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X-513-65-100

NASA TM X-55205

N65-21668
(ACCESSION NUMBER) (THRU)
28 1
(PAGES) (CODE)
TMX-55205 07
(NASA CR OR TMX OR AD NUMBER) (CATEGORY)

FACILITY FORM 602

THE RANGE RATE ERROR DUE TO THE AVERAGING TECHNIQUES OF DOPPLER MEASUREMENTS

BY
B. KRUGER

GPO PRICE $ ______

OTS PRICE(S) $ ______

Hard copy (HC) $2.00

Microfiche (MF) .50

MARCH 5, 1965

NASA — GODDARD SPACE FLIGHT CENTER —
GREENBELT, MARYLAND

X-513-65-100

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TABLE OF CONTENTS

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ABSTRACT

2/668

The averaging error in slant range rate measurements, based on Doppler frequency integration over a finite time, has been analyzed. This error is of such a magnitude that a correction is required. For example, in a near circular earth orbit the error is 0.12 m/s for 370 km (200 nm) orbit altitude and 0.5 m/s for 185 km (100 nm) orbit altitude if an integration time of 1 sec. is used. Means of correction, added noise due to correction, and the residual error after correction are discussed. Special emphasis is given to the three point correction, which is shown to be adequate in most applications. In the above case the errors are reduced to 0.0001 m/s and 0.0024 m/s respectively.

THE RANGE RATE ERROR DUE TO THE AVERAGING TECHNIQUES OF DOPPLER MEASUREMENTS

INTRODUCTION

A finite time is always required for frequency measurements. The Doppler frequency shift can therefore not be measured instantaneously. Instead, the Doppler shift is integrated over a finite time T. This integral is proportional to a finite increment in range ΔR. The Doppler measurement, therefore, yields the average range rate $\dot{R}_a$ during the time interval T

$$\dot{R}_a = \frac{1}{T} \int_t^{t+T} \dot{R} \, dt = \frac{\Delta R}{T} \tag{0.1}$$

The actual range rate at time t* is $\dot{R}$. The difference $\Delta\dot{R}$ between $\dot{R}$ and $\dot{R}_a$

$$\Delta\dot{R} = \dot{R} - \dot{R}_a \tag{0.2}$$

and suitable means for correction are discussed in this paper. The omission of the correction for $\Delta\dot{R}$ will introduce an error in the interpretation of the measurements and this error is referred to as the averaging error in this paper.

1. THE AVERAGING RANGE RATE ERROR

The averaging slant range rate error $\Delta\dot{R}$ can be calculated by expanding R in a Taylor series [1] around $t = t_0$

*All time in this paper refers to the spacecraft transponder rather than to the ground receiver in order to avoid corrections for propagation delay in this paper. The effects of propagation delay will be treated in a coming paper.

$$R(t_0 + \Delta t) = R_0 + \dot{R}_0 \Delta t + \frac{1}{2}\ddot{R}_0 \Delta t^2 + \frac{1}{6}\dddot{R}_0 \Delta t^3 + \ldots \tag{1.1}$$

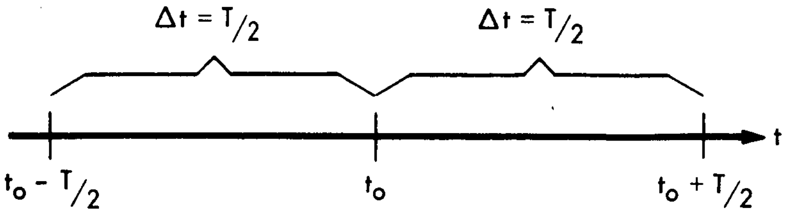

Figure 1—The Integration Interval

where

$$\left.\begin{aligned} R_0 &= R \\ \dot{R}_0 &= \dot{R} \end{aligned}\right\} \quad t = t_0 \quad \text{etc.}$$

If t_0 is the middle of integration interval T, then $\Delta T = T/2$ as shown in Figure 1. The change in R during T is

$$\Delta R = R\left(t_0 + \frac{T}{2}\right) - R\left(t_0 - \frac{T}{2}\right)$$

and applying equation (1.1) we obtain

$$\Delta R = R_0 + \dot{R}_0 \frac{T}{2} + \frac{1}{2}\ddot{R}_0 \left(\frac{T}{2}\right)^2 + \frac{1}{6}\dddot{R}_0 \left(\frac{T}{2}\right)^3 + \ldots$$

$$- \left(R_0 - \dot{R}_0 \frac{T}{2} + \frac{1}{2}\ddot{R}_0 \left(\frac{T}{2}\right)^2 - \frac{1}{6}\dddot{R}_0 \left(\frac{T}{2}\right)^3 + \ldots\right)$$

or

$$\Delta R = \dot{R}_0 T + \frac{1}{24}\dddot{R}_0 T^3 + \ldots \tag{1.2}$$

and hence

$$\dot{R}_a = \frac{\Delta R}{T} = \dot{R}_0 + \frac{1}{24}\dddot{R}_0 T^2 + \ldots \tag{1.3}$$

Using $\dot{R}_a$ instead of $\dot{R}$ means that an error $\Delta\dot{R}$ is introduced

$$\Delta\dot{R} = \dot{R} - \dot{R}_a = -\frac{1}{24}\dddot{R}_0 T^2 - \dots \tag{1.4}$$

It is seen from the above derivation, that all even derivatives $\ddot{R}_0$, d^4R/dt^4, etc. disappear in equation (1.4) if and only if, t_0 is chosen in the middle of the integration interval T. Also the coefficient for the third derivative in equation (1.4) has a minimum for this choice of t_0. We therefore conclude that interpreting $\dot{R}_a$ as $\dot{R}$ in the middle of the integration interval minimizes $\Delta\dot{R}$.

The averaging error is of such a magnitude that a correction is required in many cases. A complete correction is, of course, not possible. A practical limitation is that only one correction term is desirable, i.e., only the third derivative is used for correction and the fifth and higher order derivatives are neglected. A theoretical limitation is that all derivatives are contaminated with noise and a correction for $\Delta\dot{R}$ will therefore increase the noise in $\dot{R}$.

A practical important case is the circular orbit. A circular orbit will therefore be used as a vehicle to demonstrate the magnitude of $\Delta\dot{R}$ and of the residual error due to the neglect of the fifth and higher order derivatives. An overhead pass is chosen, because this is the worst case.

2. THE AVERAGING ERROR IN A CIRCULAR ORBIT*

A circular overhead orbit is shown in Figure 2. From Figure 2 we obtain

$$R^2 = r^2 + (r+h)^2 - 2r(r+h)\cos\theta \tag{2.1}$$

$$(r+h)^2 = r^2 + R^2 + 2rR\sin\epsilon \tag{2.2}$$

$$\frac{\sin\theta}{R} = \frac{\cos\epsilon}{r+h} \tag{2.3}$$

From these equations $d^3R/d\theta^3$ can be derived as shown in Appendix I.

*This case is treated in Ref. (2), but due to analytical errors the results are not applicable.

