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MAGNETOHYDRODYNAMIC BOUNDARY LAYER BETWEEN PARALLEL
STREAMS OF DIFFERENT MAGNETIC FIELDS AND TEMPERATURES

By H. P. Pao and C. C. Chang

December 1964

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FOREWORD

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MAGNETOHYDRODYNAMIC BOUNDARY LAYER BETWEEN PARALLEL STREAMS OF DIFFERENT MAGNETIC FIELDS AND TEMPERATURES

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ABSTRACT

An analysis and calculations of the free laminar boundary layer flow between parallel streams of different magnetic fields and temperatures were made for an incompressible, viscous, thermally and electrically conducting fluid. Small perturbation and approximate solutions were given. The approximate solution for $\alpha = 1$ gives quite a satisfactory flow field which only involves a discrepancy of 1 per cent from the exact numerical solution in most cases. The integration of the exact boundary layer equations was carried out and some sample calculations were shown in the graphs. The magnetic field thickens the flow boundary layer and thus provides a stabilizing effect on the flow field. The possibility of an extension to the parallel streams of two different fluids is also investigated.

NOMENCLATURE

u, v	=	fluid velocity
f_0, g_0	=	magnetic field strength
f, g	=	$(f_0, g_0)(\mu_e/4\pi\rho)^{1/2}$, normalized magnetic field strength or Alfvén wave velocity
T	=	temperature
p	=	fluid pressure
p	=	$(f^2 + g^2)/2 + p/\rho + g^*y$, total pressure
g^*	=	gravitational acceleration
\underline{E}	=	electrical field vector
ξ	=	$(U_1/2\nu x)^{1/2} y$, similarity variable
ψ	=	velocity stream function
Λ	=	magnetic potential function
m	=	dimensionless function associated with ψ
n	=	dimensionless function associated with Λ
θ	=	dimensionless function associated with T
R	=	$m + n$
S	=	$m - n$
U_1, U_2	=	uniform velocities at $+\infty$ and $-\infty$
H_1, H_2	=	uniform magnetic fields at $+\infty$ and $-\infty$
T_1, T_2	=	uniform temperatures at $+\infty$ and $-\infty$
$(\Delta T)_0$	=	$T_1 - T_2$

\bar{U}	=	$U_1 - U_2$
\bar{H}	=	$H_1 - H_2$
β_1	=	H_1 / U_1
β_2	=	H_2 / U_1
λ	=	U_2 / U_1
α_1	=	ν / U_1
γ_1	=	$[\alpha_1 / (1 - \beta_1)]^{1/2}$
γ_2	=	$[\alpha_1 / (1 + \beta_1)]^{1/2}$
\bar{u}	=	$U_1 - u$
\bar{f}	=	$H_1 - f$
G	=	$\bar{u} + \bar{f}$
W	=	$\bar{u} - \bar{f}$
$\bar{\psi}$	=	velocity stream function of \bar{u}
$\bar{\Lambda}$	=	magnetic potential function of \bar{f}
\bar{m}	=	dimensionless function associated with $\bar{\psi}$
\bar{n}	=	dimensionless function associated with $\bar{\Lambda}$
\bar{G}	=	$\bar{m} + \bar{n}$
\bar{W}	=	$\bar{m} - \bar{n}$
ρ	=	fluid density
μ	=	fluid viscosity
ν	=	μ / ρ , kinematic viscosity

σ	=	electrical conductivity
μ_e	=	magnetic permeability
η	=	$1 / (4\pi\mu_e\sigma)$, magnetic diffusivity
k	=	heat conductivity
c_p	=	specific heat at constant pressure
κ	=	$k / (c_p \rho)$, thermal diffusivity
γ	=	ν / κ , Prandtl number
ϵ	=	$U_i^2 / \{ c_p (\Delta T)_0 \}$, Eckert number
α	=	ν / η , magnetic Prandtl number
β^*	=	average temperature gradient
L	=	vertical length scale of the boundary layer thickness
F_i	=	$U_i^2 / g^* \beta^* (\Delta T)_0 L$, internal Froude number

Subscripts

1,2	=	upper and lower streams
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I. INTRODUCTION

The theoretical and experimental studies related to the problem of flow between two parallel streams of same or different densities have had renewed interest in the past decades. The purpose of the present investigation is to study the interaction between parallel streams and magnetic fields. In general, when a magnetic field is present, its effect upon an electrically conducting fluid has been known to be of a stabilizing nature. It is, therefore, to be expected that a magnetic field in the present case will have a stabilizing effect upon the fluid motion, and this is evidently the case.

Lessen¹ discussed the stability of the free laminar boundary layer between two uniform streams of fluids of same density and obtained the velocity distribution in the course of his investigation. He reached the conclusion that the flow is unstable even for very small Reynolds numbers. Lock^{2,3} extended the work of Lessen by considering parallel streams of different densities. Lin⁴ has considered the stability of two parallel streams for a compressible fluid. Stuart⁵ has investigated the stability of pressure flow between parallel planes under a parallel magnetic field, and Lock⁶ has considered the stability of this type of flow under a transverse magnetic field. They found that the magnetic field always has a stabilizing effect on the fluid motion.

In the present investigation, the velocity, magnetic and temperature fields are obtained. It is found that the magnetic field thickens the boundary layer, thus, indicating its stabilizing effect upon the highly unstable flow. It is hoped that a subsequent paper will present an analysis and detailed calculations on the stability of this type of flow.

2. FUNDAMENTAL EQUATIONS

For an incompressible, viscous, electrically conducting fluid, in steady two dimensional motion, the governing equations are:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{\partial P}{\partial x} + f \frac{\partial f}{\partial x} + g \frac{\partial f}{\partial y} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (1)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{\partial P}{\partial y} + f \frac{\partial g}{\partial x} + g \frac{\partial g}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (3)$$

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0, \quad (4)$$

$$\frac{\partial}{\partial y} (ug - vf) + \eta \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) = 0, \quad (5)$$

$$-\frac{\partial}{\partial x} (ug - vf) + \eta \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \right) = 0. \quad (6)$$

In the derivation of (1) - (6), it is assumed that the net charge density is zero, and that ν , σ and μ_e are constant.

With use of (4), Eq. (5) and (6) become

$$\frac{\partial}{\partial x} [ug - vf + \eta \left(\frac{\partial f}{\partial y} - \frac{\partial v}{\partial x} \right)] = 0, \quad (7)$$

$$\frac{\partial}{\partial y} [ug - vf + \eta \left(\frac{\partial f}{\partial y} - \frac{\partial v}{\partial x} \right)] = 0. \quad (8)$$

It follows that

$$ug - vf + \eta \left(\frac{\partial f}{\partial y} - \frac{\partial v}{\partial x} \right) = \text{constant}. \quad (9)^*$$

It can readily be shown by Ohms law that Eq. (9) leads to

$$E_z = \text{constant},$$

where E_z is the z-component of the electrical field. If we impose the condition that the electrical field vanishes at infinity, it then vanishes everywhere. Eq. (9) now becomes

$$ug - vf + \eta \left(\frac{\partial f}{\partial y} - \frac{\partial v}{\partial x} \right) = 0. \quad (10)$$

In this investigation we will restrict our consideration to zero electrical field.

Equations (3) and (4) can be integrated by introducing two scalar functions $\psi(x, y)$ and $\Lambda(x, y)$, such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad (11)$$

$$f = \frac{\partial \Lambda}{\partial y}, \quad g = -\frac{\partial \Lambda}{\partial x}. \quad (12)$$

* In a source free and steady state flow, the electric field obeys $\nabla \times \underline{E} = 0$ and $\nabla \cdot \underline{E} = 0$. If we impose the condition that the electrical field be constant at infinity, it follows then, from the potential theory, that \underline{E} is constant everywhere in the region. Thus from Ohms law, Eq. (9) follows immediately.

On cross-differentiating (1) and (2) to eliminate the total pressure terms, and then introducing the stream and magnetic functions, we obtain

$$\begin{aligned} \psi_y (\psi_{xxx} + \psi_{yyx}) - \psi_x (\psi_{xxy} + \psi_{yyy}) \\ = \Lambda_y (\Lambda_{xxx} + \Lambda_{yyx}) - \Lambda_x (\Lambda_{xxy} + \Lambda_{yyy}) + \nu (\psi_{xxx} + 2\psi_{xxy} + \psi_{yyy}), \end{aligned} \quad (13)$$

where the subscripts denote partial differentiations. Equation (10) becomes, after introducing stream and magnetic functions,

$$-\psi_y \Lambda_x + \psi_x \Lambda_y + \eta (\Lambda_{xx} + \Lambda_{yy}) = 0. \quad (14)$$

Now, with the boundary layer approximation,

$$\frac{\partial^2}{\partial x^2} \ll \frac{\partial^2}{\partial y^2} \quad (15)$$

equations (13) and (14) take the forms

$$\psi_y \psi_{yyx} - \psi_x \psi_{yyy} = \Lambda_y \Lambda_{yyx} - \Lambda_x \Lambda_{yyy} + \nu \psi_{yyy}, \quad (16)$$

$$-\psi_y \Lambda_x + \psi_x \Lambda_y + \eta \Lambda_{yy} = 0. \quad (17)$$

Equation (16) can be integrated once with respect to y which yields

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \Lambda_y \Lambda_{xy} - \Lambda_x \Lambda_{yy} + \nu \psi_{yy} + s(x), \quad (18)$$

where $s(x)$ is an arbitrary function of x . Comparing this equation with equation (1) we obtain

$$-\frac{\partial P}{\partial x} = s(x) ,$$

that is, the pressure gradient in x - direction is independent of y . Therefore, we can evaluate $\partial P / \partial x$ at $y = \pm \infty$.

