

17
P

N65-22208
(ACCESSION NUMBER)
17
(PAGES)
TMX 51717
(NASA CR OR TMX OR AD NUMBER)

FACILITY FORM 802

(THRU)
/ (CODE)
30
(CATEGORY)

On a Modification of Hill's Method
of General Planetary Perturbations

by

Peter Musen
Goddard Space Flight Center
Theoretical Division

GPO PRICE \$ _____
OTS PRICE(S) \$ _____
Hard copy (HC) 1.00
Microfiche (MF) .50

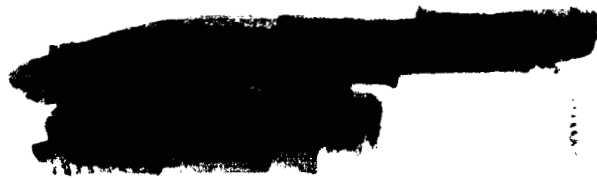


ABSTRACT

22208

In this article, a semi-analytical theory of general planetary perturbations is developed, which is somewhat akin to Hill's theory. In both methods the first order perturbations coincide, but the theories of perturbations of higher orders are different. The inconveniences of Hill's method, namely: the triple integral in the perturbations of the radius vector and the redundant constant of integration, do not appear here. The short and the long period terms containing the squares of the small divisors are localized and combined together. The existence of such a direct way of separating these important terms from the remaining perturbations constitutes a significant characteristic of a planetary theory. The form of decomposition of perturbations as used in this article leads to a system of differential equations easily integrable by Hill's procedure and to a symmetrical scheme for the computation of perturbations of higher orders.

Author



Notations

- m - the mass of the disturbed planet. The mass of the sun is taken as unity.
 m' - the mass of the disturbing planet.
 k - the Gaussian constant.
 $\mu^2 = k^2 (1+m)$.
 \vec{r} - the undisturbed position-vector of the planet m .
 $r = |\vec{r}|$.
 \vec{R} - the unit vector normal to the undisturbed orbit plane of the planet m .
 $\vec{\psi}$ - the angular velocity of rotation of the frame $(\vec{r}, \vec{R} \times \vec{r}, \vec{R})$.
 \vec{r}^0 - the unit vector in the direction of \vec{r} .
 \vec{r}' - the undisturbed position vector of the planet m' .
 $\delta\vec{r}$ - the perturbation in the position vector of the planet m .
 $\vec{r} + \delta\vec{r}$ - the disturbed position vector of the planet m .
 $\delta\vec{r}'$ - the perturbations in the position vector of the planet m' .
 $\vec{r}' + \delta\vec{r}'$ - the disturbed position vector of the planet m' .
 \vec{v} - the undisturbed velocity of the planet m .
 l - the undisturbed mean anomaly of the planet m .
 e - the undisturbed eccentricity of the planet m .
 $n = \frac{\mu}{a^{3/2}}$ - the undisturbed mean motion of the planet m .
 a - the undisturbed semi-major axis of the planet m .
 $p = a(1 - e^2)$.
 f - the undisturbed true anomaly of the planet m .
 ∇ - the del-operator with respect to \vec{r} .
 ∇' - the del-operator with respect to \vec{r}' .
 E - the base of natural logarithms.
 $\vec{\rho} = \vec{r}' - \vec{r}$.
 \vec{r}_k - the perturbations of the k^{th} order in the position vector of the planet m .
 \vec{r}'_k - the perturbations of the k^{th} order in the position vector of the planet m' .
 $\vec{\rho}_k = \vec{r}'_k - \vec{r}_k$.
 $\Omega(\vec{r}, \vec{r}') = \frac{m'}{1+m} \left(\frac{1}{\rho} - \frac{\vec{r} \cdot \vec{r}'}{r^3} \right)$ - the main part of the disturbing function.
 I - the idemfactor.

On a Modification of Hill's Method of General Planetary Perturbations

by
Peter Musen

Introduction

In this article a numerical theory of general planetary perturbations is developed. The perturbations are obtained in the standard form of series containing the periodic, the secular and the mixed terms. The coefficients of terms are numerical; they are obtained by double harmonic analysis as applied to the force components, and subsequent integration. The numerical theory of perturbations in the coordinates escapes the inconveniences of the numerical theory in the elements: for example, since the eccentricity and the sine of the inclination do not appear as divisors in the differential equations, no numerical difficulties arise in the case of nearly circular orbits.

The method presented here is somewhat akin to Hill's method (1874), at least where the first order perturbations are concerned: in both methods the first order perturbations coincide. The theories of perturbations of higher orders in both methods are different.

