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PLASMA DIFFUSION AS AN INITIAL VALUE PROBLEM

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ABSTRACT

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In describing the diffusion of a rectangular pulse of plasma in a background neutral gas the diffusion equation $\frac{\partial N}{\partial t} = D \nabla^2 N$ predicts the instantaneous appearance of plasma particles at an arbitrary distance from their initial position. By considering the problem from the moments of the Boltzmann equation it is shown that in deriving a diffusion equation it is necessary to take particle inertia into account. An integral solution of the resulting equation is obtained. This solution is interpreted as implying that the flow to an arbitrary point in space consists of particles which stream through the background gas and those that suffer a collision in transit. The problem is also discussed by expanding the distribution function into symmetric and antisymmetric parts and solving the Boltzmann equation as an initial value problem. The result can be integrated to yield the solution for the spatial and temporal development of any of the macroscopic variables. The phenomenon has been investigated experimentally by generating a pulsed plasma in a restricted volume and detecting its presence at a remote point by double probe techniques. The data demonstrate the existence of a finite diffusion velocity as well as the free streaming and collisional characteristics of the particle flow.

Author

I. INTRODUCTION

In describing the relaxation of a density gradient in a gaseous medium it is common to use the well known diffusion equation,

$$\frac{\partial N}{\partial t} = D \nabla^2 N \tag{1}$$

to predict the temporal and spatial development of the perturbation. Equation (1) yields an adequate description of diffusion phenomena under most circumstances. It is discussed in part II-A of this report. In part II-B we examine the derivation of equation (1) from the moments of the Boltzmann equation.[1] It is shown that the inertia of the particles has been neglected in the formulation of the particle flux. This omission can be shown to account for several physical anomalies predicted by equation (1), involving specifically, an instantaneous initial relaxation of any density gradients. Similar considerations have been shown to apply to heat flow.[2]

Goldstein[3] considered this problem from a statistical point of view and showed that in general, equation (1) should be modified to a form reminiscent of the telegraphers' equation. This result has been obtained by several authors.[4-6] Goldstein obtained a general solution of the initial value problem and concluded that there exists a finite velocity of propagation but did not present a means of calculating its value. Huchital and Holt[4] obtain the telegrapher's equation by considering the first two moments of the Boltzmann equation for the Lorentzian gas and present a physical justification of the solution in terms of free streaming and collisional particles, arriving at a result of $(kT/m)^{1/2}$ for the propagation speed. It is thus concluded that diffusion of a gas develops in space at the one dimensional thermal speed. The specific case of a charge neutral plasma is discussed by Shimony and Cahn[5] who obtain the result after postulating that the plasma flow is ambipolar. Their result for the propagation speed is

$(\nu_a D_a)^{1/2}$ where D_a is the ambipolar diffusion coefficient and ν_a is an "ambipolar collision frequency".

Finally, Sandler and Dahler[6] discuss the problem from the viewpoint of the fluid equations for a binary mixture and conclude that the velocity of propagation is equal to $(p/\rho)^{1/2}$ where p is the fluid pressure and ρ the fluid density.

In part II-C, Goldstein's result for the diffusion equation is obtained by retaining the effects of particle inertia in the Boltzmann equation derivation. The resulting modification of equation (1) permits a more realistic physical analysis of the relaxation of density perturbations to be made. However, in part II-D it is shown that Goldstein's result is complete only in the very special case of a mono-energetic gas. A technique is developed for the solution of the Boltzmann equation as an initial value problem which yields results that are in some qualitative agreements with Goldstein's but include the effects of a distribution of velocities. In addition, this result can be used to determine the spatial and temporal development of all the macroscopic variables.

The theoretical analysis therefore contains a description of the diffusion phenomenon at three levels:

1. the Fick's Law, or quasi-steady state approach,
2. the macroscopic approach from the moments of the Boltzmann equation, and
3. the microscopic, or Boltzmann equation approach.

Quantitative analysis of the solutions indicates, however, that they are similar under certain circumstances. An experiment has been designed to satisfy those conditions where one would expect the modified analyses to be preferable. This experiment is discussed in section IV, where it is shown that the modified analyses are required to explain the experimental data.

II. THEORETICAL DEVELOPMENT

A. The Conventional Theory of Gaseous Diffusion

If we consider the problem of a weakly ionized gas in the absence of external forces, then the momentum equation, that is the first moment of the Boltzmann Equation, can be shown to take the form[1]

$$\frac{\partial}{\partial t}\left(N_s\, m_s \langle \bar{v}_s \rangle\right) + \nabla \cdot \bar{\bar{\pi}}_s = - m_s\, \nu_s\, N^s \langle v_s \rangle \qquad (2)$$

where

N_s = number density

m_s = particle mass

$\langle \bar{v}_s \rangle$ = average particle velocity

ν_s = collision frequency with neutral particles

$\bar{\bar{\pi}}_s$ = $\int v^2 f \, dc$

and the subscript "s" denotes the sth species of particle, in this case, either electrons or ions.

We will assume the electronic and ionic distribution functions are very nearly Maxwellian, so that equation (2) can be approximated by

$$\frac{\partial}{\partial t}\left(N_s\, m_s \langle \bar{v}_s \rangle\right) + \nabla N_s\, kT_s = -m_s\, \nu_s\, N_s \langle \bar{v}_s \rangle \qquad (3)$$

It is further assumed that no temperature gradients exist, so

$$\frac{\partial}{\partial t}\left(N_s \langle \bar{v}_s \rangle\right) + \frac{kT_s}{m_s} \nabla N_s = - \nu_s\, N_s \langle \bar{v}_s \rangle \qquad (4)$$

Now the particle flux is defined by

$$\bar{\Gamma} = N \langle \bar{v} \rangle \qquad (5)$$

so equation (4) can be written

$$m_s \frac{\partial \overline{\Gamma}_s}{\partial t} + kT_s \nabla N_s = -m_s \nu_s \overline{\Gamma}_s \tag{6}$$

At this point, steady state is assumed so $\frac{\partial \overline{\Gamma}_s}{\partial t} = 0$, and equation (6) is

$$\overline{\Gamma}_s = - \frac{kT_s}{m_s \nu_s} \nabla N_s \tag{7}$$

or

$$\overline{\Gamma}_s = - D_s \nabla N_s \tag{8}$$

where the definition of the diffusion coefficient, D, is obvious.

Equation (8) is generally known as Fick's Law, and is sometimes used as a starting point in discussing the relaxation of a density perturbation.

Now the continuity equation for the s^{th} species is

$$\frac{\partial N}{\partial t} + \nabla \cdot \overline{\Gamma} = 0 \tag{9}$$

So combining equation (8) and (9) we have the usual result.

$$\frac{\partial N}{\partial t} = D \nabla^2 N \tag{10}$$

For simplicity, we consider the one dimensional case. The solution of equation (10) is easily shown to be

$$N(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} N(x',o) \sqrt{\frac{\pi}{Dt}} \, e^{-\frac{(x-x')^2}{4Dt}} \, dx' \tag{11}$$

As an example, we consider

$$N(x,o) = \begin{cases} n & |x| \leq a \\ 0 & |x| > a \end{cases}$$

Then

$$N(x,t) = \frac{n}{2\pi}\sqrt{\frac{\pi}{Dt}}\int_{-a}^{a} e^{-\frac{(x-x')^2}{4Dt}}\,dx'$$

And if we substitute

$$x-x' = w$$

we obtain

$$N(x,t) = \frac{n}{2\pi}\sqrt{\frac{\pi}{Dt}}\left\{\int_{0}^{x+a} e^{-\frac{w^2}{4Dt}}\,dw - \int_{0}^{x-a} e^{-\frac{w^2}{4Dt}}\,dw\right\}$$

which is

$$N(x,t) = \frac{n}{\sqrt{\pi}}\left\{\mathrm{erf}\left(\frac{x+a}{2\sqrt{Dt}}\right) - \mathrm{erf}\left(\frac{x-a}{2\sqrt{Dt}}\right)\right\} \tag{13}$$

The solutions are sketched for several values of time in Figure 1. This solution loses plausibility, however, when it is subjected to a physical analysis. As an initial condition, we defined a sharp pulse of gas confined in $-a \le x \le a$. However, after an arbitrarily short time interval ϵ, equation (11) predicts that plasma particles exist in all space, $-\infty < x < \infty$, as shown in Figure 2. Of course, the density of particles is still very low for $|x| >> a$, but equation (13) still implies a finite probability of finding a particle a very large distance from $x = 0$, an arbitrarily short time after $t = 0$.

Another objection can be raised after considering the physical basis of the diffusion process. Diffusion occurs only because of a gradient of particle density. We therefore would think that a gas would diffuse only when such a gradient exists. Let us again examine the situation for $t = \epsilon$ where ϵ is very small. We see that the entire plasma distribution changes in an arbitrarily short time. But is this reasonable? Certainly those particles near $x = 0$ can "see" no density gradient initially. Only those particles near the edges of the plasma are aware of the discontinuity, so only these should take part in the initial diffusion. However, equation (13) predicts that regardless of how far a particle is from a density gradient (a is arbitrary) it will still be affected by its existence.

We conclude tentatively that the solution leaves some important questions unanswered when the physical process is examined. We will show in the next section that the complete solution to the diffusion problem is free of these difficulties.

B. The Complete Solution for the Diffusion Equation

The difficulties experienced in the previous discussion can be traced to the initial stages of its development. The momentum equation was written in the form

$$\frac{\partial \overline{\Gamma}}{\partial t} + \frac{kT}{m} \nabla N = - \nu \overline{\Gamma} \qquad (6)$$

and the assumption of steady state was made, yielding

$$\overline{\Gamma} = - \frac{kT}{m\nu} \nabla N \qquad (7)$$

But the motivation for this discussion is that we are considering a non-steady state situation. Since we seek an equation in $\frac{\partial N}{\partial t}$, it seems somewhat questionable to arbitrarily put $\frac{\partial \overline{\Gamma}}{\partial t} = 0$. In this section, therefore, we will solve equation (6) together with the continuity equation,

$$\frac{\partial N}{\partial t} + \nabla \cdot \overline{\Gamma} = 0 \qquad (9)$$

without assuming $\frac{\partial \overline{\Gamma}}{\partial t} = 0$.

We take the divergence of equation (6) and the time derivative of equation (9) to yield

$$\frac{\partial}{\partial t}(\nabla \cdot \overline{\Gamma}) + \frac{kT}{m} \nabla^2 N = -\nu \nabla \cdot \overline{\Gamma} \tag{14}$$

$$\frac{\partial^2 N}{\partial t^2} + \frac{\partial}{\partial t} (\nabla \cdot \overline{\Gamma}) = 0 \tag{15}$$

Combining equations (6), (9), (14) and (15) yields

$$-\frac{\partial^2 N}{\partial t^2} + \frac{kT}{m} \nabla^2 N = \nu \frac{\partial N}{\partial t}$$

or

$$\frac{1}{\nu} \frac{\partial^2 N}{\partial t^2} + \frac{\partial N}{\partial t} = D \nabla^2 N \tag{16}$$

This is the result obtained by both Goldstein[3] and Holt and Haskell.[1]

Comparing equation (16) with the previous result, equation (10) we see that the only difference is the inclusion of the term

$$\frac{1}{\nu} \frac{\partial^2 N}{\partial t^2}$$

The next section examines the implication of the second derivative.

C. Gaseous Diffusion as an Initial Value Problem - Solution of the Diffusion Equation

For the sake of simplicity, we consider the one dimensional problem. Equation (16) is then

$$-\frac{\partial^2 N}{\partial x^2} + \frac{1}{\nu D} \frac{\partial^2 N}{\partial t^2} + \frac{1}{D} \frac{\partial N}{\partial t} = 0 \tag{17}$$

To solve equation (17) we assume a product solution for N,

$$N(x,t) = X(x)\, T(t) \tag{18}$$

Equation (17) is then

$$\frac{1}{X} X'' = \frac{1}{\nu DT} T'' + \frac{1}{DT} T' \tag{19}$$

Following the separation of variables technique we write the equation for X as

$$\frac{1}{X} X'' = -k^2$$

from which

$$X = c(k) e^{ikx} \tag{20}$$

Similarly we obtain the following equation for T,

$$T'' + \nu T' + k^2 \nu DT = 0 \tag{21}$$

the solution of which is

$$T = e^{-\frac{\nu}{2} t} \left[A(k) e^{ib(k)t} + B(k) e^{-ib(k)t} \right] \tag{22}$$

where

$$b(k) = \sqrt{k^2 \nu D - \frac{1}{4} \nu^2} \tag{23}$$

A solution of equation (17) is then

$$N(k,x,t) = e^{-\frac{\nu}{2} t} \left[A(k) e^{ib(k)t} + B(k) e^{-ib(k)t} \right] e^{ikx} \tag{24}$$

where k^2 is the separation constant.

