DEPARTMENT OF MATHEMATICS
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Multivariable Least Squares Approximation and Its Application to Solving Problems Associated with Space Vehicles

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Daniel E. Dupree

Summary Report
May 1, 1962 - February 1, 1965
Contract Number NAS-8-2642
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February 1, 1965
Prepared for George C. Marshall Space Flight Center,
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SUMMARY

This report for the period May 1, 1962 - February 1, 1965 on Contract NAS-8-2642 covers the three accomplishments which are significant in helping solve problems associated with optimized missile trajectories. These are:

1) The establishment of necessary and sufficient conditions for the existence and uniqueness of multivariable least squares approximating functions.

2) The development of a multivariable orthogonalization process which yields the least squares approximating function without solving the system of normal equations.

3) The development of a technique for obtaining an approximating function which yields an error, in the sense of least squares, that is less than a specified tolerance.

author

I. INTRODUCTION

The theory of least squares approximating polynomials in a single variable has been covered in great detail in the literature. It is the purpose of this report to generalize this theory of least squares to include not only polynomials in several variables but also multivariable transcendental functions.

II. EXISTENCE

We can state the existence problem as follows:

If $\{\beta_0, X(\beta_0)\}$, $\{\beta_1, X(\beta_1)\}, \ldots, \{\beta_n, X(\beta_n)\}$ are n + 1 pairs of values of the function $X = X(\beta)$, where $\beta = (x_1, x_2, \ldots, x_t)$, $\beta_i = (x_{1i}, x_{2i}, \ldots, x_{ti})$, and if $\varphi_0(\beta), \varphi_1(\beta), \ldots, \varphi_N(\beta)$ are N + 1 independent functions of β, under what conditions do there exist constants $A_0, A_1, \ldots, A_N$ such that

$$F(A_0, A_1, \ldots, A_N) = \sum_{i=0}^{n} \{X(\beta_i) - \sum_{j=0}^{N} A_j \varphi_j(\beta_i)\}^2$$

is a minimum?

<u>Lemma 1</u>: $F(A_0, A_1, \ldots, A_N) = \sum_{i=0}^{n} \{X(\beta_i) - \sum_{j=0}^{N} A_j \varphi_j(\beta_i)\}^2$ is a continuous function of its arguments.

<u>Proof</u>: $\left| \sum_{i=0}^{n} \{X(\beta_i) - \sum_{j=0}^{N} A_j \varphi_j(\beta_i)\}^2 - \sum_{i=0}^{n} \{X(\beta_i) - \sum_{j=0}^{N} A_j' \varphi_j(\beta_i)\}^2 \right|$

$$= \Big| \{X(\beta_0) - \sum_{j=0}^{N} A_j \varphi_j(\beta_0)\}^2 + \{X(\beta_1) - \sum_{j=0}^{N} A_j \varphi_j(\beta_1)\}^2 + \ldots$$

$$\ldots + \{X(\beta_n) - \sum_{j=0}^{N} A_j \varphi_j(\beta_n)\}^2 - \{X(\beta_0) - \sum_{j=0}^{N} A_j' \varphi_j(\beta_0)\}^2$$

$$- \{X(\beta_1) - \sum_{j=0}^{N} A_j' \varphi_j(\beta_1)\}^2 - \ldots - \{X(\beta_n) - \sum_{j=0}^{N} A_j' \varphi_j(\beta_n)\}^2 \Big|$$

$$= \Big| - 2X(\beta_0) \sum_{j=0}^{N} A_j \varphi_j(\beta_0) + \{\sum_{j=0}^{N} A_j \varphi_j(\beta_0)\}^2 - 2X(\beta_1) \sum_{j=0}^{N} A_j \varphi_j(\beta_1)$$

$$+ \{ \sum_{j=0}^{N} A_j\varphi_j(\beta_1)\}^2 - \dots - 2X(\beta_n) \sum_{j=0}^{N} A_j\varphi_j(\beta_n) + \{ \sum_{j=0}^{N} A_j\varphi_j(\beta_n)\}^2$$

$$+ 2X(\beta_0) \sum_{j=0}^{N} A'_j\varphi_j(\beta_0) - \{ \sum_{j=0}^{N} A'_j\varphi_j(\beta_0)\}^2 + 2X(\beta_1) \sum_{j=0}^{N} A'_j\varphi_j(\beta_1)$$