We determine the perturbations in rectangular coordinates directly without using the perturbations in polar coordinates as an intermediary means, as is done in Hill's method. Some other inconveniences are removed in the theory presented here: the triple integral and the seventh constant of integration so peculiar to Hill's method do not appear; the components of perturbations along the radius-vector are determined in a more direct manner; and the difficulties associated with determining the redundant constant of integration vanish.

In Hill's method the undisturbed true anomaly is taken as the independent variable. Such a choice causes numerous inconveniences if the perturbations in the motion of the disturbing body are also to be taken into account. For this reason the universal variable, time, is used in our exposition.

Since the advent of electronic machines, general perturbations theories can successfully compete with numerical integration procedures. The theory of Mars in Hansen's coordinates recently developed by Clemence (1949, 1961) brilliantly confirms this statement. The reference ellipse representing the undisturbed motion can be chosen in a variety of ways. The only restriction imposed is that the difference between the disturbed and undisturbed motions be small, of the order of the perturbations.

In the article by Musen and Carpenter (1963) a decomposition of the perturbations in the position vector along \vec{r} , \vec{v} , and \vec{R} was suggested. In the present work we suggest a decomposition along \vec{r}^0 , $\vec{R} \times \vec{r}^0$, and \vec{R} . This form of decomposition leads to a more compact and more symmetrical scheme for developing perturbations of higher order than the author's previous scheme.

Basic differential equations.

The equation of the motion of the planet m as referred to the rotating undisturbed frame \vec{r}^0 , $\vec{R} \times \vec{r}^0$, \vec{R} has the form

$$\begin{aligned} \frac{d^2 \delta \vec{r}}{dt^2} + 2\vec{\psi} \times \frac{d\delta \vec{r}}{dt} + \vec{\psi} \times (\vec{\psi} \times \delta \vec{r}) + \frac{d\vec{\psi}}{dt} \times \delta \vec{r} \\ = \mu^2 \nabla \left(\frac{1}{|\vec{r} + \delta \vec{r}|} - \frac{1}{r} \right) + \mu^2 \Omega (\vec{r} + \delta \vec{r}, \vec{r}' + \delta \vec{r}') . \end{aligned} \quad (1)$$

Substituting

$$\vec{\psi} = \frac{\mu \sqrt{p}}{r^2} \vec{R} ,$$

$$\frac{d\vec{\psi}}{dt} = - \frac{2\mu \sqrt{p}}{r^3} \frac{dr}{dt} \vec{R} ,$$

into (1), introducing the differential operator

$$D = \delta \vec{r} \cdot \nabla + \delta \vec{r}' \cdot \nabla' ,$$

and taking the equation

$$\nabla D \frac{1}{r} = - \frac{\delta \vec{r}}{r^3} + \frac{3\vec{r}\vec{r} \cdot \delta \vec{r}}{r^5}$$

into account, we can re-write (1) in the form

$$\begin{aligned} \frac{d^2 \delta \vec{r}}{dt^2} + \frac{2\mu \sqrt{p}}{r^2} \vec{R} \times \frac{d\delta \vec{r}}{dt} + \frac{\mu^2 p}{r^4} \vec{R} \times (\vec{R} \times \delta \vec{r}) \\ - \frac{2\mu \sqrt{p}}{r^3} \frac{dr}{dt} \vec{R} \times \delta \vec{r} + \mu^2 \left(\frac{\delta \vec{r}}{r^3} - \frac{3\vec{r}\vec{r} \cdot \delta \vec{r}}{r^5} \right) = \mu^2 \vec{F} , \end{aligned} \quad (2)$$

where

$$\vec{F} = \nabla (E^0 - 1 - D) \frac{1}{r} + \nabla E^0 \Omega (\vec{r}, \vec{r}') . \quad (3)$$

In applying the operators ∇ or ∇' we consider every function as a function of \vec{r} and \vec{r}' only and the perturbations $\delta\vec{r}$ and $\delta\vec{r}'$ are considered as relative constants. The operator E^D accomplishes the development of (3) into a power series with respect to the perturbations.