Apparently, each value of k leads to a different result for N. The complete solution to equation (17) is then obtained by integrating over k.

$$N(x,t) = e^{-\frac{\nu}{2} t} \int_{-\infty}^{\infty} \left\{ A(k) e^{ib(k)t} + B(k) e^{ib(k)t} \right\} e^{ikx} dk \tag{25}$$

The remainder of the problem involves relating the coefficients $A(k)$ and $B(k)$ to the initial conditions. If we consider these conditions to be $N(x,0)$ and $\lim_{t \to 0} \frac{\partial N(x,t)}{\partial t}$ (denoted by $N_t(x,0)$), then as shown in Appendix A, equation (25) becomes

$$
\begin{aligned}
N(x,t) = {} & \frac{e^{-\frac{\nu}{2}t}}{2} \left\{ N\left(x-\sqrt{\frac{kT}{m}}\, t,\, 0\right) + N\left(x+\sqrt{\frac{kT}{m}}\, t,\, 0\right) \right\} \\
& + \frac{e^{-\frac{\nu}{2}t}}{2} \sqrt{\frac{m}{kT}} \int_{x-\sqrt{\frac{kT}{m}}t}^{x+\sqrt{\frac{kT}{m}}t} N(\gamma,0) \frac{\partial}{\partial t} J_o\left[\frac{\nu}{2} \sqrt{\frac{m}{kT}} \sqrt{(x-\gamma)^2 - \frac{kT}{m}t^2} \right] d\gamma \\
& + \frac{\nu\, e^{-\frac{\nu}{2}t}}{2} \sqrt{\frac{m}{kT}} \int_{x-\sqrt{\frac{kT}{m}}t}^{x+\sqrt{\frac{kT}{m}}t} N(\gamma,0)\, J_o\left[\frac{\nu}{2} \sqrt{\frac{m}{kT}} \sqrt{(x-\gamma)^2 - \frac{kT}{m}t^2} \right] d\gamma \\
& + \frac{e^{-\frac{\nu}{2}t}}{2} \sqrt{\frac{m}{kT}} \int_{x-\sqrt{\frac{kT}{m}}t}^{x+\sqrt{\frac{kT}{m}}t} N_t(\gamma,0)\, J_o\left[\frac{\nu}{2} \sqrt{\frac{m}{kT}} \sqrt{(x-\gamma)^2 - \frac{kT}{m}t^2} \right] d\gamma
\end{aligned}
\qquad (26)
$$

It is difficult to draw any conclusions directly from equation (26). Careful evaluation of the results predicted by (26) involves choosing the initial conditions $N(x,0)$ and $N_t(x,0)$.

D. Discussion of the Theoretical Solution

In order to simplify the calculation, we shoose a particularly simple example, that of an initial δ-function,

$$N(x,0) = \delta(x); \quad N_t(x,0) = 0$$

Then equation (26) reduces to

$$N(x,t) = \frac{1}{2} e^{-\frac{\nu}{2}t} \left\{ \delta\left(x - \sqrt{\frac{kT}{m}}t\right) + \delta\left(x + \sqrt{\frac{kT}{m}}t\right) \right\}$$

$$+ \sqrt{\frac{m}{kT}} \quad \frac{1}{2} e^{-\frac{\nu}{2}t} \frac{\partial}{\partial t} J_o \left[\frac{\nu}{2} \sqrt{\frac{m}{kT}} \sqrt{x^2 - \frac{kT}{m} t^2} \right] \tag{27}$$

$$+ \sqrt{\frac{m}{kT}} \quad \frac{\nu e^{-\frac{\nu}{2}t}}{2} J_o \left[\frac{\nu}{2} \sqrt{\frac{m}{kT}} \sqrt{x^2 - \frac{kT}{m} t^2} \right]$$

It is difficult to draw any conclusions concerning the physical solution from this result. The terms in $e^{-\frac{\nu}{2}t}$ are misleading as they indicate a severe damping in the diffusion phenomenon. This variation is balanced to a certain extent by the fact that the argument of the Bessel functions is imaginary so that they represent exponentially increasing rather than oscillatory functions. However, before we proceed with an asymptotic expansion of the solution, there are several points to note. Most important we observe that equation (32) implies a propagating solution. The first term merely indicates the original pulse moving to the left and right with the one dimensional thermal velocity, $\sqrt{\frac{kT}{m}}$. It should be noted that the definite integrals leading to the second and third terms vanish for $x > \sqrt{\frac{kT}{m}}\, t$, further illustrating the propagating nature of the solution.

We see then that the first term of the expansion can be expressed as shown graphically in Figure 3.

The second and third terms are much more difficult to discuss. We will therefore consider them only for x such that $x^2 \ll \frac{kT}{m} t^2$, that is, reasonably far from the leading edges of the propagating pulse. Now since the argument of the Bessel functions is imaginary, make use of the relation

$$I_n(v) = i^{-n} J_n(iv) \tag{28}$$

where $I_n(v)$ is a modified Bessel function of the first kind.

It is relatively easy to obtain an asymptotic expansion for $I_n(v)$ as $v \to \infty$ by the method of steepest descent. The result is

$$I_n(v) \sim \left(\frac{1}{2\pi v}\right)^{\frac{1}{2}} e^{v} \qquad v \to \infty \tag{29}$$

Let us consider the third term of equation (27). The difficulty is the Bessel function

$$J_o\left(\frac{\nu}{2}\sqrt{\frac{m}{kT}}\sqrt{x^2 - \frac{kT}{m}t^2}\right)$$

By equation (28) this is

$$I_o\left(\frac{\nu}{2}\sqrt{\frac{m}{kT}}\sqrt{\frac{kT}{m}t^2 - x^2}\right)$$

So by using equation (29) we can write the entire term as

$$\sqrt{\frac{m}{kT}}\ \frac{\nu e^{-\frac{\nu}{2}t}}{2}\left(\pi\nu\sqrt{\frac{m}{kT}}\sqrt{\frac{kT}{m}t^2 - x^2}\right)^{-\frac{1}{2}} \exp\left(\frac{\nu}{2}\sqrt{\frac{m}{kT}}\sqrt{\frac{kT}{m}t^2 - x^2}\right)$$

$$\frac{kT}{m}t^2 - x^2 \to \infty$$

Now since we are taking $\frac{kT}{m}t^2 >> x^2$, we write

$$\sqrt{\frac{m}{kT}}\ \frac{1}{2}\left(\frac{1}{\pi\nu t}\right)^{\frac{1}{2}} \nu e^{-\frac{\nu}{2}t} \exp\left(\frac{1}{2}\nu t\sqrt{1 - \frac{x^2}{\frac{kT}{m}t^2}}\right)$$

and

$$\sqrt{1 - \frac{x^2}{\frac{kT}{m}t^2}} \approx 1 - \frac{1}{2}\frac{x^2}{\frac{kT}{m}t^2}$$

So the result is

$$\frac{1}{2}\left(\frac{\nu}{\pi t}\right)^{\frac{1}{2}} e^{-\frac{x^2}{4Dt}}\sqrt{\frac{m}{kT}}$$

Similarly, we can evaluate the second term in equation (27)

$$\frac{1}{2}\left(\frac{\nu}{\pi t}\right)^{\frac{1}{2}} e^{-\frac{x^2}{4Dt}} \sqrt{\frac{m}{kT}}$$

And finally

$$N(x,t) \cong \frac{1}{2} e^{-\frac{\nu}{2}t} \left\{ \delta\left(x-\sqrt{\frac{kT}{m}}\, t\right) + \delta\left(x+\sqrt{\frac{kT}{m}}\, t\right) \right\}$$

$$+ \sqrt{\frac{m}{kT}} \left(\frac{\nu}{\pi t}\right)^{\frac{1}{2}} e^{-x^2/4Dt}$$

The complete solution is as shown in Figure 4.

In order to have a good basis for comparison with the conventional theory, we should solve equation (26) for the case of an initial square pulse of plasma. However, the integrals are difficult to evaluate and their calculation will shed little additional light on this discussion. We can construct an adequate qualitative picture by considering the form of equation (26) and the preceeding example. The major points to be gathered are

1) The initial distribution splits and propagates along the positive and negative x axis.

2) These propagating groups leave behind a bell-shaped residue which flattens out slowly.

This situation can be justified by physical arguments if we keep in mind the fact that the momentum equation, equation (6) of this paper, is derived by an integration of the Boltzmann equation over velocity space. The effects of a distribution of initial velocity among the plasma particles is thus lost in the treatment above and the justification must be made from the point of view of an "average" particle moving at a speed $\sqrt{\frac{kT}{m}}$. Therefore, by virtue of the symmetry of the square pulse which we are considering, exactly half the particles are moving initially in the positive x direction and half in the negative x direction. Immediately after $t = 0$, the two clouds pass through each other and move along the x axis. This splitting, together with the previously noted attenuation, is illustrated in Figure 5.

Apparently the effect of a distribution of speeds among the particles will be to "round off" the sharp edges shown in Figure 5. This phenomenon is a complication at this stage, and further discussion of it is deferred to the next section.

We must now question the implications of the factor $e^{-\frac{\nu}{2}t}$ in the first term. We see that it is intimately related to the remaining terms of the solution since these vanish if $\nu = 0$. The important point to realize is that collisions retard the diffusion process. Diffusion occurs only because of the thermal velocity of the particles. The factor $e^{-\frac{\nu}{2}t}$ in the first term must therefore correspond to a loss from the propagating pulses due to collisions, and these particles must appear in the solution in the residue expressed by the Bessel function terms. Physically, a group of particles starts to diffuse according to its thermal velocity, and the motion of these "free streaming" particles is interrupted by collisions so that some of the particles are left behind. The complete solution is sketched in Figures 6 and 7.

We recall the major objections to the conventional theory:

1) That is predicted a finite probability of finding a particle an arbitrarily large distance from its point of origin after an arbitrarily short time, that is, an infinite velocity of propagation, and

2) That it predicted an immediate perturbation of every particle in the gas, regardless of how far a particle might be from the density gradients.

We conclude that these difficulties are no longer present.

However, an objection can be raised to the present approach. The conventional theory has existed for a considerable period of time, without having been contradicted by experiment. How then, can we justify the claims made in the preceeding sections? The answer lies in considering the geometry of most experiments. Of course, in a steady state situation, the two approaches are identical as all time derivatives vanish, so we are interested in the afterglow or pulsed plasma case. Recall that we have decided that the average velocity of propagation of the leading edges of the disturbance is $\sqrt{\frac{kT}{m}}$. A temperature of $100{,}000^{o}K$ is not at all unusual in a conventional gas discharge. The associated diffusion speed is then 1.23×10^{6} meters/sec. so that on a discharge tube of radius 1 cm, the plasma has propagated to the edges of the vessel in 8.1×10^{-9} seconds! The only diffusion phenomenon that is observed after this time is what we have called the "residue" in the preceeding discussion. And at a sufficient distance from the leading edges of the pulse, $(x << \sqrt{\frac{kT}{m}}\, t)$ the shape of the residue is asymptotic to that predicted by the conventional theory, i.e.

$$\text{residue} \sim \frac{1}{t^{1/2}} \quad e^{-\frac{x^2}{4Dt}}$$

We would, however, expect some discrepancies between the two theories when the discharge tube is so long as to allow considerable time for the disturbance to propagate. This possibility is discussed in Section IV.

III. SOLUTION OF THE BOLTZMANN EQUATION AS AN INITIAL VALUE PROBLEM

In the preceeding section we have shown that retaining the concept of particle inertia results in a significantly more complete description of the diffusion phenomenon. However, we must also concede that this extension falls short of being a complete discussion. The deficiencies of the theory are pointed out most clearly by the fact that it predicts that all the streaming particles move with velocity $\sqrt{\frac{kT}{m}}$ despite the fact that a velocity distribution was postulated. This anomaly stems directly from the nature of the macroscopic equations. The first n moments of the Boltzmann equation contain $(n+1)$ variables. For example, the first two moments, the equations of continuity and flow, contain three macroscopic variables, density, flux and pressure. In order to solve the set of macroscopic equations it is necessary to truncate the series. This is usually accomplished by assumptions of the form of the variable of highest order which relate it to the lower order variables. In our example of the first two moments, the pressure was assumed equal to nkT. It is apparent that this statement is difficult to justify for this problem. In addition, temperature gradients were neglected to avoid introducing temperature as a third variable.

In this section we will eliminate these difficulties by taking a more general view of the problem and solving the Boltzmann equation rather than the moment equations. The result of this approach is quite general and can be used to discuss the development of any of the macroscopic variables.

A. The Distribution Function

The problem of transport variables is directly related to the symmetry or anti-symmetry of the distribution function in velocity space. No flow can result in, say, the x direction if the distribution function is symmetric about the v_x axis.

However, discussion of the even and odd components of the distribution function is somewhat more complex than it might at first appear. We will show that even in the absence of external forces, the symmetric and anti-symmetric parts are coupled, so that a net flux will arise even if the distribution function is initially isotropic.