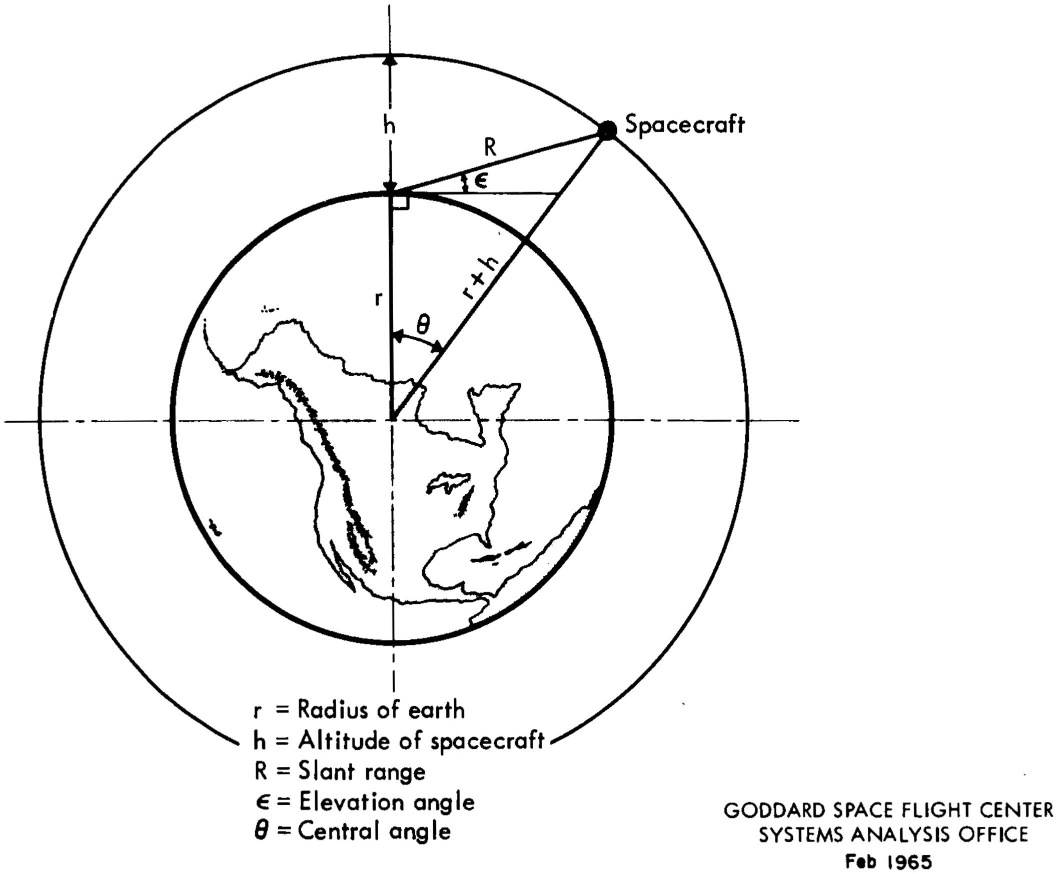

Figure 2–Orbiting Geometry

In a circular orbit $\dot{\theta}$ is constant, therefore,

$$\dddot{R} = \frac{d^3R}{d\theta^3}\,\dot{\theta}^3$$

where $\dddot{R}$ and $\dot{\theta}$ are derivatives with respect to time. $\Delta\dot{R}$ is then, using the results from Appendix I,

$$\Delta\dot{R} \approx \frac{\dot{\theta}^3\ T^2\ r^3}{8\ h^2\left(1 + \frac{h}{r}\right)}\left(\sin^4 \epsilon + \frac{2h}{r}\sin^2 \epsilon\right)\cos \epsilon \tag{2.4}$$

provided

$$\sin^2 \epsilon >> \frac{2h}{r}$$

For $\epsilon = 0$

$$\Delta \dot{R} = \frac{\dot{\theta}^3 \, T^2 \, r}{24} \tag{2.5}$$

With

$$\dot{\theta} = \frac{\mu^{1/2}}{(r + h)^{3/2}} \tag{2.6}$$

where μ = the gravitational parameter, we obtain

$$\Delta \dot{R} \approx \frac{\mu^{3/2} \, T^2}{8 \, h^2 \, r^{3/2} \left(1 + \frac{11}{2} \frac{h}{r}\right)} \left(\sin^4 \epsilon + \frac{2h}{r} \sin^2 \epsilon\right) \cos \epsilon \tag{2.7}$$

provided $\sin^2 \epsilon >> \frac{2h}{r}$.

$|d^3R/d\theta^3|_{max}$ is calculated in Appendix I, equation (A-11). Using this value, we obtain

$$|\Delta \dot{R}|_{max} = \frac{2 \, \mu^{3/2} \, T^2}{25 \sqrt{5} \, h^2 \, r^{3/2} \left(1 + 3 \frac{h}{r}\right)} \tag{2.8}$$

$|\Delta \dot{R}|_{max}$ occurs for

$$\sin \epsilon = \sqrt{\frac{4}{5} - \frac{h}{5r}} \tag{2.9}$$

Let us use the notation $\Delta\dot{R}_0$ for the averaging error for a "standard" circular earth orbit with

$$h_0 = 1.85 \text{ km} \quad (= 100 \text{ nm})$$

and a "standard" integration time

$$T_0 = 1\,\text{s}$$

The graph $\Delta\dot{R}_0$ vs. the elevation angle ϵ is shown in Figure 3. Inspecting Equation (2.7), we find that

$$\Delta\dot{R} \sim T^2$$

and

$$\Delta\dot{R} \sim \frac{1}{h^2} \left(\text{except for } \frac{h}{r} \text{ correction terms}\right)$$

For other h and T values than the "standard" values, we thus have

$$\Delta\dot{R} \approx \left(\frac{h_0}{h} \cdot \frac{T}{T_0}\right)^2 \Delta\dot{R}_0 \tag{2.10}$$

Figure 3 shows $|\Delta\dot{R}_0|_{max} = 0.5$ m/s. $|\Delta R_0|_{max}$ occurs at approximately 63° elevation angle, which is in the region most important for tracking.

A correction for $\Delta\dot{R}$ will be required in many applications. If only the third derivative is used, as in Equation (1.4), a residual error will remain. The residual error is analyzed in the next chapter.