Decomposing $\delta\vec{r}$ along the axes \vec{r}^0 , $\vec{R} \times \vec{r}^0$, \vec{R} and putting

$$\delta\vec{r} = \xi \vec{r}^0 + \eta \vec{R} \times \vec{r}^0 + \zeta \vec{R}, \quad (4)$$

we have in the relative motion

$$\frac{d\delta\vec{r}}{dt} = \frac{d\xi}{dt} \vec{r}^0 + \frac{d\eta}{dt} \vec{R} \times \vec{r}^0 + \frac{d\zeta}{dt} \vec{R}, \quad (5)$$

$$\frac{d^2\delta\vec{r}}{dt^2} = \frac{d^2\xi}{dt^2} \vec{r}^0 + \frac{d^2\eta}{dt^2} \vec{R} \times \vec{r}^0 + \frac{d^2\zeta}{dt^2} \vec{R}. \quad (6)$$

Substituting (4) through (6) into (2), we obtain

$$\frac{d^2\xi}{dt^2} - \frac{2\mu\sqrt{p}}{r^2} \frac{d\eta}{dt} - \mu^2 \left(\frac{p}{r^4} + \frac{2}{r^3} \right) \xi + \frac{2\mu\sqrt{p}}{r^3} \frac{d\zeta}{dt} \eta = \mu^2 \Xi, \quad (7)$$

$$\frac{d^2\eta}{dt^2} + \frac{2\mu\sqrt{p}}{r^2} \frac{d\xi}{dt} - \frac{2\mu\sqrt{p}}{r^3} \frac{d\zeta}{dt} \xi - \mu^2 \left(\frac{p}{r^4} - \frac{1}{r^3} \right) \eta = \mu^2 H, \quad (8)$$

$$\frac{d^2\zeta}{dt^2} + \frac{\mu^2}{r^3} \zeta = \mu^2 Z, \quad (9)$$

where we put

$$\Xi = \vec{r}^0 \cdot \vec{F},$$

$$H = \vec{R} \times \vec{r}^0 \cdot \vec{F},$$

$$Z = \vec{R} \cdot \vec{F}.$$

We shall make use of the standard method of developing the perturbations into power series with respect to the disturbing mass and put

$$\delta\vec{r} = \vec{r}_1 + \vec{r}_2 + \vec{r}_3 + \dots,$$

$$\vec{r}_k = \xi_k \vec{r}^0 + \eta_k \vec{R} \times \vec{r}^0 + \zeta_k \vec{R}$$

$$\begin{aligned}
\xi &= \xi_1 + \xi_2 + \xi_3 + \dots, \\
\eta &= \eta_1 + \eta_2 + \eta_3 + \dots, \\
\zeta &= \zeta_1 + \zeta_2 + \zeta_3 + \dots,
\end{aligned} \tag{10}$$

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots, \tag{11}$$

where \vec{r}_k , ξ_k , η_k , ζ_k , \vec{F}_k are of the order k in m' .

In the previous article by Musen and Carpenter (1963), on the basis of the formula

$$\begin{aligned}
\nabla E^D &= \nabla + (\vec{r}_1 \cdot \nabla \nabla + \vec{r}_1' \cdot \nabla' \nabla) \\
&+ \left[(\vec{r}_2 \cdot \nabla \nabla + \vec{r}_2' \cdot \nabla' \nabla) + \frac{1}{2} (\vec{r}_1 \cdot \nabla + \vec{r}_1' \cdot \nabla')^2 \nabla \right] \\
&+ \left[(\vec{r}_3 \cdot \nabla \nabla + \vec{r}_3' \cdot \nabla' \nabla) + (\vec{r}_1 \cdot \nabla + \vec{r}_1' \cdot \nabla') (\vec{r}_2 \cdot \nabla + \vec{r}_2' \cdot \nabla') \nabla \right. \\
&\left. + \frac{1}{6} (\vec{r}_1 \cdot \nabla + \vec{r}_1' \cdot \nabla')^3 \nabla \right] + \dots,
\end{aligned}$$

it was found that

$$\vec{F}_1 = \nabla \Omega = \frac{m'}{1+m} \left(\frac{\vec{\rho}}{\rho^3} - \frac{\vec{r}'}{r'^3} \right), \tag{12}$$

$$\begin{aligned}
\vec{F}_2 &= \frac{3\vec{r} \cdot \vec{r}_1 \vec{r}_1}{r^5} + \frac{3}{2} \frac{\vec{r} \vec{r}_1 \cdot \vec{r}_1}{r^5} - \frac{15}{2} \frac{\vec{r} (\vec{r} \cdot \vec{r}_1)^2}{r^7} \\
&+ \vec{r}_1 \cdot \nabla \nabla \Omega + \vec{r}_1' \cdot \nabla' \nabla \Omega,
\end{aligned} \tag{13}$$

