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by

Paul B. Davenport

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Greenbelt, Maryland

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SUMMARY

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The computational formulas used in the Brouwer theory of an artificial satellite are modified by a method similar to that suggested by R. H. Lyddane to remove the singularities at circular and equatorial orbits. The Brouwer equations are also modified to reflect the use of the Vinti form for the force function and to show all factors of e'' and $\sin i''$ explicitly.

Author

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MODIFICATION OF BROUWER'S SOLUTION FOR ARTIFICIAL SATELLITES TO INCLUDE SMALL ECCENTRICITIES AND INCLINATIONS

INTRODUCTION

As Brouwer (Reference 1) states, the singularities in his formulas for small eccentricities and small inclinations are apparent since singularities do not exist in the coordinates. However, for numeric evaluation these singularities are quite real and cause erroneous results when the eccentricity or inclination is very small. Brouwer suggests that in these singular cases the formulas be modified to obtain expressions for the perturbations in coordinates. Although this approach is feasible the modifications would be quite extensive and give rise to different algorithms for different orbits. Lyddane (Reference 2) has suggested a modification which is not too far removed from Brouwer's equations and yields a single algorithm for all orbits.

Although the primary purpose of the present modification is to remove singularities at small eccentricities and inclinations, Brouwer's formulas are also modified to meet the following requirements: (1) To show factors of e'' and $\sin i''$ explicitly rather than implicitly so that limits become obvious as either of these values approach zero, (2) To change the form of the force function to the form used by Vinti as Brouwer recommends and (3) To adapt the formulas so that they are better suited for machine calculations. In view of this last requirement it is noted that many evaluations of quadratic polynomials are required in computing the perturbations. Hence, a single function for generating quadratic evaluations (such as a macro instruction) will simplify the coding of the procedure. For this reason all quadratic evaluations are shown separately.

The development of the modified formulas is one of straightforward algebraic manipulation but tedious requiring extreme caution to avoid errors. The formulas contained herein have been verified several times algebraicly by independent checks and verified numerically by programming the procedure and comparing the numeric results with existing computer programs using the formulas of Brouwer. To avoid the possibility of typographical errors the procedure was programmed for a computer using the formulas in the review copy and the results of this program were compared with previous results.

METHOD

The basic modification to Brouwer's formulas is merely to replace the classical Keplerian elements a, e, i, h, g, l by simple functions of these elements which are nonsingular at zero eccentricity or inclination. Many such functions exist, however, we have chosen the following which vary slightly from those used by Lyddane: a, $\lambda = l + g + h$, $\mu_1 = e \cos l$, $\mu_2 = e \sin l$, $v_1 = \sin i \cos h$, and $v_2 = \sin i \sin h$. The osculating values of these functions may be obtained by Taylor series expansions about the mean values and ignoring second and higher order terms. Thus,

$$a = a'' + \delta a$$

$$\lambda = \lambda'' + \delta l + \delta g + \delta h$$

$$\mu_1 = (e'' + \delta e) \cos l'' - (e'' \delta l) \sin l''$$

$$\mu_2 = (e'' + \delta e) \sin l'' + (e'' \delta l) \cos l''$$

$$v_1 = \cos h'' (\sin i'' + \delta i \cos i'') - (\sin i'' \delta h) \sin h''$$

$$v_2 = \sin h'' (\sin i'' + \delta i \cos i'') + (\sin i'' \delta h) \cos h''$$

where the operator δ represents the sum of Brouwer's long and short period terms. As Lyddane points out, the higher terms of the Taylor series are singular and ignoring them is mathematically unjustifiable. However, the Taylor series approach yields the same results as those obtained by Lyddane. Brouwer uses l' and g' in the computation of the short period terms indicating that the l'' and g'' might be used. In the present case l'' and g'' must be used since l' and g' may be ill-defined.

