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DAMPING IN MULTI-BEAM VIBRATION ANALYSES
PART I - ANALYTICAL METHODS

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FOREWORD

This Technical Memorandum is the first of a planned series of reports dealing with the investigation of damping in vibration analyses of missile structures. This investigation is a part of the work being performed by the Lockheed Missiles & Space Company's Huntsville Research and Engineering Center for the Marshall Space Flight Center under NASA contract NAS8-11148, Task L-11.

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SUMMARY

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As a preliminary step in the investigation of vibration analyses of multi-beam structures with damping, simple analytical methods which are generally applicable for lightly damped structures are investigated. Examples are used to show applicabilities of the outlined methods.

Author

INTRODUCTION

The presence of damping in a structure influences its vibration characteristics in two aspects where significant discrepancies are noted between physical phenomena and predictions based on the assumption of zero damping. In the first place, proper damping makes finite steady-state vibration amplitude possible at resonance, while the undamped analysis predicts an infinite amplitude. Secondly, free vibrations following transient disturbances decay in time. Analyses of undamped structures predict persistent free vibrations.

It is important to point out that in almost all other respects, the undamped analysis provides accurate results for a lightly damped structure. In particular, it provides the knowledge of natural frequencies and dynamic relative displacements which is useful in obtaining approximate solutions of the damped structure at resonance.

In a large number of situations where external dampers are not intentionally employed and where vibratory aerodynamic drag is insignificant, damping must be obtained from internal sources of a structure. Major dissipative forces are then limited to the friction forces acting at structural joints, and the anelastic components of stresses acting internally of strained elements. Such dissipative forces introduce what are commonly referred to as structural damping and material damping to the system.

Energy losses due to structural damping and material damping are usually observed as areas of hysteresis loops of the force-displacement and stress-strain relationships. Differential equations of motion of a structure exhibiting hysteresis are not easily solved exactly by known methods. In this report, an approximate method of analyzing damped steady-state vibrations is described. A compatible method of transient analysis is then selected to provide a pair of techniques which is a self-contained system according to which experimental data can be generated in a format suitable for analysis; and, conversely, according to which analytical results can be meaningfully verified by experiments.

The solution methods are applied to analyses of two elementary structural components to demonstrate feasibility, as well as to form a foundation on which future analyses of complicated practical structures may be constructed.

STEADY-STATE ANALYSIS OF LIGHTLY DAMPED STRUCTURES

The Formal Mathematical Problem

A general expression of the relationship between stress and strain in a solid continuum can be written in the operator form

$$\sigma_{ij} = \mathcal{E}_{ijkl}\left\{ \epsilon_{kl} \right\}, \qquad i, j, k, l = 1, 2, \text{ or } 3, \tag{1}$$

where σ_{ij} and ϵ_{ij} are stress and strain components and $\mathcal{E}_{ijkl}$ is a tensorial modulus operator which is reduced to simple constants of multiplication in the special case of perfect elasticity. For materials which show capacity for internal damping, application of the modulus operator on strain results in a stress which is a multi-valued function of strain. If either the stress or the strain is cyclically varied between constant extremities, the relationship between them is a closed curve known as the hysteresis loop.

The mathematical statement of vibrational problems are usually derived by substitution of the stress-strain law, Equation (1), into the equation of motion

$$\sigma_{ij,j} = \rho \ddot{\xi}_i . \tag{2}$$

For elastic materials, the substitution produces the fundamental linear partial differential equations in the displacement vector ξ_i. Special geometrical considerations in many situations reduce the equations to their simplified forms, some of which are well known, e.g., the Bernouilli-Euler equation for beams and plates:

$$D \xi_{3,iijj} + \rho h \ddot{\xi}_3 = Q(x_i, t), \qquad i, j = 1, 2; \tag{3}$$

and the one-dimensional wave equation for torsional and longitudinal vibrations in bars:

$$\xi_{i,11} = \frac{1}{a^2} \ddot{\xi}_i , \qquad i = 1, 2, \text{ or } 3. \tag{4}$$

In Equations (3) and (4), the constants D and a contain, typically, elastic constants, and the mass density ρ.