3. THE RESIDUAL ERROR

Terms containing d^5R/dt^5 and higher derivatives have been neglected in derivation of equation (2.4) and (2.8). In order to estimate the residual error, we will find an upper bound for the magnitude of the term containing d^5R/dt^5. Including one more term in equation (1.4) yields

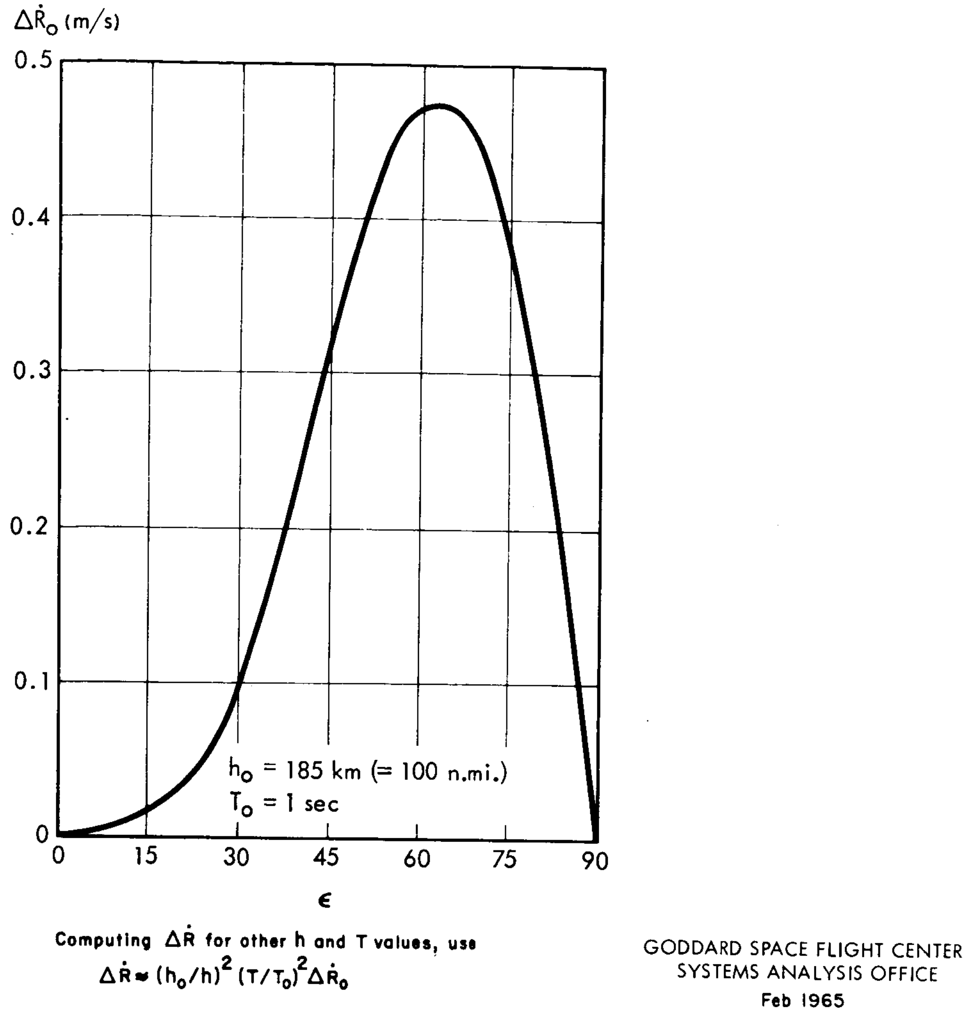

Figure 3—Averaging error $\Delta\dot{R}$ vs elevation angle ϵ for circular earth over head orbit with h_0 = 185 km (=100n. mi.) integration time $T_0 = 1$ sec.

$$\Delta\dot{R} = -\frac{1}{24}\frac{d^3R}{dt^3} - \frac{1}{1920}\frac{d^5R}{dt^5} - \cdots \tag{3.1}$$

Using δ_5 for the term containing d^5R/dt^5 we obtain

$$\delta_5 = -\frac{1}{1920}\frac{d^5R}{dt^5} \tag{3.2}$$

In Appendix I, Equation (A-11), it is shown that

$$\left|\frac{d^5R}{d\theta^5}\right| \leq 23.6 \frac{r^5}{h^4}$$

for an overhead pass. Hence

$$|\delta_5| \leq \frac{\dot{\theta}^5\ r^5\ T^4}{81.4\ h^4\ r^{5/2}\left(1 + \frac{15}{2}\frac{h}{r}\right)} \tag{3.3}$$

or

$$|\delta_5| \leq \frac{\mu^{5/2}\ T^4}{81.4\ h^4\ r^{5/2}\left(1 + \frac{15}{2}\frac{h}{r}\right)} \tag{3.4}$$

For an earth orbit with h = 185 km, the residual error is

$$|\delta_5| \leq 2.7 \cdot 10^{-4}\ T^4\ m/s \tag{3.5}$$

The upper bounds $|\delta_5|_{max}$ vs. T for overhead earth orbits are shown in Figure 4 with h as parameter. It is seen that δ_5 can be neglected for T = 1 sec. But for T = 3 sec., δ_5 is more than 0.02 m/s for h = 185 km. Figure 4 thus clearly indicates the upper limit for the integration time. These values are, however, very optimistic because of the following reasons.

The derivation of the residual error in this chapter was based on the assumption that d^3R/dt^3 is known accurately. d^3R/dt^3 is generally not known, but can be determined from a number of consecutive measurements. A minimum number of three measuring points is needed in order to determine d^3R/dt^3. Three consecutive measuring points can therefore be used for the correction of the averaging error. d^3R/dt^3 is, however, only determined to a certain accuracy by three (or any other finite number) measuring points and an increase in the residual error can therefore be expected. In the next chapter the residual error for the three point correction is analyzed and found to be 9 times larger than the theoretical residual error as given by equations (3.3) through (3.5) and as shown in Figure 4. The residual error for the three point correction is shown in Figure 7 and 8.