$$\begin{aligned}
\vec{F}_3 &= \frac{3}{r^5} (\vec{r} \cdot \vec{r}_1 \vec{r}_2 + \vec{r} \vec{r}_1 \cdot \vec{r}_2 + \vec{r} \cdot \vec{r}_2 \vec{r}_1) \\
&- \frac{15}{r^7} \vec{r} \vec{r}_1 \cdot \vec{r} \vec{r}_2 \cdot \vec{r} + \frac{3}{2} \frac{\vec{r}_1 \vec{r}_1 \cdot \vec{r}_1}{r^5} \\
&- \frac{15}{2} \frac{\vec{r} \vec{r} \cdot \vec{r}_1 \vec{r}_1 \cdot \vec{r}_1}{r^7} - \frac{15}{2} \frac{\vec{r}_1 (\vec{r} \cdot \vec{r}_1)^2}{r^7} \\
&+ \frac{35}{2} \frac{\vec{r} (\vec{r} \cdot \vec{r}_1)^3}{r^9} \\
&+ \left[(\vec{r}_2 \cdot \nabla \nabla \Omega + \vec{r}_2' \cdot \nabla' \nabla \Omega) \right. \\
&\left. + \frac{1}{2} (\vec{r}_1 \cdot \nabla + \vec{r}_1' \cdot \nabla')^2 \nabla \Omega \right]
\end{aligned} \tag{14}$$

By substituting

$$\vec{r}_1 = \xi_1 \vec{r}^0 + \eta_1 \vec{R} \times \vec{r}^0 + \zeta_1 \vec{R}$$

and

$$\begin{aligned} \nabla \nabla \Omega &= \nabla \frac{m'}{1+m} \left(\frac{\vec{\rho}}{\rho^3} - \frac{\vec{r}'}{r'^3} \right) = \frac{m'}{1+m} \left(-\frac{\vec{I}}{\rho^3} + \frac{3\vec{\rho}\vec{\rho}}{\rho^5} \right) \\ \nabla' \nabla \Omega &= \nabla' \frac{m'}{1+m} \left(\frac{\vec{\rho}}{\rho^3} - \frac{\vec{r}'}{r'^3} \right) = \frac{m'}{1+m} \left(\frac{\vec{I}}{\rho^3} - \frac{3\vec{\rho}\vec{\rho}}{\rho^5} \right) - \frac{m'}{1+m} \left(\frac{\vec{I}}{r'^3} - \frac{3\vec{r}'\vec{r}'}{r'^5} \right) \end{aligned}$$

into (13), we deduce a compact expression for \vec{F}_2 :

$$\begin{aligned} \vec{F}_2 &= \frac{1}{r^4} \left[\left(-3\xi_1^2 + \frac{3}{2}\eta_1^2 + \frac{3}{2}\zeta_1^2 \right) \vec{r}^0 + 3\eta_1\xi_1 \vec{R} \times \vec{r}^0 + 3\zeta_1\xi_1 \vec{R} \right] \\ &+ \frac{m'}{1+m} \left[\left(\frac{\vec{\rho}_1}{\rho^3} - \frac{3\vec{\rho}\vec{\rho} \cdot \vec{\rho}_1}{\rho^5} \right) - \left(\frac{\vec{r}_1'}{r'^3} - \frac{3\vec{r}'\vec{r}' \cdot \vec{r}_1'}{r'^5} \right) \right]. \end{aligned}$$

Substituting (10) and (11) into (7)-(9) we obtain the equations

$$\frac{d^2 \xi_\kappa}{dt^2} - \frac{2\mu\sqrt{p}}{r^2} \frac{d\eta_\kappa}{dt} - \mu^2 \left(\frac{p}{r^4} + \frac{2}{r^3} \right) \xi_\kappa + \frac{2\mu\sqrt{p}}{r^3} \frac{dr}{dt} \eta_\kappa = \mu^2 \Xi_\kappa,$$

$$\frac{d^2 \eta_\kappa}{dt^2} + \frac{2\mu\sqrt{p}}{r^2} \frac{d\xi_\kappa}{dt} - \frac{2\mu\sqrt{p}}{r^3} \frac{dr}{dt} \xi_\kappa - \mu^2 \left(\frac{p}{r^4} - \frac{1}{r^3} \right) \eta_\kappa = \mu^2 H_\kappa,$$

$$\frac{d^2 \zeta_\kappa}{dt^2} + \frac{\mu}{r^3} \zeta_\kappa = \mu^2 Z_\kappa,$$

$$\kappa = 1, 2, 3, \dots,$$

where

$$\Xi_\kappa = \vec{r}^0 \cdot \vec{F}_\kappa,$$

$$H_\kappa = \vec{R} \times \vec{r}^0 \cdot \vec{F}_\kappa,$$

$$Z_\kappa = \vec{R} \cdot \vec{F}_\kappa,$$

which are of the same form as (7)-(9). Thus, our problem now is to integrate the variational equations of the form (7)-(9) with the right sides known.

Integration procedure.