For materials exhibiting hysteresis damping, the substitution of the stress-strain law into the equation of motion produces differential equations which are not analytical in nature. Generally, the resulting differential equations themselves must be varied according to the solutions(Reference 1), making them difficult, if not impossible, to be solved exactly. One of the most useful mathematical tools to obtain approximate solutions of damped vibration problems via the differential-equation approach is the method developed by Krylov and Bogoliuboff (Reference 2) for nonlinear mechanics. Pisarenko (Reference 3) demonstrated the successful application of this method to numerous transient and steady-state problems. However, the approach in Reference 3 cannot be generalized easily, and when accuracies better than first approximation of the solution are desired, the technique becomes tediously complex. Difficulty also arises when the method is used to include the effects of structural damping.

A Review of Validities of Elastic Analysis

It is known that for a lightly damped structure the elastic analysis of the corresponding undamped structure provides satisfactory solutions with respect to natural frequencies and dynamic configurations (mode shapes), as well as relative amplitudes of the components of the actual displacement (or stress) solution. For the steady-state vibration of lightly damped structures, therefore, the elastic analysis provides approximate solutions to all but one (scalar) quantity: the amplitude of the magnitude of the displacement vector or stress tensor which becomes characteristically different at resonant frequencies. When the excitation frequency approaches one of the natural frequencies of the structure, the elastic analysis predicts an infinite amplitude while, actually, the amplitudes are usually finite in the damped system.

It is, therefore, desirable to seek a method of determining the resonant amplitudes of a lightly damped structure to complement, rather than to replace, the elastic analysis.

A Method of Approximate Analysis

The method which appeals to the aforementioned requirement of an analysis of finding resonant amplitudes of a structure with material damping and joint damping is based on two fundamental assumptions. The method can be outlined in four essential steps.

Step 1 Perform an elastic, free-vibration analysis of the corresponding undamped structure. Determine, in particular, mode shapes and corresponding frequencies.

Assumption 1

The amplitudes of vibration of the actual damped structure become extremely large when the frequency of excitation approaches one of the undamped natural frequencies. In addition, the displacement at resonance is primarily in the corresponding undamped mode. Justification of this assumption for material damping alone is based on experimental observations; and for combined material damping and structural damping will be made in a subsequent section.

Assumption 2

Frequencies of peak amplitudes and frequencies of 90° phase shift between excitation and response are so closely located near the undamped natural frequencies that no distinction need be made among the three types of frequencies. This assumption is, again, based on experimental knowledge.

Step 2 Calculate the total work done on the structure by the exciting force at resonance for each cycle of vibration by integrating the product of excitation and corresponding velocity of structure deformation over a cycle. The integration is to be performed at 90° -phase-shift frequencies and the result will be an expression for the work in terms of the modal frequency and the modal amplitude.

Step 3 Calculate the amount of energy dissipated by the entire structure per cycle of vibration at resonance by integrating the "specific damping" over the volume of the structure. Specific damping is the amount of energy dissipation per cycle per unit volume of the material and is usually dependent on the local stress amplitude. The resulting expression for the dissipation will contain the mode frequency and amplitude and damping properties of the material. Structural damping may be included in the calculation as a lumped amount at each structure joint.

Step 4 Equate the work done to the structure to the energy dissipated by it in each cycle. The resulting algebraic equation can then be solved for the single unknown resonant amplitude in terms of the modal frequency, material properties, and the forcing function amplitude.

If modal amplitudes in terms of the corresponding frequencies are known, an "equivalent viscous damper" can be obtained for each mode of vibration. The fractional damping coefficient of these equivalent viscous dampers can be determined as a function of the modal frequency and is not a constant for all modes.

Elementary Beam Analysis - An Example

As a first example to demonstrate the application of the procedures just outlined, the vibration of a beam with material and joint damping will be considered. The schematic of the beam is shown on Figure 1. An elementary configuration has been chosen for consideration so that the Bernouilli-Euler equation

$$a^2 \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = Q(x,t), \qquad a^2 = \frac{EI}{\rho A}, \tag{1}$$

may be appropriately used for the lateral vibrations for the corresponding elastic structure. The end conditions can be any combination of free, hinged, fixed, elastically supported or hinged, and guided conditions, which, in general, are non-dissipative. A simple riveted lap joint is used at a distance x_s from the left end to include structural damping in a representative manner in the problem.

