T(x,t) = T_0 + \frac{1}{4T_0^3}\left(1 - T_0^4\right)\left(1 - e^{-4T_0^3 t/c}\right) + \frac{x\left(1 - T_{sp}^4\right)}{\mu + 2T_0^3}\left[1 - e^{-2t\left(\mu + 2T_0^3\right)/c}\right]$$

$$+ \frac{1 - T_{sp}^4}{3}\sum_{j=1}^{\infty}\binom{\frac{3}{2}}{j+1}\frac{(4j+1)P_{2j}(x)}{\mu j(2j+1) + 2T_0^3}\left\{1 - e^{-t\left[\mu 2j(2j+1) + 4T_0^3\right]/c}\right\} \qquad (4)$$

where

$$\binom{a}{i} = \frac{a(a-1)\ldots(a-i+1)}{i!}$$

Figure 2 gives a plot of this solution in the case $\mu = 15$ and $T_0 = T_{sp} = 0$.
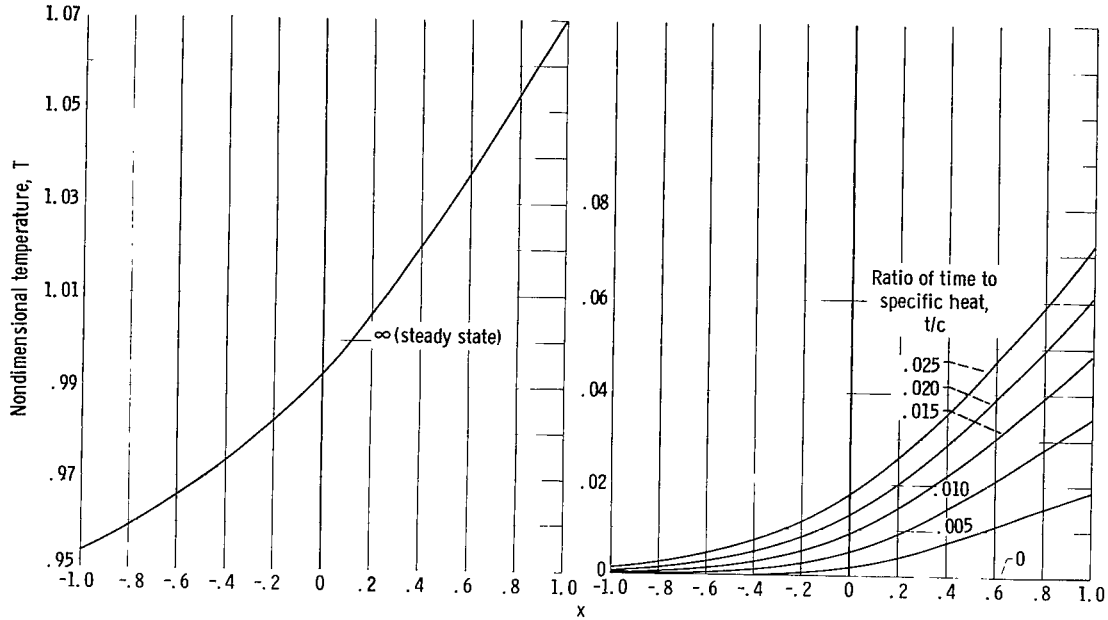
Figure 2. – Transient temperature distribution on sphere at various times for constant specific heat. Conduction parameter, 15; $T_0 = T_{sp} = 0$.

Range of validity. – The linearization of the T^4 term in equation (2) is a valid approximation only in a range where the departure from the initial temperature is small. To estimate the range of validity, $|\delta|$ can be restricted to values less than a particular value d, where d is chosen to provide the desired accuracy in approximating T^4 by $T_0^4(1 + 4d)$.

Two particular cases for which this restriction is satisfied are derived in appendix D. The first is a limit in time for which the solution curve is valid. Whenever the time $t < \tau = cdT_0/4$, the departure δ is such that $|\delta| < d$, except for $T_0 = 0$, which is also discussed in appendix D.

In the other case, if T_0 is near 1, then a limit on the conduction parameter μ guarantees validity of the solution curves for all time. Whenever $\mu > (29/24d) - (2/3)$ and $T_0 = 1$, the departure δ is such that $|\delta| < d$. In this case, letting $t \to \infty$ results in the following steady-state distribution for the sphere:

$$T(x) \equiv \lim_{t \to \infty} T(x, t) = 1 + \frac{x\left(1 - T_{sp}^4\right)}{\mu + 2} + \frac{1 - T_{sp}^4}{3} \sum_{j=1}^{\infty} \binom{\frac{3}{2}}{j + 1} \frac{(4j + 1)P_{2j}(x)}{\mu j(2j + 1) + 2} \tag{5}$$

For $T_{sp} = 0$ this is identical with the distribution derived in reference 2. Equation (5)

is used in figure 2 for comparison with the initial development of the temperature distribution.

Analysis for Large Conductivity Case

If the sphere has a large enough conductivity, then the variation in temperature about an average temperature would be expected to be small. To examine this situation, an average temperature defined by

$$T(t) = \frac{1}{2} \int_{-1}^{1} T(x, t)dx$$

can be used.

Integrating both sides of the original partial differential equation (eq. (2)) yields

$$\frac{1}{2} \int_{-1}^{1} c \frac{\partial T(x, t)}{\partial t} dx = \frac{\mu}{2} \int_{-1}^{1} \frac{\partial}{\partial x} \left[\left(1 - x^2\right) \frac{\partial T(x, t)}{\partial x} \right] dx$$

$$- \frac{1}{2} \int_{-1}^{1} \left[T^4(x, t) - T_{sp}^4 \right] dx + 2(1 - T_{sp}^4) \int_{-1}^{1} x\eta(x)dx$$

$$= 0 - \frac{1}{2} \int_{-1}^{1} T^4(x, t)dx + 1 \tag{6}$$

Assume that at any given time the temperature differences on the sphere are small because of the large value of μ. Then a perturbation technique similar to that used to get equation (3) can be used, that is, let

$$T(x, t) = T(t) + \epsilon(x, t) \qquad \epsilon(x, t) \approx |\delta| \ll T$$

Then

$$\frac{1}{2} \int_{-1}^{1} T^4(x, t)dx = \frac{1}{2} \int_{-1}^{1} \left[T^4(t) + 4T^3(t)\epsilon + O(\delta^2) \right] dx$$