We shall make use of the substitution:

$$\xi = ur + \frac{dr}{dt} \int (w - 2u) dt \quad (15)$$

$$\eta = \frac{\mu \sqrt{p}}{r} \int (w - 2u) dt \quad (16)$$

$$\zeta = \zeta, \quad (17)$$

which reduces (7)-(9) to the form integrable by Hill's procedure. Substituting (15) and (16) into (7), and making use of the relation

$$\frac{d^2 r}{dt^2} = \mu^2 \left(\frac{p}{r^3} - \frac{1}{r^2} \right)$$

We obtain

$$\frac{d^2 u}{dt^2} + \frac{\mu^2}{r^3} (u - 2w) + \frac{1}{r} \frac{dr}{dt} \frac{dw}{dt} = \frac{\mu^2 \Xi}{r}. \quad (18)$$

From (15) and (16), we have

$$w = \frac{1}{\mu \sqrt{p}} \left(r \frac{d\eta}{dt} - \eta \frac{dr}{dt} \right) + \frac{2\xi}{r}. \quad (19)$$

Differentiating (19) and taking (8) into consideration we obtain

$$\frac{dw}{dt} = \frac{\mu r H}{\sqrt{p}} \quad (20)$$

and

$$w = K_3 + \int \frac{\mu r H}{\sqrt{p}} dt, \quad (21)$$

where the integral sign represents the integral obtained in a formal manner; K_3 is the constant of integration.

Taking (20), (21) and the equation

$$\frac{dr}{dt} = \frac{\mu e \sin f}{\sqrt{p}}$$

into account, we obtain from (18):

$$\frac{d^2}{dt^2} (u - 2K_3) + \frac{\mu^2}{r^3} (u - 2K_3) = \frac{\mu^2}{r} \left(\Xi - \frac{re \sin f}{p} H \right) + \frac{2}{r^3} \int \frac{\mu^3 r H}{\sqrt{p}} dt .$$

The last equation can be integrated by using HILL's procedure and we have

$$\begin{aligned} u &= 2K_3 + K_1 \frac{r}{a} \cos f + K_2 \frac{r}{a} \sin f \\ &+ \int \frac{\mu}{\sqrt{p}} \left(\frac{\Xi}{r} - \frac{e \sin f}{p} H \right) \bar{r} r \sin(\bar{f} - f) dt \\ &+ \int \frac{\bar{r}}{r^2} \sin(\bar{f} - f) dt \int \frac{2\mu^2 r H}{p} dt , \end{aligned} \quad (22)$$

where K_1 and K_2 are constants of integration and \bar{r} , \bar{f} are considered as temporary constants; after the integration is completed they are replaced by r and f .

The double integral in (22) can be simplified through integration by parts, and we obtain

$$\int \frac{\bar{r}}{r^2} \sin(\bar{f} - f) dt \int \frac{2\mu^2 r H}{p} dt = \int \frac{2\mu r H}{\sqrt{p}} dt - \int \frac{2\mu \bar{r} r H}{p \sqrt{p}} \left[\cos(\bar{f} - f) + e \cos \bar{f} \right] dt .$$

Taking this last relation into consideration we deduce from (22) after some easy transformations

$$u = 2K_3 + K_1 \frac{r}{a} \cos f + K_2 \frac{r}{a} \sin f + A ,$$

in which

$$A = \int \left(M \bar{f} \cdot \bar{r} + N \bar{f} \cdot \bar{r} \times \bar{r} \right) dt$$

where

$$M = \frac{an}{\sqrt{1-e^2}} \frac{\bar{r}}{a} \sin(\bar{f} - f)$$

$$\begin{aligned} N &= \frac{an}{(1-e^2)^{3/2}} \frac{\bar{r}}{a} \left[+ \frac{1}{2} e \cos \bar{f} - \frac{1}{2} e \cos(\bar{f} - 2f) \right. \\ &\quad \left. - 2 \cos(\bar{f} - f) + 2 \right] . \end{aligned}$$

The expressions M and N remain the same for the perturbations of all orders. They are to be developed into a double Fourier series with respect to the mean anomaly l and with respect to the auxiliary mean anomaly \bar{l} , associated with the auxiliary true anomaly \bar{f} .