The elastic, free-vibration analysis determines the natural frequencies, p_n, n=1, 2, 3, -----, and the corresponding eigen functions $\phi_n(x)$, i.e., the elastic free-vibration solution has the form

$$w_0(x,t) = \sum_{n=0}^{\infty} \bar{w}_{on}\, \phi_n(x)\, e^{ip_n t}. \tag{2}$$

For conservative end conditions, the set of normalized eigen functions are mutually orthogonal, i.e.,

$$\int_0^{x_0} \phi_m(x)\, \phi_n(x)\, dx \begin{aligned} &= 0, \qquad m \neq n, \\ &= 1, \qquad m = n. \end{aligned} \tag{3}$$

When the actual beam (with damping) is subjected to an excitation at one of its natural frequencies, i.e., when

$$Q(x,t) = Q_0(x) \sin p_n t, \tag{4}$$

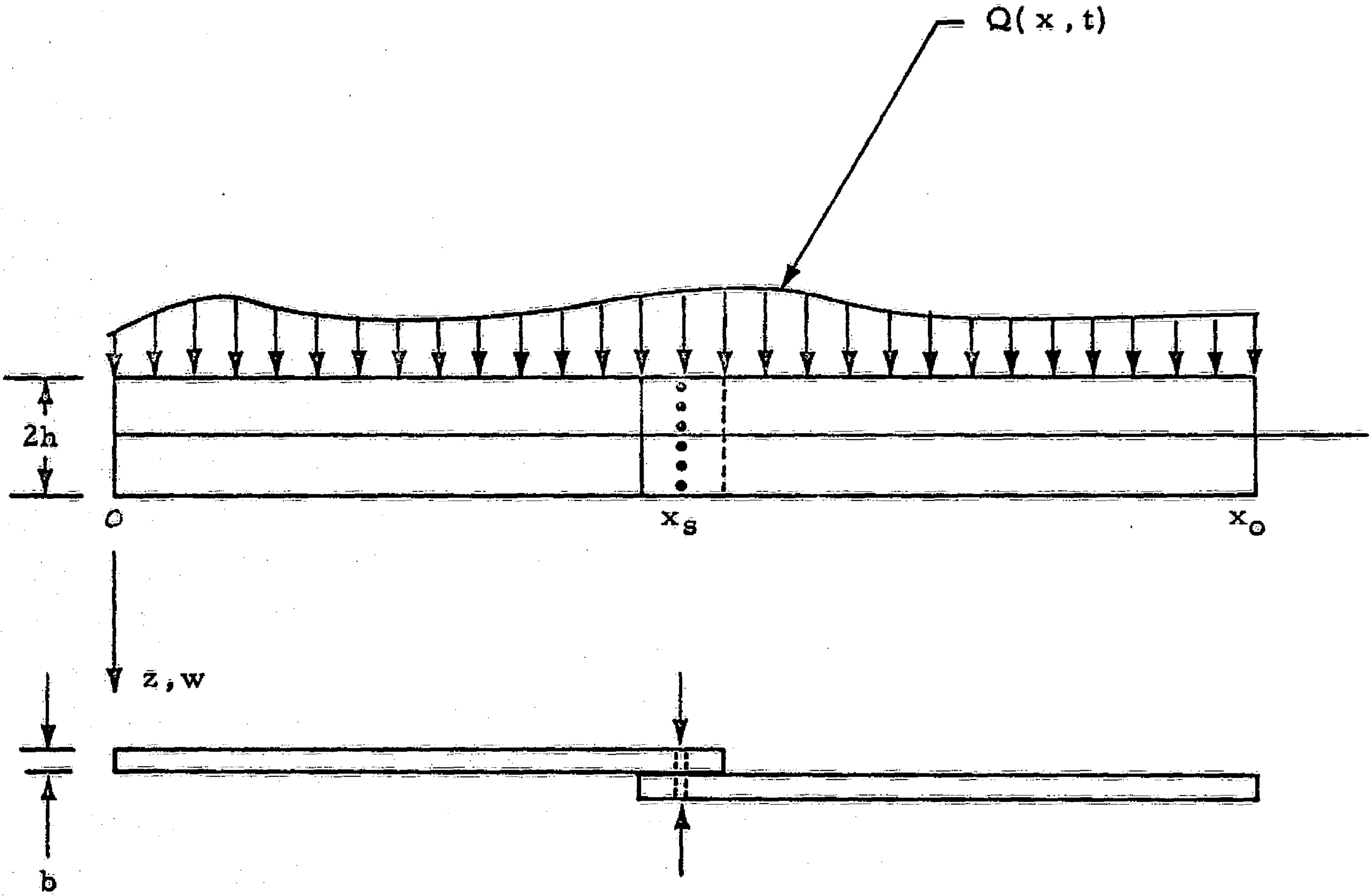


Figure 1 - Elementary Beam with Non-Dissipative End Conditions

The steady-state response is, predominantly,

$$w(x,t) = -\bar{w}_n \phi_n(x) \cos p_n t. \tag{5}$$

It is convenient to expand the forcing function into a series of $\phi_n(x)$,

$$Q(x,t) = Q_0(x) \sin p_n t = \sin p_n t \sum_m q_m \phi_m(x), \tag{6}$$

so that the work done by $Q(x,t)$ over a cycle may be a simple expression

$$\begin{aligned} \mathcal{E} &= \int_0^{2\pi/p_n} \int_0^{x_0} Q(x,t) \frac{\partial w}{\partial t} \, dx \, dt \\ &= \bar{w}_n \int_0^{2\pi/p_n} \sin^2 p_n t \; p_n dt \cdot \int_0^{x_0} \phi_n \sum_m q_m \phi_m \, dx \\ &= \pi \bar{w}_n q_n . \end{aligned} \tag{7}$$

A commonly used material damping law for a variety of metals has the form

$$d_M = J_M (\bar{\sigma})^M, \tag{8}$$

where d_m is the specific damping, $\bar{\sigma}$ the stress amplitude, and J_M and M are material damping properties. Since for an elementary beam

$$\sigma = E z \frac{\partial^2 w}{\partial x^2}, \tag{9}$$

so that in the n^{th} mode,