From the physical flow and magnetic field configuration (Fig. 1), the boundary conditions are:

$$y \rightarrow \infty : \quad \psi_y \rightarrow U_1 , \quad \Lambda_y \rightarrow H_1 , \quad (19)$$

$$y \rightarrow -\infty : \quad \psi_y \rightarrow U_2 , \quad \Lambda_y \rightarrow H_2 . \quad (20)$$

It follows that

$$s(x) = 0 \quad \text{at } y = \pm \infty , \quad x > 0 , \quad (21)$$

hence $s(x)$ is zero everywhere. Equation (18) now becomes

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \Lambda_y \Lambda_{xy} - \Lambda_x \Lambda_{yy} + \nu \psi_{yyy} . \quad (22)$$

If we suppose now that the conditions of the problem introduce no other parameters, we find, for example, that ψ is a function of $x, y, \nu, \eta, U_1, U_2, H_1$ and H_2 only, so that a dimensional argument⁷ requires that ψ be expressed as

$$\psi = (2\nu x U_1)^{1/2} m(\xi) , \quad \xi = \left(\frac{U_1}{2\nu x} \right)^{1/2} y , \quad (23)$$

where m is a function also of the parameters

$$\alpha = \frac{\nu}{\eta} , \quad \lambda = \frac{U_2}{U_1} , \quad \beta_1 = \frac{H_1}{U_1} , \quad \beta_2 = \frac{H_2}{U_1} . \quad (24)$$

The same dimensional argument leads to

$$\Lambda = (2\nu x U_1)^{1/2} n(\xi) . \quad (25)$$

By differentiation, the velocity and magnetic field components are obtained

$$\begin{aligned} u &= U_1 m' , & v &= \left(\frac{\nu U_1}{2x} \right)^{1/2} (\xi m' - m) , \\ f &= U_1 n' , & g &= \left(\frac{\nu U_1}{2x} \right)^{1/2} (\xi n' - n) , \end{aligned}$$

where the prime indicates differentiation with respect to ξ .

Substitution of (23) and (25) into (22) and (17) leads to the ordinary differential equations

$$m''' + m m'' - n n'' = 0 , \quad (26)$$

$$n'' + \alpha (m n' - n m') = 0 . \quad (27)$$

The boundary conditions are:

$$\left. \begin{aligned} \xi \rightarrow \infty : & \quad m' \rightarrow 1 , \quad n' \rightarrow \beta_1 , \\ \xi \rightarrow -\infty : & \quad m' \rightarrow \lambda , \quad n' \rightarrow \beta_2 . \end{aligned} \right\} \quad (28)$$

A fifth condition is arbitrary up to a translation along the ξ - axis.

Energy Equation

When the two streams are also of different temperatures, the boundary layer energy equation should be added which assumes the form

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \kappa \frac{\partial^2 T}{\partial y^2} + \frac{\gamma}{c_p} \left(\frac{\partial u}{\partial y} \right)^2, \quad (29)$$

in which the Joule heat is absent because the electric field is zero everywhere. We assume the temperature difference is so small that the constants of fluid properties remain uniform. We also assume that the internal Froude number F_i or the Richardson number⁸ is large, so that the buoyancy force of gravity is small in comparison with the inertia force, where

$$F_i = \frac{U_i^2}{g^* \beta^* (\Delta T)_\infty L}.$$

Thus, the flow field is essentially unaffected by the temperature field for the above assumption. If we define

$$T - T_\infty = (\Delta T)_\infty \theta(\xi), \quad (30)$$

equation (29) becomes

$$\theta'' + \gamma (m \theta' + \epsilon m''^2) = 0, \quad (31)$$

where

$$\gamma = \frac{\gamma}{\kappa} \quad \text{and} \quad \epsilon = \frac{U_i^2}{c_p (\Delta T)_\infty}.$$

are known as the Prandtl, and Eckert numbers.

The boundary conditions for the temperature field are

$$\left. \begin{array}{l} \xi \rightarrow \infty : \quad \theta \rightarrow 1 , \\ \xi \rightarrow -\infty : \quad \theta \rightarrow 0 . \end{array} \right\} \quad (32)$$

3. SOLUTION BY THE METHOD OF SMALL PERTURBATIONS

In some practical cases, the differences in both the velocity and the magnetic field between streams are small. It is advisable to find some approximate solution for this special case first.

Equations (26) and (27) in the last section correspond to the following boundary layer equations:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = f \frac{\partial f}{\partial x} + g \frac{\partial f}{\partial y} + \nu \frac{\partial^2 u}{\partial y^2} , \quad (33)$$

$$ug - vf + \eta \frac{\partial f}{\partial y} = 0 . \quad (34)$$

Now, we define

$$u = U_1 - \bar{u} \quad \text{and} \quad f = H_1 - \bar{f} , \quad (35)$$

with $\bar{u} \ll U_1$ and $\bar{f} \ll H_1$. Substituting Eq. (35) into Eqs. (33) and (34), and neglecting terms of order \bar{u}/U_1 and \bar{f}/H_1 , compared with unity, we have

$$- \frac{\partial \bar{u}}{\partial x} + \beta_1 \frac{\partial \bar{f}}{\partial x} + \alpha_1 \frac{\partial^2 \bar{u}}{\partial y^2} = 0 , \quad (36)$$

$$g - \beta_1 \nu + \frac{\alpha_1}{\alpha} \frac{\partial \bar{f}}{\partial y} = 0 , \quad (37)$$

where $\alpha = \nu/U_1$. Differentiating Eq. (37) with respect to y and using Eqs. (3) and (4), we obtain

$$\beta_1 \frac{\partial \bar{u}}{\partial x} - \frac{\partial \bar{f}}{\partial x} + \frac{\alpha_1}{\alpha} \frac{\partial^2 \bar{f}}{\partial y^2} = 0. \quad (38)$$

Adding and subtracting (36) and (38), and assuming the magnetic Prandtl's number α to be unity, we have

$$\frac{\partial G}{\partial x} = \gamma_1^2 \frac{\partial^2 G}{\partial y^2}, \quad (39)$$

$$\frac{\partial W}{\partial x} = \gamma_2^2 \frac{\partial^2 W}{\partial y^2}, \quad (40)$$

where

$$G = \bar{u} + \bar{f}, \quad W = \bar{u} - \bar{f},$$

$$\gamma_1^2 = \frac{\alpha_1}{1 - \beta_1}, \quad \gamma_2^2 = \frac{\alpha_1}{1 + \beta_1}.$$

The velocity and magnetic field distribution at the beginning of the mixing zone is

$$x = 0: \quad \left. \begin{array}{ll} \bar{u} = \bar{f} = 0, & y > 0 \\ \bar{u} = \bar{U}, \bar{f} = \bar{H}, & y < 0 \end{array} \right\} \quad (41)$$

where

$$\bar{U} = U_1 - U_2, \quad \bar{H} = H_1 - H_2.$$

The solutions of Eqs. (39) and (40) with the boundary conditions (41) are

$$G = \frac{\bar{U} + \bar{H}}{2} [1 - \phi(\xi_1)], \quad (42)$$

$$W = \frac{\bar{U} - \bar{H}}{2} [1 - \phi(\xi_2)], \quad (43)$$

where

$$t_1 = \frac{y}{2\gamma_1 x^{1/2}} = \left(\frac{1-\beta_1}{2}\right)^{1/2} \xi, \quad t_2 = \frac{y}{2\gamma_2 x^{1/2}} = \left(\frac{1+\beta_1}{2}\right)^{1/2} \xi,$$

$$\phi(t) = \frac{2}{\pi^{1/2}} \int_0^t e^{-\eta^2} d\eta. \quad (44)$$

It follows that

$$\frac{\bar{u}}{U} = \frac{1}{2} \left[1 - \frac{1}{2} (\phi_1 + \phi_2) \right] - \frac{1}{4} \frac{\bar{H}}{U} (\phi_1 - \phi_2), \quad (45)$$

$$\frac{\bar{f}_*}{H} = \frac{1}{2} \left[1 - \frac{1}{2} (\phi_1 + \phi_2) \right] - \frac{1}{4} \frac{\bar{U}}{H} (\phi_1 - \phi_2), \quad (46)$$

where

$$\phi_1 = \phi(t_1), \quad \phi_2 = \phi(t_2).$$

The plot of $(\phi_1 + \phi_2)$ and $(\phi_1 - \phi_2)$ for $\beta_1 = 1/2$ is shown in Fig. 2.

We now note that solutions (45) and (46) break down at $\beta_1 = 1$. The reason is quite obvious, since the effect of transport of vorticity is essentially annulled by the electromagnetic force, as $\beta_1 \rightarrow 1$. As a consequence, the vorticity can be diffused farther out, and in this way, the boundary layer is thickened. The boundary layer continues to thicken with increase in β_1 until the critical value $\beta_1 = 1$, at which point the inertial force is essentially annulled by the electromagnetic force. As a result, the boundary layer approximations no longer hold true for $1 - \beta_1 \ll 1$.

When $\beta_1 > 1$, γ_1^2 becomes negative, the whole formulation of the problem then breaks down. From a physical point of view, when $\beta_1 > 1$, the Alfvén speed H_1 is greater than the fluid speed U_1 ; the disturbances can, then, propagate upstream. Thus, no steady state motion is possible. The above solution is, therefore, limited to the range $\beta_1 < 1$ with $(1-\beta_1)$ not small. This is a general feature of the aligned field in the hydromagnetic boundary layer flow.^{9,10,11}

The energy equation (29), with the small perturbation approximation, becomes

$$\frac{\partial T}{\partial x} = \frac{\kappa}{U_1} \frac{\partial^2 T}{\partial y^2}, \quad (47)$$

in which we have also neglected the dissipation heat. Equation (47) gives, with the boundary condition (32)

$$\begin{aligned} \theta &= \frac{1}{2} \left[1 + \phi \left\{ y/2 (\kappa x / U_1)^{1/2} \right\} \right] \\ &= \frac{1}{2} \left[1 + \phi \left\{ (\gamma/2)^{1/2} \xi \right\} \right]. \end{aligned} \quad (48)$$

4. APPROXIMATE SOLUTION FOR $\alpha=1$

As in Section 3, we define

$$u = U_1 - \bar{u}, \quad f = H_1 - \bar{f},$$

where \bar{u} and \bar{f} are not necessarily small here in comparison to U_1 and H_1 .