$$= T^4(t) + 2T^3(t) \int_{-1}^{1} \left[T(x, t) - T(t) \right] dx + O(\delta^2)$$

$$= T^4(t) + 0 + O(\delta^2)$$

$$\approx T^4(t)$$

In general, the specific heat c may depend on temperature; however, since the temperature differences on the sphere at any given time are assumed to be small, $\partial c(T)/\partial x \approx 0$ for $-1 \leq x \leq 1$ at any given time. Therefore, integrating by parts results in

$$\frac{1}{2} \int_{-1}^{1} c(T) \frac{\partial T(x,t)}{\partial t} dx \approx \frac{1}{2} \left[c(T) \int \frac{\partial T(x,t)}{\partial t} dx \right]_{-1}^{1} - 0$$

$$= c(T) \frac{d}{dt} \left[\frac{1}{2} \int_{-1}^{1} T(x,t) dx \right]$$

$$= c(T) \frac{dT(t)}{dt}$$

By substituting this into equation (6), the equation for the average temperature becomes
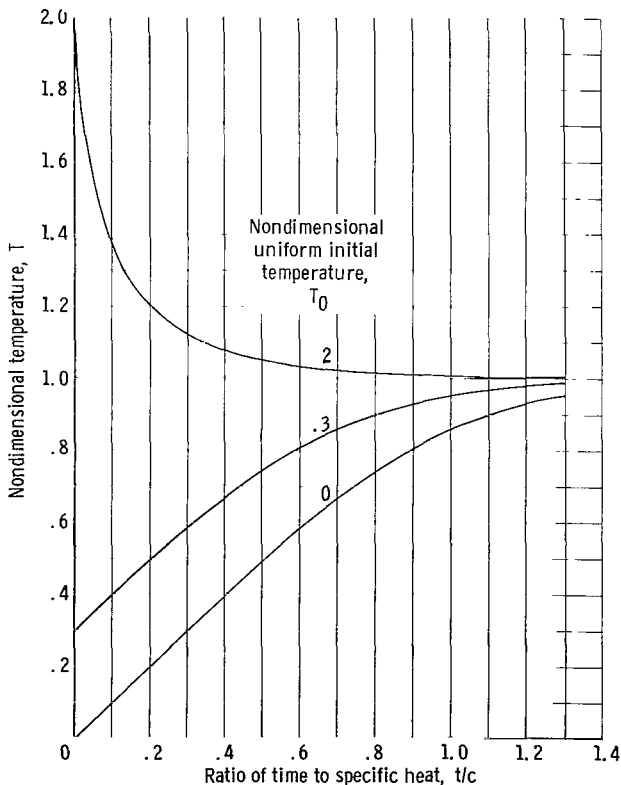
$$c(T) \frac{dT(t)}{dt} = 1 - T^4(t) \tag{7}$$

Figure 3. – Variation of average temperature with time for constant specific heat.

If the radiating source is absent (i. e., $\Phi = 0$), then the temperature on the sphere remains uniform. In this special case, equation (2) with the same nondimensionalization reduces immediately to equation (7). Therefore the solutions of this section apply also to this special case.

Solution for constant specific heat. - If c(T) is taken as a constant c, then equation (7) for the average temperature can be solved directly. The solution is

$$t + C_0 = c\left(\frac{1}{2} \tan^{-1} T + \frac{1}{4} \ln \left| \frac{1+T}{1-T} \right| \right) \tag{8}$$

where C_0 is determined from the initial temperature by $t(T_0) = 0$. The branch of the solution where $0 \leq T < 1$ applies if $T_0 < 1$. If $T_0 > 1$, the branch where $T > 1$ applies.

Figure 3 is a graph of this solution for the cases $T_0 = 0$, $T_0 = 0.3$, and $T_0 = 2$. Solutions for other initial temperatures can

be obtained simply by translating the time axis. For example, the curve for $T_0 = 0.3$ is exactly the curve for $T_0 = 0$ with the abscissa shifted to the right by

$$\frac{t}{c} = \frac{1}{2}\tan^{-1}(0.3) + \frac{1}{4}\ln\frac{1.3}{0.7} = 0.3$$

Solution for Debye specific heat. - In general, the specific heat c may depend on temperature. In order to examine this case, the Debye model for specific heat will be used since it represents a good approximation for many materials, such as elemental metals.

The theory and formulas used are derived in most statistical mechanics texts, including references 3 and 4. The agreement of the Debye model with experimental results is also discussed in these references. Tables of Debye characteristic temperatures Θ^* for elemental metals and some crystals can be found in references 3 and 4. For the lower temperature range, the characteristics are presented more completely in reference 5.

The specific heat for the Debye model is given by

$$c^* = 3c_\infty^*\left(\frac{T^*}{\Theta^*}\right)^3 \int_0^{\Theta^*/T^*} \frac{e^x x^4}{(e^x - 1)^2}\,dx$$

where Θ^* is the Debye characteristic temperature. Since $c = c^*/c_\infty^*$ and $T^*/\Theta^* = T/\Theta$, the time variational equation (7) for large conductivity becomes

$$3\left(\frac{T}{\Theta}\right)^3 \int_0^{\Theta/T} \frac{e^x x^4}{(e^x - 1)^2}\,dx\,\frac{dT}{dt} = 1 - T^4 \tag{9}$$

The solution to this equation, derived in appendix E, is

$$t + C_0 = 3\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\frac{2n-1}{2n+3}\overline{B}_n\Theta^{2n}$$

$$\times \left[\frac{(-1)^n}{2}\tan^{-1}T + \frac{1}{4}\log\left|\frac{1+T}{1-T}\right| - \sum_{j=0}^{m}\frac{1}{(2n-4j-1)T^{2n-4j-1}}\right] \tag{10}$$

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where C_0 is determined by boundary conditions; $m = (n - 1)/2$ or $m = (n - 2)/2$ depending on whether n is odd or even, except when $n = 0$, in which case the summation on j is omitted and $\overline{B}_n$ are the following nonzero modified Bernoulli numbers (altered by setting $\overline{B}_0 = -1$):

$$\overline{B}_0 = -1 \qquad \overline{B}_1 = \frac{1}{6} \qquad \overline{B}_2 = \frac{1}{30}$$

$$\overline{B}_3 = \frac{1}{42} \qquad \overline{B}_4 = \frac{1}{30} \qquad \overline{B}_5 = \frac{5}{66}$$

$$\overline{B}_6 = \frac{691}{2730} \qquad \overline{B}_7 = \frac{7}{6} \qquad \cdot \; \cdot \; \cdot$$

If $T_0 > 1$, the branch of the solution where $T > 1$ applies. If $\Theta/2\pi < T_0 < 1$, the solution holds where $\Theta/2\pi < T < 1$.

