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A NOTE ON DIELECTRIC LENSES

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In the following we consider dielectric lenses with spherical or cylindrical symmetry, in which the index of refraction N is a function of the distance r from the center of the sphere or from the axis of the cylinder. Only the geometrical properties of optical ray paths are investigated; amplitude, phase and polarization of the electromagnetic wave are neglected. In the spherical case, the problem always reduces to the consideration of ray paths in a plane passing through the center of the sphere; in the cylindrical case, we shall limit ourselves to the case of rays lying in a plane perpendicular to the cylinder axis.

The following assumptions are made:

- 1) the radius of the sphere, or of the cylinder, is equal to unity;
- 2) the index of refraction $N(r)$ of the dielectric material relative to the surrounding medium is finite, except possibly at $r=0$, and $N(1) = 1$;
- 3) $N(r)$ is continuous with its first derivative in $0 < r < 1$.

The geometry of the problem is the same for both spherical and cylindrical lenses, and is illustrated in Fig. 1. An optical ray enters the lens at P_1 with an angle of incidence α , it is smoothly deviated along the path $P_1 P_2$, and leaves the lens at P_2 . Let us indicate by R the optical length of the path $P_1 P_2$, by θ its angular extension as seen from the center $r = 0$, and by δ the deviation of the ray at P_2 from the direction of incidence; it is:

$$\delta = \theta + 2\alpha - \pi \quad (1)$$

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Following the usual procedure (see e.g. Toraldo di Francia, 1957 and Huynen, 1958), we set

$$r^2 N^2 = 1 - \xi^2, \quad (2)$$

and introduce a new function $f(\xi)$ satisfying the relation:

$$2N^2 r dr = -f(\xi) d\xi. \quad (3)$$

Then it is:

$$R = \int_0^{\cos \alpha} \frac{f(\xi) d\xi}{\sqrt{\cos^2 \alpha - \xi^2}}, \quad (4)$$

$$\theta = \sin \alpha \int_0^{\cos \alpha} \frac{f(\xi) d\xi}{(1 - \xi^2) \sqrt{\cos^2 \alpha - \xi^2}}, \quad (5)$$

$$\ln r = -\frac{1}{2} \int_0^{\xi} \frac{f(\eta) d\eta}{1 - \eta^2}. \quad (6)$$

Relations (2) and (6) give N as a function of r , in parametric form.

Formulas (4), (5) and (6) have been employed by various authors. Huynen (1958) has discussed the case in which

$$f(\xi) = a_0 + a_1 \xi, \quad (7)$$

where a_0 and a_1 are constants.

In this note, the expressions for R , θ and $\ln r$ are derived for the case in which $f(\xi)$ is an analytic function of ξ in a neighborhood of $\xi = 0$. It is then pointed out that restrictions have to be placed on the coefficients of the power series representing $f(\xi)$, if one wants to have solutions with physical significance. In particular, the limitations on a_0 and a_1 for the simple case (7) are investigated in detail.

Let us split $f(\xi)$ into its even and odd components:

$$f(\xi) = f_e(\xi) + f_o(\xi), \quad (8)$$

where:

$$f_e(\xi) = \frac{f(\xi) + f(-\xi)}{2}, \quad f_o(\xi) = \frac{f(\xi) - f(-\xi)}{2}. \quad (9)$$

Assume that for the range of values of ξ of interest to us, f_e and f_o may be represented by the uniformly convergent power series:

$$f_e(\xi) = \sum_{m=0}^{\infty} a_{2m} \xi^{2m}, \quad f_o(\xi) = \sum_{m=0}^{\infty} a_{2m+1} \xi^{2m+1}, \quad (10)$$

where a_{2m} and a_{2m+1} are constants.

Substituting expansions (10) into relations (4), (5) and (6) and integrating term by term, it is found that

$$\begin{aligned} R = & \frac{\pi}{2} \left\{ a_0 + \sum_{m=1}^{\infty} \frac{(2m-1)!!}{m! 2^m} a_{2m} (\cos \alpha)^{2m} \right\} + \\ & + \sum_{m=0}^{\infty} a_{2m+1} (\cos \alpha)^{2m+1} \sum_{h=0}^m \frac{(-1)^h}{2h+1} \binom{m}{h}, \quad (11) \\ \theta = & \frac{\pi}{2} a_0 + \left(\frac{\pi}{2} - \alpha \right) a_1 + \frac{\pi}{2} \sum_{m=1}^{\infty} \frac{a_{2m} (\cos \alpha)^{2m}}{\sum_{k=0}^m \binom{m}{k} (-\sin^2 \alpha)^k} \left\{ 1 + \frac{1}{(m-1)! 2^{m-1} \sin \alpha} \times \right. \\ & \times \sum_{h=0}^{m-1} (2h-1)!! (2m-2h-3)!! \sum_{l=1}^{m-h} \binom{m}{h+l} (-\sin^2 \alpha)^l \left. \right\} + \end{aligned}$$

$$+ \sum_{m=1}^{\infty} a_{2m+1} (\cos \alpha)^{2m} \left\{ \left(\frac{\pi}{2} - \alpha \right) \sum_{h=0}^m \binom{m}{h} (\tan \alpha)^{2m-2h} - \sum_{h=0}^{m-1} \sum_{\ell=1}^{m-h} \binom{m}{h} \frac{(-1)^{m-h-\ell}}{2(m-h-\ell)+1} (\tan \alpha)^{2\ell-1} \right\}, \quad (12)$$