$$\bar{\sigma}_n = E z \bar{w}_n \phi_n''(x), \tag{10}$$

and

$$d_M = J_M (E z \bar{w}_n \phi_n'')^M. \tag{11}$$

The total dissipation due to material damping is then

$$D_M = b \int_{-h}^{h} \int_0^{x_0} d_M(x,z) \, dx \, dz. \tag{12}$$

Inserting Equation (11) into Equation (12) and carrying out the indicated integration in z,

$$D_M = \frac{b E^M J_M \left[h^{M+1} - (-h)^{M+1} \right] \overline{W}_n^M}{M+1} \int_0^{x_0} [\phi'']^M dx \tag{13}$$

The integral in x can be carried out for specified end conditions. In general

$$\int_0^{x_0} [\phi_n''(x)]^M dx = N_M(M, x_0)\, p_n^M, \tag{14}$$

and from the last two equations,

$$D_M = L_M(b, h, x_0, E, e, J_M, M)\,(p_n \overline{W}_n)^M, \tag{15}$$

with L_M being a known constant determined by every geometrical, mechanical, and damping property of the beam.

For the time being, structural damping due to a simple transverse lap-joint of the type shown on Figure 1 is assumed to be governed by the following law:

$$D_S = J_S [\overline{m}(x_s)]^S, \tag{16}$$

where $\overline{m}$ is the maximum value of the bending moment during a cycle at the joint location. The bending moment is to be computed according to elementary beam theory as if the joint were not present, that is

$$\overline{m} = E I \overline{W}_n \phi''(x_s). \tag{17}$$

Equation (16) then becomes

$$D_S = L_S(b, h, x_0, E, e, S, J_S, x_s)\,(p_n \overline{W}_n)^S. \tag{18}$$

Equating the work done by the excitation to the total dissipation of the beam, an algebraic equation in $\overline{W}_n$ is obtained:

$$\pi \overline{W}_n q_n = L_M (p_n \overline{W}_n)^M + L_S (p_n \overline{W}_n)^S. \tag{19}$$

The solution of $\overline{W}_n$ from Equation (19) usually does not present any essential problem.