$$- \{ \sum_{j=0}^{N} A'_j\varphi_j(\beta_1)\}^2 + \dots + 2X(\beta_n) \sum_{j=0}^{N} A'_j\varphi_j(\beta_n) - \{ \sum_{j=0}^{N} A'_j\varphi_j(\beta_n)\}^2 \Big|$$

$$= \Big| - 2X(\beta_0) \{ \sum_{j=0}^{N} A_j\varphi_j(\beta_0) - \sum_{j=0}^{N} A'_j\varphi_j(\beta_0)\} - 2X(\beta_1) \{ \sum_{j=0}^{N} A_j\varphi_j(\beta_1)$$

$$- \sum_{j=0}^{N} A'_j\varphi_j(\beta_1)\} - \dots - 2X(\beta_n) \{ \sum_{j=0}^{N} A_j\varphi_j(\beta_n) - \sum_{j=0}^{N} A'_j\varphi_j(\beta_n)\}$$

$$+ \Big\{ \{ \sum_{j=0}^{N} A_j\varphi_j(\beta_0)\}^2 - \{ \sum_{j=0}^{N} A'_j\varphi_j(\beta_0)\}^2 \Big\} +$$

$$\Big\{ \{ \sum_{j=0}^{N} A_j\varphi_j(\beta_1)\}^2 - \{ \sum_{j=0}^{N} A'_j\varphi_j(\beta_1)\}^2 \Big\}$$

$$+ \dots + \Big\{ \{ \sum_{j=0}^{N} A_j\varphi_j(\beta_n)\}^2 - \{ \sum_{j=0}^{N} A'_j\varphi_j(\beta_n)\}^2 \Big\} \Big|$$

$$= \Big| \{ \sum_{j=0}^{N} A_j\varphi_j(\beta_0) - \sum_{j=0}^{N} A'_j\varphi_j(\beta_0)\} \cdot \Big\{ - 2X(\beta_0)$$

$$+ \{ \sum_{j=0}^{N} A_j\varphi_j(\beta_0) + \sum_{j=0}^{N} A'_j\varphi_j(\beta_0)\} \Big\}$$

$$+ \{ \sum_{j=0}^{N} A_j\varphi_j(\beta_1) - \sum_{j=0}^{N} A'_j\varphi_j(\beta_1)\} \cdot \Big\{ - 2X(\beta_1)$$

$$+ \{ \sum_{j=0}^{N} A_j\varphi_j(\beta_1) + \sum_{j=0}^{N} A'_j\varphi_j(\beta_1)\} \Big\}$$

$$+ \dots + \left(\sum_{j=0}^{N} A_j\varphi_j(\beta_n) - \sum_{j=0}^{N} A_j'\varphi_j(\beta_n)\right) \cdot \Big\{ - 2X(\beta_n) + \left(\sum_{j=0}^{N} A_j\varphi_j(\beta_n) + \sum_{j=0}^{N} A_j'\varphi_j(\beta_n)\right) \Big\}\Big|$$

$$\leq \left| \sum_{j=0}^{N} (A_j - A_j')\varphi_j(\beta_0) \cdot \Big\{ - 2X(\beta_0) + \sum_{j=0}^{N} (A_j + A_j')\varphi_j(\beta_0) \Big\} \right|$$

$$+ \left| \sum_{j=0}^{N} (A_j - A_j')\varphi_j(\beta_1) \cdot \Big\{ - 2X(\beta_1) + \sum_{j=0}^{N} (A_j + A_j')\varphi_j(\beta_1) \Big\} \right|$$

$$+ \dots + \left| \sum_{j=0}^{N} (A_j - A_j')\varphi_j(\beta_n) \cdot \Big\{ - 2X(\beta_n) + \sum_{j=0}^{N} (A_j + A_j')\varphi_j(\beta_n) \Big\} \right|$$

$$\leq \left| \sum_{j=0}^{N} (A_j - A_j')\varphi_j(\beta_0) \right| \cdot \Big\{ |2X(\beta_0)| + \left| \sum_{j=0}^{N} (A_j + A_j')\varphi_j(\beta_0) \right| \Big\}$$

$$+ \left| \sum_{j=0}^{N} (A_j - A_j')\varphi_j(\beta_1) \right| \cdot \Big\{ |2X(B_1)| + \left| \sum_{j=0}^{N} (A_j + A_j')\varphi_j(\beta_1) \right| \Big\}$$