In the absence of external forces, the Boltzmann equation takes the form

$$\frac{\partial f}{\partial t} + \bar{v} \cdot \nabla f = \left(\frac{\partial f}{\partial t}\right)_c \tag{30}$$

and for gradients in the x direction only, it becomes

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial t}\right)_c \tag{31}$$

Let us consider splitting the distribution function into odd and even components with respect to v_x, i.e.,

$$f(v_x) = f^o(v_x) + f^e(v_x) \tag{32}$$

where

$$f^o(-v_x) = -f^o(v_x) \tag{33}$$

$$f^e(-v_x) = f^e(v_x) \tag{34}$$

It may not be obvious that the expansion defined by equations (32), (33) and (34) is complete. We can prove completeness rigorously be defining the quantities $\tilde{f}^e(k)$ and $\tilde{f}^o(k)$ as follows:

$$\tilde{f}^o(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\alpha) \sin k\alpha \, d\alpha \tag{35}$$

$$\tilde{f}^e(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\alpha) \cos k\alpha \, d\alpha \tag{36}$$

Then $f^e(v_x)$ and $f^o(v_x)$ are determined by

$$f^o(v_x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}^o(k) \sin kv_x \, dk = \frac{1}{2\pi} \iint_{-\infty}^{\infty} f(\alpha) \sin k\alpha \sin kv_x \, dk \, d\alpha \tag{37}$$

$$f^e(v_x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}^e(k) \cos kv_x \, dk = \frac{1}{2\pi} \iint_{-\infty}^{\infty} f(\alpha) \cos k\alpha \cos kv_x \, dk \, d\alpha \tag{38}$$

Equations (37) and (38) apparently satisfy the conditions of equations (33) and (34). Finally we can readily prove the completeness of the expansion as follows:

$$f^o(v_x) + f^e(v_x) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} f(\alpha) \cos k(\alpha - v_x) \, dk \, d\alpha$$

$$= \int_{-\infty}^{\infty} f(\alpha) \, \delta \, (\alpha - v_x) \, d\alpha = f(v_x)$$

We can therefore write the Boltzmann equation in the form

$$\frac{\partial f^o}{\partial t} + \frac{\partial f^e}{\partial t} + v_x \frac{\partial f^o}{\partial x} + v_x \frac{\partial f^e}{\partial x} = \left\{ \frac{\partial}{\partial t} \left(f^o + f^e \right) \right\}_c \tag{39}$$

And as shown in Appendix 3

$$\left\{ \frac{\partial}{\partial t} \left(f^o + f^e \right) \right\}_c = -\nu f^o$$

So equation (39) becomes

$$\frac{\partial f^o}{\partial t} + \frac{\partial f^e}{\partial t} + v_x \frac{\partial f^o}{\partial x} + v_x \frac{\partial f^e}{\partial x} = -\nu f^o \tag{40}$$

If we now multiply equation (40) by $\frac{1}{2\pi}$ $\sin k\alpha \sin kv_x$ and integrate over k and α, we have

$$\frac{\partial f^o}{\partial t} + v_x \frac{\partial f^e}{\partial x} = -\nu f^o \tag{41}$$

and similarly, multiplying by $\frac{1}{2\pi}$ cos $k\alpha$ cos kv_x and integrating yields

$$\frac{\partial f^e}{\partial t} + v_x \frac{\partial f^o}{\partial x} = 0 \tag{42}$$

Equations (41) and (42) enable us to solve the Boltzmann equation as an initial value problem.

B. Solutions for f^o and f^e

We can combine equations (41) and (42) to obtain partial differential equations in f^o and f^e individually,

$$\frac{\partial^2 f^o}{\partial t^2} + \nu \frac{\partial f^o}{\partial t} = v^2 \frac{\partial^2 f^o}{\partial x^2} \tag{43}$$

$$\frac{\partial^2 f^e}{\partial t^2} + \nu \frac{\partial f^e}{\partial t} = v^2 \frac{\partial^2 f^e}{\partial x^2} \tag{44}$$

However, it is important to note that f_o and f_1 are still related by the initial conditions. If these conditions are

$$f^o(v,x,0)$$

$$f^e(v,x,0)$$

$$\lim_{t \to 0} \frac{\partial}{\partial t} f^o(v,x,t) \qquad \text{(denoted by } f_t^{\,o}(v,x,0)\text{)}$$

$$\lim_{t \to 0} \frac{\partial}{\partial t} f^e(v,x,t) \qquad \text{(denoted by } f_t^{\,e}(v,x,0)\text{)}$$

then according to equations (41) and (42), they are related by

$$f_t^{\,o}(v,x,0) + v_x \frac{\partial}{\partial x} f^e(v,x,0) = - \nu f^o(v,x,0) \tag{45}$$

$$f_t^{\,e}(v,x,0) + v_x \frac{\partial}{\partial x} f^o(v,x,0) = 0 \tag{46}$$

Let us consider the problem of an initially isotropic distribution function. Then

$$f^e(v,x,0) = n(x)\, F(v)$$

$$f^o(v,x,0) = 0$$

The remaining initial conditions are determined from equations (45) and (46) to be

$$f_t^{\,o}(v,x,0) = - v_x\, n'(x)\, F(v)$$

$$f_t^{\,e}(v,x,0) = 0$$

Equations (43) and (44) are of the same form as that discussed in Appendix A. The solutions can be expressed as

$$\begin{aligned} f^e(v,x,t) &= \frac{1}{2}\, e^{-\frac{\nu}{2}t}\, F(v) \left\{ n(x-vt) + n(x+vt) \right\} \\ &+ \frac{e^{-\frac{\nu}{2}t}}{2v}\, F(v) \int_{x-vt}^{x+vt} n(\gamma)\, \frac{\partial}{\partial t}\, J_o\left[\frac{\nu}{2}\frac{1}{v} \sqrt{(x-\gamma)^2 - v^2t^2} \right] d\gamma \\ &+ \frac{\nu e^{-\frac{\nu}{2}t}}{2v}\, F(v) \int_{x-vt}^{x+vt} n(\gamma)\, J_o\left[\frac{\nu}{2}\frac{1}{v} \sqrt{(x-\gamma)^2 - v^2t^2} \right] d\gamma \end{aligned} \tag{47}$$

$$f^o(v,x,t) = - \frac{e^{-\frac{\nu}{2}t}}{2v}\, F(v) \int_{x-vt}^{x+vt} n'(\gamma)\, J_o\left[\frac{\nu}{2}\frac{1}{v} \sqrt{(x-\gamma)^2 - v^2t^2} \right] d\gamma \tag{48}$$

If we integrate equation (48) by parts, we find

$$f^{o}(v,x,t) = -\frac{e}{2}^{-\frac{\nu}{2}t} F(v) \left\{ n(x+vt) - n(x-vt) \right\}$$

$$+ \frac{\nu e^{-\frac{\nu}{2}t}}{4v} F(v) \int_{x-vt}^{x+vt} n(\gamma) \frac{x-\gamma}{\sqrt{(x-\gamma)^2 - v^2t^2}} J_1\left[\frac{\nu}{2} \frac{1}{v} \sqrt{(x- \ )^2 - v^2t^2} \right] d\gamma \qquad (49)$$

So finally,

$$f(x,v,t) = e^{-\frac{\nu}{2}t} F(v)\, n(x-vt)$$

$$+ \frac{\nu e^{-\frac{\nu}{2}t}}{2v} F(v) \int_{x-vt}^{x+vt} n(\gamma) \sqrt{\frac{x-\gamma-vt}{x-\gamma+vt}}\; J_1\left[\frac{\nu}{2} \frac{1}{v} \sqrt{(x-\gamma)^2 - v^2t^2} \right] d\gamma$$

$$+ \frac{\nu e^{-\frac{\nu}{2}t}}{2v} F(v) \int_{x-vt}^{x+vt} n(\gamma)\, J_o\left[\frac{\nu}{2} \frac{1}{v} \sqrt{(x-\gamma)^2 - v^2t^2} \right] d\gamma \qquad (50)$$

There are several important aspects of equation (50) that should be noted. In contrast with the previous result, equation (26), it may appear that equation (50) predicts propagation of the "fast" group only along the positive x axis. In this connection, it is necessary to recall that v takes both positive and negative values so that propagation in both directions is implied. Second and most important, equation (50) predicts that each velocity class diffuses at its own intrinsic speed. Therefore, a diffusing gas smears out due to a distribution of initial velocities.

Equation (50) is essentially a correction to the result of the macroscopic approach of Section II-C. This correction is important if the spread of initial velocities is "large" compared with the "average" speed. A further implication of equation (50) that might, under some circumstances, be important is the point that "hot" particles diffuse more quickly than "cool" ones. Therefore temperature gradients are immediately set up in a diffusing gas so that it is unrealistic, in general, to discuss density gradients without considering the associated temperature gradients. However, the most important implication of equation (50) is as follows. In postulating that the pressure was equal to nkT, it was explicitly assumed that the distribution function was factorable into a density in configuration space and a density in velocity space. Equation (50) shows that this is not, in general, true.

Finally, we should note that given the initial distribution function, any macroscopic variable can be determined by taking the appropriate moment of equation (50). This approach represents a considerable simplification, as the higher order moments of the Boltzmann equation are usually non-linear and therefore fairly difficult to solve.

IV. EXPERIMENTAL ANALYSIS

A. Objects of the Experiment

The object of the experiment was to establish the necessity, under certain conditions, for using equation (16) or equation (50) rather than equation (1) to describe the diffusion phenomenon. To this end, a discharge tube was designed permitting pulsed ionization of the gas in a specific volume and the observation of the ionization at a remote point. The experiment was based on the following specific objectives:

1) Equations (1) and (16) predict differing traces of density versus time at a point remote from the point of origin of a freely diffusing plasma. Reference to Figures 1 and 6 shows that equations (1) and (16) predict results as shown qualitatively in Figures 8 and 9 respectively. Therefore, a major objective of the experiment was to analyze the results under conditions designed to produce a curve of the Figure 9 type.

2) One of the major points of the preceeding discussion is that the modified theory predicts a propagating solution. Therefore, another major objective of the experiment has been to observe and measure a finite velocity of diffusion.

3) The preceeding interpretation of the predicted phenomenon in terms of "free streaming" and "residue" particles calls for at least qualitative verification. If, as has been argued, the initial peak of density does indeed represent those particles that have travelled the complete path from ionization point to sensing point without suffering a collision, then the magnitude of the peak should exhibit the characteristic exponential dependence upon pressure and length of path.

In view of these objectives, the apparatus shown in Figure 10 was constructed.

B. Experimental Apparatus

The experimental apparatus is designed for the production of a repetitive pulsed plasma between a pair of electrodes and the detection of the plasma by probes placed some distance away. Five sets of ionization electrodes are provided and numbered as in Figure 10 so as to provide variability of the path length between the ionization and sensing ports. The distances from these electrode sets to probe set A_2A_2' are as follows:

Electrode set E - 6.7 centimeters
D - 9.8 centimeters
C - 13 centimeters
B - 17 centimeters
A - 21 centimeters

A number of points concerning the mechanics of the experiment are worthy of mention. The presence and approximate density of the plasma are detected by the classical floating double probe technique. The presence of a conducting medium between probes A and B causes a potential to appear across R_1. It was found that the clarity of the results was considerably improved by differentiating this potential.

Pulsed ionization of the gas was accomplished by exterior electrodes for two reasons:

1) Mounting the electrodes exterior to the cell permits great flexibility in determining their distance relative to the probes.

2) Exterior mounting eliminates direct current conducting paths from the ionization to sensing electrodes.

However, in order that ionization by external electrodes be possible without prohibitive pulse sizes, it was necessary to provide a source of primary electrons from an independent source. This was accomplished by maintaining a very low level dc discharge across the tube between electrode B_o and ground. The dc current was maintained at approximately .01 ma. At this level, no visible glow was apparent. Finally, the dc discharge was operated with the anode grounded in order that the potential at which the probes float, i.e. the plasma space potential, be as close to ground potential as possible.

C. Experimental Results

The first series of measurements was aimed at comparing the shape of the probe curve with that predicted in Figure 8. As previously explained, the shape is a sensitive function of both the neutral gas pressure and the distance between the ionizing electrodes and sensing probes. Figure 11 is presented as an example of the results obtained. In this case, the system pressure was 0.35 Torr and the electrode spacing was 13 cm. It should be noted that both a sharp initial group indicating the free streaming particles, and a smooth secondary group indicating the residue particles, are observed. The interplay between these two groups as a function of pressure is demonstrated by the series of photographs in Figure 12. As expected, since the initial peak represents these particles which traverse the path without any collisions, the size of the peak is greatest for low pressures and seems to disappear altogether at higher pressures.

The effect of increasing the length of the diffusion path is very similar to increasing the pressure as both procedures result in greater probability of collisions with neutrals. The series of photographs in Figure 13 was taken at a pressure of 0.3 Torr. The length of the diffusion path for each sample is indicated. The similarity between Figures 12 and 13 should be noted.

Finally, an attempt to determine the velocity of propagation has been made by detecting the appearance of the leading edge of the free streaming particles originating from the various sets of electrodes. This attempt was complicated by the fact that the initial condition included some electrons with very high velocities. And according to the discussion of Section III, these stream between the probes too quickly to be measured. Despite this situation it was possible to determine a propagation speed for the bulk of particles. The result is shown in Figure 14. The velocity was calculated to be approximately 8×10^5 meters/second. Further, the change in the slope of the leading edge is evidence of the effect of a distribution of velocities.

D. Discussion

The results presented in the preceeding section would appear to verify, at least qualitatively, both the differential diffusion equation, equation (16), and its integration to equation (31). The two representative groupings, into "fast" and "slow" particles can be observed for a variety of pressures and diffusion path lengths.

As presented here, the effect of increasing the path length, as shown in Figure 13, is an immediate consequence of equation (31) as this equation predicts an exponential damping of the leading edges of the diffusing gas. The interpretation of this data as "fast" particles representing free streaming and "slow" particles indicating collisions is justified qualitatively by the data in Figure 12. Here the number of fast particles reaching the probes was shown to decrease rapidly with increasing pressure, while the number of particles in the residue is greatly enhanced. Interpreting Figure 12 according to the preceeding discussion, we conclude that at a pressure of 0.2 Torr, almost all the particles reach the probes without suffering a collision, while at 0.4 Torr, the reverse is true.