If structural damping is not present, an explicite solution is possible,

$$\overline{W}_n = \left(\frac{\pi}{L_M}\right)^{\frac{1}{M-1}} \frac{g_n^{\frac{1}{M-1}}}{p_n^{\frac{M}{M-1}}} \; . \tag{20}$$

To put in a more familiar form, let $M=2+K$, ($0 \le K \le 1$ for most materials), then

$$\overline{W}_n = \left(\frac{\pi}{L_M}\right)^{\frac{1}{1+K}} \left(\frac{g_n}{p_n^2}\right) \left(\frac{p_n}{g_n}\right)^{\frac{K}{1+K}} \tag{21}$$

The first factor, $(\pi/L_M)^{\frac{1}{1+K}}$, is independent of the forcing function and the frequency and is constant for a given beam. It can be determined if all properties of the beam are known. It can also be experimentally determined at any convenient modal frequency, or statically. The last factor, $(p_n/g_n)^{\frac{K}{1+K}}$, differentiates hysteresis damping from viscous damping. For a linear viscoelastic beam at resonance, the proper factor to use would be $1/p_n$. Therefore, an equivalent modal damper obtained for a given forcing amplitude g_n in the n^{th} mode would predict modal amplitudes which are too high for modes at lower frequencies, and too low for higher modes. For example, if $g_1 = g_2$ and the equivalent viscous damper is derived on the basis of experimental data obtained in the first mode, i.e., at p_1, then the predicted second mode amplitude would be only $(p_1/p_2)^{1.5}$ of the actual value, (assuming K=1). If p_2 is approximately $2p_1$, the equivalent viscous damper would produce a predicted amplitude which is only 35% of the actual amplitude, a grossly nonconservative estimate.

An assumed structural damping law was used in the above analysis primarily to demonstrate the simplicity which might have been realized if such a law had existed. A justification of the form of such an empirical relationship will be made soon. Before that is attempted, however, it should be noted that structural damping has been investigated in the past in a purely analytical fashion (References 4, 5, and 6). Theoretical predictions of damping in idealized joint configurations were made and experimental verification of such predictions were then performed in the laboratory on carefully prepared joint specimens. Application of such theoretical results to a practical system, however, involves either gross simplification of the actual joint conditions and environments, or tremendous generalization of the theory to such an extent that the task becomes insurmountable. Empirical damping laws for typical structural joints would be, therefore, not only the preferred approach, but also the only approach before the state-of-the-art of a rational analysis is improved to a satisfactory stage.

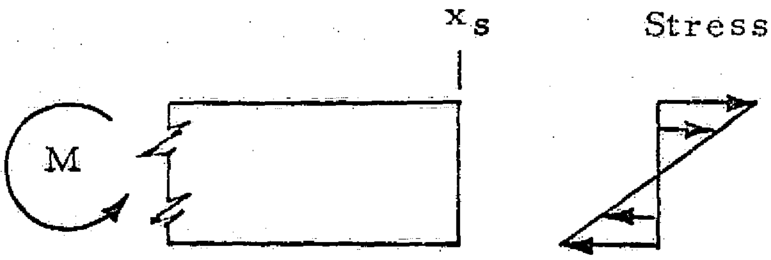
To justify Equation (16), consider the simplified joint by which the elementary beam of Figure 1 is constructed. The idealized stress distribution at $x = x_s$ on the left-hand half for various bending moments are shown on Figure 2. If it is assumed that microscopic slip is physically permissible, i.e., rivet holes are bigger than the rivet diameters, then slip will take place when the transmitted stress exceeds that determined by friction. Ideally, slip is initiated at the outer regions of the joint at a prescribed bending moment and propogates towards the middle as the bending moment is increased. The stress distribution at $x = x_s$ is, then, not linearly distributed over the depth. On unloading, the decrease of stress follows an elastic distribution until reverse slip is produced. Steady-state cycling of the bending moment above the initial slip moment would, therefore, result in a moment-differential rotation diagram such as that shown on Figure 2(d).The area of the closed loop represents the total energy dissipation at the joint per cycle of variation. The relationship between the enclosed area and the bending moment can, for this simple case, be derived. However, (if a more direct approach is chosen) it is empirically possible to express this relationship by an exponential expression such as Equation (16) over a bounded range of variation.