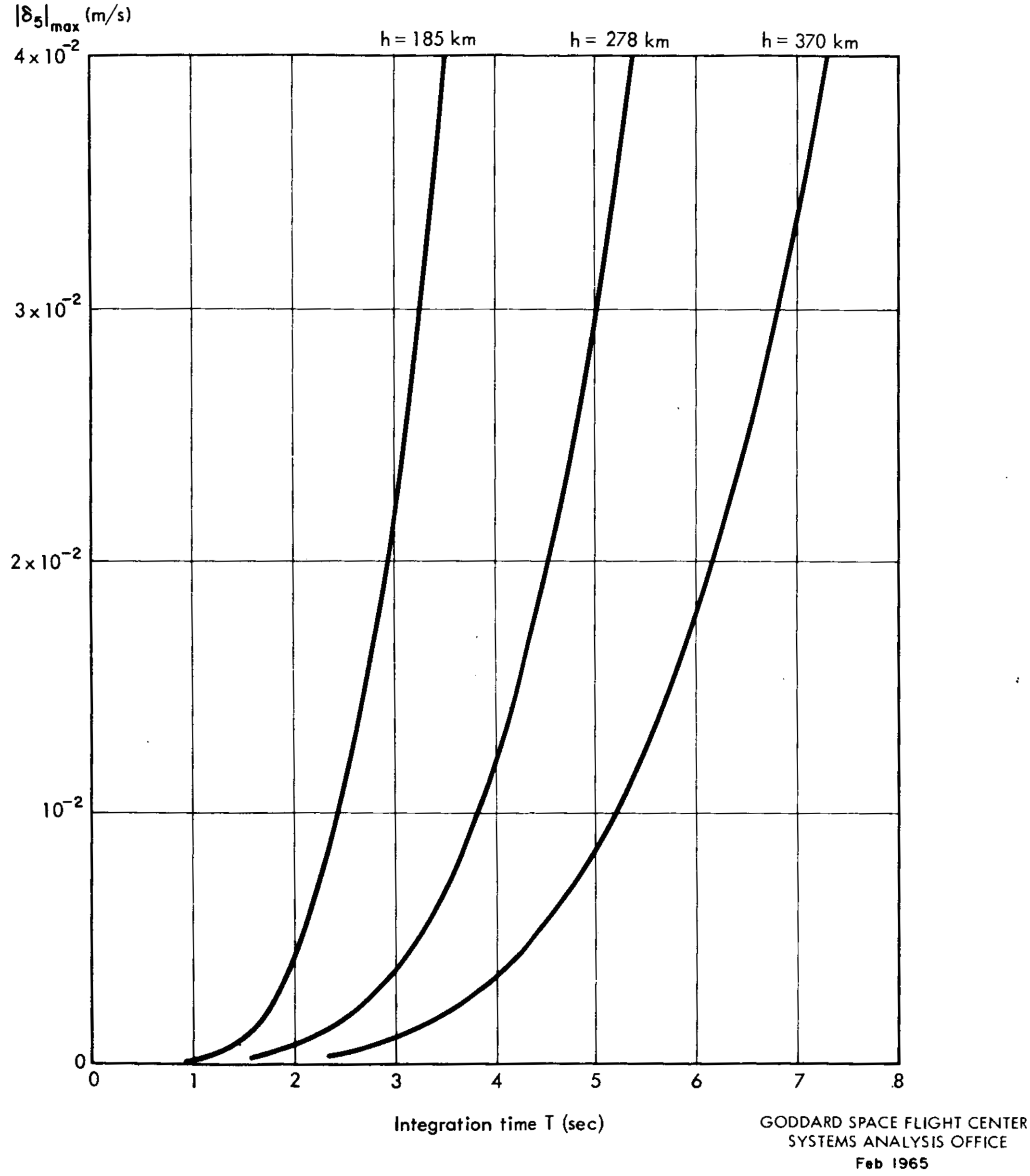

Figure 4–Max. residual error $|\delta_5|_{max}$ after correction for d^3R/dt^3. $|\delta_5|_{max}$ is due to neglecting d^5R/dt^5. d^3R/dt^3 is assumed to be accurately known.

4. THREE POINT CORRECTION

It was assumed in the previous chapters that the third derivative of R is known. This is generally not the case, but from three consecutive measurements $\dddot{R}$ can be calculated and $\Delta\dot{R}$ can be corrected for. The correction equation, the increase in noise, and the residual error for the three point correction are discussed in this chapter.

The Correction Equation

The equation for the three point correction can be obtained in the following way. It is assumed that three consecutive measurements are made with repetition time T_1 as shown in Figure 5. The measured average range rates are $\dot{R}_{a1}$, $\dot{R}_{a2}$, and $\dot{R}_{a3}$ respectively. The integration time is assumed to be T for all three measurements. This is not strictly true for the N-counting technique (3) in which T is a function of $\dot{R}_a$. During three consecutive measurements the variation in T is, however, small enough to be neglected. It is shown in Appendix II, Equation (A-33) that under the above conditions the range rate at time $(t_0 + \tau)$ is

$$\dot{R}(t_0 + \tau) = \left(-\frac{T^2}{24\,T_1^2} - \frac{\tau}{2T_1} + \frac{\tau^2}{2\,T_1^2}\right)\dot{R}_{a1}$$

$$+ \left(1 + \frac{T^2}{12\,T_1^2} - \frac{\tau^2}{T_1^2}\right)\dot{R}_{a2}$$

$$+ \left(-\frac{T^2}{24\,T_1^2} + \frac{\tau}{2T_1} + \frac{\tau^2}{2T_1^2}\right)\dot{R}_{a3} + \delta_4 + \delta_5 \qquad (4.1)$$

where δ_4 and δ_5 are the residual errors due to the neglection of fourth and fifth order terms. It is also shown in Appendix II that

$$\delta_4 = \frac{1}{6}\,\frac{d^4R}{dt^4}\,T_1^3\left(-1 - \frac{T^2}{4T_1^2} + \frac{\tau^2}{T_1^2}\right)\frac{\tau}{T_1} \qquad (4.2)$$

and

$$\delta_5 = \frac{1}{24}\frac{d^5R}{dt^5}T_1^4\left[\frac{T^2}{12\,T_1^2} + \frac{7\,T^4}{240\,T_1^2} - \left(1 + \frac{T^2}{2\,T_1^2}\right)\frac{\tau^2}{T_1^2} + \frac{\tau^4}{T_1^4}\right] \tag{4.3}$$

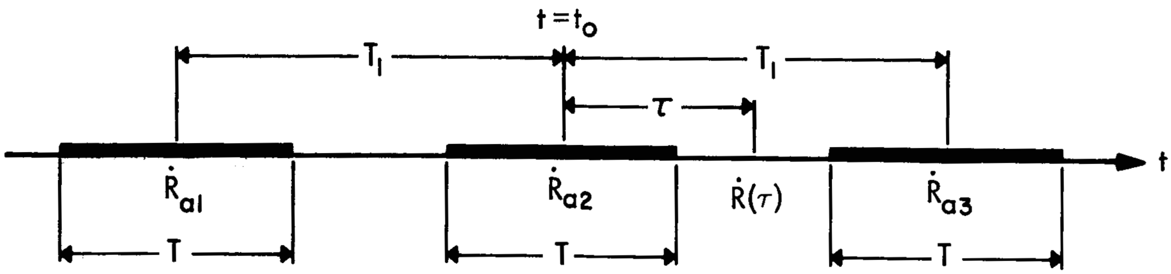

Figure 5–Three point correction based on three consecutive measurements

The Residual Error

It is shown in Appendix I that for a circular overhead path

$$\left|\frac{d^4R}{d\theta^4}\right| \leq 3\,\frac{r^4}{h^3} \tag{4.4}$$

and

$$\left|\frac{d^5R}{d\theta^5}\right| \leq 23.6\,\frac{r^5}{h^4} \tag{4.5}$$

For a circular orbit $d\theta/dt$ is constant and therefore

$$\frac{d^nR}{dt^n} = \frac{d^nR}{d\theta^n}\cdot\left(\frac{d\theta}{dt}\right)^n \tag{4.6}$$

Using Equation (2.6)

$$\frac{d\theta}{dt} = \frac{\mu^{1/2}}{(r + h)^{3/2}}$$

and Equations (4.2), (4.3), (4.4), (4.5), and (4.6), the upper bounds for the residual errors can be obtained.