Figure 14 shows the difference in the arrival time of fast particles for two different electrode sets. An accurate determination of the velocity of propagation is difficult because of its large magnitude, and the tendency of the leading edge to damp quickly with distance.

The magnitude of this velocity (8×10^5 meters/sec.) leads to some tentative conclusions regarding the nature of the flow. If the detected flow is electronic, we must set $v_{electron} = \sqrt{\frac{kT^e}{m^e}} = 8 \times 10^5$ meters/second. This implies an electron temperature of some $43{,}000^oK$. Though this may seem somewhat high for a room temperature experiment where the electric fields are perpendicular to the travel, it should be recalled that the large charge to mass ratio of an electron makes only a very small electric field necessary for electrons to reach this temperature. And one would suppose that the large applied fields involve sufficient fringing to provide the necessary energy.

On the other hand, it might be postulated that the flow is ambipolar in nature. This situation is discussed in Appendix B. It is shown that the velocity of ambipolar flow is approximately within the mass ratio

$$v_a \approx \sqrt{\frac{kT^+}{M}} \sqrt{1 + \frac{T^-}{T^+}}$$

where M is the ion mass and T^- and T^+ are the electron and ion temperatures. From this, it can be easily shown that in order for ambipolar flow to reach a velocity of 8×10^5 meters/second, the electron or ion temperature would have to be of the order of $10^9\ ^oK$. We conclude therefore, that the detected response indicates mainly electronic flow.

V. CONCLUSIONS

This paper has presented an extension of the conventional theory of nonequilibrium gaseous diffusion. It has been shown that the development of the conventional theory incorporates certain assumptions and the proposed modification involves a derivation without them. The conventional theory runs into two difficulties when the initial condition of a square pulse of gas is considered. The standard differential equation integrates to indicate first an infinite velocity of diffusion, and second an instantaneous deformation of the entire gas, regardless of how distant any density perturbations might be. Neither of these difficulties is experienced with the modified theory.

The modified diffusion equation differs from the conventional theory in being second rather than first order in time. Integration of this equation has led to the following physical description of the diffusion process. Diffusion occurs solely because of the thermal motion of the particles. The outer limits of the diffusing gas therefore expand at the thermal speed. The chief effect of collisions with massive neutrals is to remove colliding particles from the "fast" expanding edges and leave them in a bell-shaped residue which continues to diffuse outward. The observation of the diffusion of a gas at a point remote from the point of origin, can then be divided into three time periods:

1. where no diffusing gas is detected until gaseous particles, moving at the thermal speed, can reach the observation point,
2. where the "fast" particles, i.e. those particles that do not suffer a collision in transit, arrive, and,
3. where the "slow" residue, representing those particles that have suffered a collision(s), arrive.

Any discussion of gaseous diffusion is basically an initial value problem. This consideration is important in the conception of an experimental verification of the above because of the considerable difficulty one would experience in establishing known initial conditions. The experiment described in this paper is thus largely qualitative in nature.

The experiment provides for the establishment of a repetitive, pulsed plasma in a specific region of space and its detection by means of double floating probes a finite distance away. The length of the diffusion path, that is, the distance between the ionizing electrodes and detecting probes is a variable.

As shown in Figure 11, the "fast" and "slow" particle groups predicted theoretically are, in fact, detected. And the characteristics of these groups as the background pressure and diffusion path are varied, justifies their interpretation as "free streaming" and "collisional" particles.

A finite time gap before the arrival of the initial particles has been detected, justifying another aspect of the theoretical argument. The velocity of propagation has been measured by considering the difference in this time delay for two different lengths of diffusion path. The velocity has been determined to be approximately 8×10^5 m/s in this experiment. The magnitude of the streaming velocity indicates electronic, as opposed to ambipolar, flow.

VI. APPENDICES

A. Integration of the Diffusion Equation

In Section II-C, we showed that a general solution of the diffusion equation takes the form

$$N(x,t) = e^{-\frac{\nu}{2}t} \int_{-\infty}^{\infty} \left\{ A(k)\, e^{ib(k)t} + B(k)\, e^{-ib(k)t} \right\} e^{ikx}\, dk$$

The remainder of the problem involves relating the coefficients $A(k)$ and $B(k)$ to the boundary conditions.

Let us denote the initial density and time rate of change of density by

$$N(x,0) \qquad \text{and} \qquad N_t(x,0) = \lim_{t \to 0} \frac{\partial N(x,t)}{\partial t}$$

Then

$$N(x,0) = \int_{-\infty}^{\infty} \left\{ A(k) + B(k) \right\} e^{ikx}\, dk$$

$$N_t(x,0) = \int_{-\infty}^{\infty} \left\{ A(k) \left(- \frac{\nu}{2} + ib\right) + B(k) \left(- \frac{\nu}{2} - ib\right) \right\} e^{ikx}\, dk$$

Combining these two relations,

$$N_t(x,0) + \left(\frac{\nu}{2} + ib\right) N(x,0) = 2i \int_{-\infty}^{\infty} A(k)\, b(k)\, e^{ikx}\, dk \qquad (51)$$

$$N_t(x,0) + \left(\frac{\nu}{2} - ib\right) N(x,0) = -2i \int_{-\infty}^{\infty} B(k)\, b(k)\, e^{ikx}\, dk \qquad (52)$$

We now multiply equations (25) and (26) by $e^{-i\alpha x}$ and integrate over x, while noting that

$$\int_{-\infty}^{\infty} e^{ix(k-\alpha)}\, dx = 2\pi\, \delta\, (k-\alpha) \qquad (53)$$

So we find

$$A(\alpha) = \frac{1}{4\pi b(\alpha)} \int_{-\infty}^{\infty} \left\{ (b-i\,\frac{\nu}{2})\, N(x,0) - i\, N_t(x,0) \right\} e^{-i\alpha x}\, dx \tag{54}$$

$$B(\alpha) = \frac{1}{4\pi b(\alpha)} \int_{-\infty}^{\infty} \left\{ (b+i\,\frac{\nu}{2})\, N(x,0) + i\, N_t(x,0) \right\} e^{-i\alpha x}\, dx \tag{55}$$

And after some minor simplification, we can write

$$\begin{aligned} N(x,t) = {} & \frac{e^{-\frac{\nu}{2}t}}{4\pi} \iint_{-\infty}^{\infty} N(\alpha,0)\, e^{ik(x-\alpha)} \cos bt\, dk\, d\alpha \\ & + \frac{e^{-\frac{\nu}{2}t}}{4\pi} \iint_{-\infty}^{\infty} \nu\, N(\alpha,0)\, e^{ik(x-\alpha)}\, \frac{\sin bt}{b}\, dk\, d\alpha \\ & + \frac{e^{-\frac{\nu}{2}t}}{4\pi} \iint_{-\infty}^{\infty} 2\, N_t(\alpha,0)\, e^{ik(x-\alpha)}\, \frac{\sin bt}{b}\, dk\, d\alpha \end{aligned} \tag{56}$$

where α is a dummy variable.

For the sake of clarity, we represent this by

$$N(x,t) = I_1 + I_2 + I_3 \tag{57}$$

Now by definition

$$\frac{\sin bt}{b} = \frac{\sin \sqrt{\nu D}\; t\; \sqrt{k^2 - \frac{\nu}{4D}}}{\sqrt{\nu D}\; \sqrt{k^2 - \frac{\nu}{4D}}} \tag{58}$$

Stratton[7] shows that this can be written

$$\frac{\sin bt}{b} = \frac{1}{2\sqrt{\nu D}} \int_{-\sqrt{\nu D}t}^{\sqrt{\nu D}t} J_o \left[\frac{1}{2} \sqrt{\frac{\nu}{D}} \sqrt{\beta^2 - \nu D t^2} \right] e^{-ik\beta}\, d\beta$$

Therefore

$$I_3 = \frac{e^{-\frac{\nu}{2}t}}{4\pi} \int_{-\sqrt{\nu D}t}^{\sqrt{\nu D}t} d\beta \left\{ \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} \phi(\alpha,\beta)\, e^{ik(x-\alpha-\beta)}\, d\alpha \right\} \qquad (59)$$

where

$$\phi(\alpha,\beta) = \nu\, N_t(\alpha,0)\, J_o\left[\frac{1}{2}\sqrt{\frac{\nu}{D}} \sqrt{\beta^2 - \nu Dt^2} \right]$$

The quantity inside the brackets is in double transform form and is

$$\frac{1}{2\pi}\, \phi(x-\beta,\beta)$$

So

$$I_3 = \frac{e^{-\frac{\nu}{2}t}}{2} \int_{-\sqrt{\nu D}t}^{\sqrt{\nu D}t} N_t(x-\beta,0)\, J_o\left[\frac{1}{2}\sqrt{\frac{\nu}{D}} \sqrt{\beta^2 - Dt^2} \right] d\beta \qquad (60)$$

And if we substitute $\gamma = x-\beta$

$$I_3 = \frac{e^{-\frac{\nu}{2}t}}{2} \int_{x-\sqrt{\nu D}t}^{x+\sqrt{\nu D}t} N_t(\gamma,0)\, J_o\left[\frac{1}{2}\sqrt{\frac{\nu}{D}} \sqrt{(x-\gamma)^2 - Dt^2} \right] d\gamma \qquad (61)$$

By inspection, we can write further,

$$I_2 = \frac{e^{-\frac{\nu}{2}t}}{4} \int_{x-\sqrt{\nu D}t}^{x+\sqrt{\nu D}t} N(\gamma,0)\, J_o\left[\frac{1}{2}\sqrt{\frac{\nu}{D}} \sqrt{(x-\gamma)^2 - Dt^2} \right] d\gamma \qquad (62)$$

Further, we note that

$$I_1 = 2\, e^{-\frac{\nu}{2}t}\, \frac{\partial}{\partial t}\, \frac{1}{\nu}\, e^{\frac{\nu}{2}t}\, I_2$$

so

$$I_1 = \frac{1}{2}\left\{ N(x-\sqrt{\nu D}t,0) + N(x+\sqrt{\nu D}t,0) \right\}$$
$$+ \frac{1}{2} \int_{x-\sqrt{\nu D}t}^{x+\sqrt{\nu D}t} N(\gamma,0)\, \frac{\partial}{\partial t}\, J_o\left[\frac{1}{2}\sqrt{\frac{\nu}{D}} \sqrt{(x-\gamma)^2 - \nu dT^2} \right] d\gamma$$

and finally

$$N(x,t) = \frac{e^{-\frac{\nu}{2}t}}{2}\left\{N(x-\sqrt{\nu D}t,0) + N(x+\sqrt{\nu D}t,0)\right\}$$

$$+ \frac{e^{-\frac{\nu}{2}t}}{2}\sqrt{\frac{m}{kT}}\int_{x-\sqrt{\nu D}t}^{x+\sqrt{\nu D}t} N(\gamma,0)\,\frac{\partial}{\partial t}\,J_o\left[\frac{1}{2}\sqrt{\frac{\nu}{D}}\sqrt{(x-\gamma)^2-\nu D t^2}\right]d\gamma$$

$$+ \frac{\nu e^{-\frac{\nu}{2}t}}{2}\sqrt{\frac{m}{kT}}\int_{x-\sqrt{\nu D}t}^{x+\sqrt{\nu D}t} N(\gamma,0)\,J_o\left[\frac{1}{2}\sqrt{\frac{\nu}{D}}\sqrt{(x-\gamma)^2-\nu D t^2}\right]d\gamma$$

$$+ \frac{e^{-\frac{\nu}{2}t}}{2}\sqrt{\frac{m}{kT}}\int_{x-\sqrt{\nu D}t}^{x+\sqrt{\nu D}t} N_t(\gamma,0)\,J_o\left[\frac{1}{2}\sqrt{\frac{\nu}{D}}\sqrt{(x-\gamma)^2-\nu D t^2}\right]d\gamma \qquad (63)$$

B. The Collision Terms

If we assume that the particles described by the distribution function f suffer collisions only with particles described by a second distribution function f', then the collision term can be written as[8]

$$\left(\frac{\partial f}{\partial t}\right)_c = \iiint (\tilde{f}\,\tilde{f}' - f\,f')\,gp\,dp\,d\phi\,dc' \qquad (64)$$

where

$\tilde{f}$ ($\tilde{f}'$) is the post collision value of f (f')

g is the relative velocity of the primed and unprimed particles

p is the impact parameter

ϕ is the azimuthal angle

In order to simplify equation (64) it is necessary to make further assumptions concerning the types of particles involved. If f is to be the distribution function of an electron gas colliding with a gas of massive neutral molecules described by f', then it is reasonable to assume that

$$\tilde{f}' = f'$$

That is, that the neutral gas is not effected by collisions with electrons. Noting further that

$$n' = \int f' \, dc'$$

we can write equation (64) as

$$\left(\frac{\partial f}{\partial t}\right)_c = n' \iint (\tilde{f} - f) \, gp \, dp \, d\phi \tag{65}$$

If we now follow the method of section III and expand f into symmetric and antisymmetric parts, we see

$$\left(\frac{\partial f}{\partial t}\right)_c = n' \iint (\tilde{f}^e - f^e) \, gp \, dp \, d\phi + n' \iint (\tilde{f}^o - f^o) \, gp \, dp \, d\phi$$

That is

$$\left(\frac{\partial f}{\partial t}\right)_c = \left(\frac{\partial f^o}{\partial t}\right)_c + \left(\frac{\partial f^e}{\partial t}\right)_c \tag{66}$$

Now it has been shown[9] that spherical harmonics constitute a set of proper functions for the collision operator, that is, if we expand f according to

$$f = \sum_{\ell} f_{\ell} P_{\ell} (\cos \theta) \tag{67}$$

that

$$\left(\frac{\partial f}{\partial t}\right)_c = - \sum_{\ell} \nu_{\ell} f_{\ell} P_{\ell} (\cos \theta) \tag{68}$$

where

$$\nu_{\ell} = 2\pi n' \omega \int_0^{\infty} \left[1 - P_{\ell} (\cos \chi) \right] p \, dp \tag{69}$$

and χ is the scattering angle.