Appendix E outlines a method for obtaining a solution for smaller values of T. The Debye T^3 approximation shall be used for very low temperatures (say $T*/\Theta* \leq 1/6$) where $c*(T)$ is given approximately by

$$c* = \frac{4}{5} \pi^4 c_\infty^* \left(\frac{T*}{\Theta*}\right)^3$$

With $c = \dfrac{c*}{c_\infty^*}$ and $\dfrac{T*}{\Theta*} = \dfrac{T}{\Theta}$, the time variational equation (7) becomes

$$\frac{4}{5} \pi^4 \left(\frac{T}{\Theta}\right)^3 \frac{dT}{dt} = 1 - T^4$$

which can be solved explicitly to yield

$$T(t) = \left[1 - \left(1 - T_0^4\right) e^{-5\Theta^3 t/\pi^4}\right]^{1/4} \qquad (11)$$

Figure 4 is a graph of the average temperature for various Debye specific heats. For $T_0 = 0$, formula (11) is used up to $T/\Theta = 1/6$ and formula (10) from there on. The constant C_0 in formula (10) is determined from the boundary condition when $T/\Theta = 1/6$.
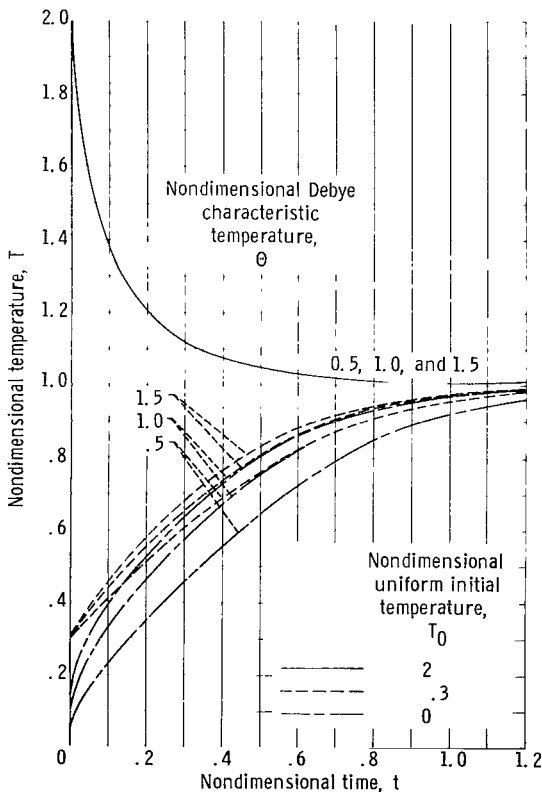
Figure 4. – Variation of average temperature with time for various Debye specific heats.

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Approximate Solution for Space-Dependent Temperature

Expressions for the average temperature of the sphere for large conductivity have been developed in the preceding section. An approximation to the spatial distribution of temperature, also for large conductivity, can be found as follows. From equation (5), which is valid for $\mu > (29/24d) - (2/3)$ and constant spherical heat, the temperature difference between points symmetric with respect to the $\theta = \pi/2$ plane at steady state can be expressed as

$$T(x) - T(-x) = \frac{2x(1 - T_{sp}^4)}{\mu + 2} + \frac{1 - T_{sp}^4}{3} \sum_{j=1}^{\infty} \binom{\frac{3}{2}}{j+1} \frac{4j+1}{\mu j(2j+1)+2} \left[P_{2j}(x) - P_{2j}(-x)\right]$$

Since the Legendre polynomials $P_{2j}(x)$ are even functions,

$$T(x) - T(-x) = \frac{2x(1 - T_{sp}^4)}{\mu + 2} \cong \frac{2x(1 - T_{sp}^4)}{\mu} \tag{12}$$

For large μ, the same temperature difference as a function of time can be expressed from appendix C as follows (where again the even Legendre polynomials drop out):

$$T(x,t) - T(-x,t) = \frac{2x(1 - T_{sp}^4)}{\mu} \left(1 - e^{-2\mu t/c}\right) \tag{13}$$

Define the ratio of these two temperature differences as the ratio of formation $R(x,t)$, that is,

$$R(x,t) \equiv \frac{T(x,t) - T(-x,t)}{T(x) - T(-x)} \tag{14}$$

From equations (12) to (14), the ratio is

$$R(x,t) = 1 - e^{-2\mu t/c} \tag{15}$$

The fact that $R(x,t)$ is independent of x indicates that the temperature distribution on the sphere grows in proportion to the steady-state distribution.

Expression (15) is valid in the same range that the approximate solution (eq. (4)) is valid provided μ is always large $\mu > (29/24d) - (2/3)$. Figure 5 illustrates the