$$|\delta_4| \leq \frac{1}{2} \frac{\mu^2 T_1^3}{h^3 r^2 \left(1 + 6 \frac{h}{r}\right)} \cdot \left| 1 + \frac{T^2}{4 T_1^2} - \frac{\tau^2}{T_1^2} \right| \cdot \left| \frac{\tau}{T_1} \right| \tag{4.7}$$

$$|\delta_5| \leq 0.98 \frac{\mu^{5/2} T_1^4}{h^4 r^{5/2} \left(1 + 7.5 \frac{h}{r}\right)} \cdot \left| \frac{T^2}{12 T_1^2} + \frac{7 T^4}{240 T_1^4} - \left(1 + \frac{T^2}{2 T_1^2}\right) \frac{\tau^2}{T_1^2} + \frac{\tau^4}{T_1^4} \right| \tag{4.8}$$

which also can be written

$$|\delta_4| \leq |\delta_4|_{max}$$

$$|\delta_5| \leq |\delta_5|_{max}$$

For a circular earth orbit with $\mu = 3.99 \cdot 10^{14}$ m^3/s^2, $r = 6.38 \cdot 10^6$ m, $T_1 = 1s$ and $h = 185$ km we obtain

$$|\delta_4|_{max} = 0.263 \left| 1 + \frac{T^2}{4T_1^2} - \frac{\tau^2}{T_1^2} \right| \cdot \left| \frac{\tau}{T_1} \right| \tag{4.9}$$

$$|\delta_5|_{max} = 0.0212 \left| \frac{T^2}{12 T_1^2} + \frac{7 T^4}{240 T_1^4} - \left(1 + \frac{T^2}{2 T_1^2}\right) \frac{\tau^2}{T_1^2} + \frac{\tau^4}{T_1^4} \right| \text{ m/s} \tag{4.10}$$

Equations (4.9) and (4.10) show the relationship between the residual errors and the times T, T_1, and τ. The equations can also be shown graphically. Figure 6 shows the residual maximum errors $|\delta_4|_{max}$ and $|\delta_5|_{max}$ for the case that $\dot{R}$ is referred to the time τ. For convenient reference, the total measuring interval is shown at the bottom of Figure 6. It is seen, that if $\tau = 0$, i.e. $\dot{R}$ is referred to the middle of the measuring interval, then $|\delta_4|_{max} = 0$. For other τ values $|\delta_4|_{max}$ increases and reaches a maximum at approximately $\tau = \pm 0.6\ T_1$ and decreases to zero at approximately $\tau = \pm 1.1\ T_1$. The solid graphs are for the case $T = T_1$ and the dotted for $T = 1/2\ T_1$. The graphs are computed for a circular orbit and an overhead pass with an altitude of 185 km and $T_1 = 1$ sec.

It is seen from Figure 6 that $\dot{R}$ should be referred to the middle of the measuring interval, i.e. $\tau = 0$, because $|\delta_4|_{max}$ is zero. The two other zeros at $\tau = \pm 1.1\ T_1$ are functions of T and the slopes of the curves are very steep in the neighborhood of these two zeros. Small variations in T or T_1 will therefore produce relatively large residual errors. If, for instance, N-counting techniques [3] are employed, considerable variations in T have to be expected.

We see also from Figure 6 that there is some residual error $|\delta_5|$ for $\tau = 0$. Equation (4.8) shows that $|\delta_5|_{max}$ is proportional to $(T_1)^4$. The rapid increase of $|\delta_5|_{max}$ for increasing T_1 is shown in Figure 7 for $\tau = 0$, $T = T_1$ and h as a parameter. For h = 185 km, $|\delta_5|_{max}$ is $4 \cdot 10^{-2}$ m/s if the integration time T is 2 seconds. This residual error exceeds already the accuracy claimed for some range rate systems. Figure 8 shows $|\delta_5|_{max}$ vs. h with T_1 as parameter for the case $T = T_1$. The rapid increase of $|\delta_5|_{max}$ for low values of h is evident from this graph.

The R. M. S. Noise

The total R. M. S. noise η_t after correction is obtained by adding the noise η from each measurement in a R. M. S. sense. If we write Equation (4.1) in the form

$$\dot{R}(\Delta t) = \sum_{\nu} a_\nu \dot{R}_{a\nu} + \delta_4 + \delta_5 \tag{4.11}$$

where

$$\sum_{\nu} a_\nu = 1 \tag{4.12}$$

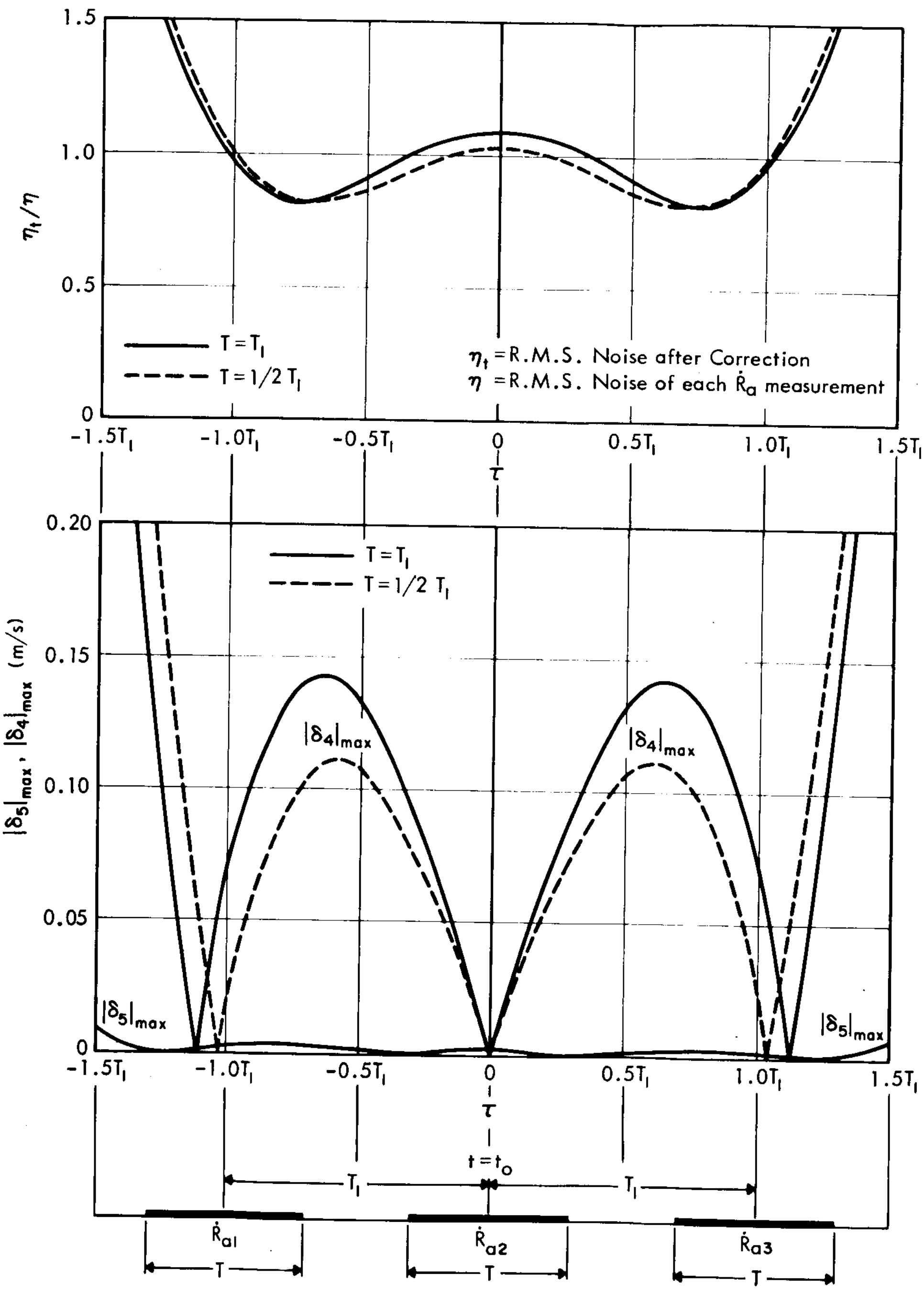

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Figure 6—Three point correction of $R(t_0 + \tau)$. Relative noise η_t/η and max. residual errors $|\delta_4|_{max}$ and $|\delta_5|_{max}$ after correction due to neglecting 4th and 5th order terms.