We therefore expand f^o and f^e according to

$$f^e = \sum_{\ell} f_{\ell}^e \; P_{2\ell} \; (\cos \theta) \tag{70}$$

$$f^o = \sum_{\ell} f_{\ell}^o \; P_{2\ell+1} \; (\cos \theta) \tag{71}$$

so that

$$\left(\frac{\partial f^e}{\partial t}\right)_c = - \sum_{\ell} \nu_{2\ell} \; f_{\ell}^e \; P_{\ell} \; (\cos \theta) \tag{72}$$

$$\left(\frac{\partial f^o}{\partial t}\right)_c = - \sum_{\ell} \nu_{2\ell+1} \; f_{\ell}^o \; P_{2\ell+1} \; (\cos \theta) \tag{72}$$

where the ν's are determined by equation (69). Now if the collision process favors small angle scattering, the higher order anisotropics, i.e., hieher values of ℓ, are destroyed quickly so that we can write

$$f^e \approx f_e^{\,o} \tag{73}$$

$$f^o \approx f_o^{\,o} \; P_1 \; (\cos \theta) \tag{74}$$

and

$$\left(\frac{\partial f^e}{\partial t}\right)_c \approx - \nu_o \; f^e$$

$$\left(\frac{\partial f^o}{\partial t}\right)_c \approx - \nu_1 \; f^o$$

Now from equation (69) we see immediately

$$\nu_o = 0 \tag{75}$$

$$\nu_1 = 2\pi \; N\omega \int_0^{\infty} (1-\cos\chi) \; p \; dp \tag{76}$$

It is interesting to note that ν_1 is the collision frequency for momentum transfer.[5] Therefore

$$\left(\frac{\partial f^e}{\partial t}\right)_c \approx 0 \tag{77}$$

$$\left(\frac{\partial f^o}{\partial t}\right)_c \approx -\nu f^o \tag{78}$$

C. Non-Equilibrium Ambipolar Diffusion

We write the time dependent forms of the continuity and flow equations for both electrons and ions:

Electrons

$$\frac{\partial N^-}{\partial t} + \nabla \cdot \Gamma^- = 0 \tag{42}$$

$$m \frac{\partial \Gamma^-}{\partial t} + kT^- \nabla N^- + N^- e\, E = - m \nu^- \Gamma^- \tag{43}$$

Ions

$$\frac{\partial N^+}{\partial t} + \nabla \cdot \Gamma^+ = 0 \tag{44}$$

$$M \frac{\partial \Gamma^+}{\partial t} + kT^+ \nabla N^+ - N^+ eE = - M \nu^+ \Gamma^+ \tag{45}$$

Here, ν^- and ν^+ represent the electron-neutral and ion-neutral collision frequencies respectively.

We make the standard assumptions of quasi-neutrality and ambipolar flow,

$$N^- = N^+ = N$$
$$\Gamma^- = \Gamma^+ = \Gamma$$

So equations (43) and (45) become

$$m \frac{\partial \Gamma}{\partial t} + kT^{-} \nabla N + N e E = -m \nu^{-} \Gamma \tag{46}$$

$$M \frac{\partial \Gamma}{\partial t} + kT^{+} \nabla N - N e E = -M \nu^{+} \Gamma \tag{47}$$

Adding equations (46) and (47)

$$(m + M) \frac{\partial \Gamma}{\partial t} + k (T^{-} + T^{+}) \nabla N = -(m \nu^{-} + M \nu^{+}) \Gamma \tag{48}$$

Now combining equation (48) with the continuity equation,

$$\frac{\partial N}{\partial t} + \nabla \cdot \Gamma = 0 \tag{49}$$

yields

$$\left[\frac{m + M}{m \nu^{-} + M \nu^{+}} \right] \frac{\partial^2 N}{\partial t^2} + \frac{\partial N}{\partial t} = \frac{k(T^{-} + T^{+})}{m \nu^{-} + M \nu^{+}} \nabla^2 N \tag{50}$$

This is of the form

$$\frac{1}{\nu^a} \frac{\partial^2 N}{\partial t^2} + \frac{\partial N}{\partial t} = D^a \nabla^2 N \tag{51}$$

where

$$D^a = \frac{kT^{-} + kT^{+}}{m \nu^{-} + M \nu^{+}} = \frac{\mu^{+} D^{-} + \mu^{-} D^{+}}{\mu^{-} + \mu^{+}} \tag{52}$$

is the usual result for the ambipolar diffusion coefficient, and we define the effective ambipolar collision frequency by

$$\nu^a = \frac{M \nu^{+} + m \nu^{-}}{M + m} \approx \nu^{+} + \frac{m}{M} \nu^{-} \tag{53}$$

Comparing equation (51) with equation (16), we see that the ambipolar diffusion speed is

$$v^a = \sqrt{\nu^a D^a}$$

$$= \sqrt{k \frac{T^- + T^+}{m+M}} \approx \sqrt{\frac{kT^+}{M}} \sqrt{1 + \frac{T^-}{T^+}} \tag{54}$$

In the case of an isothermal plasma we have

$$v^a \approx \sqrt{2} \sqrt{\frac{kT}{M}} = \sqrt{2}\, v^+ \tag{55}$$

and the gas diffuses approximately at the ion thermal speed.

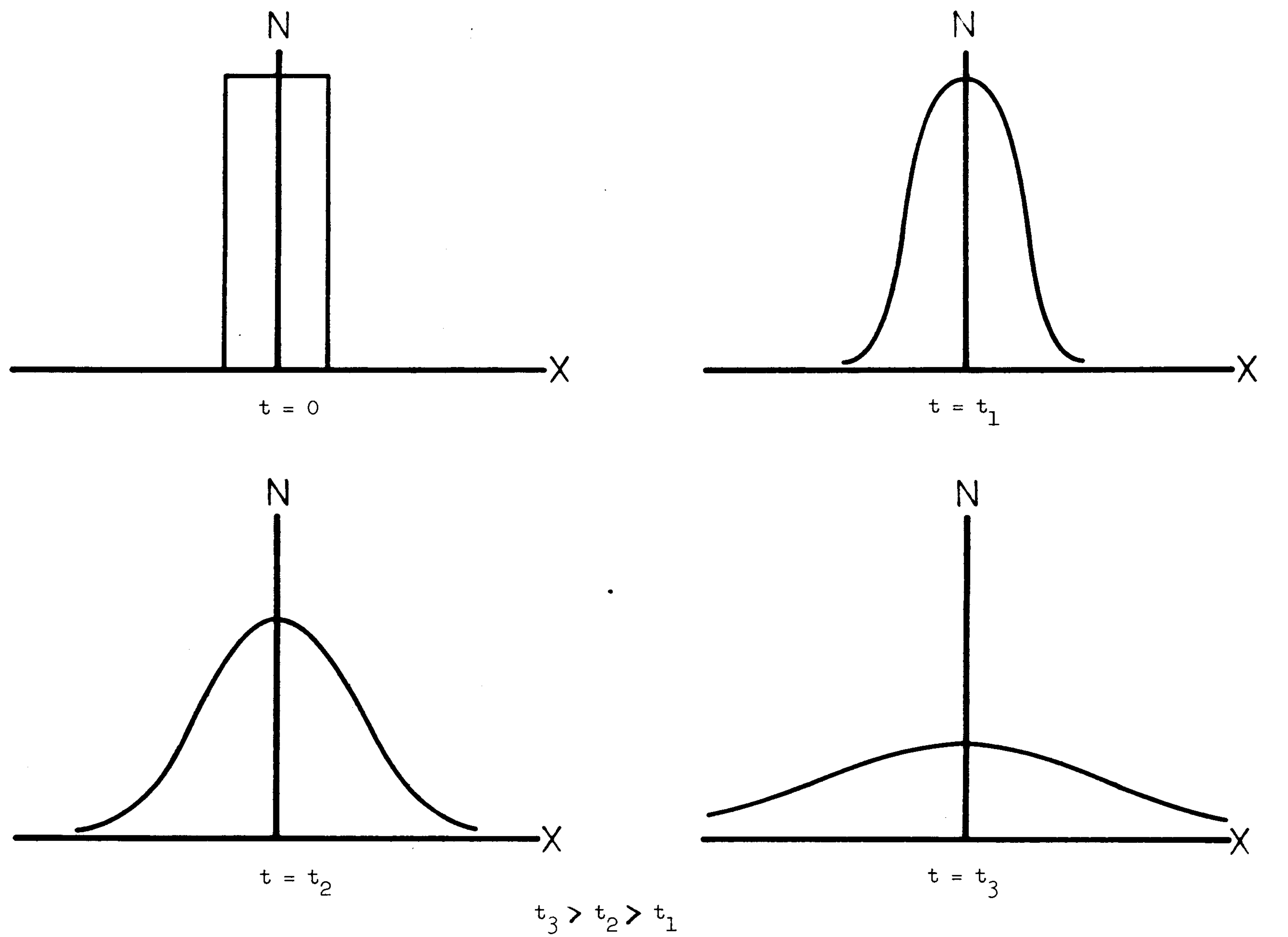


Figure 1. Conventional Theory for the Diffusion of a Rectangular Pulse of Gas.

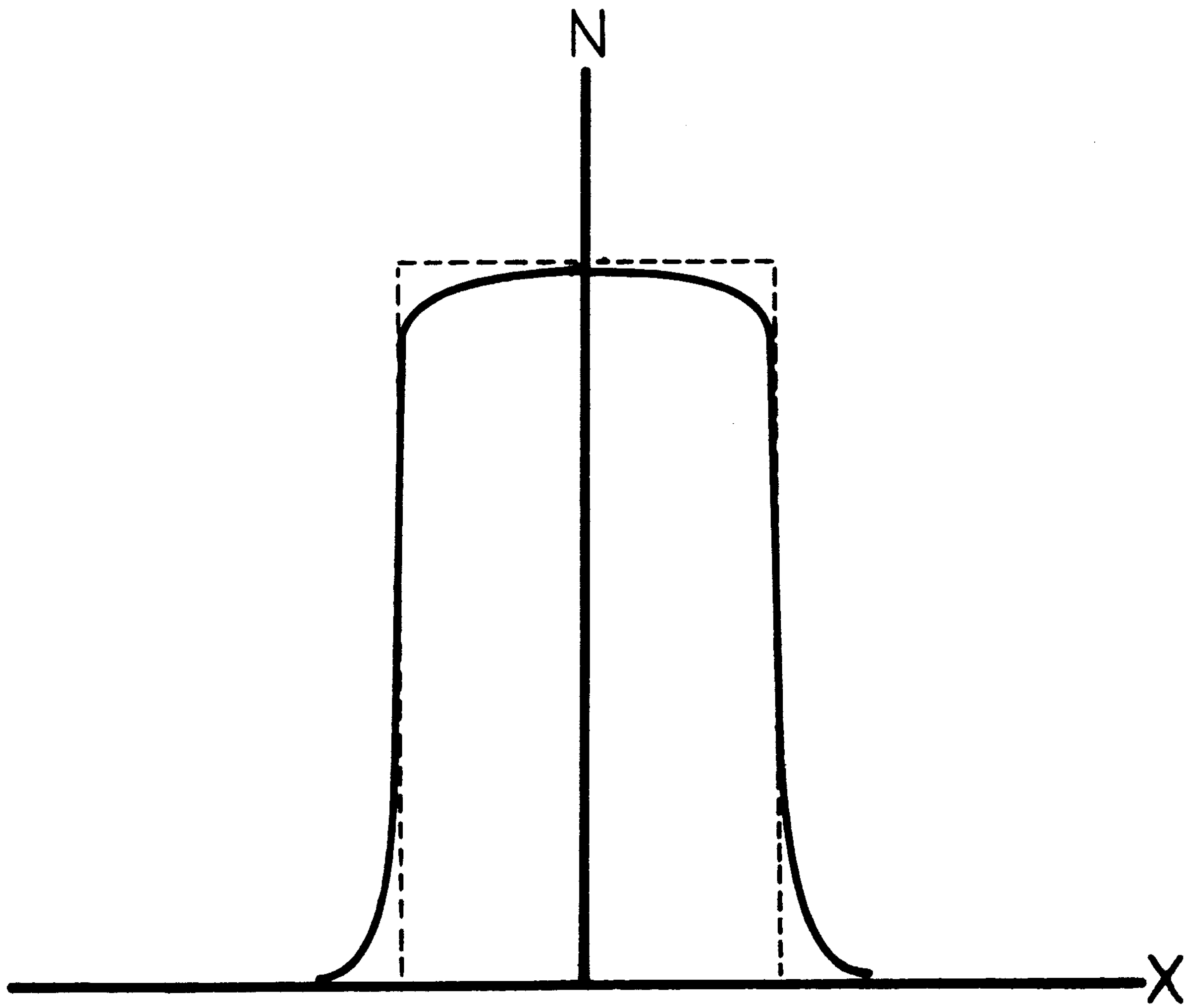


Figure 2. Conventional Theory for Diffusion of a Rectangular Pulse of Gas at Time $t=\epsilon$.