$$+ \dots + \left| \sum_{j=0}^{N} (A_j - A_j')\varphi_j(\beta_n) \right| \cdot \Big\{ |2X(\beta_n)| + \left| \sum_{j=0}^{N} (A_j + A_j')\varphi_j(\beta_n) \right| \Big\}$$

$$\leq \sum_{j=0}^{N} |A_j - A_j'| \, |\varphi_j(\beta_0)| \cdot \Big\{ |2X(\beta_0)| + \sum_{j=0}^{N} |A_j + A_j'| \, |\varphi_j(\beta_0)| \Big\}$$

$$+ \sum_{j=0}^{N} |A_j - A_j'| \, |\varphi_j(\beta_1)| \cdot \Big\{ |2X(\beta_1)| + \sum_{j=0}^{N} |A_j + A_j'| \, |\varphi_j(\beta_1)| \Big\}$$

$$+ \dots + \sum_{j=0}^{N} |A_j - A_j'| \, |\varphi_j(\beta_n)| \cdot \Big\{ |2X(\beta_n)| + \sum_{j=0}^{N} |A_j + A_j'| \, |\varphi_j(\beta_n)| \Big\}$$

$$\leq \max_{0 \leq j \leq N} |A_j - A_j'| \cdot \Big\{ \sum_{j=0}^{N} |\varphi_j(\beta_0)| \; (\, |2X(\beta_0)| + \max_{0 \leq j \leq N} |A_j + A_j'| \cdot$$

$$\sum_{j=0}^{N} |\varphi_j(\beta_0)| \}$$

$$+ \sum_{j=0}^{N} |\varphi_j(\beta_1)| \{ |2X(\beta_1)| + \max_{0 \leq j \leq N} |A_j + A_j'| \sum_{j=0}^{N} |\varphi_j(\beta_1)| \}$$

$$+ \dots + \sum_{j=0}^{N} |\varphi_j(\beta_n)| \{ |2X(\beta_n)| + \max_{0 \leq j \leq N} |A_j + A_j'| \sum_{j=0}^{N} |\varphi_j(\beta_n)| \} \Big\}.$$

Hence, $F(A_0, A_1, \dots, A_N)$ is a continuous function of its arguments.

<u>Lemma 2</u>: The function

$$Q(A_0, A_1, \dots, A_N) = \sum_{i=0}^{n} \{ -2X(\beta_i) \sum_{j=0}^{N} A_j \varphi_j(\beta_i) \}$$

is a continuous function of its arguments.

<u>Proof</u>: $\left| \sum_{i=0}^{n} \{ -2X(\beta_i) \sum_{j=0}^{N} A_j \varphi_j(\beta_i) \} - \sum_{i=0}^{n} \{ -2X(\beta_i) \sum_{j=0}^{N} A_j' \varphi_j(\beta_i) \} \right|$

$$= \left| \sum_{i=0}^{n} \{ -2X(\beta_i) \sum_{j=0}^{N} A_j \varphi_j(\beta_i) + 2X(\beta_i) \sum_{j=0}^{N} A_j' \varphi_j(\beta_i) \} \right|$$

$$= \left| \sum_{i=0}^{n} \{ -2X(\beta_i) \sum_{j=0}^{N} (A_j - A_j') \varphi_j(\beta_i) \} \right| \leq \sum_{i=0}^{n} \{ |2X(\beta_i)| \sum_{j=0}^{N} |A_j - A_j'| |\varphi_j(\beta_i)| \}$$

$$\leq \max_{0 \leq j \leq N} |A_j - A_j'| \sum_{i=0}^{n} \Big\{ |2X(\beta_i)| \sum_{j=0}^{N} |\varphi_j(\beta_i)| \Big\} .$$

<u>Lemma 3</u>: If $\sum_{i=0}^{n} X(\beta_i) \varphi_j(\beta_i) \neq 0$, for some j, $0 \leq j \leq N$, then