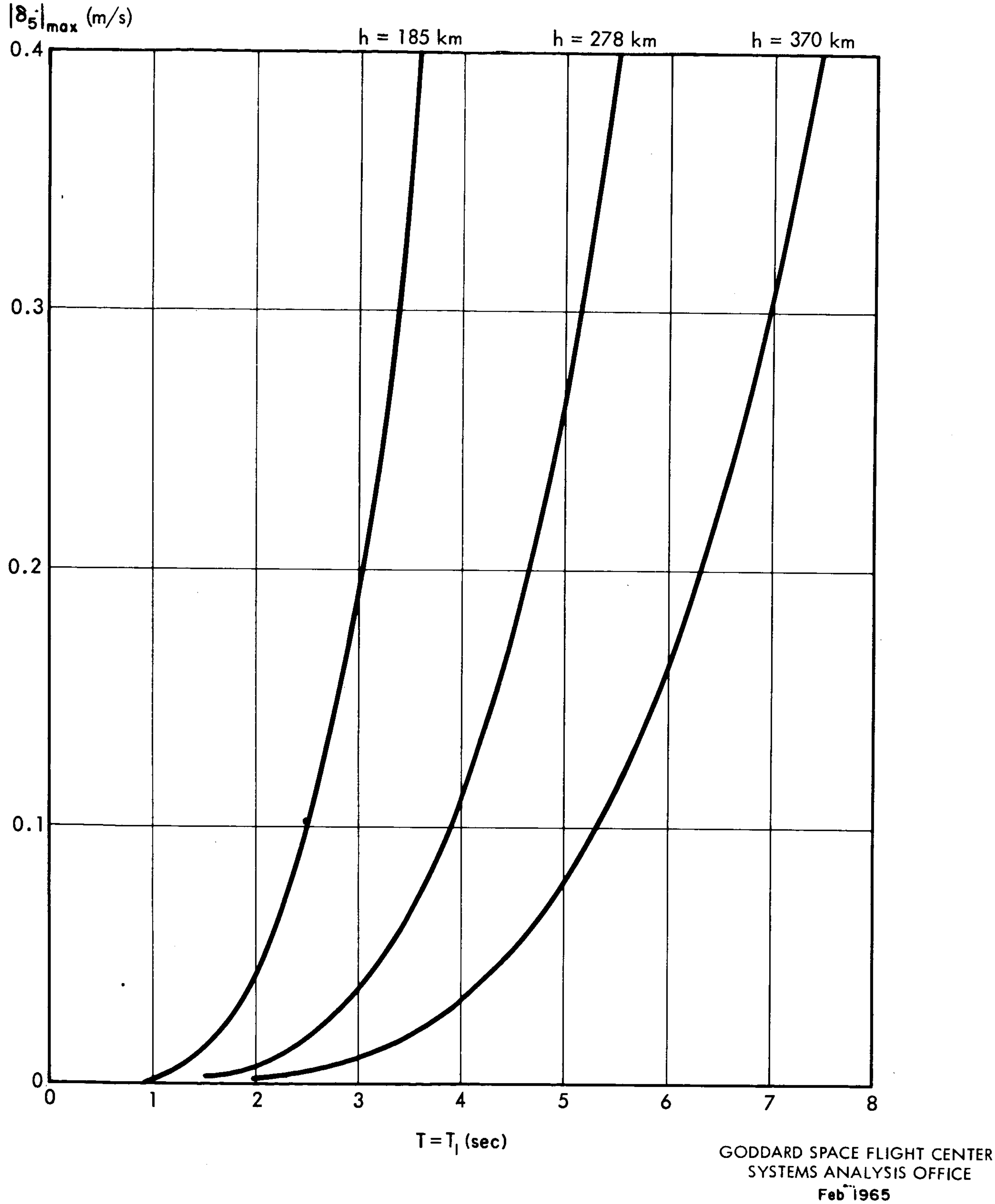

Figure 7—Three point correction of $\dot{R}(t_0)$ max. residual error $|\delta_5|_{max}$ due to neglecting 5th and higher order terms. $T = T_1$.