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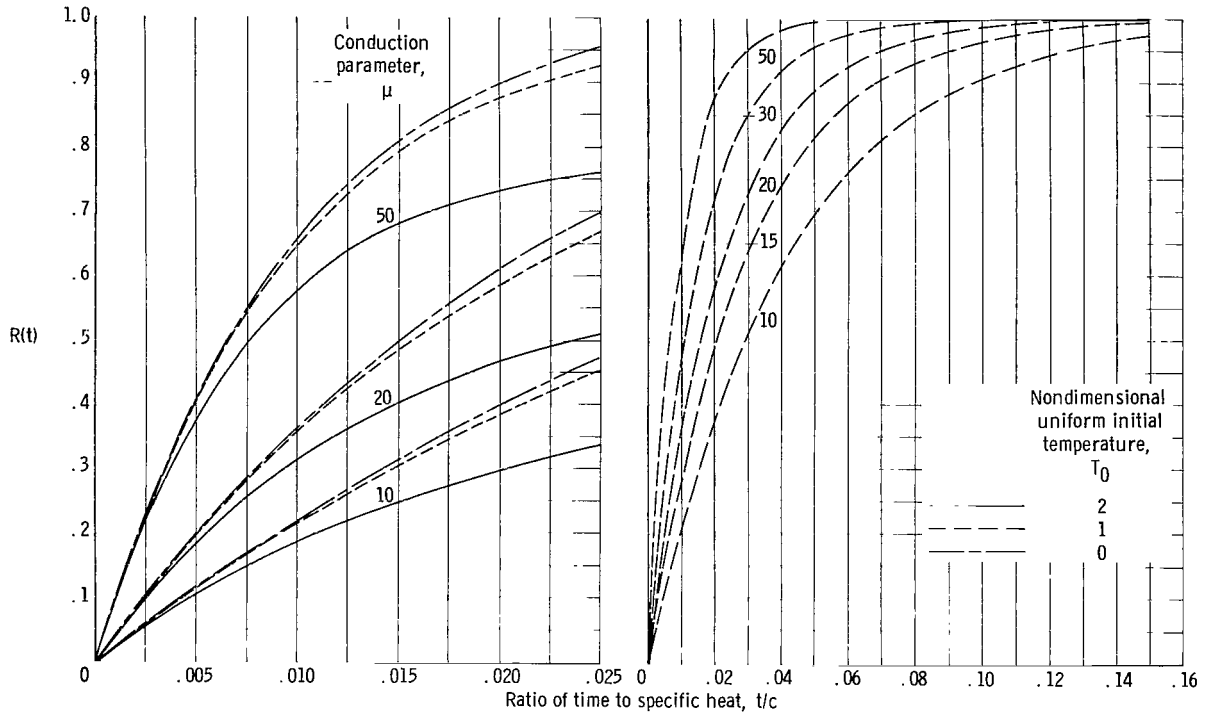
(a) For various nondimensional uniform initial temperatures.   (b) Nondimensional uniform initial temperature, 1.

Figure 5. - Ratio of formation (proportion of steady-state distribution formed).

ratio of formation for several values of the conduction parameter μ and initial temperature T_0. Figure 5(a) shows that the distribution depends mostly upon the conduction parameter μ when μ is large and T_0 is small. The time required to approach the steady state for various μ values is shown in figure 5(b).

The fact that $R(x, t)$ is independent of x indicates that, in the temperature expansion,

$$T(x, t) = T(t) + \epsilon(x, t)$$

it should be possible to express $\epsilon(x, t)$ as a separable function of x and t. Such a separability is shown for constant specific heat in the following section.

Solution for constant specific heat. - The temperature for this case is shown in appendix C to be

$$T(x, t) = T(t) + \frac{1 - T_{sp}^4}{\mu} \left(1 - e^{-2\mu t/c}\right) P_1(x) + \frac{1}{3} \frac{1 - T_{sp}^4}{\mu} \sum_{i=1}^{\infty} \binom{\frac{3}{2}}{i + 1} \frac{4i + 1}{i(2i + 1)} \left[1 - e^{-2\mu i(2i+1)t/c}\right] P_{2i}(x) \quad (16a)$$

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For large μ, equation (5) is

$$T(x) = 1 + \frac{1 - T_{sp}^4}{\mu} P_1(x) + \frac{1}{3} \frac{1 - T_{sp}^4}{\mu} \sum_{i=1}^{\infty} \binom{\frac{3}{2}}{i+1} \frac{4i+1}{i(2i+1)} P_{2i}(x) \qquad (16b)$$

and

$$T(x,t) = T(t) + \left(1 - e^{-2\mu t/c}\right)\left[T(x) - 1\right]$$

$$+ \frac{1}{3} \frac{1 - T_{sp}^4}{\mu} \sum_{i=1}^{\infty} \binom{\frac{3}{2}}{i+1} \frac{4i+1}{i(2i+1)} \left[e^{-2\mu t/c} - e^{-2\mu i(2i+1)t/c}\right] P_{2i}(x) \qquad (16c)$$

For large μ, the last terms can be neglected in comparison to the other terms. Consequently,

$$T(x,t) = T(t) + \left(1 - e^{-2\mu t/c}\right)\left[T(x) - 1\right] \qquad (17)$$

which is the desired separable form of the temperature distribution. The time dependent portion of $\epsilon(x,t)$ is seen to be equal to $R(x,t)$ from equation (15).

Solution for Debye specific heat. - In general, the specific heat will not be a constant but may be assumed to follow a Debye model. To approximate the complete solution in this case, a solution of the same type as that used in the constant specific heat case should be used.

In order to justify the form of equation (17) for a Debye specific heat, it should be stated that the conduction has been assumed large so that at any instant the temperature does not vary greatly on the sphere. Therefore at any instant the specific heat is nearly constant for the sphere. As a result, the departure from the average temperature can be calculated at any given time by equation (17).