After the integration is performed \bar{l} is replaced by l . We have

$$\int \frac{r}{a} \cos f dl = -\frac{3}{2} \text{ent} + \frac{1}{2} \sqrt{1-e^2} \frac{r}{a} \sin f + \frac{1}{2\sqrt{1-e^2}} \frac{r^2}{a^2} \sin f, \quad (25)$$

$$\int \frac{r}{a} \sin f dl = -\sqrt{1-e^2} \frac{r^2}{a^2} \left(\cos f + \frac{1}{2} e \cos^2 f \right). \quad (26)$$

Putting

$$S = \int (w - 2u) dt$$

and taking (21), (24), (25) and (26) into account, we deduce

$$S = -\frac{3K_3}{n} nt + \frac{K_1}{n} \left(3\text{ent} - \sqrt{1-e^2} \frac{r}{a} \sin f - \frac{1}{\sqrt{1-e^2}} \frac{r^2}{a^2} \sin f \right) + \sqrt{1-e^2} \frac{K_2}{n} \frac{r^2}{a^2} (2 \cos f + e \cos^2 f) + K_4 + B$$

and, after some easy transformations,

$$S = \frac{3}{n} (-K_3 + eK_1) nt - \frac{K_1}{n\sqrt{1-e^2}} \frac{r^2}{a^2} \left(2 \sin f + \frac{1}{2} e \sin 2f \right) + \frac{K_2\sqrt{1-e^2}}{n} \frac{K_2}{n} \frac{r^2}{a^2} \left(2 \cos f + \frac{1}{2} e + \frac{1}{2} e \cos 2f \right) + K_4 + B, \quad (27)$$

where

$$B = \iint \frac{narH}{\sqrt{1-e^2}} dt - \int 2A dt. \quad (28)$$

The formulas for the computation of ξ , η and ζ become

$$\xi = ru + \frac{nae \sin f}{\sqrt{1-e^2}} S, \quad (29)$$

$$\eta = \frac{na^2 \sqrt{1-e^2}}{r} S, \quad (30)$$

$$\zeta = K_5 \frac{r}{a} \cos f + K_6 \frac{r}{a} \sin f + C, \quad (31)$$

where C is the standard expression

$$C = \int \frac{na}{\sqrt{1-e^2}} \vec{R} \cdot \vec{F} \vec{r} \sin(\vec{F} - f) dt \quad (32)$$

The expressions of A, B and C as given here are reducible to the form given in our previous article. The present form, however, facilitates the comparison with the perturbations already computed by Hill's method (Froeschke, 1933) if the substitution

$$dt = \frac{r^2}{a^2} \frac{df}{n\sqrt{1-e^2}}$$

is made.

Determination of constants of integration.

We consider here the determination of constants of integration for the case when the elements are osculating at the epoch $t = 0$ and for the case when they are mean. If the elements are osculating, then we have

$$(\delta \vec{r})_0 = 0, \quad \left(\frac{d\delta \vec{r}}{dt} \right)_0 = 0, \quad (33)$$

where the zero-subscript designates the value of the expression at the epoch. From (33) we deduce

$$u_0 = 0, \quad w_0 = 0, \quad \left(\frac{du}{dt} \right)_0 = 0, \quad S_0 = 0. \quad (34)$$

Taking (21) into account, we obtain

$$bK_3 = - \left[\int \frac{naH}{\sqrt{1-e^2}} dt \right]_0. \quad (35)$$

Differentiating (24) and taking the equations

$$\begin{aligned} \frac{d}{dt} \frac{r}{a} \cos f &= - \frac{\sin f}{\sqrt{1-e^2}} \\ \frac{d}{dt} \frac{r}{a} \sin f &= + \frac{\cos f + e}{\sqrt{1-e^2}} \end{aligned}$$

into account, we deduce from (34):

$$+ K_1 \frac{r_0}{a} \cos f_0 + K_2 \frac{r_0}{a} \sin f_0 = -A_0 - 2K_3$$

$$- K_1 \frac{\sin f_0}{\sqrt{1-e^2}} + K_2 \frac{\cos f_0 + e}{\sqrt{1-e^2}} = -A_0'$$

where we put

$$A_0' = \left(\frac{dA}{dt} \right)_0$$

From these last equations we obtain

$$K_1 = - \frac{\cos f_0 + e}{1 - e^2} (A_0 + 2K_3) + \frac{A_0'}{\sqrt{1-e^2}} \frac{r_0}{a} \sin f_0$$

$$K_2 = - \frac{\sin f_0}{1 - e^2} (A_0 + 2K_3) - \frac{A_0'}{\sqrt{1-e^2}} \frac{r_0}{a} \cos f_0$$