$Q(A_0, A_1, \dots, A_N)$ has non-zero extreme values on the sphere

$$A_0^2 + A_1^2 + \dots + A_N^2 = 1.$$

Proof: Let $R(A_0, A_1, \dots, A_N) = A_0^2 + A_1^2 + \dots + A_N^2 - 1$ and let γ be an undetermined Lagrangian multiplier. Then the extreme values of $Q(A_0, A_1, \dots, A_N)$ will occur at the zeros of the following system of equations:

$$\frac{\partial Q}{\partial A_0} + \gamma \frac{\partial R}{\partial A_0} = 0$$

$$\frac{\partial Q}{\partial A_1} + \gamma \frac{\partial R}{\partial A_1} = 0$$

$$\vdots$$

$$\frac{\partial Q}{\partial A_N} + \gamma \frac{\partial R}{\partial A_N} = 0.$$

But $Q(A_0, A_1, \dots, A_N) = \sum_{i=0}^{n} (- 2X(\beta_i) \sum_{j=0}^{N} A_j \varphi_j(\beta_i))$

$$= \sum_{i=0}^{n} (- 2X(\beta_i) \left\{ A_0\varphi_0(\beta_i) + \dots + A_N\varphi_N(\beta_i) \right\})$$

$$= \sum_{i=0}^{n} (- 2X(\beta_i)A_0\varphi_0(\beta_i) - 2X(\beta_i)A_1\varphi_1(\beta_i) - \dots - 2X(\beta_i)A_N\varphi_N(\beta_i))$$

$$= - 2A_0 \sum_{i=0}^{n} X(\beta_i)\varphi_0(\beta_i) - 2A_1 \sum_{i=0}^{n} X(\beta_i)\varphi_1(\beta_i) - \dots - 2A_N \sum_{i=0}^{n} X(\beta_i)\varphi_N(\beta_i).$$

Thus, the system above becomes

$$-2\sum_{i=0}^{n} X(\beta_i)\varphi_0(\beta_i) + 2\gamma\Lambda_0 = 0$$

$$-2\sum_{i=0}^{n} X(\beta_i)\varphi_1(\beta_i) + 2\gamma\Lambda_1 = 0$$

$$\vdots$$

$$-2\sum_{i=0}^{n} X(\beta_i)\varphi_N(\beta_i) + 2\gamma\Lambda_N = 0.$$

Now $\gamma \neq 0$, since $\sum_{i=0}^{n} X(\beta_i)\varphi_j(\beta_i) \neq 0$ for some j, $0 \leq j \leq N$.

Therefore, $\Lambda_j = 1/\gamma \{ \sum_{i=0}^{n} X(\beta_i)\varphi_j(\beta_i)\}$, $j = 0,1,\ldots,N$, and

$\Lambda_0^2 + \Lambda_1^2 + \ldots + \Lambda_N^2 = 1$ becomes

$$1/\gamma^2 \{ \sum_{i=0}^{n} X(\beta_i)\varphi_0(\beta_i)\}^2 + \{ \sum_{i=0}^{n} X(\beta_i)\varphi_1(\beta_i)\}^2 + \ldots$$

$$\ldots + \{ \sum_{i=0}^{n} X(\beta_i)\varphi_N(\beta_i)\}^2 = 1.$$

Hence, $1/\gamma^2 \cdot \sum_{j=0}^{N} \{ \sum_{i=0}^{n} X(\beta_i)\varphi_j(\beta_i)\}^2 = 1$, or

$\gamma^2 = \sum_{j=0}^{N} \{ \sum_{i=0}^{n} X(\beta_i)\varphi_j(\beta_i)\}^2$. Thus,

$$\gamma = \pm \sqrt{\sum_{j=0}^{N} \left(\sum_{i=0}^{n} X(\beta_i)\varphi_j(\beta_i) \right)^2} \text{ , and}$$

$$A_j = 1/\gamma \sum_{i=0}^{n} X(\beta_i)\varphi_j(\beta_i) = \pm \frac{\sum_{i=0}^{n} X(\beta_i)\varphi_j(\beta_i)}{\sqrt{\sum_{j=0}^{N} \left(\sum_{i=0}^{n} X(\beta_i)\varphi_j(\beta_i) \right)^2}}$$

Therefore, the extreme values of $Q(A_0, A_1, \ldots, A_N)$ are

$$Q(A_0, A_1, \ldots, A_N) = \pm 2 \sum_{j=0}^{N} \left[\frac{\sum_{i=0}^{n} X(\beta_i)\varphi_j(\beta_i)}{\sqrt{\sum_{j=0}^{N} \left(\sum_{i=0}^{n} X(\beta_i)\varphi_j(\beta_i) \right)^2}} \right] . \left(\sum_{i=0}^{n} X(\beta_i)\varphi_j(\beta_i) \right)$$

$$= \pm 2 \sum_{j=0}^{N} \left[\frac{\left(\sum_{i=0}^{n} X(\beta_i)\varphi_j(\beta_i) \right)^2}{\sqrt{\sum_{j=0}^{N} \left(\sum_{i=0}^{n} X(\beta_i)\varphi_j(\beta_i) \right)^2}} \right] .$$

These extreme values are not zero if $\sum_{i=0}^{n} X(\beta_i)\varphi_j(\beta_i)$ fails to vanish for some j, $0 \leq j \leq N$.