Prior to gross relative slip rotation of the two halves of the beam, the amount of partial slip is directly governed by the deviation of the stress distribution from linearity, and must be of the order of the bending strain. Hence, both the assumption that the deformation of the made-up beam can be approximated by the elastic counterpart, and the assumption that microscopic slip is permissible is justified if the gross slip condition is not reached. When the level of excitation is so high that gross slip is possible statically, the increase of energy dissipation with amplitude is not of a sufficiently high rate to compensate for the increasing amount of work input. Infinite resonant amplitudes can therefore occur for such large excitations (Reference 7).

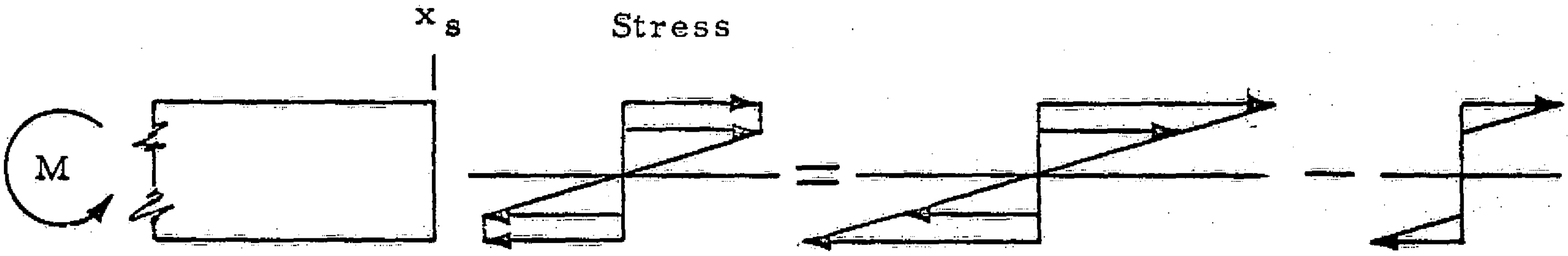
One-Dimensional Vibration - A Second Example

The torsional and longitudinal vibrations of a prismatic bar are governed by similar equations and present essentially identical problems. Longitudinal vibration in a uniform bar will be used to demonstrate the analytical technique.

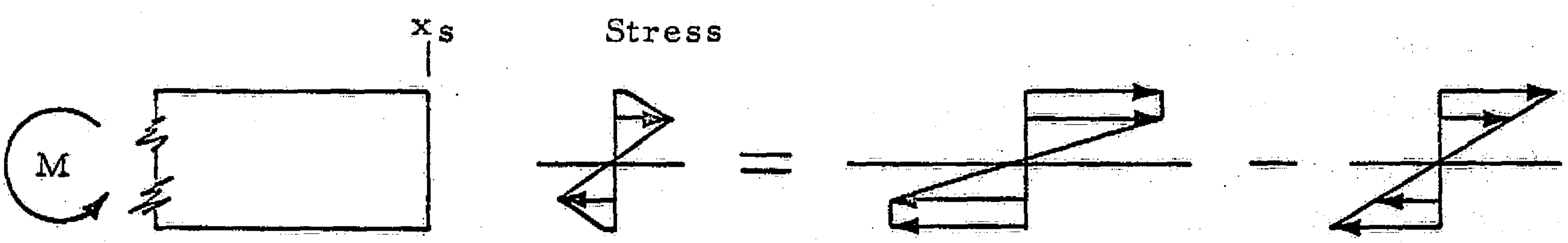
The natural frequencies f_n and modes $\phi_n(x)$ for a free-free bar are