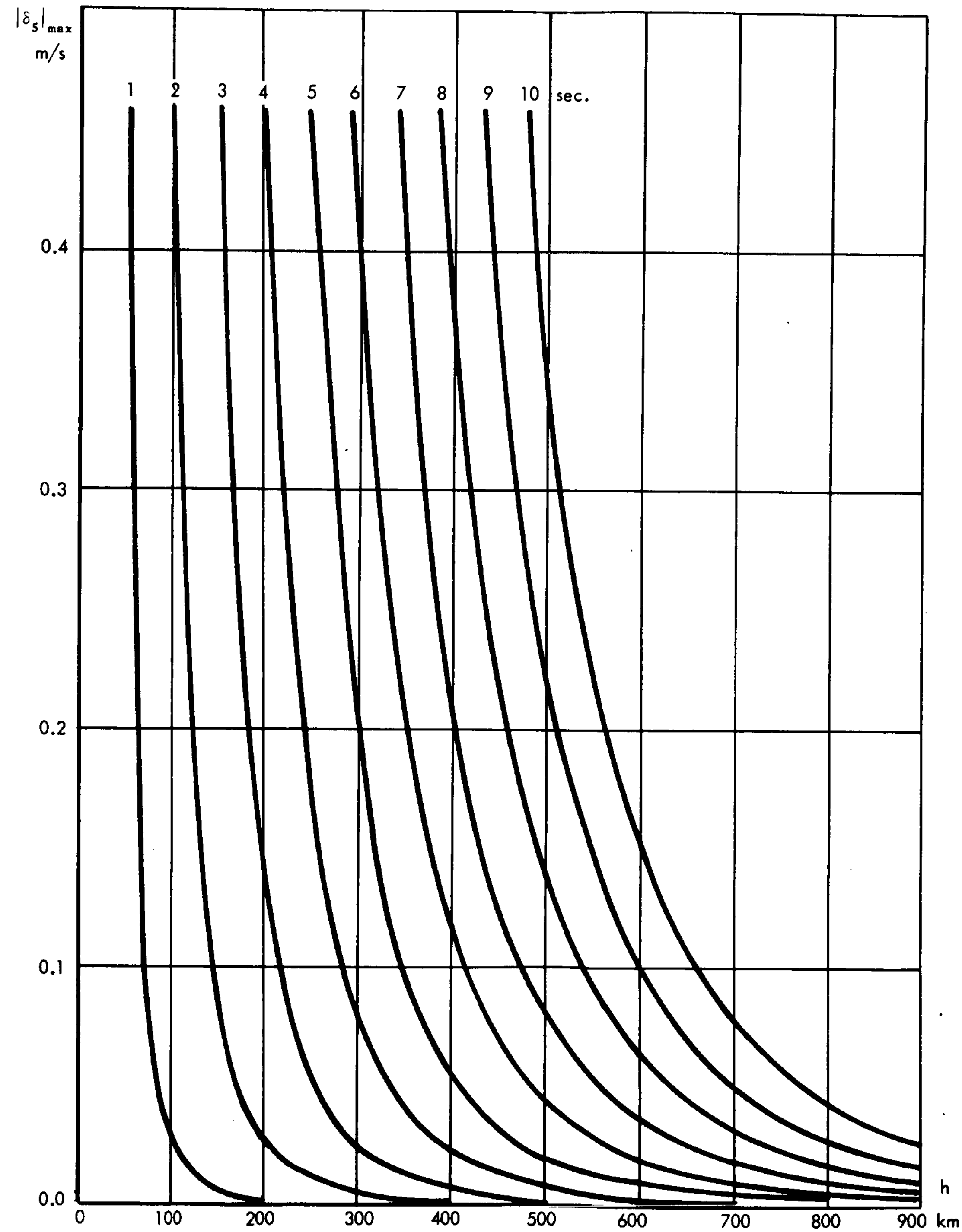

Figure 8–Maximum residual error $|\delta_5|_{max}$ after three point correction vs. altitude h for the case of a circular orbit and an overhead pass. $\dot{R}$ is referred to the mid point of the measuring intervals (τ=O) and the integration time is equal to the repetition time of the measurements ($T = T_1$).

we obtain the total noise

$$\eta_t = \eta \sqrt{\Sigma a_\nu^2} \qquad (4.13)$$

assuming that each measurement has the same noise η. The ratio η_t/η is shown in Figure 6 for two values of T: $T = T_1$ and $T = 1/2\ T_1$. It is seen that the noise increases 8.6% for $T = T_1$ and $\tau = 0$. The decrease of R. M. S. noise of 19% at $\tau \approx 0.7\ T_1$ can generally not be utilized because of the large residual error $|\delta_4|$.

5. CONCLUSIONS

It has been shown that the averaging error for a circular earth orbit is 0.5 m/s for 1 second integration time and 185 km orbit altitude. The averaging error is proportional to the square of the integration time and could therefore be reduced by shortening the integration time. The disadvantage of shortening the integration time is the increase in other errors such as quantization errors. The averaging error can also be reduced by applying corrections. A three point correction, based on three consecutive measurements, can reduce the residual error to below 0.01 m/s, provided that the corrected $\dot{R}$ is referred to a time corresponding to the middle of the three measurements. The noise is shown not to increase more than 8.6% in this case. The three point correction is therefore adequate in most applications.

The time to which $\dot{R}$ is referred is rather critical in a three point correction. If the integration time varies or is not known, then a four or five point correction may have to be used.

ACKNOWLEDGMENT

The author is indebted to Dr. F. O. Vonbun for many helpful suggestions and encouragement.

APPENDIX I

Derivation of R^{III}

Differentiating equation (2.1) with respect to θ yields

$$RR^{I} = r(r+h)\sin\theta \qquad \text{(A-1)}$$

$$(R^{I})^2 + RR^{II} = r(r+h)\cos\theta \qquad \text{(A-2)}$$

$$3R^{I}R^{II} + RR^{III} = -r(r+h)\sin\theta \qquad \text{(A-3)}$$

thus

$$R^{III} = -R^{I}\left(1 + 3\frac{R^{II}}{R}\right) \qquad \text{(A-4)}$$

From equation (A-1) and (2.3)

$$R^{I} = r\cos\epsilon \qquad \text{(A-5)}$$

is obtained. Solving for R^{II}, using equation (2.1) and (A-2)

$$R^{II} = \frac{r^2\sin^2\epsilon + rh + \frac{1}{2}h^2}{R} - \frac{R}{2} \qquad \text{(A-6)}$$

hence

$$R^{III} = -r\left\{-\frac{1}{2} + 3\,\frac{r^2\sin^2\epsilon + rh + \frac{1}{2}h^2}{R^2}\right\}\cos\epsilon \qquad \text{(A-7)}$$

Equation (2.2) can be solved for $1/R$

$$\frac{1}{R} = \frac{r \sin \epsilon + \sqrt{r^2 \sin^2 \epsilon + 2hr + h^2}}{2hr + h^2}$$

and

$$\frac{1}{R^2} = \frac{2 r^2 \sin^2 \epsilon + 2hr + h^2 + 2r \sin \epsilon \sqrt{r^2 \sin^2 \epsilon + 2hr + h^2}}{(2hr + h^2)^2}$$

or

$$\frac{1}{R^2} \approx \frac{1}{h^2} \frac{\sin^2 \epsilon + \frac{h}{r}}{1 + \frac{h}{r}} \tag{A-8}$$

for $\sin^2 \epsilon >> \frac{2h}{r}$

and thus

$$R^{III} = -\frac{3 r^2}{h^2} \left(\frac{\sin^4 \epsilon + \frac{2h}{r} \sin^2 \epsilon}{1 + \frac{h}{r}} \right) \cos \epsilon \tag{A-9}$$

valid for $\sin^2 \epsilon >> \frac{2h}{r}$

For $\sin \epsilon = 0$

we find

$$R^{III} = -r \tag{A-10}$$

The maximum of R^{III} is found from Equation (A-9)

$$|R^{III}|_{max} = \frac{48}{25\sqrt{5}} \frac{r^3}{h^2} \left(1 + \frac{3h}{2r}\right) \tag{A-11}$$

$$\text{for } \sin \epsilon = \sqrt{\frac{4}{5} - \frac{h}{5r}} \tag{A-12}$$

Estimation of $|R^{IV}|$ and $|R^{V}|$

Differentiating (A-3) two more times with respect to θ yields

$$3(R^{II})^2 + 4R^{I}R^{III} + RR^{IV} = -r(r+h)\cos\theta \tag{A-13}$$

$$10\,R^{II}R^{III} + 5R^{I}R^{IV} + RR^{V} = r(r+h)\sin\theta \tag{A-14}$$

R^{IV} has extreme values for $R^{V} = 0$. From Equation (A-1) and (A-11) we obtain for $R^{V} = 0$

$$10\,R^{II}\,R^{III} + 5\,R^{I}R^{IV} = RR^{I}$$

Substituting R^{III} from Equation (A-4) yields

$$-10\,R^{I}\,R^{II}\left(1 + 3\frac{R^{II}}{R}\right) + 5\,R^{I}\,R^{IV} = RR^{I} \tag{A-15}$$