From Eqs. (3) and (4) we have

$$\bar{u} = \frac{\partial \bar{\psi}}{\partial y}, \quad v = -\frac{\partial \bar{\psi}}{\partial x}, \quad \bar{f} = \frac{\partial \bar{\Lambda}}{\partial y}, \quad g = -\frac{\partial \bar{\Lambda}}{\partial x}.$$

The same dimensional argument, as in Section 2, leads to

$$\begin{aligned} \bar{\psi} &= \frac{\bar{U}}{U_1} (2\nu x U_1)^{1/2} \bar{m}(\xi), \\ \bar{\Lambda} &= \frac{\bar{U}}{U_1} (2\nu x U_1)^{1/2} \bar{n}(\xi). \end{aligned}$$

By differentiation, we have

$$\bar{u} = \bar{U} \bar{m}'(\xi), \quad v = \bar{U} \left(\frac{\nu}{2xU_1} \right)^{1/2} (\bar{m}'\xi - \bar{m}), \quad (49)$$

$$\bar{f} = \bar{U} \bar{n}'(\xi), \quad g = \bar{U} \left(\frac{\nu}{2xU_1} \right)^{1/2} (\bar{n}'\xi - \bar{n}). \quad (50)$$

The boundary layer equations (26) and (27) become

$$\bar{m}''' + \bar{m}''(\xi - s\bar{m}) - \bar{n}''(\beta_1\xi - s\bar{n}) = 0, \quad (51)$$

$$\bar{n}''/\alpha + \bar{n}''(\xi - s\bar{m}) - \bar{m}''(\beta_1\xi - s\bar{n}) = 0, \quad (52)$$

where $s = \bar{U}/U_1 = 1 - \lambda$. It is easily seen that Eqs. (51) and (52) reduce to the equations for small perturbation approximation if $s \ll 1$ and $s r \ll \beta_1$, where $r = \bar{H}/\bar{U}$.

The boundary conditions in the present case are

$$\left. \begin{array}{ll} \xi \rightarrow \infty : & \bar{m}' \rightarrow 0, \quad \bar{n}' \rightarrow 0, \\ \xi \rightarrow -\infty : & \bar{m}' \rightarrow 1, \quad \bar{n}' \rightarrow r. \end{array} \right\} \quad (53)$$

Adding and subtracting (51) and (52), and assuming $\alpha = 1$, we obtain

$$\bar{G}''' + \bar{G}''[\xi(1 - \beta_1) - s\bar{W}] = 0, \quad (54)$$

$$\bar{W}''' + \bar{W}''[\xi(1 + \beta_1) - s\bar{G}] = 0, \quad (55)$$

where $\bar{G} = \bar{m} + \bar{n}$ and $\bar{W} = \bar{m} - \bar{n}$. Since a fifth boundary condition is arbitrary up to a translation along the ξ -axis, we require that

$$\bar{m}'(0) = \frac{1}{2}. \quad (56)$$

Being suggested by the small perturbation solution, we now assume that

$$\bar{\pi}'(0) = \frac{1}{2}r. \quad (57)$$

Integration gives

$$\bar{m} = \frac{1}{2}\xi + a \quad \text{and} \quad \bar{n} = \frac{1}{2}r\xi + b \quad \text{as} \quad \xi \rightarrow 0.$$

From the conditions (53) we also have

$$\bar{m} = c_1, \quad \bar{n} = d_1, \quad \text{as} \quad \xi \rightarrow \infty,$$

and

$$\bar{m} = \xi + c_2, \quad \bar{n} = r\xi + d_2 \quad \text{as} \quad \xi \rightarrow -\infty.$$

In the actual case, a and b are very small and the constants c_1 , d_1 , c_2 and d_2 are positive. Therefore, we will approximate the terms involving s in Eqs. (54) and (55) by

$$s\bar{W} = \frac{s\xi}{2} (1-r), \quad (58)$$

$$s\bar{G} = \frac{s\xi}{2} (1+r). \quad (59)$$

Eqs. (54) and (55) then become

$$\bar{G}'' + l_1 \xi \bar{G}' = 0, \quad (60)$$

$$\bar{W}'' + l_2 \xi \bar{W}' = 0, \quad (61)$$

where

$$l_1 = 1 - \beta_1 - \frac{s}{2}(1-r),$$

$$l_2 = 1 + \beta_1 - \frac{s}{2}(1+r).$$

It follows immediately that

$$\bar{G}' = \frac{1+r}{2} \left[1 - \phi \left\{ \left(\frac{\ell_1}{2} \right)^{1/2} \xi \right\} \right], \quad \bar{W}' = \frac{1-r}{2} \left[1 - \phi \left\{ \left(\frac{\ell_2}{2} \right)^{1/2} \xi \right\} \right].$$

Therefore, we obtain

$$\frac{\bar{u}}{\bar{U}} = \frac{1}{2} \left[1 - \frac{1}{2} (\bar{\phi}_1 + \bar{\phi}_2) \right] - \frac{r}{4} (\bar{\phi}_1 - \bar{\phi}_2), \quad (62)$$

$$\frac{\bar{f}}{\bar{H}} = \frac{1}{2} \left[1 - \frac{1}{2} (\bar{\phi}_1 + \bar{\phi}_2) \right] - \frac{1}{4r} (\bar{\phi}_1 - \bar{\phi}_2), \quad (63)$$

where $\bar{\phi}_1 = \phi \left\{ \left(\frac{\ell_1}{2} \right)^{1/2} \xi \right\}$ and $\bar{\phi}_2 = \phi \left\{ \left(\frac{\ell_2}{2} \right)^{1/2} \xi \right\}$. It can easily be shown that the approximate energy equation is

$$\theta'' + \ell_3 \xi \theta' = 0, \quad \ell_3 = \gamma \left(1 - \frac{1}{2} s \right). \quad (64)$$

Integration of (64) gives

$$\theta = \frac{1}{2} \left[1 + \phi \left\{ \left(\frac{\ell_3}{2} \right)^{1/2} \xi \right\} \right]. \quad (65)$$

The approximate solutions (62), (63) and (65) differ from the small perturbation solutions (45), (46) and (48) only by the argument in the error function ϕ defined in (44). A comparison of the approximate solutions with the exact boundary layer solutions is shown in Fig. 3. Eq. (62) gives the values of u and f which is almost indistinguishable from the exact solution in this particular case. Small perturbation solutions are also shown in dotted lines. Fig. 4 shows that for $\lambda < 1/2$, the small perturbation solution has quite a discrepancy with the exact solution while the approximate solution involves much smaller errors. For instance, with $\lambda = 1/3$, the maximum error for the approximate solution is only 1 per cent while that for the small perturbation solution is 4.5 per cent. Even when $\lambda = 0$, the average error is about 2 per cent for the approximate solution. It is noted that the approximate solution is very helpful in the numerical integration.

5. THE INTEGRATION OF THE EXACT BOUNDARY LAYER EQUATIONS

5.1 Asymptotic Expansions

The governing equations (26), (27) and (31) have the property that, if $m(\xi)$, $n(\xi)$ and $\theta(\xi)$ are any solutions, then $m(\omega)$, $n(\omega)$ and $\theta(\omega)$ are also solutions, provided that

$$\omega = a\xi + b,$$

$$m(\xi) = am_1(\omega), \quad n(\xi) = an_1(\omega), \quad \theta(\xi) = a^4\theta_1(\omega).$$

Restricted to the case $a = 1$, it follows from Eqs. (26) and (27) that

$$R''' + R''S = 0, \quad (66)$$

$$R'' - S'' = RS' - SR', \quad (67)$$

in which we have written

$$R = m + n \text{ and } S = m - n.$$

Differentiating (67) and then subtracting this from (66) we obtain

$$S''' + S''R = 0. \quad (68)$$

First, let us consider the asymptotic expansions when ξ is large and negative.

The boundary conditions require that

$$R' \rightarrow \lambda + \beta_2 \text{ and } S' \rightarrow \lambda - \beta_2 \text{ as } \xi \rightarrow -\infty.$$

There are two cases to be considered, $\lambda > \beta_2$ and $\lambda = \beta_2 = 0$.

No steady state cases will correspond to $\lambda < \beta_2$, since, then, the Alfvén

speed H_2 is greater than the fluid speed U_2 at $\xi \rightarrow -\infty$.

In the first case, when $\lambda > \beta_2$, we can assume that

$$R \rightarrow (\lambda + \beta_2)(\xi + A), \quad (69)$$

$$S \rightarrow (\lambda - \beta_2)(\xi + A), \quad (70)$$

as $\xi \rightarrow -\infty$. Eqs. (66) and (68) then become approximately

$$R''' + R''(\lambda - \beta_2)(\xi + A) = 0, \quad (71)$$

$$S''' + S''(\lambda + \beta_2)(\xi + A) = 0, \quad (72)$$

the solution of which may be written

$$R'' \approx 2 c_1 (\lambda - \beta_2) \exp \left\{ -\frac{1}{2} (\lambda - \beta_2) (\xi + A)^2 \right\}, \quad (73)$$

$$S'' \approx 2 c_1 (\lambda - \beta_2) \exp \left\{ -\frac{1}{2} (\lambda + \beta_2) (\xi + A)^2 \right\}, \quad (74)$$

so that

$$R' \approx \lambda + \beta_2 + 4 c_1 \left(\frac{\lambda - \beta_2}{2} \right)^{1/2} \operatorname{erf} \left\{ -\left(\frac{\lambda - \beta_2}{2} \right)^{1/2} (\xi + A) \right\}, \quad (75)$$

$$S' \approx \lambda - \beta_2 + 4 c_2 \left(\frac{\lambda + \beta_2}{2} \right)^{1/2} \operatorname{erf} \left\{ -\left(\frac{\lambda + \beta_2}{2} \right)^{1/2} (\xi + A) \right\}, \quad (76)$$

where

$$\operatorname{erf} z = \int_z^\infty e^{-t^2} dt,$$

and

$$R \approx (\lambda + \beta_2)(\xi + A) + 4 c_1 \left(\frac{\lambda - \beta_2}{2} \right)^{1/2} \int_{-\infty}^{\xi} \operatorname{erf} \left\{ -\left(\frac{\lambda - \beta_2}{2} \right)^{1/2} (\xi + A) \right\} d\xi, \quad (77)$$

$$S \approx (\lambda - \beta_2)(\xi + A) + 4 c_2 \left(\frac{\lambda + \beta_2}{2} \right)^{1/2} \int_{-\infty}^{\xi} \operatorname{erf} \left\{ -\left(\frac{\lambda + \beta_2}{2} \right)^{1/2} (\xi + A) \right\} d\xi. \quad (78)$$