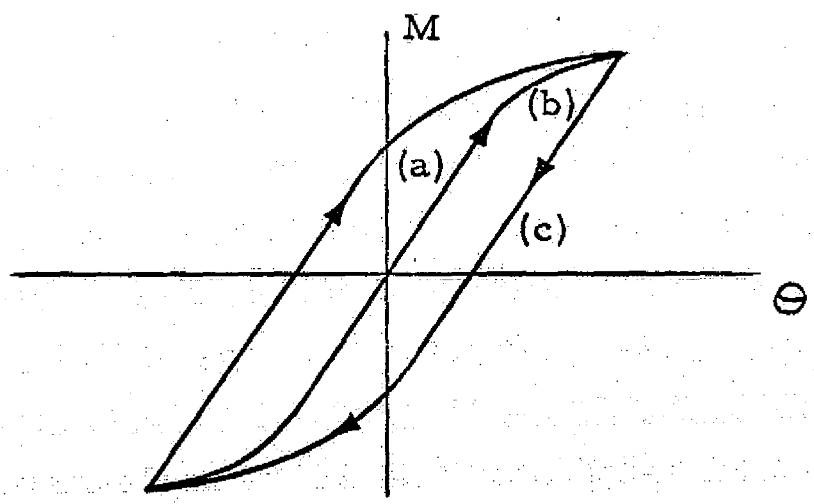
(a) $M \leqq M_s$, $\dot{M} > 0$ - No Slip



(b) $M > M_s$, $\dot{M} > 0$ - Partial Slip



(c) $\dot{M} < 0$,



(d) Moment vs. Differential Rotation at $x = x_s$

Figure 2 - Slip of Structural Joint

$$p_n = \frac{n\pi a}{x_0} \quad , \qquad a = \sqrt{E/\rho} \; ; \tag{1}$$

$$\phi_n(x) = \cos \frac{n\pi x}{x_0} = \cos {}^{p_n}/_a \, x \; , \tag{2}$$

and the solution under an axial excitation, $q_0 \sin p_n t$ @ $x = 0$ is

$$u = - \bar{u}_n \cos {}^{p_n}/_a \, x \; \cos p_n t \, . \tag{3}$$

The work done by the applied end force during a cycle in the n^{th} mode is

$$\mathcal{E} = \int_0^{2\pi/p_n} \int_0^{x_0} q_0 \sin p_n t \; u_n \sin p_n t \; p_n \, dx \, dt = \pi \bar{u}_n q_0 x_0 \tag{4}$$

The stress amplitude is

$$\bar{\sigma} = E \bar{u}_n \frac{p_n}{a} \sin {}^{p_n}/_a \, x \, , \tag{5}$$

and the total energy dissipation is

$$\begin{aligned} D_M &= A \int_0^{x_0} J_M \left(E \bar{u}_n \, {}^{p_n}/_a \sin {}^{p_n}/_a \, x \right)^M dx \\ &= L_M' \left(p_n \bar{u}_n \right)^M . \end{aligned} \tag{6}$$

Hence, from $\quad \mathcal{E} = D_M ,$

$$\bar{u}_n = \left(\frac{\pi x_0}{L_M'} \right)^{\frac{1}{M-1}} \frac{q_0^{\frac{1}{M-1}}}{p_n^{\frac{M}{M-1}}} , \tag{7}$$

which shows an identical frequency dependency as in the flexural vibration of a beam.

TRANSIENT VIBRATIONS OF LIGHTLY DAMPED STRUCTURES

General Discussion

For structures possessing small amounts of structural damping and material damping, it is convenient to divide the study of transient responses into two stages. During the first stage, transient excitation is being applied and the structural response is very much the same as that of a perfectly elastic system. That is to say that hysteresis damping, unlike viscous damping, has very little to do with the initial disturbances since damping is obtained via slight, anelastic deviation from linearity. During the second stage, which begins when excitation ceases, the response is a free vibration with a decaying amplitude, with the rate of decay governed by the damping.