Figure 6 illustrates a sample calculation using equation (17) for the case $\mu = 15$ and $T_{sp} = 0$ for two initial temperatures, $T_0 = 0$ and $T_0 = 2$. Figure 6(a) shows the results for constant specific heat, while figure 6(b) shows the results for Debye specific heat.
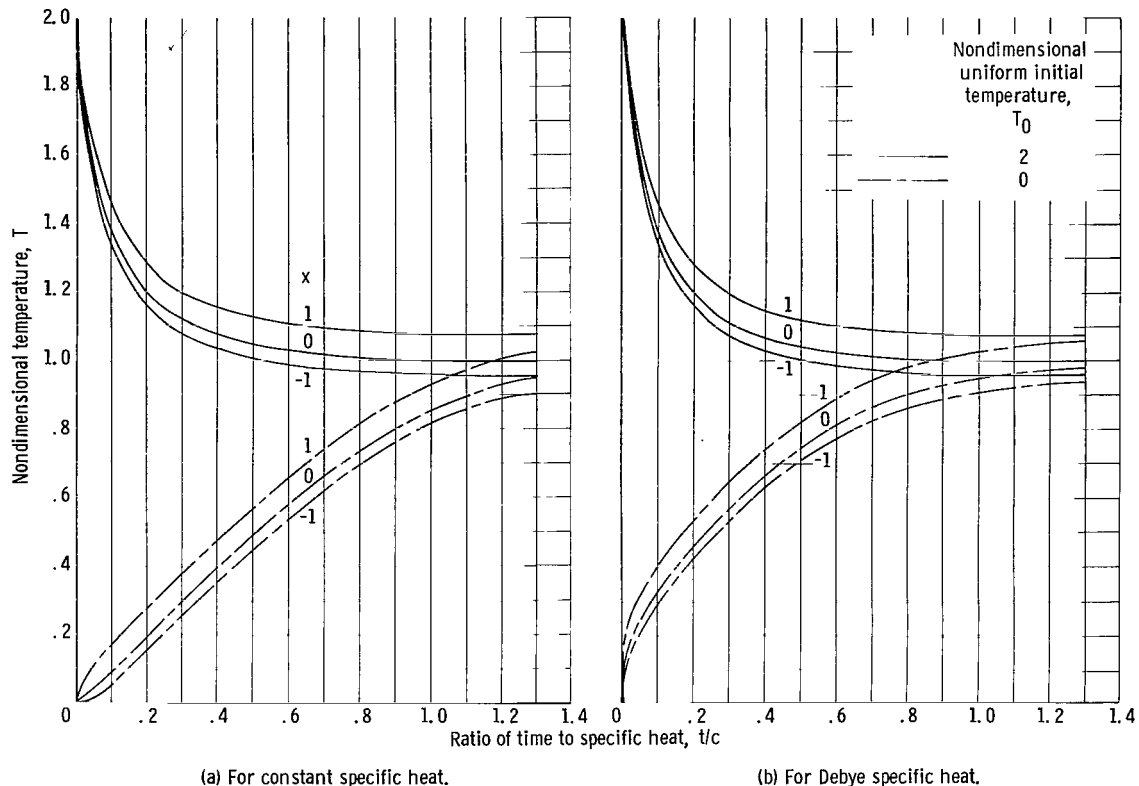
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Figure 6. – Transient temperature distribution on sphere. Conduction parameter, 15; nondimensional background temperature, 0.

CONCLUDING REMARKS

The equation governing the transient temperature distribution on a sphere in space was solved for certain cases wherein the perturbation of the temperature is small compared to some space independent value. In one case, this temperature was the initial temperature of the sphere, and in another case, it was the average temperature. In the first case, the validity of the approximation was limited by the time; that is, as time increases, the temperature may depart significantly from the initial temperature. In the latter case, the validity was also limited by the conductivity, which must be large in order to keep the space variation of temperature small.

In the second case, a ratio of formation (eq. (15)) was shown to be independent of x. Hence, the solution T(x, t) could reasonably be written as two terms – the superposition of a time-dependent average temperature and a space- and time-dependent perturbation. As a result of this analysis, at least two conclusions can be drawn:

(1) The assumption of a temperature-dependent specific heat (Debye model), as opposed to a constant specific heat, in general, will give a different theoretical curve

for the transient temperature distribution. Both the rate of change and the temperature level may be different. For temperatures greater than the Debye characteristic temperature, the differences are negligible; however, for lower temperatures, the differences become significant. As expected, the differences become most significant for temperatures near absolute zero.

(2) When the conduction is large (e. g., $\mu \geq 10$), the form of the final temperature distribution on the sphere is obtained very rapidly in comparison with the time to approach the average steady-state temperature level.

Lewis Research Center,
    National Aeronautics and Space Administration,
        Cleveland, Ohio, December 22, 1964.

APPENDIX A

SYMBOLS

A_p — projected area for parallel radiation source

A_t — total surface area

A_Φ — area of surface receiving radiation

$\overline{B}_n$ — modified Bernoulli number

b — wall thickness

c — nondimensional specific heat, c^*/c^*_∞

c^* — actual specific heat

c^*_∞ — classical specific heat (3R for elemental material)

d — maximum allowable perturbation (e.g., $d = 0.1$)

k — thermal conductivity

$O(\delta^2)$ — terms of order of δ^2

P_n — Legendre polynomial

$R(x,t)$ — ratio of formation (defined by eq. (14))

r — radius of sphere

$T(t)$ — average temperature

$T(x)$ — equilibrium temperature

$T(x,t)$ — nondimensional temperature, T^*/T^*_∞

T_0 — nondimensional uniform initial temperature

T_{sp} — nondimensional background temperature

T^* — actual absolute temperature

T^*_0 — actual uniform initial absolute temperature

T^*_{sp} — actual background absolute temperature

T^*_∞ — reference temperature (defined by eq. (B4))

t — nondimensional time, t^*/t^*_r

t^* — actual time

t^*_r — reference time, $\rho b c^*_\infty / \epsilon \sigma T^{*3}_\infty$

x — $\cos\theta$

α — absorptivity (defined in eq. (B2))

α^* — absorptivity for radiation at angle θ

δ — perturbation

ϵ — total hemispherical emissivity

$\epsilon(x,t)$ — temperature departure

ζ — variable in eq. (E3)

Θ — nondimensional Debye characteristic temperature, Θ^*/T^*_∞

Θ^* — Debye characteristic temperature

θ — colatitude angle

μ — conduction parameter, $kb/r^2 \epsilon \sigma T^{*3}_\infty$

ρ — density

σ — Stefan-Boltzmann constant

τ — characteristic time

Φ — source flux density

APPENDIX B

DETERMINATION OF REFERENCE TEMPERATURE

The following treatment (see fig. 7) is similar to that used in reference 2. An infinitely conducting, opaque sphere radiating to a sink at temperature T^*_{sp}, while receiving radiation from a source, has an equilibrium temperature T^*_∞ that may be determined by the following form of the principle of energy conservation:
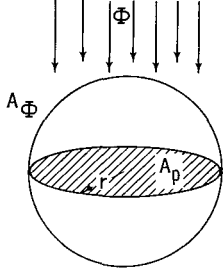
$$\int_{A_\Phi} \alpha^* \Phi \, dA_\Phi = \int_{A_t} \epsilon \sigma \left(T^*_\infty{}^4 - T^*_{sp}{}^4\right) dA_t \tag{B1}$$

If ϵ is defined as the total hemispherical emissivity for the surface and

$$\alpha \Phi \equiv \frac{1}{A_p} \int_{A_\Phi} \alpha^* \Phi \, dA_\Phi \tag{B2}$$

Figure 7. – Emitting and absorbing areas for sphere subjected to radiation flux Φ.

then equation (B1) becomes

$$\alpha \Phi = \frac{A_t}{A_p} \epsilon \sigma \left(T^*_\infty{}^4 - T^*_{sp}{}^4\right) \tag{B3}$$

For the sphere $A_t = 4\pi r^2$ and $A_p = \pi r^2$, so that the reference temperature T^*_∞ to be used in the derivations is defined by

$$\alpha \Phi \equiv 4\epsilon \sigma \left(T^*_\infty{}^4 - T^*_{sp}{}^4\right) \tag{B4}$$

A special case occurs when the radiating source is absent or not considered. Then $\alpha \Phi = 0$, and the normalization is just $T^*_\infty = T^*_{sp}$. The case when $T^*_\infty = T^*_{sp} = 0$ is treated in appendix F.