Since it is known

$$\operatorname{erf} z \approx \frac{1}{2} \frac{e^{-z^2}}{z} \left(1 - \frac{1}{2z^2} + \frac{3}{4z^4} - \dots \right), \quad (79)$$

integration of this expression leads to

$$R \approx (\lambda + \beta_2)(\xi + A) + \frac{c_1 e^{-z_1^2}}{z_1^2} \left(1 - \frac{3}{2 z_1^2} + \frac{15}{4 z_1^4} - \dots \right), \quad (80)$$

$$S \approx (\lambda - \beta_2)(\xi + A) + \frac{c_2 e^{-z_2^2}}{z_2^2} \left(1 - \frac{3}{2 z_2^2} + \frac{15}{4 z_2^4} - \dots \right), \quad (81)$$

$$z_1 = - \left(\frac{\lambda - \beta_2}{2} \right)^{1/2} (\xi + A), \quad (82)$$

$$z_2 = - \left(\frac{\lambda + \beta_2}{2} \right)^{1/2} (\xi + A). \quad (83)$$

A second approximation can be obtained by substituting $\{(\lambda + \beta_2)(\xi + A) + c_1 e^{-z_1^2}/z_1^2\}$ for R and $\{(\lambda - \beta_2)(\xi + A) + c_2 e^{-z_2^2}/z_2^2\}$ for S in Eqs. (66) and (68). We get eventually, when ξ is large and negative,

$$R'' \approx 2 c_1 (\lambda - \beta_2) e^{-z_1^2} - c_1 c_2 b \{2(\lambda + \beta_2)\}^{1/2} \frac{e^{-(z_1^2 + z_2^2)}}{z_2^3} + \dots, \quad (84)$$

$$R' \approx \lambda + \beta_2 + 4 c_1 \left(\frac{\lambda - \beta_2}{2} \right)^{1/2} \operatorname{erf} z_1 - \frac{c_1 c_2 b}{1+b} \frac{e^{-(z_1^2 + z_2^2)}}{z_1^4} + \dots, \quad (85)$$

$$R \approx (\lambda + \beta_2)(\xi + A) + \frac{c_1 e^{-z_1^2}}{z_1^2} \left(1 - \frac{3}{2 z_1^2} + \frac{15}{4 z_1^4} - \dots \right) - \frac{c_1 c_2 b}{\{2(\lambda + \beta_2)\}^{1/2} (1+b)^2} \frac{e^{-(z_1^2 + z_2^2)}}{z_2^5} + \dots, \quad (86)$$

$$S'' \approx 2 c_2 (\lambda + \beta_2) e^{-z_2^2} - \frac{c_1 c_2 \{2(\lambda - \beta_2)\}^{1/2}}{b} \frac{e^{-(z_1^2 + z_2^2)}}{z_1^3} + \dots, \quad (87)$$

$$S' \approx \lambda - \beta_2 + 4 c_2 \left(\frac{\lambda + \beta_2}{2} \right)^{1/2} \operatorname{erf} z_2 - \frac{c_1 c_2}{1+b} \frac{e^{-(z_1^2 + z_2^2)}}{z_1^4} + \dots, \quad (88)$$

$$S \approx (\lambda - \beta_2)(\xi + A) + \frac{c_2 e^{-z_2^2}}{z_2^2} \left(1 - \frac{3}{2 z_2^2} + \frac{15}{4 z_2^4} - \dots \right) - \frac{c_1 c_2 b}{\{2(\lambda - \beta_2)\}^{1/2} (1+b)^2} \frac{e^{-(z_1^2 + z_2^2)}}{z_1^5} + \dots, \quad (89)$$

where

$$b = (\lambda - \beta_2) / (\lambda + \beta_2).$$

In the second case, if $\lambda = \beta_2 = 0$, we can suppose that $R(\xi) \rightarrow -a$ and $S(\xi) \rightarrow -a$ as $\xi \rightarrow -\infty$, where a is a constant. The reason that R and S approach the same constant $-a$, is quite obvious. Since, if we let $\beta_2 \rightarrow 0$ first, Eqs. (86) and (89) yield the conclusion that R and S approach the same constant. Now we let $\lambda \rightarrow 0$, we should expect they approach the same constant at $\lambda = 0$. If we substitute $-a$ for R and S in Eqs. (66) and (68) and integrate three times, we get

$$R(\xi) \simeq -a + a_1 e^{a\xi},$$

$$S(\xi) \simeq -a + b_1 e^{a\xi}.$$

This suggests that we try the expansions

$$R(\xi) \simeq -a + a_1 e^{a\xi} + a_2 e^{2a\xi} + a_3 e^{3a\xi} + \dots, \quad (90)$$

$$S(\xi) \simeq -a + b_1 e^{a\xi} + b_2 e^{2a\xi} + b_3 e^{3a\xi} + \dots \quad (91)$$

Substituting in Eqs. (66) and (68), and equating to zero the coefficients of successive powers of $e^{a\xi}$, we obtain the relations

$$a_2 = b_2 = -\frac{a_1 b_1}{4a},$$

$$a_3 = -\frac{a_2(4b_1 + a_1)}{18a},$$

$$b_3 = -\frac{a_2(4a_1 + b_1)}{18a},$$

$$\dots$$

and in general all coefficients are functions of a , a_1 and b_1 . If we let $\omega = a\xi$, then we have

$$R_1(\omega) = R(\xi)/a = -1 + A_1 e^{\omega} + A_2 e^{2\omega} + A_3 e^{3\omega} + \dots, \quad (92)$$

$$S_1(\omega) = S(\xi)/a = -1 + B_1 e^{\omega} + B_2 e^{2\omega} + B_3 e^{3\omega} + \dots, \quad (93)$$

with

$$\begin{aligned} A_2 &= B_2 = -A_1 B_1 / 4, \\ A_3 &= -A_2 (4B_1 + A_1) / 18, \\ B_3 &= -A_2 (4A_1 + B_1) / 18, \\ &\dots \end{aligned}$$

The corresponding asymptotic expansions for $R(\xi)$ and $S(\xi)$ when ξ is $+\infty$, and when $R' \rightarrow 1 + \beta_1$, $S' \rightarrow 1 - \beta_1$, can be obtained in a similar way, or from (86) and (89) by using the fact that $-R(-\xi)$ and $-S(-\xi)$ are also solutions of (66) and (68). Putting $\lambda=1$, $\beta_2 = \beta_1$, and B, d_1, d_2 for A, c_1, c_2 , they are

$$\begin{aligned} R \approx (1 + \beta_1)(\xi - B) - \frac{d_1 e^{-\eta_1^2}}{\eta_1^2} \left(1 - \frac{3}{2\eta_1^2} + \frac{15}{4\eta_1^4} - \dots \right) + \\ + \frac{d_1 d_2 \bar{b}}{\{2(1 + \beta_1)\}^{1/2} (1 + \bar{b})^2} \frac{e^{-(\eta_1^2 + \eta_2^2)}}{\eta_2^5} + \dots, \end{aligned} \quad (94)$$

$$\begin{aligned} S \approx (1 - \beta_1)(\xi - B) - \frac{d_2 e^{-\eta_2^2}}{\eta_2^2} \left(1 - \frac{3}{2\eta_2^2} + \frac{15}{4\eta_2^4} - \dots \right) + \\ + \frac{d_1 d_2 \bar{b}}{\{2(1 - \beta_1)\}^{1/2} (1 + \bar{b})^2} \frac{e^{-(\eta_1^2 + \eta_2^2)}}{\eta_1^5} + \dots, \end{aligned} \quad (95)$$

where

$$\begin{aligned} \eta_1 &= \left(\frac{1 - \beta_1}{2} \right)^{1/2} (\xi - B), \\ \eta_2 &= \left(\frac{1 + \beta_1}{2} \right)^{1/2} (\xi - B), \\ \bar{b} &= (1 - \beta_1) / (1 + \beta_1). \end{aligned}$$

Since m and n are related with R and S in the following manner

$$m = \frac{1}{2}(R + S) \quad \text{and} \quad n = \frac{1}{2}(R - S), \quad (96)$$

the asymptotic expansions for m and n are readily obtained from above equations.

5.2 Numerical Solution

We expand the solution in a Taylor series about $\xi = 0$

$$m = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 + \dots, \quad (97)$$

$$n = b_0 + b_1 \xi + b_2 \xi^2 + b_3 \xi^3 + \dots, \quad (98)$$

where all coefficients are functions of a_0, a_1, a_2, b_0 and b_1 .

The asymptotic solution reveals that the constants occurring in (86) and (89), or (94) and (95), are three in number, namely A, c_1, c_2 , (a, a_1, b_1 for the case $\lambda = \beta_2 = 0$) and B, d_1, d_2 . Therefore, the total number of these arbitrary constants is eleven. If we now seek to join the asymptotic solution and the power series solution at some intermediate positive and negative values of ξ , say ξ_1 and $-\xi_1$, inspection of the differential equations (26) and (27) reveals that continuity of m, n , and all derivatives, at ξ_1 and $-\xi_1$, is insured by equating m, m', m'', n, n' . This yields ten algebraic equations in eleven unknowns, $a_0, a_1, a_2, b_0, b_1, A, c_1, c_2, B, d_1$, and d_2 . Hence one constant is arbitrary. However, the boundary conditions and the governing equations reveal that this arbitrariness is only up to a translation along the ξ -axis.

We can, therefore pre-assign a fixed value to one of these constants, say $a_0 = 0$, or $a_1 = (1 + \lambda)/2$, which corresponds to $m(0) = 0$ or $m'(0) = (1 + \lambda)/2$.