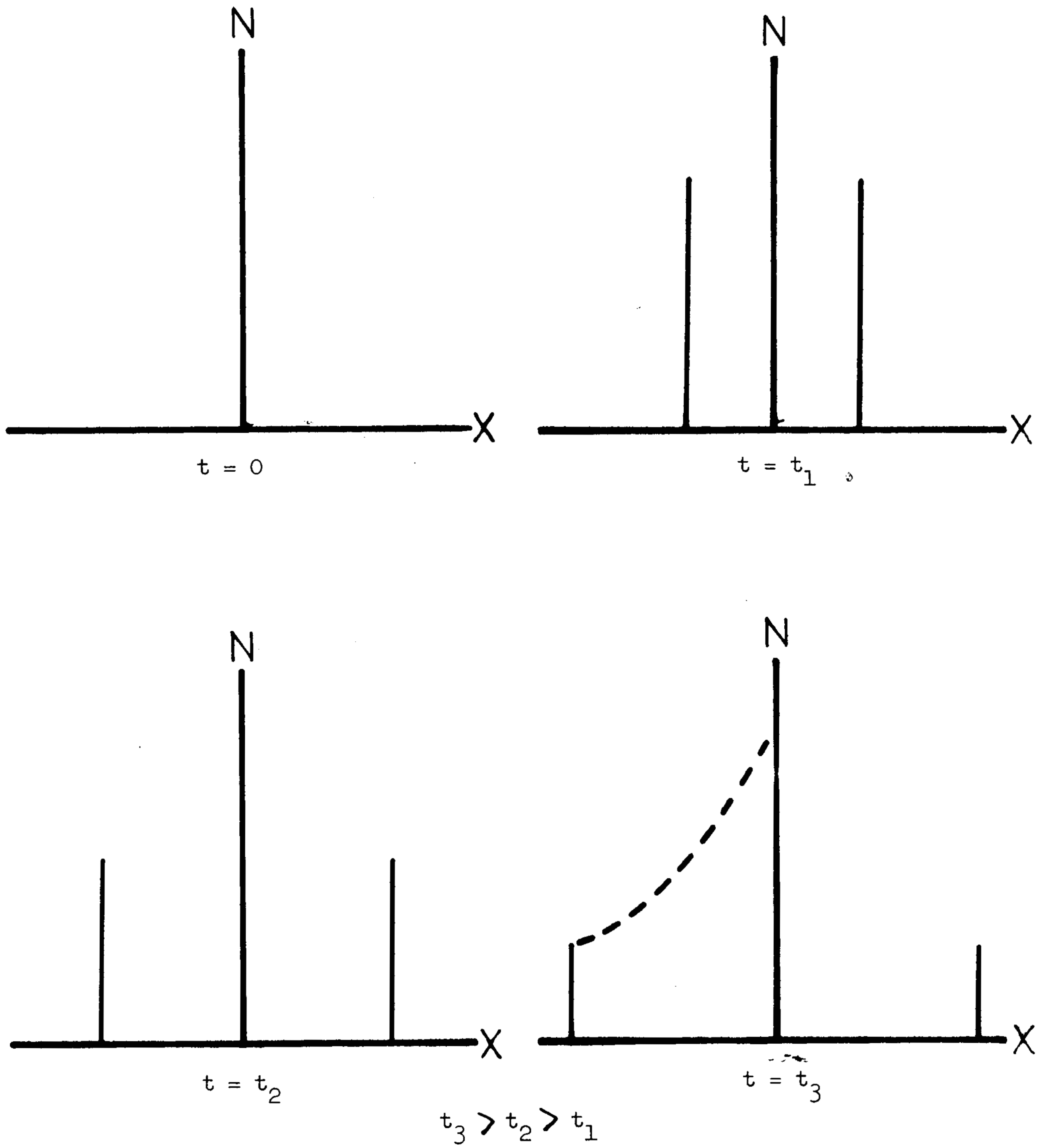


Figure 3. Partial Solution for the Diffusion of an Initial δ-Function.

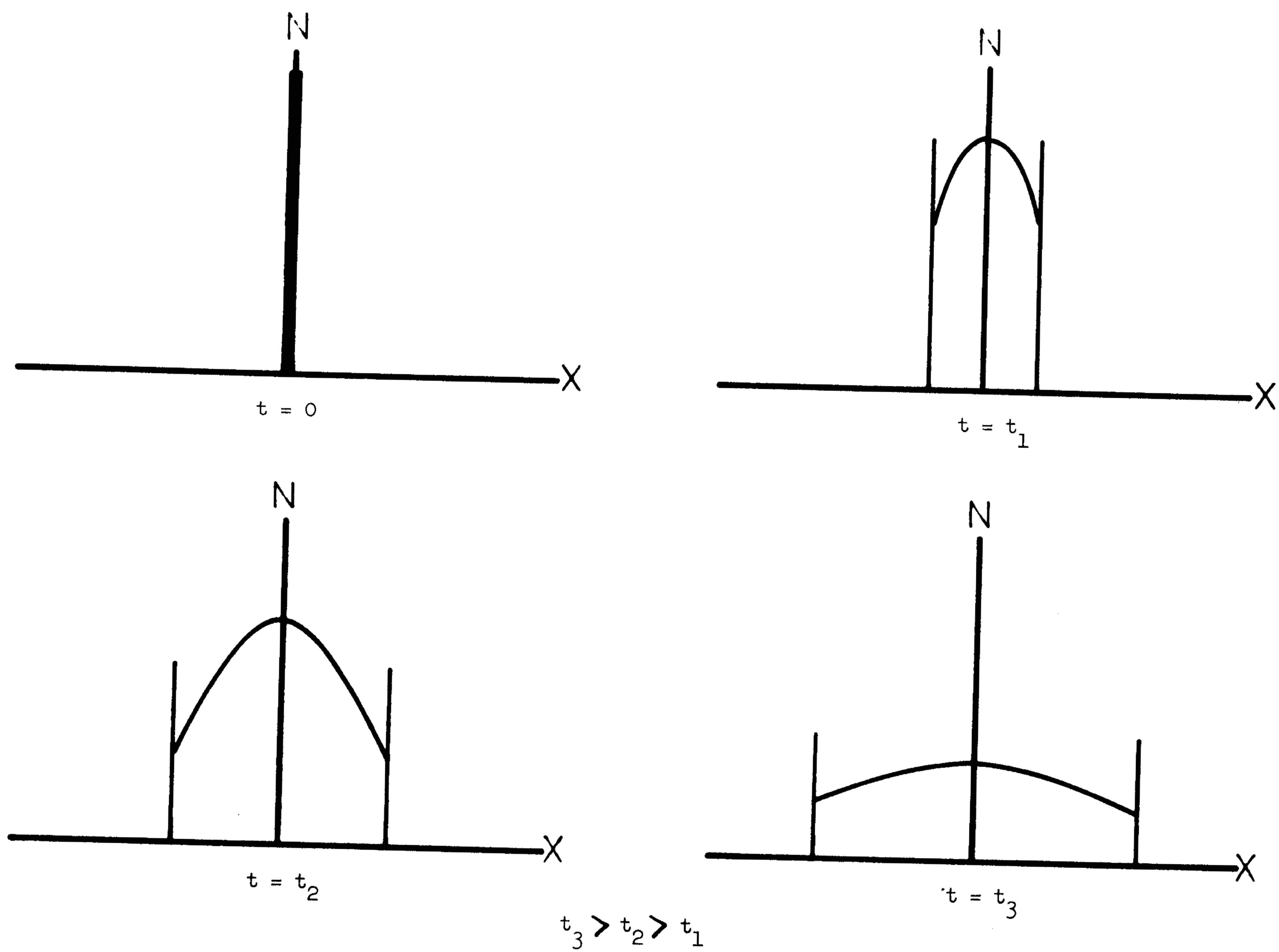


Figure 4. Complete Solution for the Diffusion of an Initial δ-Function.

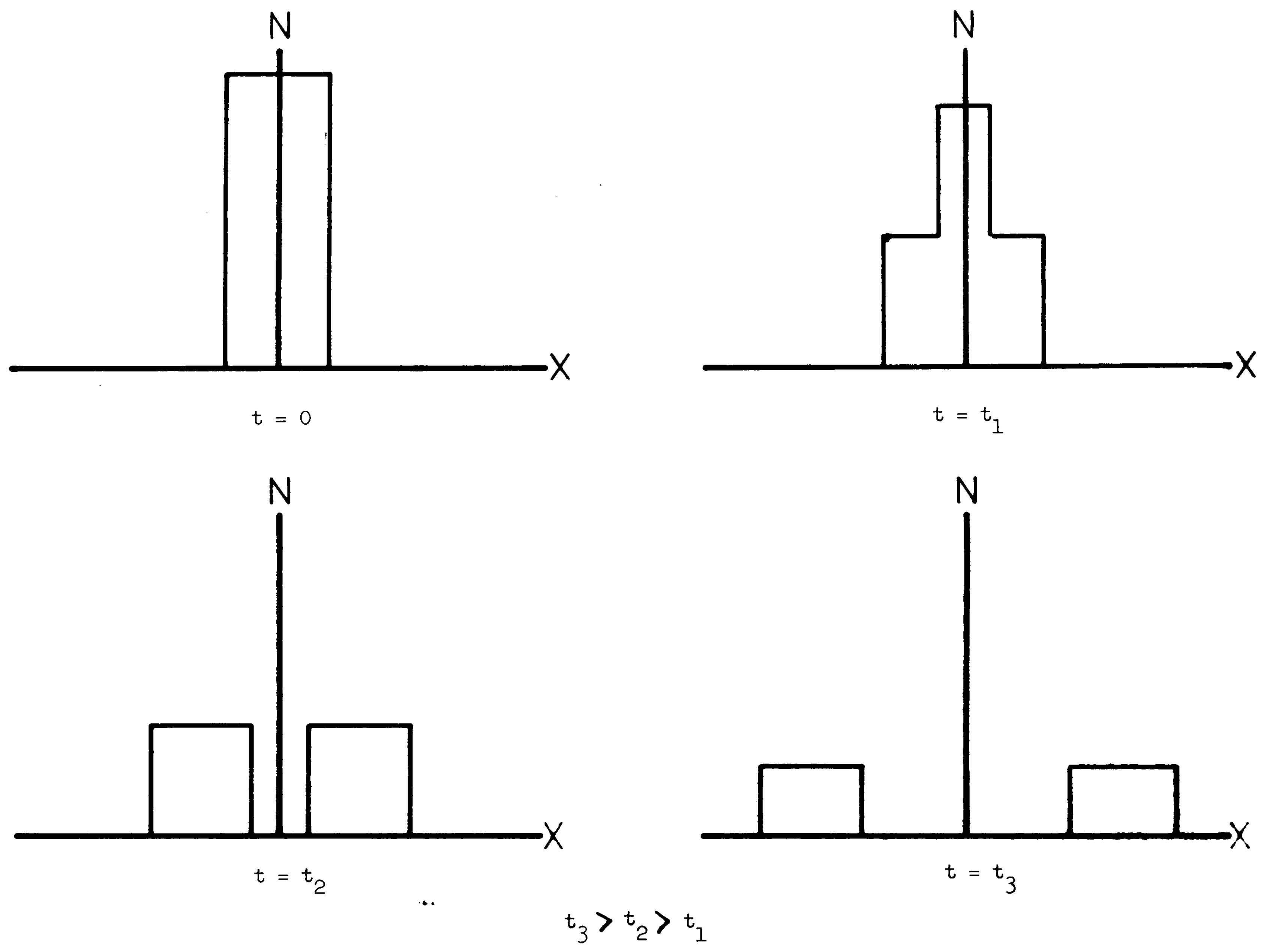


Figure 5. Partial Solution for the Diffusion of an Initial Rectangular Pulse.