In a similar way we deduce

$$K_5 = - \frac{\cos f_0 + e}{1 - e^2} C_0 + \frac{C_0'}{\sqrt{1-e^2}} \frac{r_0}{a} \sin f_0$$

$$K_6 = - \frac{\sin f_0}{1 - e^2} C_0 - \frac{C_0'}{\sqrt{1-e^2}} \frac{r_0}{a} \cos f_0$$

and putting $t = 0$ in (27) we deduce the following value for K_4 :

$$K_4 = + \frac{K_1}{n \sqrt{1-e^2}} \frac{r_0^2}{a^2} \left(2 \sin f_0 + \frac{1}{2} e_0 \sin 2f_0 \right)$$

$$- \frac{K_2 \sqrt{1-e^2}}{n} \frac{r_0^2}{a^2} \left(2 \cos f_0 + \frac{1}{2} e + \frac{1}{2} e \cos 2f_0 \right) - B_0$$

The mean elements can be defined in several ways. We accept here the following definition: the elements are mean if

- (1) the perturbations of the true longitude λ with respect to the orbit-plane defined by these elements do not contain the terms of the form

$$K_0, Kt, K^{(c)} \cos l \text{ and } K^{(s)} \sin l; \text{ and} \quad (36)$$

(2) the expression for the "third coordinate" ζ does not contain the terms of the form

$$K^{(*)} \cos l, \quad K^{(*)} \sin l \quad .$$

We have

$$\left(\frac{r}{a}\right)^n \cos mf = \frac{1}{2} C_0^{n,m} + C_1^{n,m} \cos l + C_2^{n,m} \cos 2l + \dots \quad ,$$

$$\left(\frac{r}{a}\right)^n \sin mf = S_1^{n,m} \sin l + S_2^{n,m} \sin 2l + \dots \quad .$$

where

$$C_i^{n,m} = X_i^{n,m} + X_{-i}^{n,m}$$

$$S_i^{n,m} = X_i^{n,m} - X_{-i}^{n,m} \quad ,$$

and $X_i^{n,m}$ are Hansen's coefficients.

Let us start with the determination of constants of integration in the perturbations of the first order. Perturbations of the first order in the true longitude are given by the expression η/r , where $\eta = \eta_1$, and consequently the terms (36) must be absent in the expression

$$\begin{aligned} \frac{\eta}{nr\sqrt{1-e^2}} &= \frac{3}{n} \left(\frac{a}{r}\right)^2 (-K_3 + eK_1) nt \\ &- \frac{K_1}{n\sqrt{1-e^2}} \left(2 \sin f + \frac{1}{2} e \sin 2f\right) \\ &+ \frac{K_2 \sqrt{1-e^2}}{n} \left(2 \cos f + \frac{1}{2} e + \frac{1}{2} e \cos 2f\right) \\ &+ \frac{a^2}{r^2} K_4 + \frac{a^2}{r^2} B \quad . \end{aligned} \tag{37}$$

We have to separate the terms of the form (36) in the development of $a^2/r^2 B$:

$$\frac{a^2}{r^2} B = \alpha_0 + \beta_0 nt + \alpha_1 \cos l + \beta_1 \sin l + \dots \quad . \tag{37'}$$

The condition for absence of the constant term in (37') leads to the equation

$$+ \frac{K_2 \sqrt{1-e^2}}{n} \left(C_0^{0,1} + \frac{1}{2} e + \frac{1}{4} e C_0^{0,2} \right) + \frac{1}{2} C_0^{-2,0} K_4 + \alpha_0 = 0 \quad (38)$$

In a similar way we deduce

$$- \frac{K_1}{n \sqrt{1-e^2}} \left(2S_1^{0,1} + \frac{1}{2} e S_1^{0,2} \right) + \beta_1 = 0 \quad (39)$$

$$+ \frac{K_2 \sqrt{1-e^2}}{n} \left(2C_1^{0,1} + \frac{1}{2} e C_1^{0,2} \right) + \alpha_1 = 0 \quad (40)$$

$$+ \frac{3}{2n} C_0^{-2,0} (-K_3 + e K_1) + \beta_0 = 0 \quad (41)$$

Separating in C the terms with the argument l ,

$$C = c_1 \cos l + s_1 \sin l + \dots$$

we obtain, taking (31) into account,

$$K_2 C_1^{1,1} + c_1 = 0 \quad (42)$$

$$K_6 S_1^{1,1} + s_1 = 0 \quad (43)$$

In the planetary case the solutions of equations (38)-(43) can be found without any difficulty, because the coefficients only of one unknown in each equation are not small. The coefficients $C_i^{n,m}$ can be computed either by using the classical analytical expressions or by Cayley's tables (1861), or by means of harmonic analysis. The latter procedure is preferable if the eccentricity is not very small.