FORCE FUNCTION AND BASIC CONSTANTS

Using the form

$$U = \frac{\mu}{r}\left[1 - \sum_{k=2}^{5} J_k \left(\frac{R}{r}\right)^k P_k (\sin \beta)\right]$$

as the adopted force function and the basic constants

$$a_0 = a'' = \text{semi-major axis constant},$$

$$e_0 = e'' = \text{eccentricity constant},$$

$$i_0 = i'' = \text{inclination constant},$$

$$h_0 = \text{right ascension of ascending node constant},$$

$$g_0 = \text{argument of perigee constant},$$

$$l_0 = \text{mean anomaly constant},$$

$$n_0 = \mu^{1/2} a_0^{-3/2} ,$$

$$\lambda_0 = h_0 + g_0 + l_0 = \text{mean longitude constant},$$

$$R = \text{equatorial radius},$$

and comparision with Brouwer's form for the force function gives

$$k_2 = \frac{1}{2} J_2 R^2 , \qquad A_{3.0} = - J_3 R^3 ,$$

$$k_4 = - \frac{3}{8} J_4 R^4 , \qquad A_{5.0} = - J_5 R^5 .$$

The Brouwer abbreviations become

$$\eta = \left(1 - e_0^2\right)^{1/2} , \qquad \theta = \cos i_0 ,$$

$$\gamma_2 = \frac{1}{2} J_2 \left(\frac{R}{a_0}\right)^2 , \qquad \gamma_3 = - J_3 \left(\frac{R}{a_0}\right)^3 ,$$

$$\gamma_4 = - \frac{3}{8} J_4 \left(\frac{R}{a_0}\right)^4 , \qquad \gamma_5 = - J_5 \left(\frac{R}{a_0}\right)^5 ,$$

$$\gamma_2' = \frac{1}{2} J_2 \left(\frac{R}{p}\right)^2 , \qquad \gamma_3' = - J_3 \left(\frac{R}{p}\right)^3 ,$$

$$\gamma_4' = - \frac{3}{8} J_4 \left(\frac{R}{p}\right)^4 , \qquad \gamma_5' = - J_5 \left(\frac{R}{p}\right)^5 ,$$

where

$$p = a_0 \eta^2 .$$

The γ_i and γ_i' do not appear in the present development these being replaced by the J_i and the abbreviation $\gamma = - 1/2\,(R/p)$.

SECULAR TERMS

$$p_{l1} = 25\,\eta^2 + 16\,\eta - 15 , \qquad p_{l2} = - 90\,\eta^2 - 96\,\eta + 30 ,$$

$$p_{l3} = 25\,\eta^2 + 144\,\eta + 105 , \qquad q_{l1} = 3\,\theta^2 - 1 ,$$

$$q_{l2} = p_{l3}\,\theta^4 + p_{l2}\,\theta^2 + p_{l1} , \qquad q_{l3} = 3e_0^2 \left(35\,\theta^4 - 30\,\theta^2 + 3\right) ,$$

$$\Delta \dot{l} = \eta \left[J_2\, q_{l1} + \frac{1}{8} \gamma^2 \left(J_2^2\, q_{l2} - 5 J_4\, q_{l3} \right) \right] ,$$

$$p_{g1} = 25\,\eta^2 + 24\,\eta - 35 , \qquad p_{g2} = - 126\,\eta^2 - 192\,\eta + 90 ,$$

$$p_{g3} = 45\eta^2 + 360\eta + 385 , \qquad q = 5\theta^2 - 1 ,$$

$$q_{g2} = p_{g3}\theta^4 + p_{g2}\theta^2 + p_{g1} , \qquad p_{g4} = -9\eta^2 + 21 ,$$

$$p_{g5} = 126\eta^2 - 270 , \qquad p_{g6} = -189\eta^2 + 385 ,$$

$$q_{g3} = p_{g6}\theta^4 + p_{g5}\theta^{\cancel{4}2} + p_{g4}$$

$$\Delta\dot{g} = J_2 q + \frac{1}{8}\gamma^2\left(J_2^2 q_{g2} - 5 J_4 q_{g3}\right)$$