Some sample numerical solutions are shown in Figs. (5) - (8). The asymptotic values are reached quite rapidly, especially when β_1 and β_2 are small. In most cases, the velocity and magnetic field approach their asymptotic values when $|\xi| \approx 4$. The approximate solution in Section 4 gives a good guide in the numerical integration.

The numerical solutions in Figs. (5) - (8) correspond to $\lambda = 0., 0.333, 0.50,$ and 0.80 respectively. As we can see that the thickness of flow boundary layer increases as the strength of the magnetic field increases; that is, the magnetic field has a diffusing effect on the flow field. This effect is much profound when β_1 or β_2 is comparable to one, as shown in Figs. 6 and 7. Indeed, the approximate and small perturbation solutions have also shown this effect and some discussion has been given in these sections. It will be interesting to note that the flow field becomes more stable when the strength of the magnetic field increases, since then the boundary layer is thickened and the velocity shear is reduced. A detailed study of the stability of the magnetohydrodynamic flow between parallel streams will give some answer to the question: how the flow between parallel streams can be stabilized by a magnetic field.

6. TEMPERATURE FIELD

When the fluid motion is determined, the temperature field follows readily from Eq. (31). Since the asymptotic expansion for m is known, the asymptotic expansion for θ can similarly be obtained. The numerical solutions for the temperature field are also shown in Figs. (5) - (8).

As a special case when the magnetic field strength is zero, Eqs. (26) and (31) become, with $\gamma = 1$

$$m''' + m m'' = 0 , \quad (99)$$

$$\theta'' + m \theta' = 0 , \quad (100)$$

where we have neglected the dissipation heat in the energy equation. The boundary conditions are

$$\left. \begin{array}{l} \xi \rightarrow \infty : m' \rightarrow 1 , \quad \theta \rightarrow 1 , \\ \xi \rightarrow -\infty : m' \rightarrow \lambda , \quad \theta \rightarrow 0 . \end{array} \right\} \quad (101)$$

If we define

$$\theta = \frac{1}{1-\lambda} (\bar{\theta} - \lambda) , \quad (102)$$

it follows immediately that

$$\bar{\theta}(\xi) = m'(\xi) , \quad (103)$$

whence

$$\theta = \frac{1}{1-\lambda} (m' - \lambda). \quad (104)$$

In this case, the temperature field is related to the flow field through this simple relation. A comparison between Eq. (104) and the exact solution with dissipation heat included, is shown in Fig. 9. As we can see that the dissipation heat is important when $\lambda < 1/2$. On the other hand, the dissipation heat may be neglected safely when $\lambda > 0.8$. In Fig. 9, Eq. (104) is almost indistinguishable from the exact solution for $\lambda = 0.8$.

7. PARALLEL STREAMS OF TWO FLUIDS

When the parallel streams are of two fluids so that the constants of fluid properties are different in the two streams. If the mass diffusion is neglected, there are, then, some additional conditions to be satisfied at the interface. Since the problem is arbitrary up to a translation along the ξ - axis and the interface is one of the stream surfaces, we will take $\xi = 0$ as the interface and

$$m = 0 \quad \text{at} \quad \xi = 0. \quad (105)$$

Other conditions at the interface are: the shearing stress, velocity, magnetic field strength, temperature and heat flux must be continuous, which yields

$$\xi = 0 : \quad \left. \begin{aligned} m_1 &= m_2, & m'_1 &= m'_2, & f_1 \nu_1^{1/2} m''_1 &= f_2 \nu_2^{1/2} m''_2, \\ n_1 &= n_2, & n'_1 &= n'_2, & \theta_1 &= \theta_2, & k_1 \theta'_1 &= k_2 \theta'_2. \end{aligned} \right\} \quad (106)$$

Eqs. (105) and (106) together with the conditions at infinity

$$\left. \begin{aligned} \xi \rightarrow \infty : m' \rightarrow 1, \quad n' \rightarrow \beta_1, \quad \theta \rightarrow 1, \\ \xi \rightarrow -\infty : m' \rightarrow \lambda, \quad n' \rightarrow \beta_2, \quad \theta \rightarrow 0, \end{aligned} \right\} \quad (107)$$

provide adequate boundary conditions for the governing equations (26), (27) and (31).

Therefore, asymptotic expansions, Taylor series and numerical integration can be carried out in a similar manner.

8. CONCLUSIONS

From a study of the interaction between the flow and magnetic fields, the following conclusions may be drawn:

1. The thickness of the flow boundary layer increases as the strength of magnetic field increases. In other words, the magnetic field has a diffusing effect upon the flow field.
2. The thickening of the flow boundary layer reduces the velocity shear, thus, provides a stabilizing effect upon fluid motion.
3. The approximate solution for the case of $\alpha = 1$ gives a quite satisfactory flow field, which only involves a discrepancy of 1 per cent from the exact numerical solution in most cases.
4. The extension to the two fluids may be readily carried out in a similar manner.

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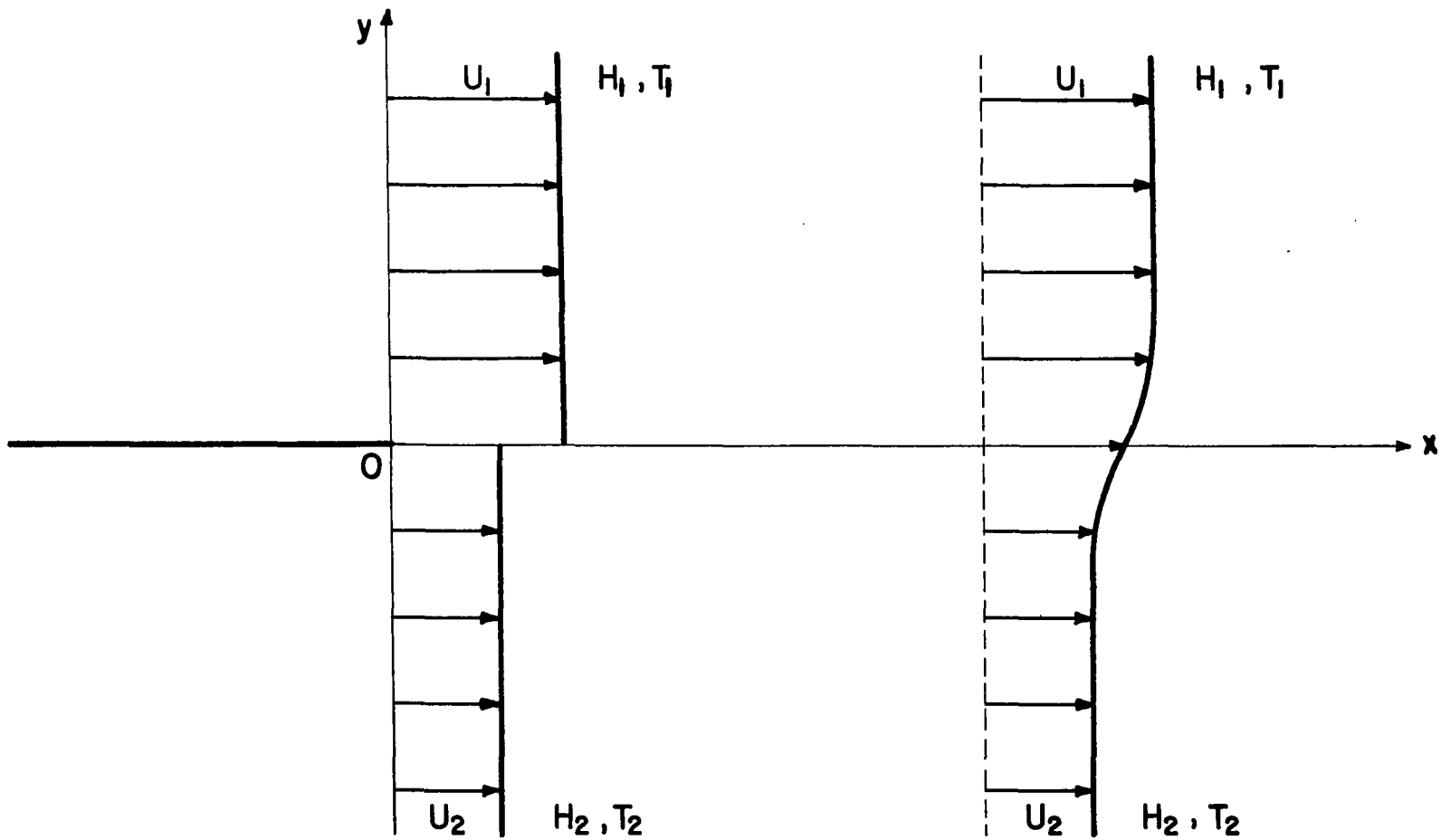


Figure 1. Configuration of the flow, magnetic and temperature fields.