<u>Theorem</u>: If $\varphi_0(\beta), \varphi_1(\beta), \ldots, \varphi_N(\beta)$ are N + 1 functions satisfying $\sum_{i=0}^{n} X(\beta_i)\varphi_j(\beta_i) \neq 0$, for some j, $0 \leq j \leq N$, then there

exists constants $A_0, A_1, \ldots, A_N$ such that

$$F(A_0,A_1,\ldots,A_N) = \sum_{i=0}^{n} \{X(\beta_i) - \sum_{j=0}^{N} A_j\varphi_j(\beta_i)\}^2$$

is a minimum.

<u>Proof</u>: Let $\pi \neq 0$ be the minimum value of $Q(A_0,A_1,\ldots,A_N)$ on the unit sphere $\sum_{j=0}^{N} A_j^2 = 1$.

<u>Case 1</u>: $\pi < 0$. If $\pi < 0$, then $\sqrt{\sum_{j=0}^{N} A_j^2} > 1/\pi \ (\alpha + 1)$, where α = g.l.b. of $F(A_0,A_1,\ldots,A_N)$. α exists since $F \geq 0$. Therefore,

$$\begin{aligned}
F(A_0,A_1,\ldots,A_N) &= \sum_{i=0}^{n} \{X(\beta_i) - \sum_{j=0}^{N} A_j\varphi_j(\beta_i)\}^2 \\
&= \sum_{i=0}^{n} X^2(\beta_i) + \sum_{i=0}^{n} \{ - 2X(\beta_i) \sum_{j=0}^{N} A_j\varphi_j(\beta_i)\} \\
&\quad + \sum_{i=0}^{n} \{ \sum_{j=0}^{N} A_j\varphi_j(\beta_i)\}^2 \\
&\geq \sum_{i=0}^{n} \{ - 2X(\beta_i) \sum_{j=0}^{N} A_j\varphi_j(\beta_i)\} \geq \pi \sqrt{\sum_{j=0}^{N} A_j^2} \\
&> \pi \ (1/\pi) \ (\alpha + 1) = \alpha + 1 > \alpha,
\end{aligned}$$

a contradiction. Hence, this case cannot occur.

Case 2: $\pi > 0$. We assume that

$$\sqrt{\sum_{j=0}^{N} A_j^2} > 1/\pi \quad (\alpha + 1) \quad \text{and obtain the same}$$

contradiction as in Case 1. Thus,

$$\sqrt{\sum_{j=0}^{N} A_j^2} \leq 1/\pi \quad (\alpha + 1) = R$$

is a closed bounded set of points $(A_0, A_1, \ldots, A_N)$ in $N + 1$ dimensional space. $F(A_0, A_1, \ldots, A_N)$ is a continuous function of $(A_0, A_1, \ldots, A_N)$ and has a minimum value in or on the set.

III. UNIQUENESS

We might reconsider the existence problem in a different fashion which will enable us to solve the uniqueness problem as well. Recall that we desire to minimize the function

$$F(A_0,A_1,\dots,A_N) = \sum_{i=0}^{n} \{X(\beta_i) - \sum_{j=0}^{N} A_j\varphi_j(\beta_i)\}^2 .$$