$$p_{h1} = 9\eta^2 + 12\eta - 5 , \qquad p_{h2} = 5\eta^2 + 36\eta + 35 ,$$

$$q_{h1} = -2 , \qquad q_{h2} = -4\left(p_{h2}\theta^2 - p_{h1}\right) ,$$

$$q_{h3} = 4\left(3\eta^2 - 5\right)\left(7\theta^2 - 3\right) ,$$

$$\Delta\dot{h} = \theta\left[J_2 q_{h1} + \frac{1}{8}\gamma^2\left(J_2^2 q_{h2} - 5 J_4 q_{h3}\right)\right]$$

$$\dot{l} = n_0\left(1 + 3\gamma^2\Delta\dot{l}\right) , \qquad l'' = \dot{l}\,t + l_0 ,$$

$$\dot{g} = 3 n_0\gamma^2\Delta\dot{g} , \qquad g'' = \dot{g}\,t + g_0 ,$$

$$\dot{h} = 3 n_0\gamma^2\Delta\dot{h} , \qquad h'' = \dot{h}\,t + h_0 ,$$

$$\dot{\lambda} = n_0\left[1 + 3\gamma^2\left(\Delta\dot{l} + \Delta\dot{g} + \Delta\dot{h}\right)\right] , \qquad \lambda'' = \dot{\lambda}\,t + \lambda_0 .$$

LONG-PERIOD TERMS

$$q_1 = 1 - 15\theta^2 \qquad q_2 = 1 - 7\theta^2$$

$$q_3 = 21\theta^4 - 14\theta^2 + 1 \qquad q_4 = 1 - 9\theta^2$$

$$q_5 = 75\theta^4 - 30\theta^2 + 11 \qquad q_6 = 35\theta^4 - 14\theta^2 + 3$$

$$q_7 = 135\theta^4 - 50\theta^2 + 11 \qquad q_8 = 45\theta^4 - 18\theta^2 + 5$$

$$p_1 = 3e_0^2 + 4 \qquad p_2 = 9e_0^2 + 4$$

$$p_3 = \eta^2 + \eta + 1 \qquad p_4 = -9\eta^2 - 6\eta + 7$$

$$b_1 = -\frac{1}{4}\gamma e_0 \sin i_0 \left[J_2 q_1 + 5\frac{J_4}{J_2} q_2\right]$$

$$b_2 = \frac{J_3}{J_2} q - \frac{5}{4}\frac{J_5}{J_2}\gamma^2 p_1 q_3$$

$$b_3 = \frac{35}{24}\frac{J_5}{J_2}\gamma^2 e_0^2 q_4 \sin^2 i_0$$

$$b_4 = -\frac{J_3}{J_2} q + \frac{5}{4}\frac{J_5}{J_2}\gamma^2 p_2 q_3$$

$$b_5 = \frac{\frac{1}{4}\gamma e_0 \sin i_0}{q}\left[J_2 q_5 + 5\frac{J_4}{J_2} q_6\right]$$

$$b_6 = -\frac{J_3}{J_2} q + \frac{5}{4}\frac{J_5}{J_2}\gamma^2 p_1 \left[q_3 - 6\,\frac{q_6 \sin^2 i_0}{q}\right]$$

$$b_7 = \frac{35}{72}\frac{J_5}{J_2}\gamma^2 e_0^2 \,\frac{q_7 \sin^2 i_0}{q}$$

$$b_8 = \frac{1}{8}\gamma e_0 \sin i_0 \left\{ J_2\left[\frac{2p_3 q_1}{\eta + 1} - \frac{225\theta^5 + 75\theta^4 - 80\theta^3 - 20\theta^2 + 23\theta + 1}{q(\theta + 1)}\right] - 5\frac{J_4}{J_2}\left[\frac{2p_3 q_2}{\eta + 1} - \frac{105\theta^5 + 35\theta^4 - 40\theta^3 - 12\theta^2 + 7\theta + 1}{q(\theta + 1)}\right]\right\}$$