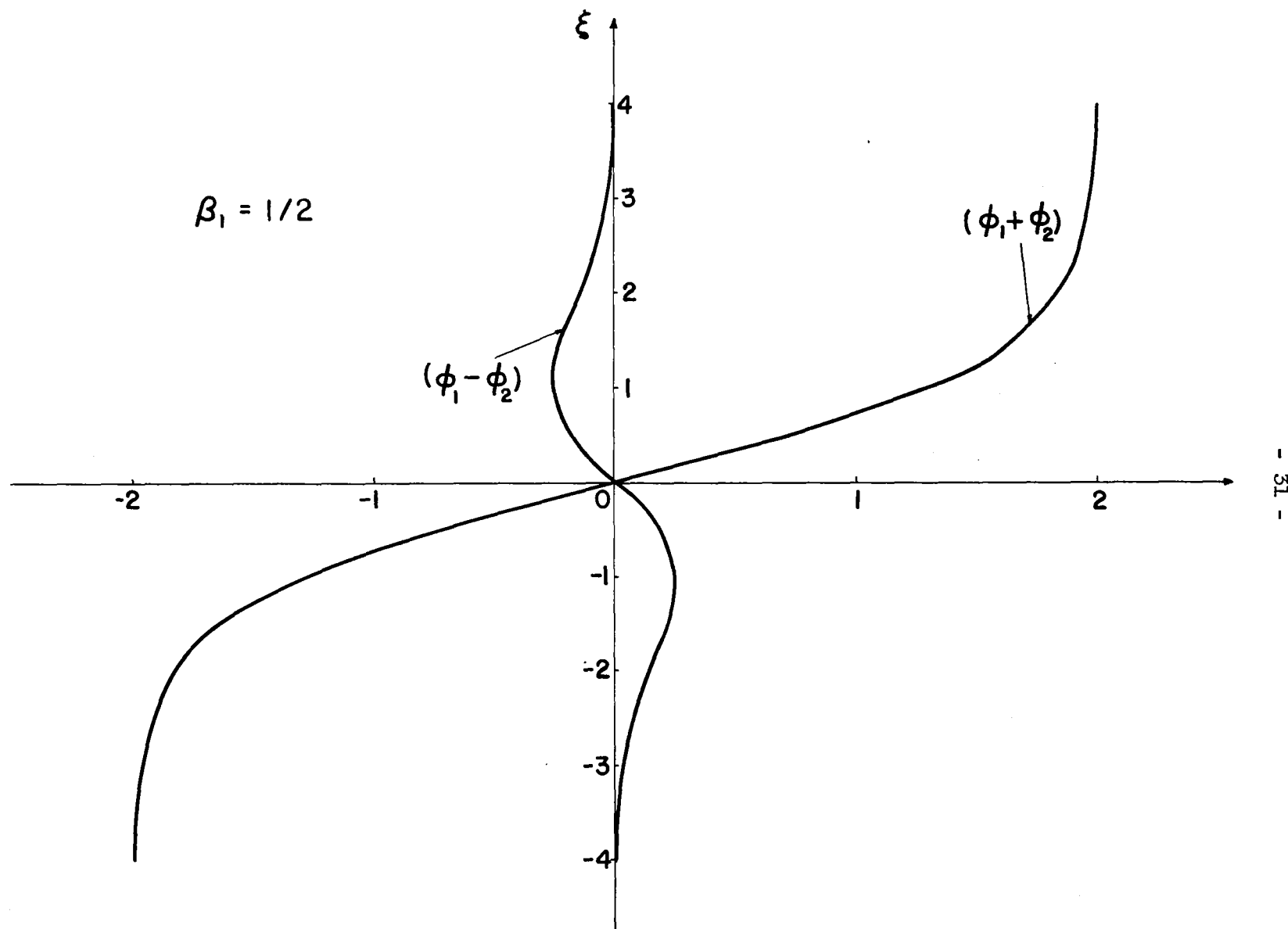


Figure 2. $(\phi_1 + \phi_2)$ -and $(\phi_1 - \phi_2)$ - diagram for small perturbation solution with $\beta_1 = 1/2$.

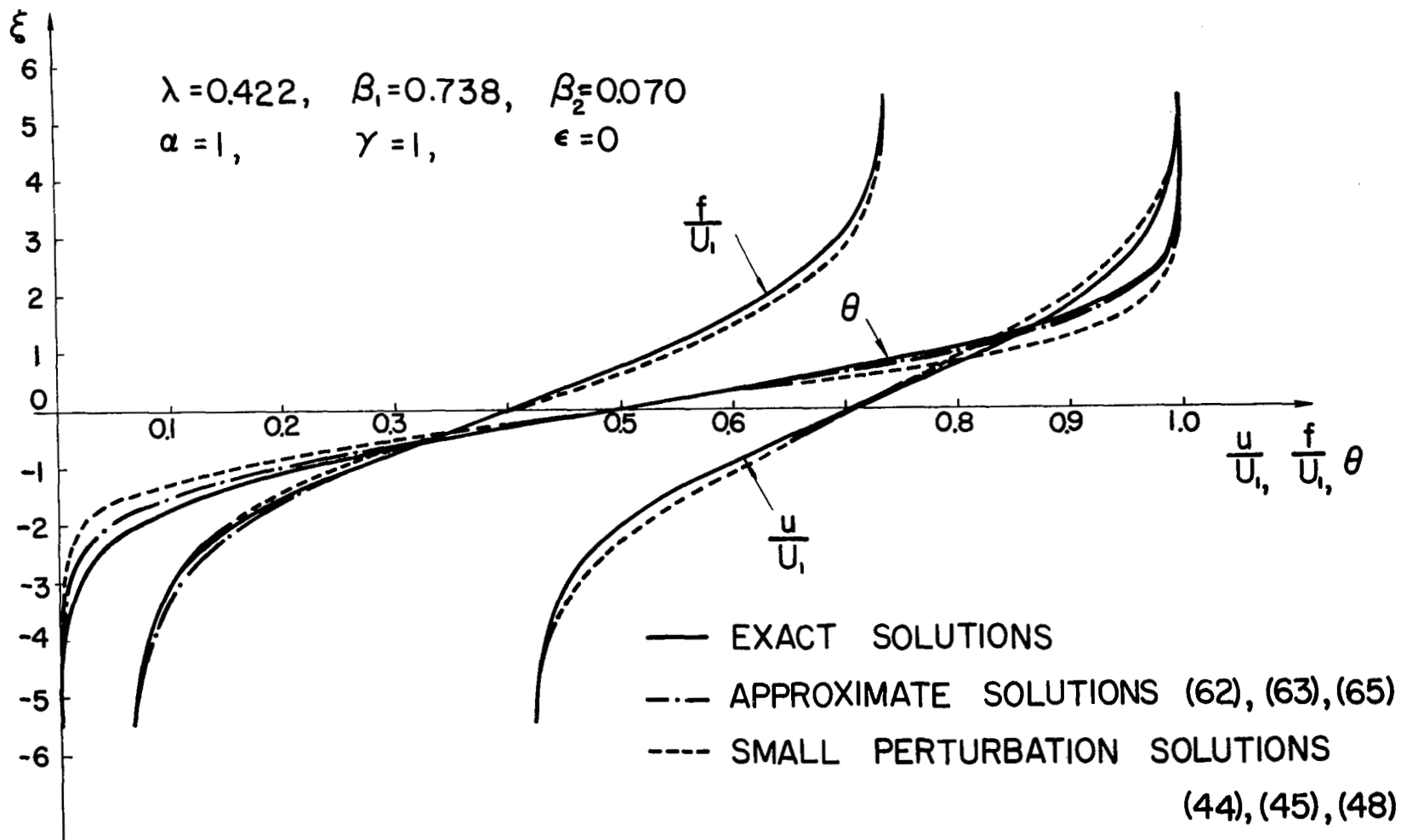


Figure 3. Comparison of small perturbation solution, approximate solution and exact solution.

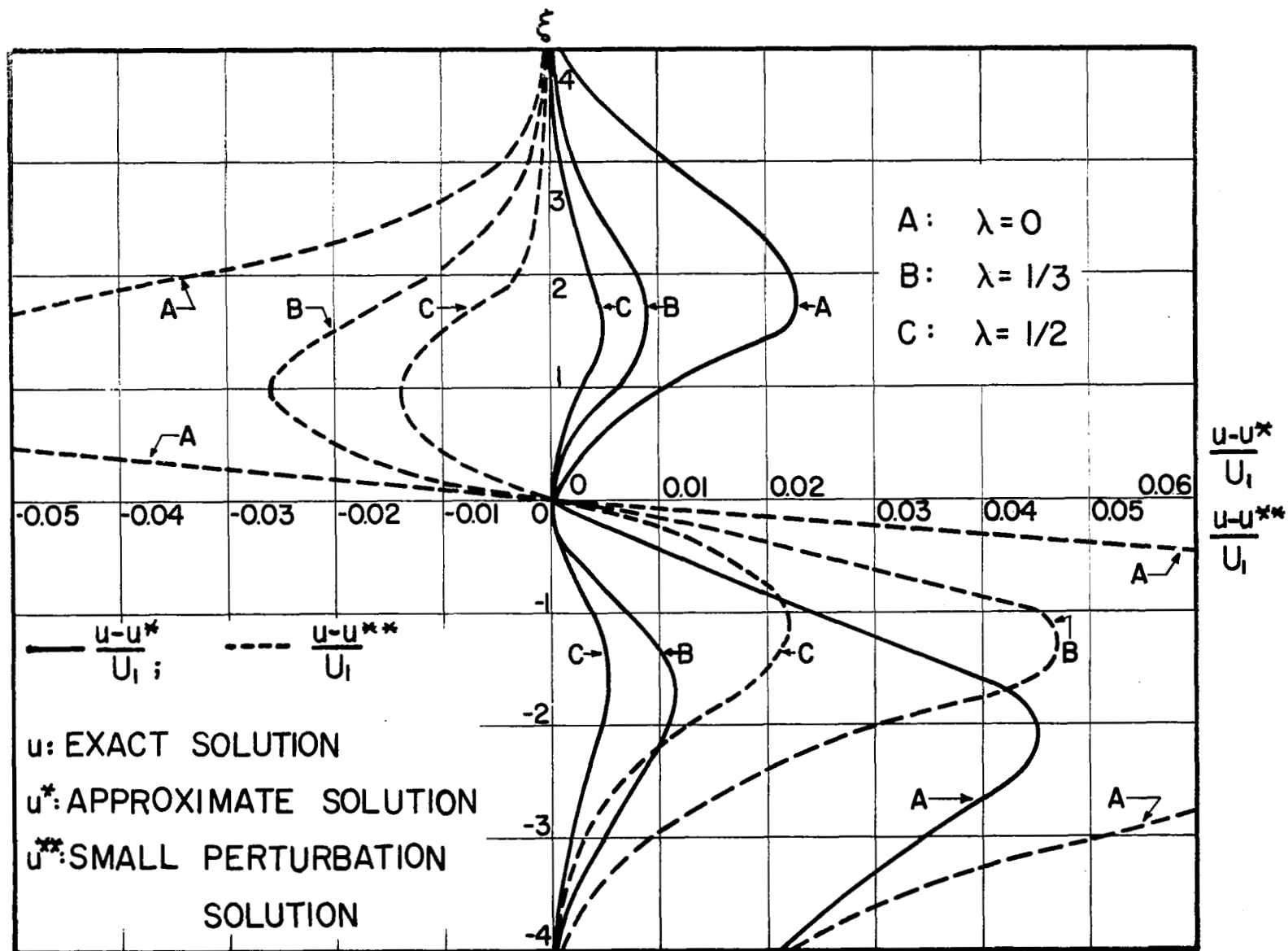


Figure 4. Comparison of small perturbation solution, approximate solution and exact solution for non-magnetic case .