Notice that F can be written as the following dot product:

$$F = \bar{B}\cdot\bar{B}, \text{ where}$$

$$\bar{B} = (X(\beta_0) - \sum_{j=0}^{N} A_j\varphi_j(\beta_0),\ X(\beta_1) - \sum_{j=0}^{N} A_j\varphi_j(\beta_1),\dots,$$

$$X(\beta_n) - \sum_{j=0}^{N} A_j\varphi_j(\beta_n)).$$

Thus, $F = \bar{B}\cdot\bar{B} = \|\bar{B}\|^2$

$$= \|(X(\beta_0),\ X(\beta_1),\dots,X(\beta_n)) - A_0(\varphi_0(\beta_0),\ \varphi_0(\beta_1),\dots,$$

$$\varphi_0(\beta_n)) - A_1(\varphi_1(\beta_0),\ \varphi_1(\beta_1),\dots,\varphi_1(\beta_n)) - \dots -$$

$$A_N(\varphi_N(\beta_0),\ \varphi_N(\beta_1),\dots,\ \varphi_N(\beta_n))\|^2$$

$$= \|\bar{X} - A_0\bar{\varphi}_0 - A_1\bar{\varphi}_1 - \dots - A_N\bar{\varphi}_N\|^2, \text{ where}$$

$$\bar{X} = (X(\beta_0),\ X(\beta_1),\dots,\ X(\beta_n)) \text{ and}$$

$$\bar{\varphi}_j = (\varphi_j(\beta_0),\ \varphi_j(\beta_1),\dots,\ \varphi_j(\beta_n)),\quad j = 0,1,\dots,N.$$

Theorem: If f has a relative minimum value at point P in Euclidean k space and if $f > 0$, then f^2 also has a relative minimum value at P.

Proof: Since f has a relative minimum value at P, then $f(P) < f(Q)$ for all Q contained in some neighborhood of P. But $f > 0$, and this implies that $0 < f^2(P) \leq f(P)f(Q)$ and $0 < f(P)f(Q) \leq f^2(Q)$ in this neighborhood. Thus, $f^2(P) \leq f^2(Q)$ in this neighborhood.

Therefore, in order to minimize $F(A_0,A_1,\ldots,A_N)$ it is sufficient to minimize the function $H(A_0,A_1,\ldots,A_N) = \|\bar{X} - A_0\bar{\varphi}_0 - A_1\bar{\varphi}_1 - \cdots - A_N\bar{\varphi}_N\|$. Now H is a continuous function, since

$$\left| H(A_0', A_1',\ldots,A_N') - H(A_0,A_1,\ldots,A_N)\right|$$

$$= \left| \left\| \bar{X} - \sum_{j=0}^{N} A_j' \bar{\varphi}_j \right\| - \left\| \bar{X} - \sum_{j=0}^{N} A_j\bar{\varphi}_j \right\| \right| \leq \left\| \sum_{j=0}^{N} (A_j' - A_j)\bar{\varphi}_j \right\|$$

$$\leq \sum_{j=0}^{N} \left| A_j' - A_j \right| \left\| \bar{\varphi}_j \right\| \leq \max_{0 \leq j \leq N} \left| A_j' - A_j \right| \sum_{j=0}^{N} \left\| \bar{\varphi}_j \right\| .$$

Similarly, the function $K(A_0,A_1,\ldots,A_N) = \left\| \sum_{j=0}^{N} A_j\bar{\varphi}_j \right\|$ is continuous, and has a minimum λ on the sphere, $\sum_{j=0}^{N} A_j^2 = 1$. Now $\lambda \neq 0$, since the vectors $\bar{\varphi}_0,\bar{\varphi}_1,\ldots,\bar{\varphi}_N$ are linearly independent. Hence, $\lambda > 0$.

Let π denote the g.l.b. of H. Then $\pi \geq 0$ and if

$\sqrt{\sum_{j=0}^{N} A_j^2} > \frac{1}{\lambda} (\pi + 1 + \|\bar{X}\|) = R$, then

$$H \geq K - \|\overline{X}\| \geq \sqrt{\sum_{j=0}^{N} A_j^2} \cdot \lambda - \|\overline{X}\| > \pi + 1,$$ a

contradiction. Therefore, in seeking the minimum of H, we can restrict ourselves to the consideration of H in the closed and bounded region

$\sqrt{\sum_{j=0}^{N} A_j^2} \leq \frac{1}{\lambda}(\pi + 1 + \|\overline{X}\|) = R.$ In such a region the continuous function H has a minimum.