Determination of constants of integration in higher order perturbations requires some additional considerations. Let r_1, r_2, \dots be the perturbations in the radius-vector r and $\lambda_1, \lambda_2, \dots$ be the perturbations in the true longitude λ of the first, second, etc., orders. From

$$\begin{aligned} (\xi_1 + \xi_2 + \dots) \vec{r}^0 + (\eta_1 + \eta_2 + \dots) \vec{R} \times \vec{r}^0 &= \left(r_1 \frac{\partial \vec{r}}{\partial r} + \lambda_1 \frac{\partial \vec{r}}{\partial \lambda} \right) \\ &+ \left[\left(r_2 \frac{\partial \vec{r}}{\partial r} + \lambda_2 \frac{\partial \vec{r}}{\partial \lambda} \right) + \frac{1}{2} \left(r_1^2 \frac{\partial^2 \vec{r}}{\partial r^2} + 2r_1 \lambda_1 \frac{\partial^2 \vec{r}}{\partial r \partial \lambda} + \lambda_1^2 \frac{\partial^2 \vec{r}}{\partial \lambda^2} \right) \right] + \dots \end{aligned}$$

and substituting

$$\frac{\partial \vec{r}}{\partial t} = \vec{r}^0, \quad \frac{\partial \vec{r}}{\partial \lambda} = \vec{R} \times \vec{r}, \quad \frac{\partial^2 \vec{r}}{\partial t^2} = 0, \quad \frac{\partial^2 \vec{r}}{\partial t \partial \lambda} = \vec{R} \times \vec{r}^0, \quad \frac{\partial^2 \vec{r}}{\partial \lambda^2} = -\vec{r}$$

we obtain

$$\xi_1 = r_1, \quad \eta_1 = r\lambda_1$$

$$\xi_2 = r_2 - \frac{1}{2}\lambda_1^2 r, \quad \eta_2 = r\lambda_2 + r_1\lambda_1, \dots$$

consequently,

$$\lambda_2 = \frac{\eta_2 - \xi_1 \eta_1}{r}$$

or, taking (30) into account,

$$\lambda_2 = n\sqrt{1-e^2} \frac{a^2}{r^2} (S_2 - \xi_1 S_1),$$

where S_1 corresponds to the first and S_2 to the second order perturbations. As we see, before the determination of the constants of integration of the second order in the case of mean elements a correction term $-\xi_1 S_1$ must be added to B_2 . For the perturbation of the third order a similar correction term will depend upon ξ_1, S_1, ξ_2, S_2 .

Conclusion

A revival of the general interest in planetary theories can be observed in our time. Several scientific institutions are dedicating their time and efforts to the astronomical solution of the planetary problem. The results by Brouwer (1944), Gontkovskaya (1958) and Danby (1962) must especially be mentioned. A considerable amount of work on the theoretical exposition as well as on programming has also been done at the Theoretical Division of Goddard Space Flight Center. In the present article we suggest a new scheme which is convenient for computing the perturbations of the first as well as of higher orders.

The determination of constants of integration in the case of both the osculating and the mean elements is a straightforward process in the proposed scheme. An important feature of the scheme is that the squares of small divisors, as caused by the commensurability of mean motions, are introduced by integration of only one expression, namely $w - 2u$. The short and the long period terms containing the squares of small divisors constitute a significant part in the perturbations ξ, η, ζ . The existence of such a direct way of separating these terms from the remaining perturbations

constitutes a significant part of a planetary theory. The development here is kept in the form which facilitates comparison with the results obtained on the basis of the classical form of Hill's theory, if necessary.

References

Brouwer, D., Integration of the equations of general planetary theory in rectangular coordinates, Astr. Journ. 51, 37-43, 1944

Cayley, A., Tables of the developments of functions in the theory of elliptic motion, Mem. Roy. Astron. Soc., 29, 191, 1861

Clemence, G. M., First-order theory of Mars, Astr. Papers of Amer. Ephemer., Vol. XI, Part II, Washington, 1949

Clemence, G. M., Theory of Mars, Completion, Vol. XVI, Part II, 1961

Danby, J. M. A., Integration of the equations of planetary motion in rectangular coordinates, Astr. Journ. 67, 287-299, 1962

Gontkovskaya, V. T., The application of modern computational techniques to the analytical methods of celestial mechanics (Russian). Bull. Inst. Theoret. Astron. 82, 592-629, 1958

Hill, G. W., A method of computing absolute perturbations, Astr. Nachr. 83, 209, 1874

Hill, G. W., Collected works, 1, 151, Washington, 1905

Musen, P. and Carpenter, L. On the general planetary perturbations in rectangular coordinates, Journ. of Geoph. Res., vol. 68, No. 9, p. 2727, 1963

Proskurin, V. F., The theory of motion of Ceres, Trudy Inst. Theor. Astronomy, II, Leningrad, 1952