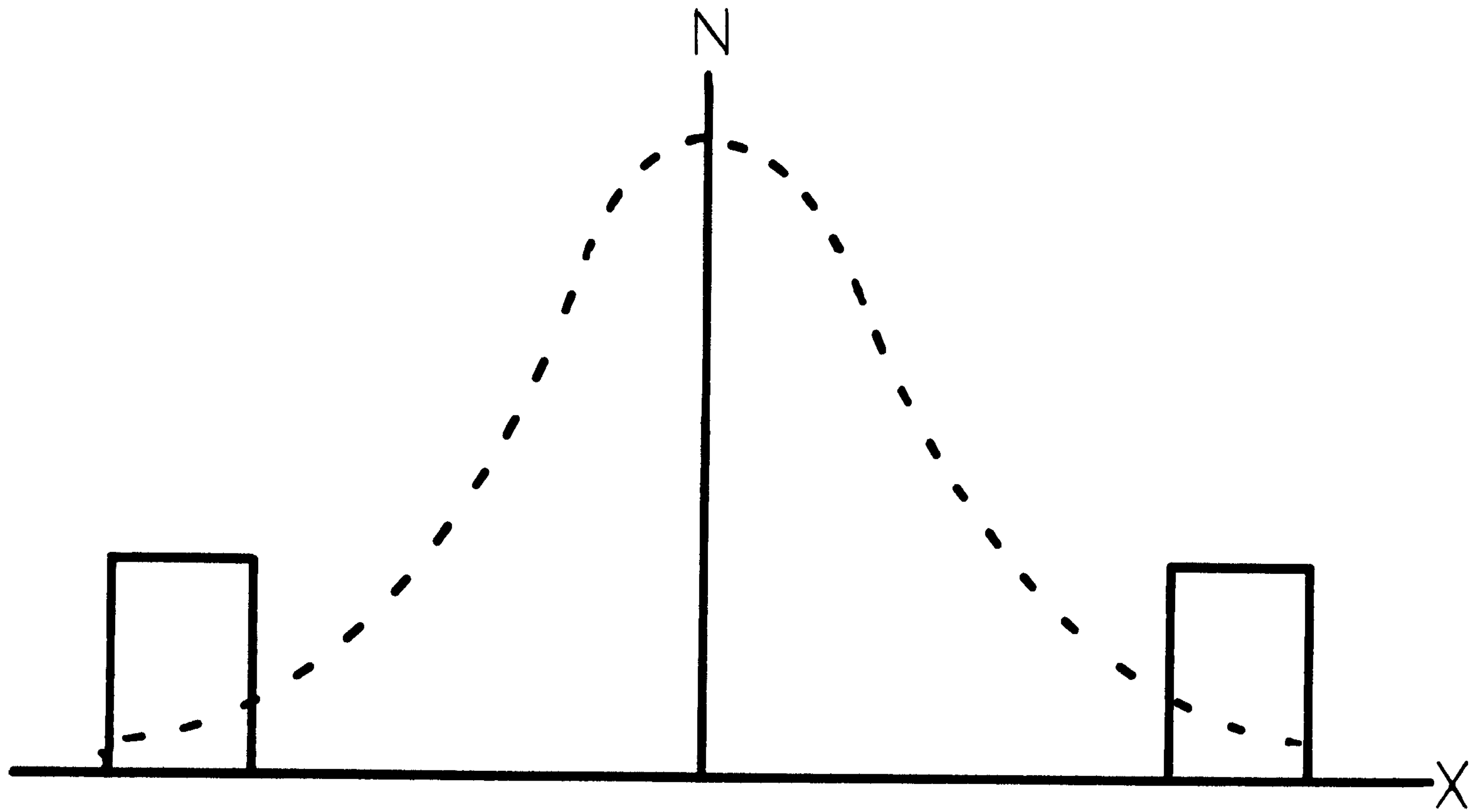


Figure 6. The Free Streaming Group and its Collisional Residue. (dotted line)

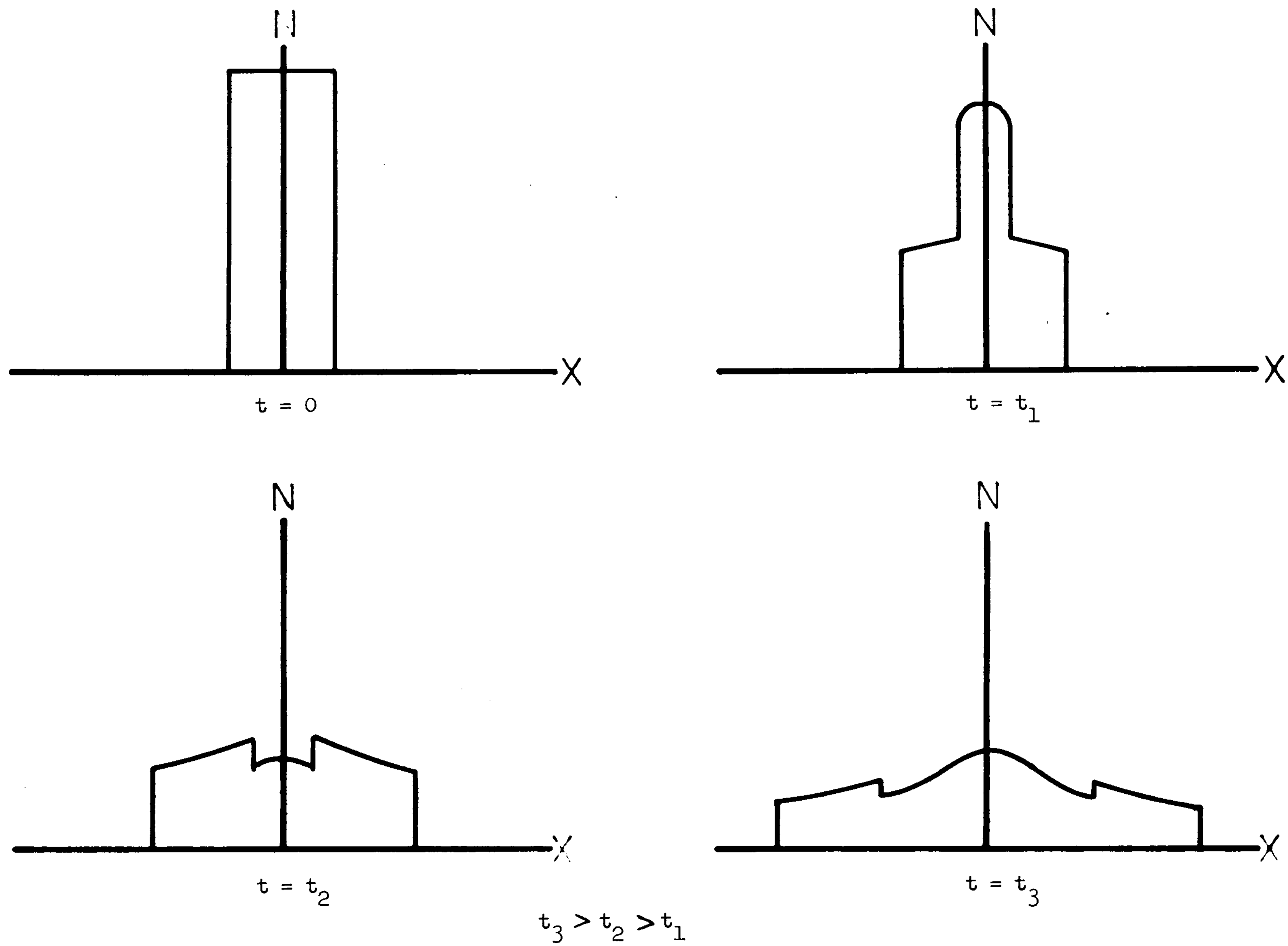


Figure 7. Complete Solution from the Macroscopic Equations for Diffusion of an Initial Rectangular Pulse

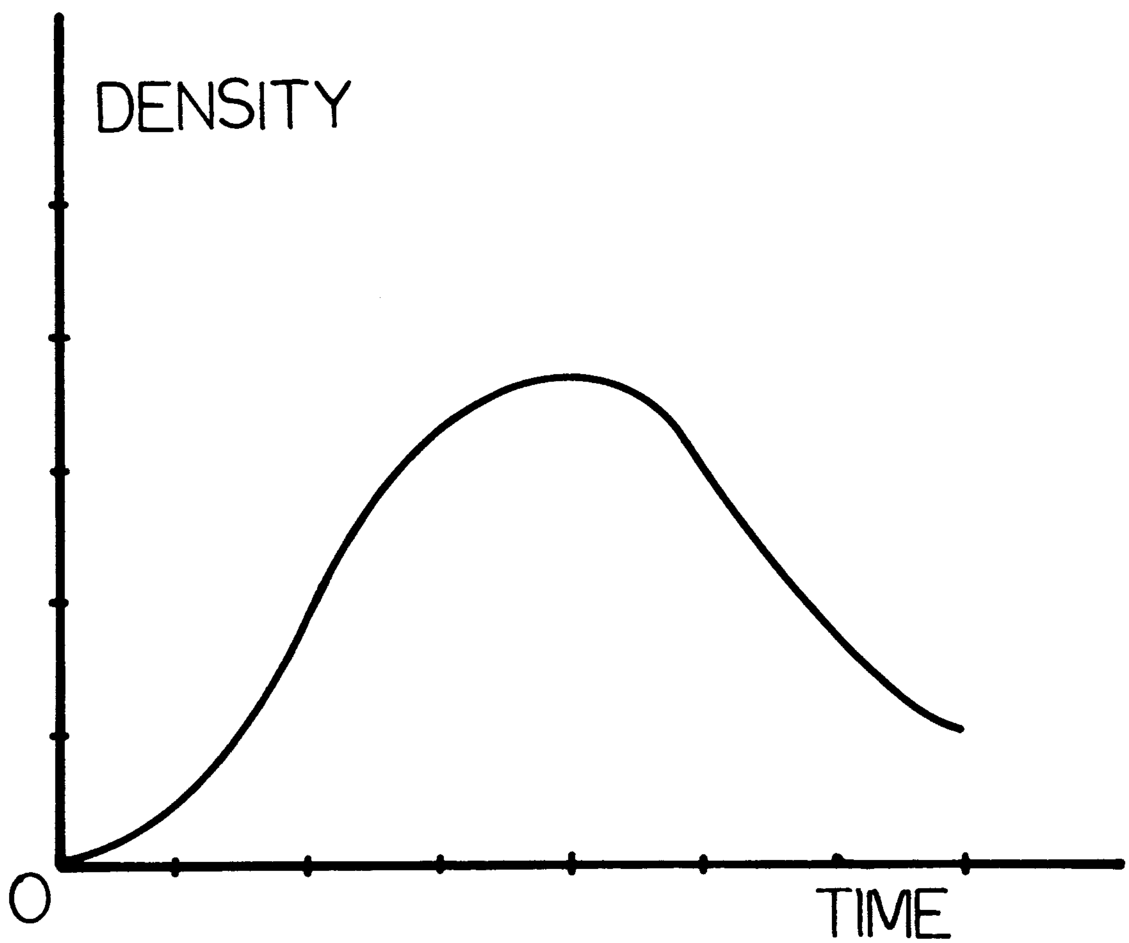


Figure 8. Development of Density vs. Time as Predicted by the Conventional Theory.

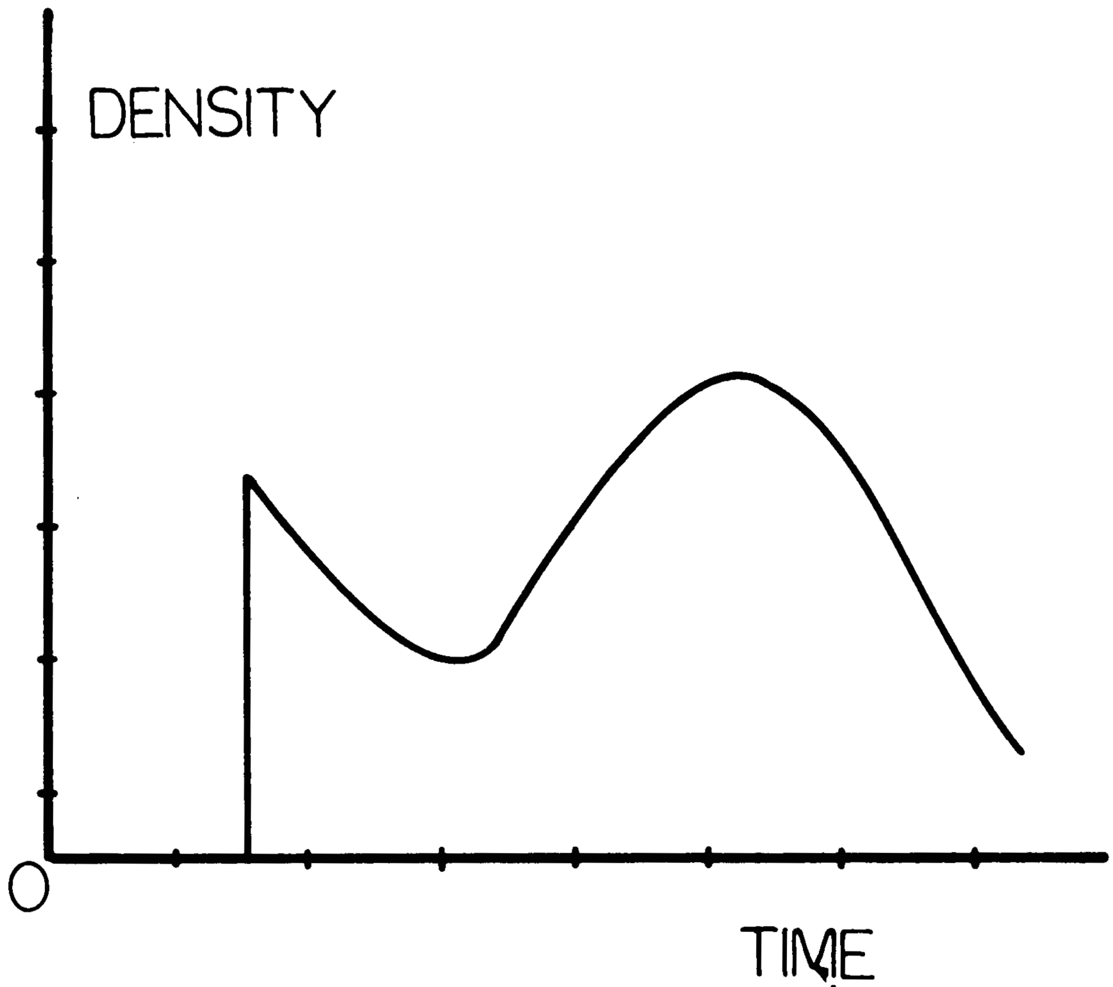


Figure 9. Development of Density vs. Time as Predicted by the Modified Theory.