$$b_9 = \frac{J_3}{J_2}\left[\frac{p_3}{\eta+1} + \frac{\theta}{\theta+1}\right]q - \frac{5}{4}\frac{J_5}{J_2}\gamma^2\left\{q_3\left[\frac{p_1\theta}{\theta+1} + \frac{p_4\eta^2}{\eta+1} + 9e_0^2 + 26\right] - \frac{6p_1\theta q_6 \sin^2 i_0}{q(\theta+1)}\right\}$$

$$b_{10} = \frac{35}{72}\frac{J_5}{J_2}\gamma^2 e_0^2 \sin^2 i_0 \left\{q_4\left[\frac{3p_3}{\eta+1} + 2 + \frac{\theta}{\theta+1}\right] - \frac{2\theta q_8}{q(\theta+1)}\right\}$$

$$\delta_1 e = \frac{\gamma\eta^2 \sin i_0}{q}\left(b_1 \cos 2g'' + b_2 \sin g'' + b_3 \sin 3g''\right)$$

$$\delta_1 i = -\frac{\gamma e_0 \theta}{q}\left(b_1 \cos 2g'' + b_2 \sin g'' + b_3 \sin 3g''\right)$$

$$e_0 \delta_1 l = \frac{\gamma\eta^3 \sin i_0}{q}\left(b_1 \sin 2g'' + b_4 \cos g'' - b_3 \cos 3g''\right)$$

$$\sin i_0 \delta_1 h = -\frac{\gamma e_0 \theta}{q}\left(b_5 \sin 2g'' + b_6 \cos g'' + b_7 \cos 3g''\right)$$

$$\delta_1 \lambda = \frac{\gamma e_0 \sin i_0}{q}\left(b_8 \sin 2g'' + b_9 \cos g'' + b_{10} \cos 3g''\right)$$

SHORT-PERIOD TERMS

$$E'' - e_0 \sin E'' = l'', \qquad f'' = \tan^{-1}\left(\frac{\eta \sin E''}{\cos E'' - e_0}\right)$$

$$\epsilon = 1 + e_0 \cos f''$$

$$a = a_0\left\{1 + 2J_2\frac{\gamma^2}{\eta^2}\left[q_{l1}(\epsilon-\eta)\left(\epsilon^2 + \epsilon\eta + \eta^2\right) + 3\epsilon^3 \sin^2 i_0 \cos(2g'' + 2f'')\right]\right\}$$

$$\delta_2 e = J_2\gamma^2\left\{q_{l1}\left[\frac{e_0 p_3}{\eta+1} + 3\cos f'' + 3e_0\cos^2 f'' + e_0^2\cos^3 f''\right]\right.$$

$$+ \sin^2 i_0\left[3\left(e_0 + 3\cos f'' + 3e_0\cos^2 f'' + e_0^2\cos^3 f''\right)\cos(2g'' + 2f'')\right.$$

$$\left.\left. - \eta^2\left(3\cos(2g'' + f'') + \cos(2g'' + 3f'')\right)\right]\right\}$$

$$\delta_2 i = J_2 \gamma^2 \theta \sin i_0 \left\{ 3 \cos (2g'' + 2f'') + e_0 \left[3 \cos (2g'' + f'') + \cos (2g'' + 3f'') \right] \right\}$$

$$e_0 \, \delta_2 l = -\frac{1}{2} J_2 \gamma^2 \eta \left\{ 2g_{l1} \left(\epsilon^2 + \epsilon + \eta^2 \right) \sin f'' \right.$$