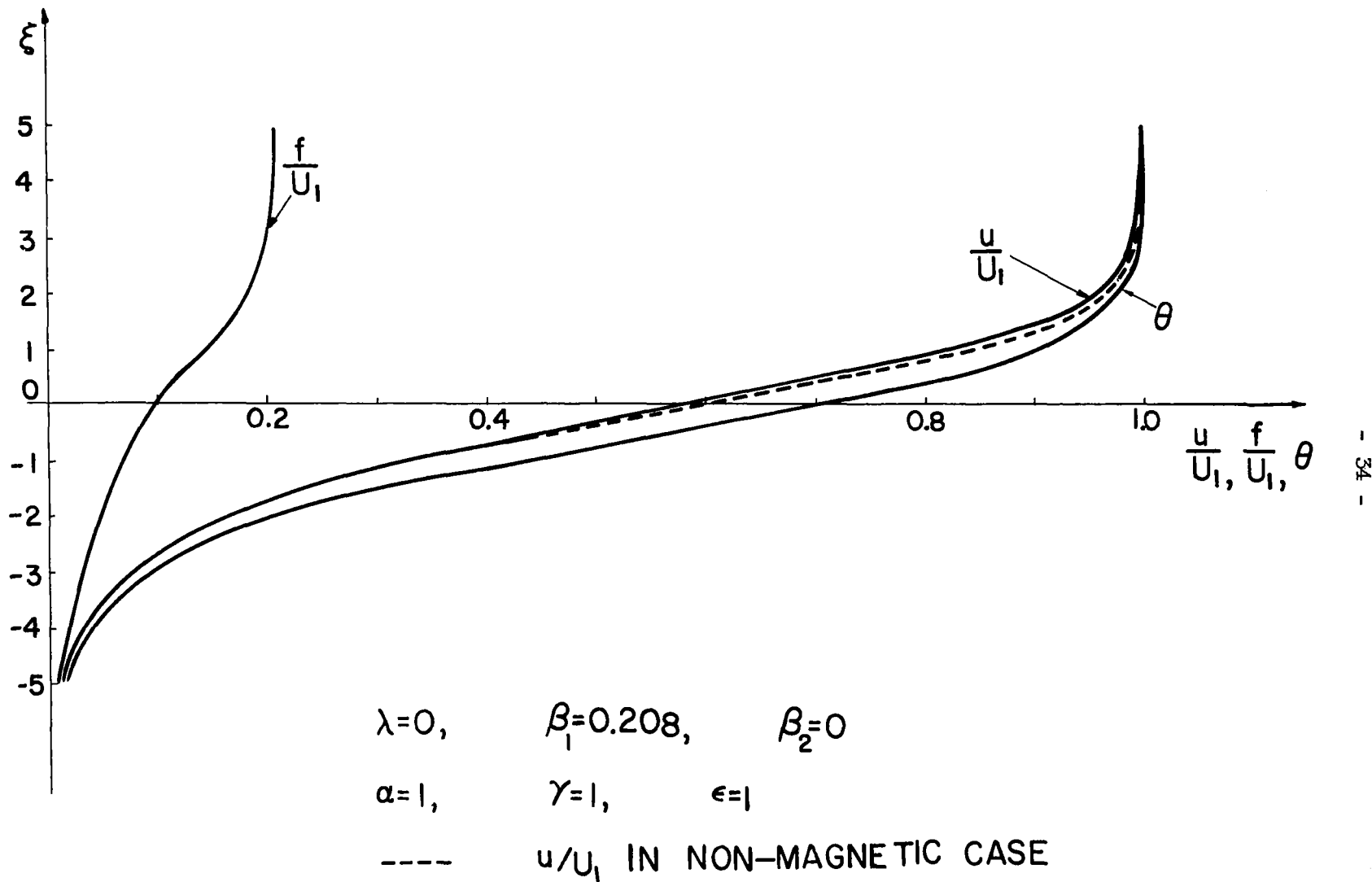


Figure 5. Velocity, magnetic and temperature fields .

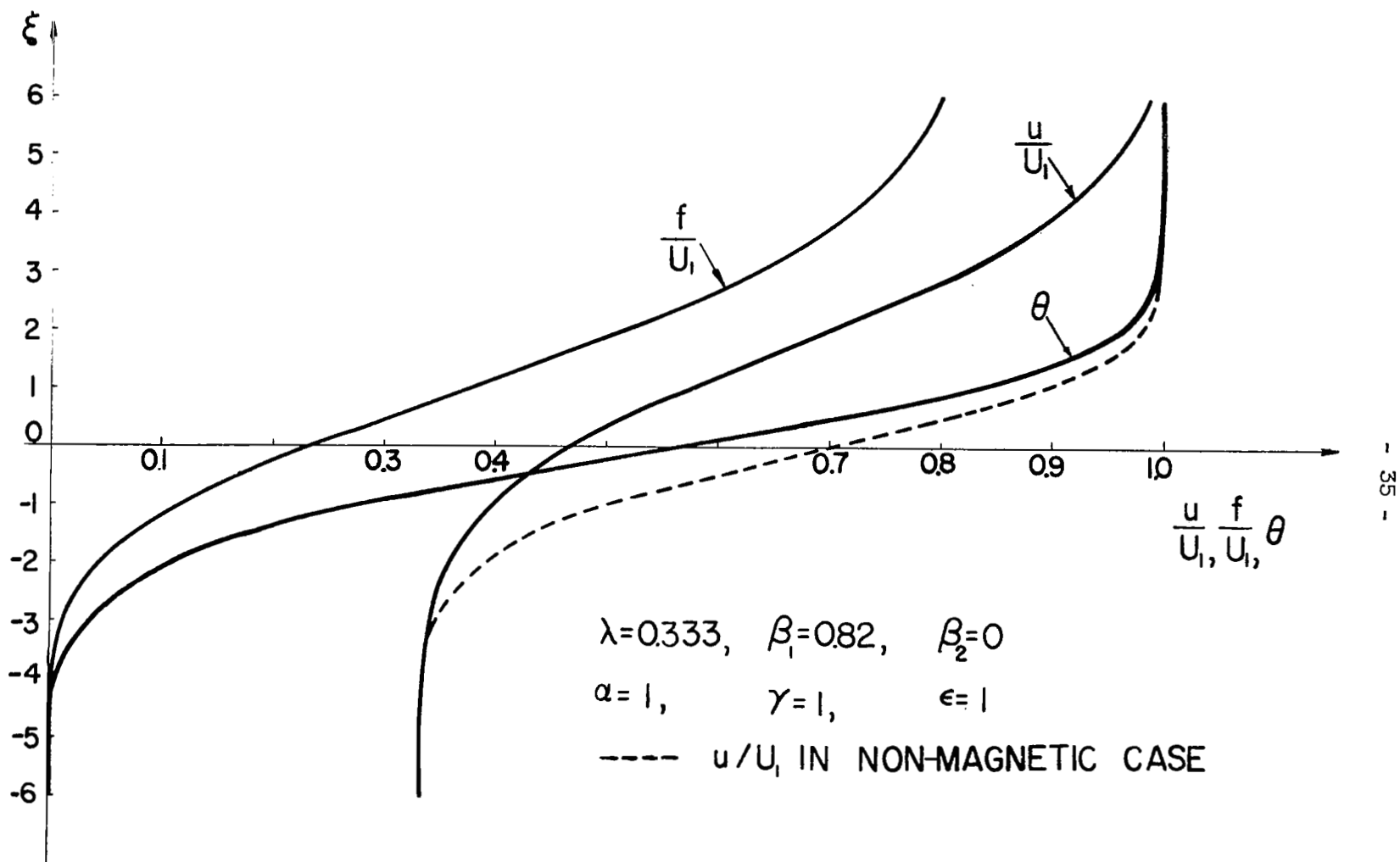


Figure 6. Velocity, magnetic and temperature fields.