APPENDIX C

SOLUTION TO LINEARIZED PARTIAL DIFFERENTIAL EQUATION

ASSUMING CONSTANT SPECIFIC HEAT

The linearized differential equation is the following:

$$c \frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x} \left[\mu(1 - x^2) \frac{\partial T(x,t)}{\partial x} \right] - \left[4T_0^3 T(x,t) - 3T_0^4 - T_{sp}^4 \right] + 4(1 - T_{sp}^4)x\eta(x) \tag{3}$$

Assume that the specific heat is a constant, and let the sphere be initially at a uniform temperature (i.e., $T(x,0) = T_0$).

Apply the Laplace transformation with respect to t, and define

$$g(x,s) \equiv L \left[T(x,t) \right]$$

Then proceed formally

$$c \left[sg(x,s) - T_0 \right] = \frac{\partial}{\partial x} \left[\mu(1 - x^2) \frac{\partial g(x,s)}{\partial x} \right] - 4T_0^3 g(x,s) + \frac{1}{s}(3T_0^4 + T_{sp}^4) + \frac{4}{s}(1 - T_{sp}^4)x\eta(x) \tag{C1}$$

Substitute

$$y(x,s) = s(sc + 4T_0^3)g(x,s) - (scT_0 + 3T_0^4 + T_{sp}^4)$$

Then equation (C1) becomes

$$\frac{\mu}{sc + 4T_0^3} \frac{\partial}{\partial x} \left[(1 - x^2) \frac{\partial y(x,s)}{\partial x} \right] = y(x,s) - 4(1 - T_{sp}^4)x\eta(x) \tag{C2}$$

The function $x\eta(x)$ may be expanded in a series of Legendre polynomials

$$x\eta(x) = \frac{1}{2} P_1(x) + \frac{1}{6} \sum_{j=0}^{\infty} (4j + 1) \binom{\frac{3}{2}}{j + 1} P_{2j}(x) \tag{C3}$$

where

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$$\binom{a}{i} = \frac{a(a-1)\ldots(a-i+1)}{i!}$$

Also write

$$y(x,s) = \sum_{i=0}^{\infty} a_i(s)P_i(x)$$

Then, for constant μ,

$$\frac{\mu}{sc + 4T_0^3} \sum_{i=0}^{\infty} a_i(s) \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_i(x)\right] = \sum_{i=0}^{\infty} a_i(s)P_i(x) - 2(1 - T_{sp}^4)P_1(x)$$

$$- \frac{2}{3}(1 - T_{sp}^4) \sum_{j=0}^{\infty} (4j+1)\binom{\frac{3}{2}}{j+1}P_{2j}(x) \qquad (C4)$$

The Legendre polynomials satisfy

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_i(x)}{dx}\right] + i(i+1)P_i(x) = 0$$

Therefore,

$$\frac{\mu}{sc + 4T_0^3} \sum_{i=0}^{\infty} a_i(s)i(i+1)P_i(x) = -\sum_{i=0}^{\infty} a_i(s)P_i(x) + 2(1 - T_{sp}^4)P_1(x)$$

$$+ \frac{2}{3}(1 - T_{sp}^4) \sum_{j=0}^{\infty} (4j+1)\binom{\frac{3}{2}}{j+1}P_{2j}(x) \qquad (C5)$$

Since the Legendre polynomials form a complete orthogonal set on $-1 \leq x \leq 1$, the coefficients may be equated; then

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$$a_0(s) = 1 - T_{sp}^4$$

$$a_1(s) = \frac{2sc + 8T_0^3}{sc + 2\mu + 4T_0^3}(1 - T_{sp}^4)$$

and for $j \geq 1$

$$a_{2j}(s) = \frac{2}{3}(1 - T_{sp}^4)(4j + 1)\binom{\frac{3}{2}}{j+1}\frac{sc + 4T_0^3}{sc + \mu 2j(2j + 1) + 4T_0^3}$$

$$a_{2j+1}(s) = 0$$

Now

$$y(x, s) = (1 - T_{sp}^4)\left[1 + \frac{2sc + 8T_0^3}{sc + 2\mu + 4T_0^3}x + \frac{2}{3}\sum_{j=1}^{\infty}\binom{\frac{3}{2}}{j+1}(4j + 1)\frac{(sc + 4T_0^3)P_{2j}(x)}{sc + \mu 2j(2j + 1) + 4T_0^3}\right]$$

$$\text{(C6)}$$

and

$$g(x, s) = \frac{scT_0 + 3T_0^4 + 1}{s(sc + 4T_0^3)} + \frac{2x(1 - T_{sp}^4)}{s(sc + 2\mu + 4T_0^3)}$$

$$+ \frac{2}{3}(1 - T_{sp}^4)\sum_{j=1}^{\infty}\binom{\frac{3}{2}}{j+1}\frac{(4j + 1)P_{2j}(x)}{s\left[sc + \mu 2j(2j + 1) + 4T_0^3\right]} \qquad \text{(C7)}$$

The inverse transform yields

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$$T(x,t) = T_0 + \frac{1}{4T_0^3}(1 - T_0^4)\left(1 - e^{-4T_0^3 t/c}\right) + \frac{x(1 - T_{sp}^4)}{\mu + 2T_0^3}\left[1 - e^{-2(\mu + 2T_0^3)t/c}\right]$$

$$+ \frac{1 - T_{sp}^4}{3}\sum_{j=1}^{\infty}\binom{\frac{3}{2}}{j+1}\frac{(4j+1)P_{2j}(x)}{\mu j(2j+1) + 2T_0^3}\left\{1 - e^{-\left[\mu 2j(j+1) + 4T_0^3\right]t/c}\right\} \quad (4)$$

where

$$\binom{a}{i} = \frac{a(a-1)\ldots(a - i + 1)}{i!}$$

Since this expression converges uniformly in x for all $t \geq 0$ and satisfies equation (3), it is indeed the solution.