$$\ln r^4 = a_0 \ln \left| \frac{1-\xi}{1+\xi} \right| + a_1 \ln(1-\xi^2) + \sum_{m=1}^{\infty} a_{2m} \left\{ \ln \left| \frac{1-\xi}{1+\xi} \right| + 2 \sum_{h=1}^m \frac{\xi^{2h-1}}{2h-1} \right\} + \sum_{m=1}^{\infty} a_{2m+1} \left\{ \ln(1-\xi^2) + 2 \sum_{h=1}^m \frac{\xi^{2h}}{2h} \right\}, \quad (13)$$

where

$$(2h-1)!! = 1 \times 3 \times 5 \times \dots \times (2h-1),$$

is the semi-factorial of $(2h-1)$ (in particular $(-1)!! = 1$).

Huynen (1958) derived an expression equivalent to (13), for the case in which $f(\xi)$ is a polynomial of degree n in ξ ; he also gave the first few terms of expansions (11) and (12).

The coefficients a_{2m} and a_{2m+1} cannot be arbitrarily chosen; in fact, the angle θ must obviously be positive for the entire range $0 \leq \alpha < \frac{\pi}{2}$. Suppose that $f(\xi)$ is a polynomial of degree n in ξ , then the fundamental limitation on the choice of the coefficients is:

$$\theta(a_0, a_1, \dots, a_n; \alpha) > 0, \text{ for all } 0 \leq \alpha < \frac{\pi}{2}, \quad (14)$$

i. e. the representative point $(a_0, a_1, \dots, a_n)$ must belong to a certain region of the $(n+1)$ -dimensional space $(a_0, a_1, \dots, a_n)$.

From formulas (2) and (3) it is seen that the relationship between N and r remains unchanged when both ξ and $f(\xi)$ change sign, i. e. when ξ and all the coefficients a_{2m} change sign (see also formula (13)). The ambiguity that thus arises in the choice of the representative point $(a_0, a_1, \dots)$ for a given lens $N=N(r)$ is eliminated by requiring that inequality (14) be satisfied.

In order to elucidate this concept, let us examine in detail the case of formula (7), for which relation (12) reduces to

$$\theta = \frac{\pi}{2} a_0 + \left(\frac{\pi}{2} - \alpha\right) a_1 ; \quad (15)$$

then relation (14) gives:

$$a_0 + a_1 > 0, \quad a_0 \geq 0, \quad (16)$$

and therefore the representative point (a_0, a_1) must belong to the shaded region of Fig. 2.

If we also require that $N(0)$ be finite and non-zero and that $(dN/dr)_{r=0}=0$, then the representative point must belong either to the straight line DGF for which

$$a_0 + a_1 = 2, \quad N(0) = 4^{a_0/4}, \quad (17)$$

or to the straight line DS for which

$$a_1 - a_0 = 2, \quad N(0) = 4^{-a_0/4}. \quad (18)$$

In Fig. 2, the point $D(0, 2)$ corresponds to the free space lens ($N=1, \delta=0$), the point $G(1, 1)$ to the Luneberg lens ($N = \sqrt{2-r^2}$; Luneberg, 1944) and the point $F(2, 0)$ to Maxwell's fish eye ($N=2/(1+r^2)$), while the point $H(2, 2)$ represents

the isotropic lens ($N = \sqrt{(2-r)/r}$, $\delta = \pi$; see e.g. Eaton, 1953).

Let us now impose the further restriction that the optical rays make less than one complete turn around the origin, i. e. that $\theta < 2\pi$; then the representative points (a_0, a_1) cannot be chosen outside the rhomboid $0AB_0C$ of Fig. 3. The permissible points of Fig. 2, which are external to the rhomboid $0AB_0C$ of Fig. 3 correspond to the so-called higher-order lenses, which were first considered by Stettler (1955).

If we require that all incident rays make at least ℓ , but less than $(\ell+1)$, complete turns around the origin before leaving the lens (ℓ is any positive integer), then $2\pi\ell \leq \theta < 2\pi(\ell+1)$, and therefore the representative point (a_0, a_1) must belong to the rhomboid $B_{\ell-1}A_{\ell}B_{\ell}C_{\ell}$, which is obtained by shifting $0AB_0C$, an amount 4ℓ along the positive a_0 axis (see Fig. 3).

That portion of the permissible region of Fig. 2 which is not covered by the rhomboids of Fig. 3 represents lenses for which the number of complete turns of the optical ray around the origin may be changed by varying the angle α of incidence.

The investigation that we developed for the simple case of formula (7) could be repeated for the more complicated cases in which $f(\xi)$ is a polynomial of degree $n = 2, 3$, etc. In the case $n = 2$, the angle θ is given by the relation:

$$\theta = \frac{\pi}{2} a_0 + \left(\frac{\pi}{2} - \alpha\right) a_1 + \frac{\pi}{2} (1 - \sin \alpha) a_2, \quad (19)$$

which follows from (12).

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Captions for Figures

- 1 Geometry for the Optical Ray Path
- 2 The Permissible Domain in the (a_0, a_1) Plane
- 3 Fundamental Lenses (shaded area) and Higher-Order Lenses