More than one method of obtaining damped free-vibration solutions is available. For example, there are the approach used by Pisarenko in Reference 3, the graphical or numerical solution method in the phase plane, and that of obtaining a description of free vibration decay rate. The basis of selecting one method over another depends on the ultimate purpose of the analysis. In the present case, the logarithmic-decrement method of description is selected because it is based on the energy dissipation per cycle of free oscillation, making it compatible with the selected steady-state analysis. In particular, experimental data in free vibrations reduced by the following selected analytical method can be directly applied to forced vibration analysis.

Method of Analysis

As in the case of the steady state, the damped free-vibration analysis will depend heavily on analytical results of the undamped structure. The natural frequencies and mode shapes of the undamped structure are, again, assumed to be good representations of the actual dynamic behavior of the lightly damped structure. However, the assumption that the response is dominated by a single mode cannot be justified in free vibrations. An alternate assumption must be employed. The assumption that the damping law holds in general for arbitrary stress bias permits the superpositioning of results of separate single mode analyses, and will be used.

Knowing the mode shape and the corresponding frequency, the maximum strain energy of the structure in each mode during a cycle at the corresponding mode frequency can be computed as a volume integral of a function of the local stress amplitude. The maximum strain energy during the subsequent cycle is then smaller by the amount of dissipation, again expressible as a volume integral of a (different) function of the stress amplitude. Logarithmic decrements can, therefore, be obtained for the decay of either the energy, stress, or displacement by taking the logarithm of the ratio of successive amplitudes.

Decay of Elementary Beam Vibration - An Example

The free vibration of the beam shown in Figure 1 will again be used as an example.

In the n^{th} mode, the maximum total strain energy is

$$e(t) = \frac{b}{2} E \bar{w}_n^2 \int_{-h}^{h} z^2 dz \int_0^{x_0} (\phi_n'')^2 dx = L_E (\rho_n \bar{w}_n)^2, \tag{1}$$

where the constant $L_E = L_E(b, h, x_0, E, \rho)$. The dissipation per cycle is given by Equations (15) and (18) on Page 8 and is

$$D(t) = L_M (\rho_n \bar{w}_n)^M + L_S (\rho_n \bar{w}_n)^S \tag{2}$$

The maximum strain energy in the previous cycle is

$$e(t - 2\pi/p_n) = e(t) + D(t) = L_E (\rho_n \bar{w}_n)^2 + L_M (\rho_n \bar{w}_n)^M + L_S (\rho_n \bar{w}_n)^S,$$

and the logarithmic decrement for the energy decay is

$$\begin{aligned} \delta_e &= \ln \left\{ \frac{e(t - 2\pi/p_n)}{e(t)} \right\} = \ln \left\{ 1 + \frac{L_M}{L_E} (\rho_n \bar{w}_n)^{M-2} + \frac{L_S}{L_E} (\rho_n \bar{w}_n)^{S-2} \right\} \\ &\cong \frac{L_M}{L_E} (\rho_n \bar{w}_n)^{M-2} + \frac{L_S}{L_E} (\rho_n \bar{w}_n)^{S-2} . \end{aligned} \tag{3}$$

For stress or displacement decay the logarithmic decrement, $\delta_{\sigma, w}$, is simply $\frac{1}{2} \delta_e$. In Equation (3), the displacement amplitude $\bar{w}_n$ appears on the right-hand side. Since free vibration is being considered, $\bar{w}_n$ is not a constant (for M = 2 or S = 3) but a function of time. It is seen that the rate of decay in the case of structural damping and material damping decreases with decreasing amplitude. This phenomenon is the principal difference between hysteresis damping and viscous damping (for which δ remains a constant).

CONCLUSIONS

Simple methods of vibration analyses have been outlined for structures possessing structural damping and material damping. By the use of examples, the application of the methods are demonstrated, and the major differences between hysteresis damping and viscous damping are shown.

Damping laws for frictional dissipations at typical structural joints are not generally available at the present time. The major difficulty which looms over any practical analysis of damped structures is, therefore, not the theory, but the lack of raw physical data on joint damping behavior. The analytical methods described in this report form a foundation for establishing the format of required damping data according to which experimentation may be conducted.

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