If equation (2) is now linearized with $T(x,t) = T(t) + \epsilon(x,t)$ where $\epsilon(x,t) \ll T(t)$ and $cT' + T^4 = 1$ (eq. (7)), the equation for ϵ becomes

$$c\frac{\partial \epsilon(x,t)}{\partial t} = \mu \frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial \epsilon(x,t)}{\partial x}\right] - (1 - T_{sp}^4)\left[1 - 4x\eta(x)\right] - 4T^3(t)\epsilon(x,t) \quad (C8)$$

If it is assumed that

$$\epsilon(x,t) = \sum_{i=0}^{\infty} a_i(t)P_i(x)$$

then

$$a_0 = 0$$

$$a_1 = 2(1 - T_{sp}^4)\exp - \left[\int_0^t \frac{2\mu + 4T^3(t)}{c}dt\right]\int_0^t \frac{1}{c}\exp\left[\int_0^t \frac{2\mu + 4T^3(t)}{c}dt\right]dt$$

and the coefficients of the even terms are

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$$a_{2i} = \binom{\frac{3}{2}}{i+1} (4i+1) \frac{2}{3} (1 - T_{sp}^4) \exp\left[-2\int_0^t \frac{\mu i(2i+1) + 2T^3(t)}{c} dt\right]$$

$$\int_0^t \frac{1}{c} \exp\left[2\int_0^t \frac{\mu i(2i+1) + 2T^3(t)}{c} dt\right] dt$$

and all other a_i are zero. If $\mu \gg T^3(t)$ and if c is constant, then

$$T(x,t) - T(-x,t) = \epsilon(x,t) - \epsilon(-x,t)$$

$$= \frac{2x(1 - T_{sp}^4)}{\mu} \left(1 - e^{-2\mu t/c}\right) \tag{13}$$

and the coefficients a_{2i} can be evaluated. For this situation, $T(x,t)$ can be written in the form given by equation (16a).

APPENDIX D

RANGE OF VALIDITY FOR LINEARIZED SOLUTION

The purpose of this appendix is to derive conditions under which the linearization of equation (2) gives a good approximation. Therefore, restrict $\delta = (T - T_0)/T_0$ such that $|\delta| < d$, where d is chosen to provide any desired degree of approximation to the exact solution for $T_0 > 0$. (The exceptional case $T_0 = 0$ is also examined.)

If t is small, equation (4) yields

$$|\delta| = \frac{|T(x,t) - T_0|}{T_0} \leq \frac{1 - T_0^4}{T_0}\frac{t}{c} + \frac{2}{T_0}\frac{t}{c} + \frac{2}{3T_0}\sum_{j=1}^{\infty}\binom{\frac{3}{2}}{j+1}(4j+1)\frac{t}{c} + O(t^2)$$

where the $O(t^2)$ terms are negative. Thus

$$|\delta| \leq (1 + 2 + 1)\frac{t}{T_0 c} = \frac{4}{T_0}\frac{t}{c}$$

Set $(4/T_0)(t/c) = d$ and $\tau = cdT_0/4$. Then $t < \tau = cdT_0/4$ implies $|\delta| < d$.

If the initial temperature is near 1, then for certain values of μ, equation (4) will always be valid. That is, if $T_0 = 1$, then equation (4) leads to

$$|\delta| = |T(x,t) - 1| \leq \frac{1}{\mu + 2} + \frac{1}{3}\sum_{j=1}^{\infty}\binom{\frac{3}{2}}{j+1}\frac{4j+1}{\mu j(2j+1)+2}$$

$$= \frac{1}{\mu + 2} + \frac{5}{8(3\mu + 2)} - \frac{3}{32(5\mu + 1)} + \cdots$$

$$\leq \frac{3}{3\mu + 6} + \frac{5}{8(3\mu + 2)} \leq \left(3 + \frac{5}{8}\right)\frac{1}{3\mu + 2} = \frac{29}{24}\frac{1}{\mu + \frac{2}{3}}$$

Set $(29/24)/[\mu + (2/3)] = d$; then $\mu > (29/24d) - (2/3)$ implies $|\delta| < d$.

For the special case $T_0 = 0$, a different approach must be taken to approximate the

solution to equation (2). Rather than letting $T = T_0(1 + \delta)$, the term T^4 is assumed to be negligible. In this case, equations (2) and (3) are identical and the method of appendix C gives the solution

$$T(x, t) = \frac{t}{c} + \frac{x}{\mu} \left(1 - T_{sp}^4\right) \left(1 - e^{-2\mu t/c}\right)$$

$$+ \frac{1 - T_{sp}^4}{3} \sum_{j=1}^{\infty} \binom{\frac{3}{2}}{j+1} \frac{(4j + 1)P_{2j}(x)}{\mu j(2j + 1)} \left[1 - e^{-\mu 2j(2j+1)t/c}\right] \qquad \text{(D1)}$$

This solution is the same as that obtained by taking the limit of equation (4) as T_0 approaches zero. To determine when T^4 can be neglected, all the terms in equation (2) can be evaluated from equation (D1). For small values of t, they may be written as

$$-T^4(x, t) \approx -\left(\frac{t}{c}\right)^4 \left[1 + 4x\eta(x) \left(1 - T_{sp}^4\right)\right]^4 \qquad \text{(D2)}$$

$$c \frac{\partial T}{\partial t} - T_{sp}^4 \approx \left(1 - T_{sp}^4\right)\left[1 + 4x\eta(x) \left(1 - T_{sp}^4\right)\right] \qquad \text{(D3)}$$

$$\mu \frac{\partial}{\partial x} \left[(1 - x^2) \frac{\partial T}{\partial x}\right] = -2\mu \frac{t}{c} \left\{4x\eta(x) + \frac{2}{3} \sum_{j=0}^{\infty} \binom{\frac{3}{2}}{j+1} (4j + 1)\left[j(2j + 1) - 1\right]P_{2j}(x)\right\} \qquad \text{(D4)}$$

Therefore, in order that T^4 be negligible in equation (2) it must be negligible compared to the larger of the other terms. Thus, since the term given by equation (D3) is greater than the terms given by equation (D4), the condition that T^4 is negligible becomes

$$\frac{t}{c} << \frac{1 - T_{sp}^4}{1 + 4x\eta(x) \left(1 - T_{sp}^4\right)}$$

Then the approximation $T^4 \approx 0$ is consistent and can be considered valid.