Equation (A-15) has two roots

$$R^{I} = 0$$

or (A-16)

$$\theta = 0$$

and

$$-10\, R^{II}\left(1 + 3\frac{R^{II}}{R}\right) + 5\, R^{IV} = R \tag{A-17}$$

For $R^{I} = 0$ we obtain from Equation (A-13)

$$R^{IV} = \left.\frac{-3\,(R^{II})^2 - r^2 - rh}{R}\right|_{\theta=0}$$

and thus

$$|R^{IV}|_{max\,1} = 3\frac{r^4}{h^3} + \frac{r^2}{h} + r$$

After elimination of R^{II} between Equation (A-13) and (A-17) we obtain

$$\frac{3}{2}R^{IV} = \frac{R}{10} - 4\frac{R^{I}R^{III}}{R} - \frac{R^{I2}}{R} \tag{A-18}$$

or

$$|R^{IV}| \leq \frac{R}{15} + \frac{8}{3}\left|\frac{R^{I}R^{III}}{R}\right| + \frac{3}{2}\left|\frac{(R^{I})^2}{R}\right| \tag{A-19}$$

From

$$|R| \geq h, \qquad |R^{I}| \leq r$$

and

$$|R^{III}| \leq 0.86 \frac{r^3}{h^2}$$

we find that

$$|R^{IV}|_{max\ 2} \leq 2.29 \frac{r^4}{h^3}$$

if lower order terms of h/r are neglected. $|R^{IV}|_{max\ 1}$ is larger than $|R^{IV}|_{max\ 2}$ and again neglecting lower order terms we thus have

$$|R^{IV}|_{max\ 1} = |R^{IV}|_{max} \leq 3 \frac{r^4}{h^3} \tag{A-20}$$

Because of the complex nature of the equations we will establish an upper bound for $|R^{V}|$ rather than deriving the exact expressions. From Equation (A-1) and (A-14) we obtain

$$|R^{V}| \leq |R^{I}| + 10 \left| \frac{R^{II}\, R^{III}}{R} \right| + 5 \left| \frac{R^{I}\, R^{IV}}{R} \right|$$

Using Equation (A-5) and (A-11) we find

$$|R^{V}| \leq 23.6 \frac{r^5}{h^4} \tag{A-21}$$

neglecting the lower order terms of h/r.

APPENDIX II

Three Point Correction

Expand R in a Taylor series around t_0, including up to fifth order terms:

$$R(\Delta t) = R_0 + \dot{R}_0 \Delta t + \frac{1}{2!} \ddot{R}_0 \Delta t^2 + \frac{1}{3!} \dddot{R}_0 \Delta t^3 + \frac{1}{4!} R_0^{IV} \Delta t^4 + \frac{1}{5!} R_0^{V} \Delta t^5 + \cdots$$

The derivatives are

$$\dot{R}(\Delta t) = \dot{R}_0 + \ddot{R}_0 \Delta t + \frac{1}{2!} \dddot{R}_0 \Delta t^2 + \frac{1}{3!} R_0^{IV} \Delta t^3 + \frac{1}{4!} R_0^{V} \Delta t^4 + \cdots \tag{A-22}$$

$$\ddot{R}(\Delta t) = \ddot{R}_0 + \dddot{R}_0 \Delta t + \frac{1}{2!} R_0^{IV} \Delta t^2 + \frac{1}{3!} R_0^{V} \Delta t^3 \tag{A-23}$$

$$\dddot{R}(\Delta t) = \dddot{R}_0 + R_0^{IV} \Delta t + \frac{1}{2!} R_0^{V} \Delta t^2 \tag{A-24}$$

$$R^{IV}(\Delta t) = R_0^{IV} + R_0^{V} \Delta t \tag{A-25}$$

$$R^{V}(\Delta t) = R_0^{V} \tag{A-26}$$

For the first measuring interval, see Figure 4 we have

$$\dot{R}_{a1} = \frac{R\left(-T_1 + \frac{T}{2}\right) - R\left(-T_1 - \frac{T}{2}\right)}{T}$$

and thus

$$\dot{R}_{a1} = \dot{R}(-T_1) + \frac{1}{3!}\dddot{R}(-T_1)\frac{T^2}{4} + \frac{1}{5!}R^V(-T_1)\frac{T^4}{16} \tag{A-27}$$

and in the same manner

$$\dot{R}_{a2} = \dot{R}_0 + \frac{1}{3!}\dddot{R}_0\frac{T^2}{4} + \frac{1}{5!}R^V_0\frac{T^4}{16} \tag{A-28}$$

$$\dot{R}_{a3} = \dot{R}(T_1) + \frac{1}{3!}\dddot{R}(T_1)\frac{T^2}{4} + \frac{1}{5!}R^V(T_1)\frac{T^4}{16} \tag{A-29}$$

$\dot{R}(\pm T_1)$, $\dddot{R}(\pm T_1)$ and $R^V(\pm T_1)$ can be expressed in $\dot{R}_0$, $\dddot{R}_0$ etc. with the aid of Equation (A-22) through (A-24) by putting $\Delta t = \pm T_1$. Solving for $\dot{R}_0$, $\ddot{R}_0$ and $\dddot{R}_0$ we obtain

$$\dot{R}_0 = \dot{R}_{a2} - \frac{1}{4!}T^2\dddot{R}_0 - \frac{1}{16\cdot 5!}T^4 R^V_0 \tag{A-30}$$

$$\ddot{R}_0 = \frac{\dot{R}_{a3} - \dot{R}_{a1}}{2T_1} - \frac{1}{3!}\left(T_1^2 + \frac{T^2}{4}\right)R^{IV} \tag{A-31}$$

$$\dddot{R}_0 = \frac{\dot{R}_{a3} - 2\dot{R}_{a2} + \dot{R}_{a1}}{T_1^2} - \frac{1}{4!}\left(2T_1^2 + T^2\right)R^V_0 \tag{A-32}$$

Substitution into Equation (A-22) yields

$$\dot{R}(\Delta t) = \left(\frac{\Delta t^2}{2T_1^2} - \frac{\Delta t}{2T_1} - \frac{T^2}{24T_1^2}\right)\dot{R}_{a1} + \left(1 - \frac{\Delta t^2}{T_1^2} + \frac{2T^2}{24T_1^2}\right)\dot{R}_{a2}$$

$$+ \left(\frac{\Delta t^2}{2T_1^2} + \frac{\Delta t}{2T_1} - \frac{T^2}{24T_1^2}\right)\dot{R}_{a3}$$

$$- \frac{1}{6}\left(1 + \frac{T^2}{4T_1^2} - \frac{\Delta t^2}{T_1^2}\right)\frac{\Delta t}{T_1} R_0^{IV} T_1^3$$

$$+ \frac{1}{24}\left\{\frac{7T^4}{240T_1^4} + \frac{T^2}{12T_1^2} - \left(1 + \frac{T^2}{2T_1^2}\right)\frac{\Delta t^2}{T_1^2} + \frac{\Delta t^4}{T_1^4}\right\} R_0^{V} T_1^4 \qquad \text{(A-33)}$$

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