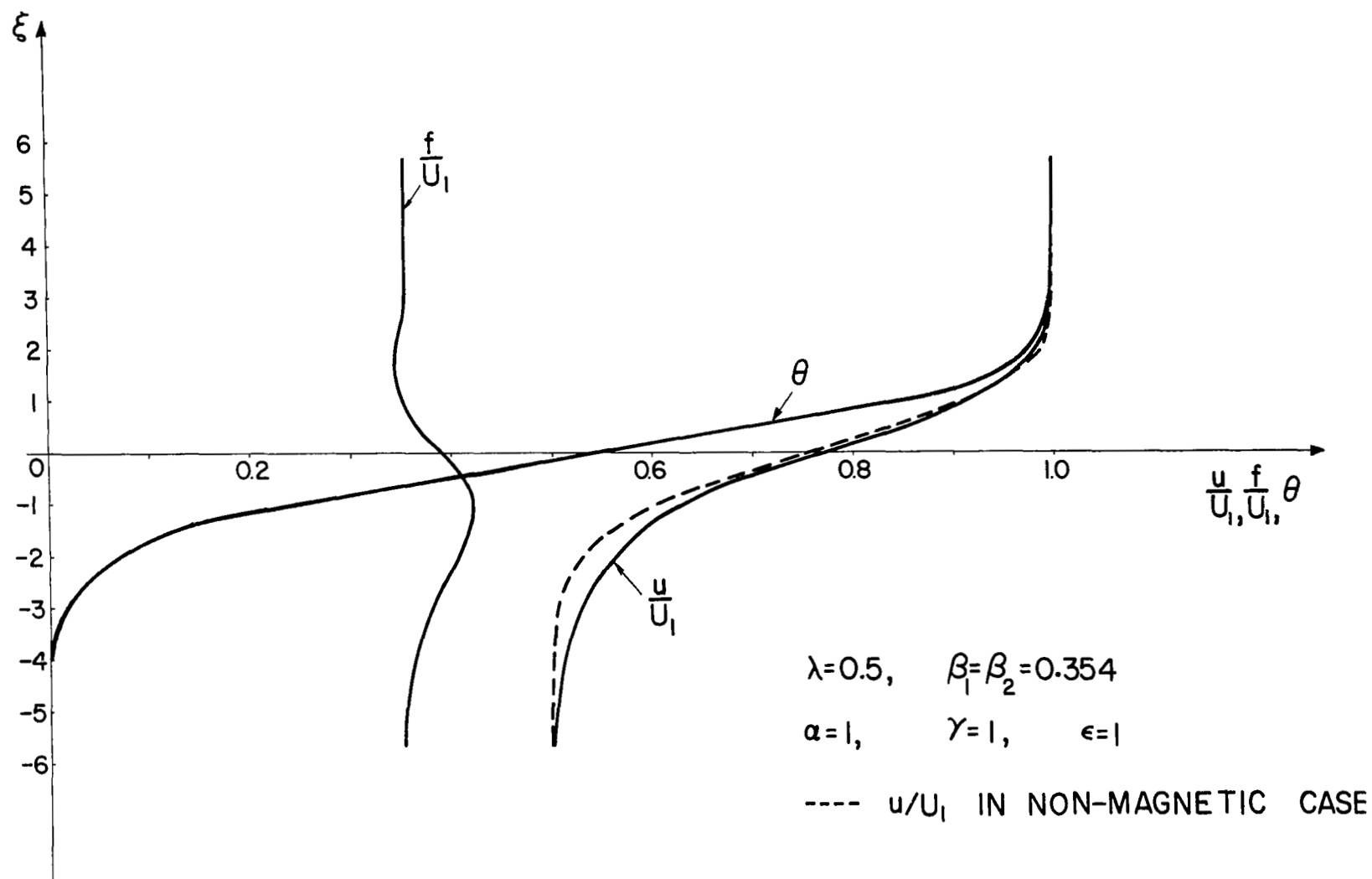


Figure 7. Velocity, magnetic and temperature fields.

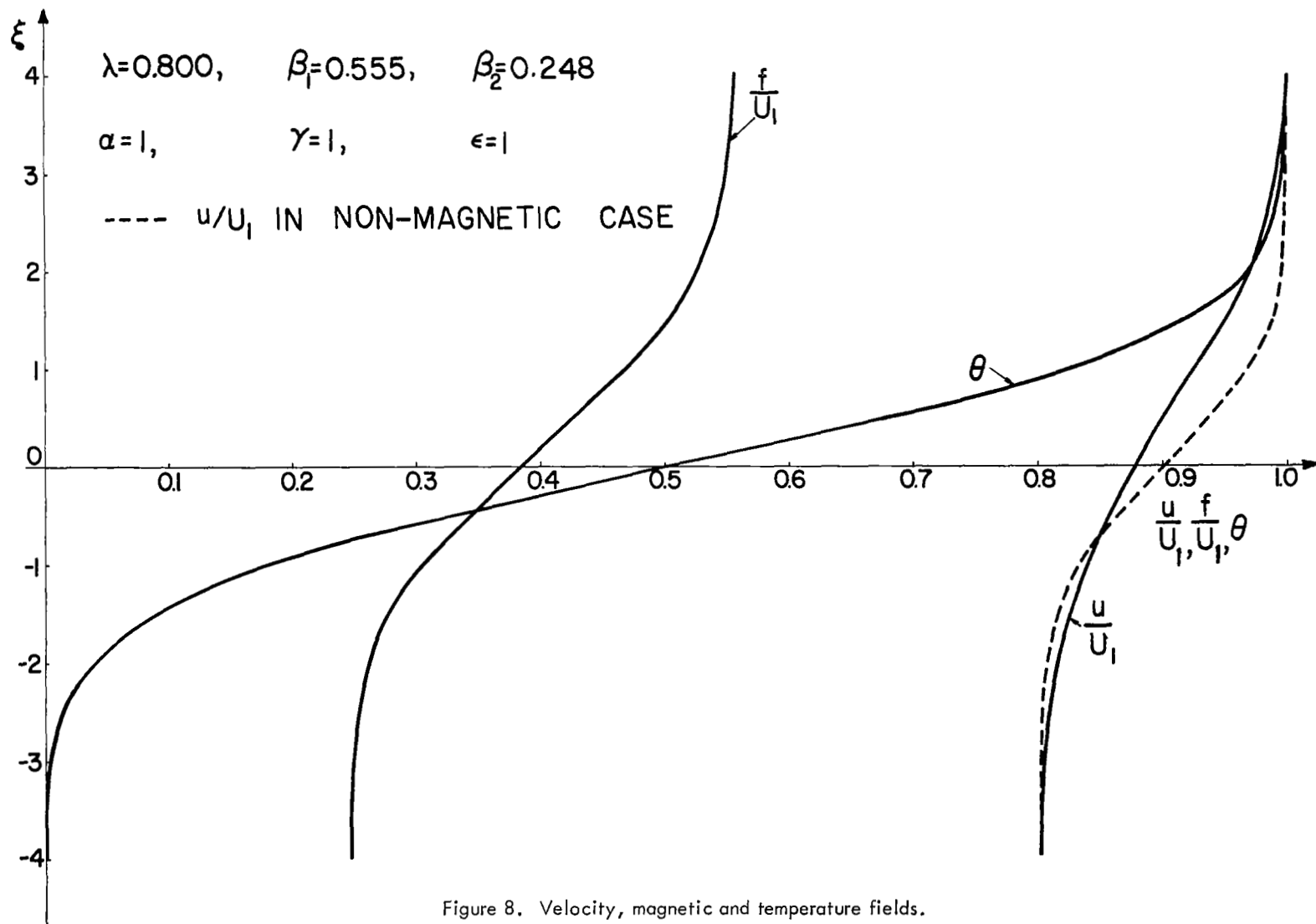


Figure 8. Velocity, magnetic and temperature fields.

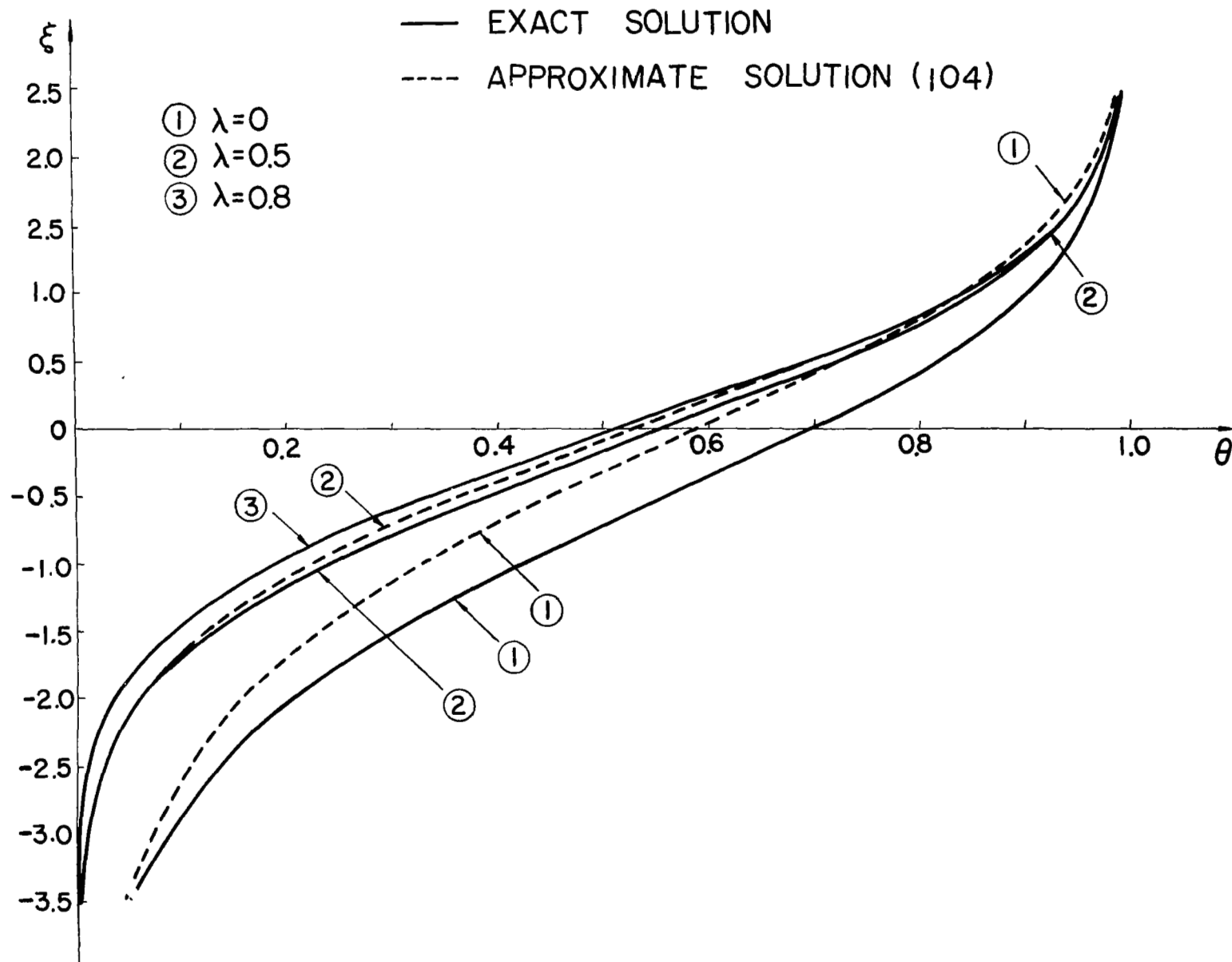


Figure 9. Comparison of Temperature field given by Eq. (104) and the exact solution.