APPENDIX E

SOLUTION TO EQUATION FOR AVERAGE TEMPERATURE

WITH DEBYE SPECIFIC HEAT

To evaluate equation (9) first evaluate

$$\int_0^{\Theta/T} \frac{e^x x^4}{(e^x - 1)^2} dx \tag{E1}$$

Consider $e^z z^4/(e^z - 1)^2$ as a function of the complex variable z. The function has poles at $z = \pm 2\pi ni$ ($n = 1, 2, 3, \ldots$) and a removable singularity at $z = 0$. Therefore, the power series of the function about $z = 0$ has a radius of convergence of 2π. The convergence is uniform in any smaller circle so that equation (E1) can be integrated termwise; the resulting series converges for $|z| < 2\pi$. Thus a purely formal procedure may be followed, and the resulting series will converge for $|z| < 2\pi$. Thus

$$\frac{e^z z^4}{(e^z - 1)^2} = \frac{z^4}{(e^{z/2} - e^{-z/2})^2} = \frac{1}{4} z^4 \operatorname{csch}^2\left(\frac{z}{2}\right) \tag{E2}$$

Integrating along any path in $|z| < 2\pi$ results in

$$\int_0^\zeta \frac{e^z z^4}{(e^z - 1)^2}\, dz = \frac{1}{4} \int_0^\zeta z^4 \operatorname{csch}^2\!\left(\frac{z}{2}\right) dz$$

$$= -\frac{1}{2}\zeta^4 \coth \frac{\zeta}{2} + 2\int_0^\zeta z^3 \coth\!\left(\frac{z}{2}\right) dz$$

$$= -\frac{\zeta^4}{2}\sum_{n=0}^\infty \frac{(-1)^{n+1}2^{2n}}{(2n)!}\overline{B}_n\!\left(\frac{\zeta}{2}\right)^{2n-1} + 2\int_0^\zeta \sum_{n=0}^\infty \frac{(-1)^{n+1}2^{2n}}{(2n)!}\overline{B}_n \frac{z^{2n+2}}{2^{2n-1}}\, dz$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}\overline{B}_n \zeta^{2n+3} + \sum_{n=0}^\infty \int_0^\zeta \frac{(-1)^{n+1}4}{(2n)!}\overline{B}_n z^{2n+2}\, dz$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}\frac{2n-1}{2n+3}\overline{B}_n \zeta^{2n+3} \tag{E3}$$

where $\overline{B}_n$ are the modified Bernoulli numbers defined in the text. (The series expansion $\coth u$ can be found in ref. 6.)

Since the integration is independent of path for $|z| < 2\pi$, let the path be $z = x$ from 0 to $\Theta/T < 2\pi$. Then

$$\int_0^{\Theta/T} \frac{e^x x^4}{(e^x - 1)^2}\, dx = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}\frac{2n-1}{2n+3}\overline{B}_n\!\left(\frac{\Theta}{T}\right)^{2n+3} \tag{E4}$$

The differential equation (eq. (9)), therefore, becomes

$$\left[3\sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}\frac{2n-1}{2n+3}\overline{B}_n\!\left(\frac{\Theta}{T}\right)^{2n}\right]\frac{dT}{dt} = 1 - T^4 \tag{E5}$$

The solution to equation (E5) is given as equation (10) in the text.

To obtain the solution for smaller values of T, equation (E2) can be expanded in a

series about a large value of z on the positive real axis rather than the origin. Then the radius of convergence will be larger and include larger values of Θ/T. The rest of the aforementioned method could then be applied.

APPENDIX F

EQUATION WHEN $T_\infty^* = 0$

The case $T_\infty^* = 0$ occurs only in the absence of a radiating source and then only when $T_{sp}^* = 0$. Since the initial temperature is uniform and there is no radiating source, there is no conduction and equation (1) becomes

$$\rho bc^* \frac{dT^*}{dt^*} = -\epsilon \sigma T^{*4}$$

The solutions for the various assumptions concerning specific heat are found easily without nondimensionalizing. They are as follows:

For constant specific heat:

$$T^* = \left(\frac{3\epsilon\sigma}{\rho bc^*} t^* + \frac{1}{T_0^{*3}} \right)^{-1/3}$$

For Debye specific heat (general case):

$$t^* + C_0 = \frac{3\rho bc_\infty^*}{\epsilon \sigma} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)}{(2n)! \, (2n+3)^2} \bar{B}_n \Theta^{*2n} \frac{1}{T^{*2n+3}}$$

where

C_0 is determined by boundary conditions, Θ^* is the Debye characteristic temperature, and $\bar{B}_n$ are the nonzero Bernoulli numbers B_n altered by setting $\bar{B}_0 = -1$. The solution is valid for $T^* > \Theta^*/2\pi$.

For Debye specific heat with the T^3 law:

$$T^* = T_0^* \exp\left(-\frac{5\epsilon\sigma\Theta^{*3}}{4\pi^4 \rho bc_\infty^*} t^* \right)$$

REFERENCES

1. Mark, H.: Operation of a Solar Space Environment Simulator. Proc. Inst. of Environmental Sci., Wash., D.C., Apr. 5-7, 1961, pp. 521-531.

2. Nichols, Lester D.: Surface-Temperature Distribution on Thin-Walled Bodies Subjected to Solar Radiation in Interplanetary Space. NASA TN D-584, 1961.

3. Fowler, R. H.: Statistical Mechanics. Second ed., Cambridge Univ. Press (London), 1936, pp. 119-138.

4. Kittel, Charles: Introduction to Solid State Physics. Second ed., John Wiley & Sons, Inc., 1956, pp. 125-132.

5. Gray, Dwight E., ed.: American Institute of Physics Handbook. McGraw-Hill Book Co., Inc., 1957, tables 4e-6 to 4e-8.

6. Dwight, Herbert B.: Tables of Integrals and Other Mathematical Data. Fourth ed., The Macmillan Co., 1961, p. 154.

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