$$\left. + 3 \sin^2 i_0 \left[\left(\eta^2 - \epsilon - \epsilon^2 \right) \sin (2g'' + f'') + \left(\epsilon^2 + \epsilon + \frac{1}{3} \eta^2 \right) \sin (2g'' + 3f'') \right] \right\}$$

$$\sin i_0 \, \delta_2 h = - J_2 \gamma^2 \theta \sin i_0 \left\{ 6 \left(f'' - l'' + e_0 \sin f'' \right) \right.$$

$$\left. - 3 \sin (2g'' + 2f'') - e_0 \left[3 \sin (2g'' + f'') + \sin (2g'' + 3f'') \right] \right\}$$

$$\delta_2 \lambda = J_2 \gamma^2 \left\{ \left(15 \theta^2 - 6 \theta - 3 \right) (f'' - l'') + e_0 \left[q_{l1} \frac{\left(\epsilon^2 + \epsilon + \eta^2 \right)}{\eta + 1} \right. \right.$$

$$\left. + \left(15 \theta^2 - 6 \theta - 3 \right) \right] \sin f'' + \frac{3}{2} e_0 \sin^2 i_0 \left[\frac{\eta^2 - \epsilon - \epsilon^2}{\eta + 1} + \frac{5 \theta + 3}{\theta + 1} \right] \sin (2g'' + f'')$$

$$+ \frac{1}{2} e_0 \sin^2 i_0 \left[\frac{3 \left(\epsilon^2 + \epsilon + \frac{1}{3} \eta^2 \right)}{\eta + 1} + \frac{5 \theta + 3}{\theta + 1} \right] \sin (2g'' + 3f'')$$

$$\left. + \frac{3}{2} \frac{(5 \theta + 3)}{\theta + 1} \sin^2 i_0 \sin (2g'' + 2f'') \right\}$$

OSCULATING ELEMENTS

The formula for the semi-major axis, a, is given above. The remaining osculating elements are obtained by the following:

$$\delta e = \delta_1 e + \delta_2 e , \qquad \sin i_0 \, \delta h = \sin i_0 \, \delta_1 h + \sin i_0 \, \delta_2 h ,$$

$$\delta i = \delta_1 i + \delta_2 i , \qquad e_0 \, \delta l = e_0 \, \delta_1 l + e_0 \, \delta_2 l ,$$

$$\delta\lambda = \delta_1\lambda + \delta_2\lambda ,$$

$$\mu_1 = \left(e_0 + \delta e\right) \cos l'' - e_0\, \delta l \sin l'' ,$$

$$\mu_2 = \left(e_0 + \delta e\right) \sin l'' + e_0\, \delta l \cos l'' ,$$

$$v_1 = \cos h'' \left(\sin i_0 + \delta i \cos i_0\right) - \sin i_0\, \delta h \sin h'' ,$$

$$v_2 = \sin h'' \left(\sin i_0 + \delta i \cos i_0\right) + \sin i_0\, \delta h \cos h'' ,$$

$$e = \sqrt{\mu_1^2 + \mu_2^2} ,$$

$$i = i_0 + \delta i ,$$

$$h = \tan^{-1}\left(\frac{v_2}{v_1}\right) ,$$

$$l = \tan^{-1}\left(\frac{\mu_2}{\mu_1}\right) ,$$

$$g = \lambda'' + \delta\lambda - h - l .$$

The coordinates and velocity components are then computed from the osculating elements in the usual manner.

CONCLUSION

The formulas contained herein are valid for all eccentricities and inclination (except inclinations near the critial inclination and $i = \pi$) yielding the same results (to the order of J_2) as Brouwer's formulas when neither the eccentricity nor the inclination are small. When the eccentricity is small l and g may be ill-defined, and when the inclination is small h, l, and g may be ill-defined. These cases cause no numeric problems, however, since $\lambda = g + l + h$ is always well defined.

REFERENCES

1. Brouwer, D. 1959, The Astronomical Journal, 64,378.

2. Lyddane, R. H. 1963, ibid. 68,555.