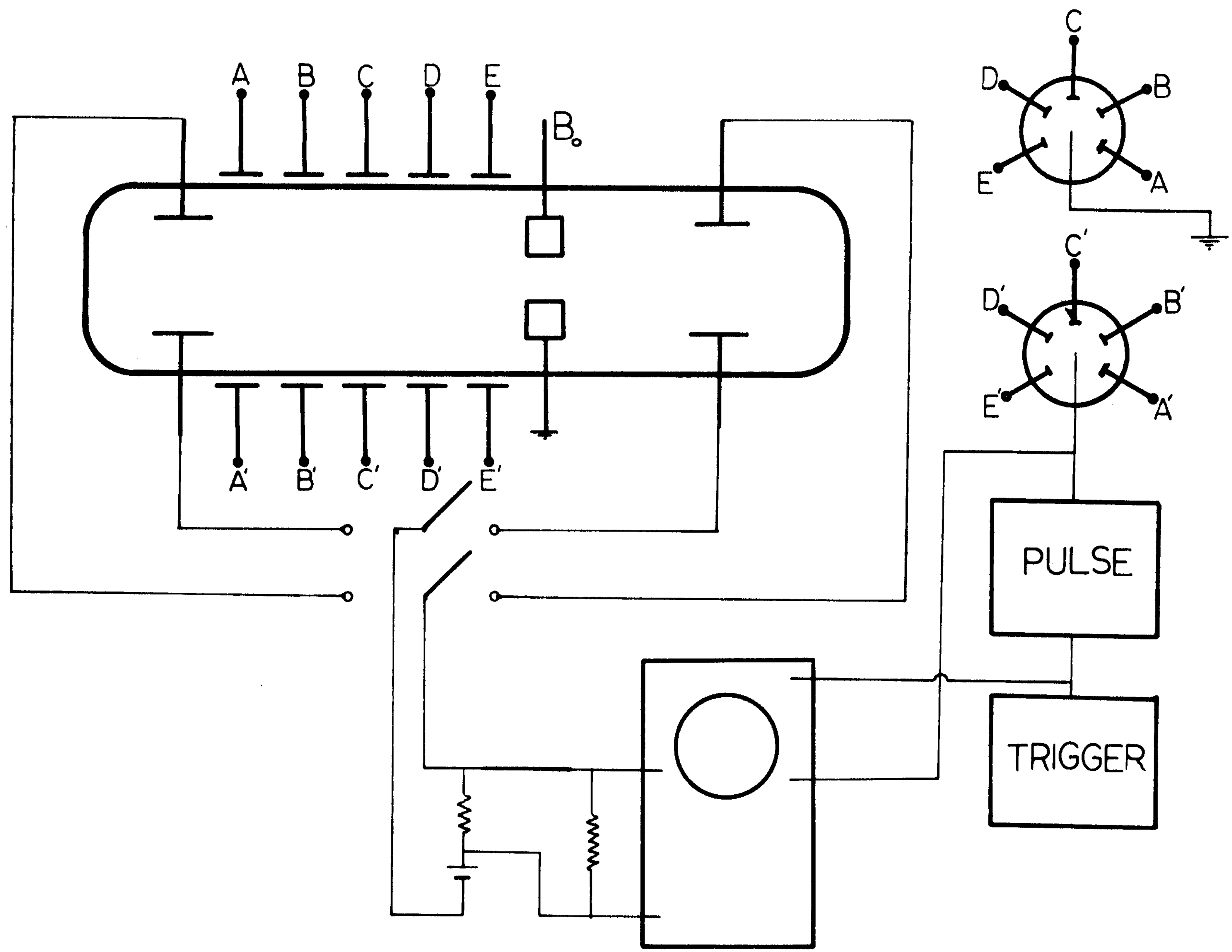
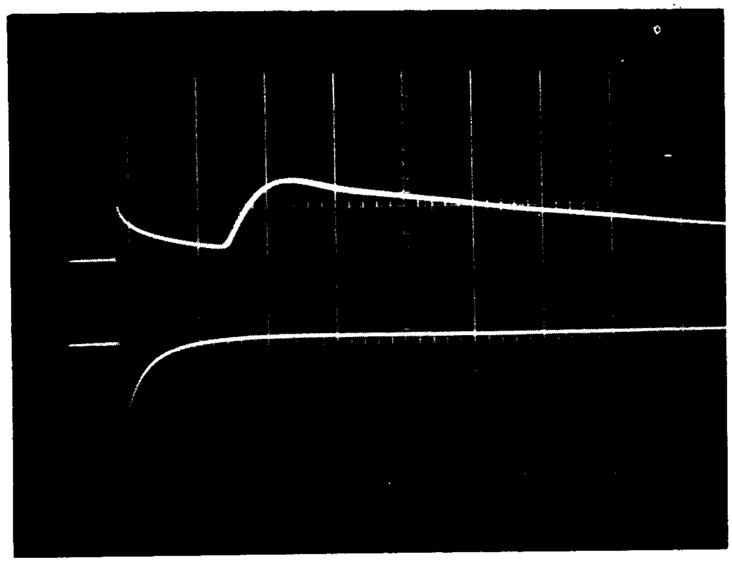


Figure 10. Experimental Apparatus



Top Trace: Probe Response
Bottom Trace: Ionizing Pulse
Scale: 20 microseconds/cm.

Figure 11. Sample of Data Obtained.

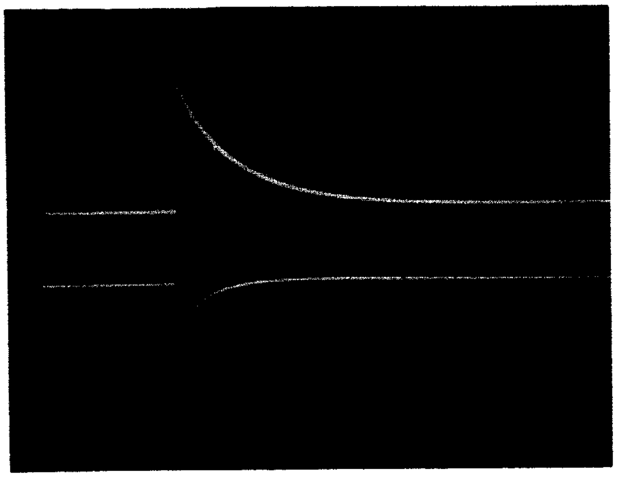


Figure 12a: Pressure = 0.20 Torr
Top Trace: Probe Response
Bottom Trace: Ionizing Pulse
Scale: 20 microseconds/cm

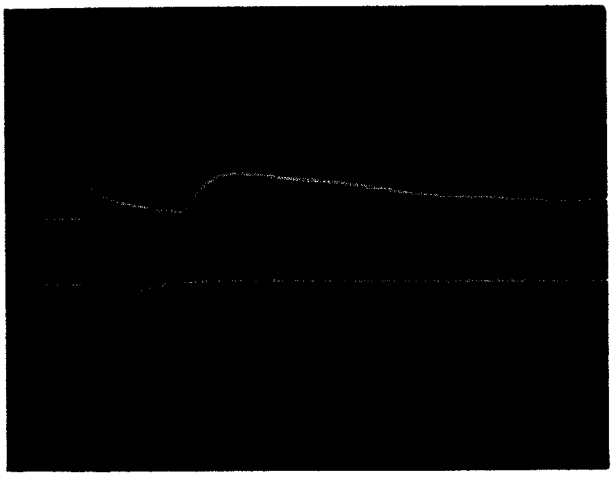


Figure 12b. Pressure = 0.35 Torr

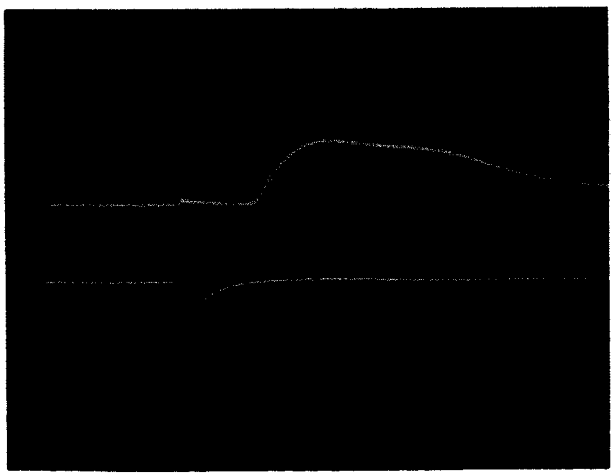


Figure 12c. Pressure = 0.40 Torr

Figure 12. Probe Response for Various Gas Pressures.

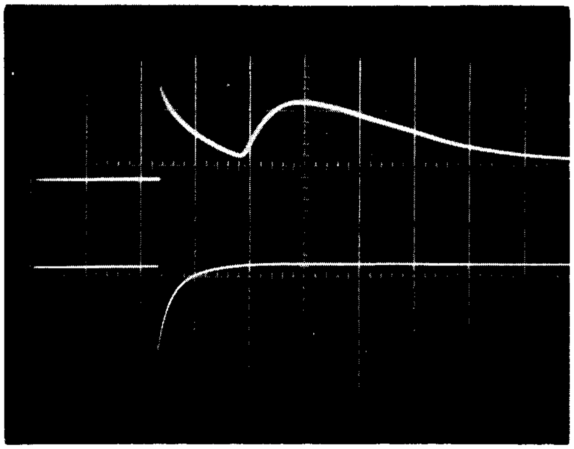


Figure 13a: 9.8 cm. separation
Top Trace: Probe Response
Bottom Trace: Ionizing Pulse
Scale: 20 microseconds/cm

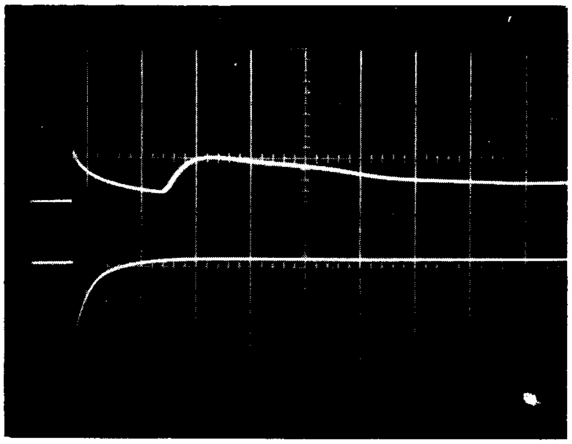


Figure 13b. 13 cm separation

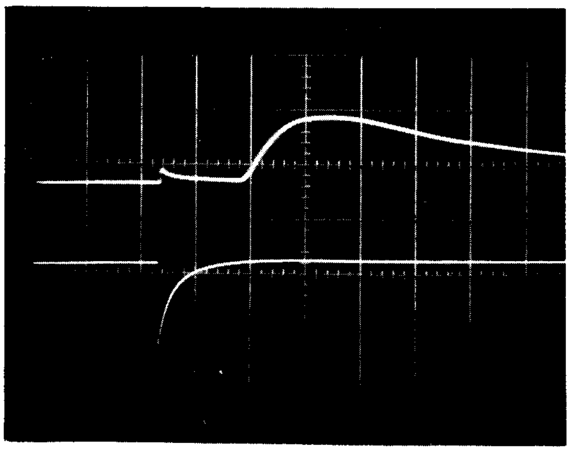


Figure 13c: 17 cm separation

Figure 13. Probe Response for Various Electrode Separations.

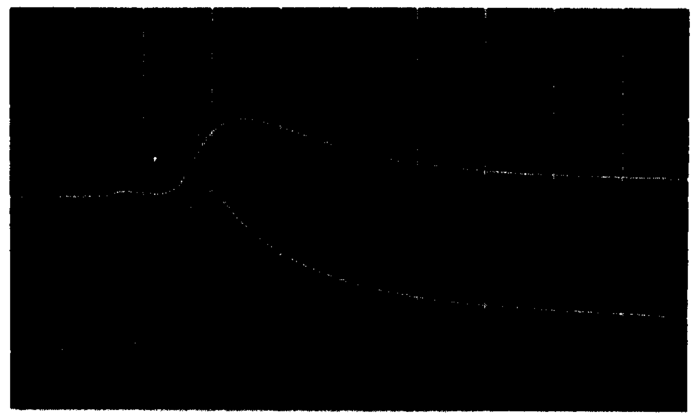


Figure 14. Determination of Propagation Velocity

Simultaneous observation of the leading edge by two sets of probes at different distances from the initial pulse. The lower trace shows the response of the closer set of probes. The difference between the shapes of the traces is evidence of the distribution of velocities.

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