Now let us assume that the two polynomials $\sum_{j=0}^{N} A_j \varphi_j(\beta)$ and $\sum_{j=0}^{N} A_j' \varphi_j(\beta)$ yield the best approximation in the sense of least squares to the function $X = X(\beta)$. Then

$$\|\overline{X} - A_0\overline{\varphi}_0 - A_1\overline{\varphi}_1 - \dots - A_N\overline{\varphi}_N\| = \|\overline{X} - A_0'\overline{\varphi}_0 - A_1'\overline{\varphi}_1 - \dots - A_N'\overline{\varphi}_N\| = \theta \text{ ; i.e., } \|\overline{X} - \sum_{j=0}^{N} A_j\overline{\varphi}_j\| = \|\overline{X} - \sum_{j=0}^{N} A_j'\varphi_j\| = \theta,$$

where $\theta \neq 0$. For if $\theta = 0$, then $A_j = A_j'$, $j = 0,1,\dots,N$. Moreover,

$$\|\overline{X} - \sum_{j=0}^{N}\left\{\frac{A_j + A_j'}{2}\right\}\overline{\varphi}_j\| = \|(\frac{\overline{X}}{2} - \frac{1}{2}\sum_{j=0}^{N} A_j\overline{\varphi}_j) + (\frac{\overline{X}}{2} - \frac{1}{2}\sum_{j=0}^{N} A_j'\overline{\varphi}_j)\| \leq \frac{1}{2}\|\overline{X} - \sum_{j=0}^{N} A_j\overline{\varphi}_j\| + \frac{1}{2}\|\overline{X} - \sum_{j=0}^{N} A_j'\overline{\varphi}_j\| = \theta.$$

Therefore, $\|\overline{X} - \sum_{j=0}^{N}\left\{\frac{A_j + A_j'}{2}\right\}\overline{\varphi}_j\| = \theta$, and

$$\left\| \overline{X} - \sum_{j=0}^{N} \left\{ \frac{\Lambda_j + \Lambda_j'}{2} \right\} \overline{\varphi}_j \right\| = \frac{1}{2} \left\| \overline{X} - \sum_{j=0}^{N} \Lambda_j \overline{\varphi}_j \right\| +$$

$\frac{1}{2} \left\| \overline{X} - \sum_{j=0}^{N} \Lambda_j' \overline{\varphi}_j \right\|$. Thus, $\overline{X} - \sum_{j=0}^{N} \Lambda_j \overline{\varphi}_j = \alpha \left\{ \overline{X} - \sum_{j=0}^{N} \Lambda_j' \overline{\varphi}_j \right\}$, $\alpha \geq 0$, since the collection of all vectors having real components is a strictly normalized system.

Now if $\alpha \neq 1$, $\overline{X}$ is a linear combination of the vectors $\overline{\varphi}_0, \overline{\varphi}_1, \ldots, \overline{\varphi}_N$ and $\theta \neq 0$ yields a contradiction. Thus, $\alpha = 1$, $\sum_{j=0}^{N} (\Lambda_j - \Lambda_j') \overline{\varphi}_j = 0$, and $\Lambda_j = \Lambda_j'$, $j = 0,1,\ldots,N$, since the $\overline{\varphi}_j$ are linearly independent.

IV. GENERATION OF MULTIVARIABLE ORTHOGONAL POLYNOMIALS

Again let $\{\beta_0, X(\beta_0)\}$, $\{\beta_1, X(\beta_1)\}, \ldots, \{\beta_n, X(\beta_n)\}$ be $n + 1$ tabular points for the function $X = X(\beta)$, where $\beta = (x_1, x_2, \ldots, x_t)$ and let $\varphi_0(\beta)$, $\varphi_1(\beta), \ldots, \varphi_N(\beta)$ be $N + 1$ independent functions of β. We require a polynomial of the form $\sum_{j=0}^{N} A_j\varphi_j(\beta)$ satisfying the property that $\sum_{i=0}^{n} \{X(\beta_i) - \sum_{j=0}^{N} A_j\varphi_j(\beta_i)\}^2$ is minimum. A necessary condition for this to be minimum is that

$\frac{\partial F}{\partial A_0} = \frac{\partial F}{\partial A_1} = \ldots = \frac{\partial F}{\partial A_N} = 0$. This yields the system of equations

$$
\begin{aligned}
A_0\bar{\varphi}_0 \cdot \bar{\varphi}_0 + A_1\bar{\varphi}_1 \cdot \bar{\varphi}_0 + \ldots + A_N\bar{\varphi}_N \cdot \bar{\varphi}_0 &= \bar{X} \cdot \bar{\varphi}_0 \\
A_0\bar{\varphi}_0 \cdot \bar{\varphi}_1 + A_1\bar{\varphi}_1 \cdot \bar{\varphi}_1 + \ldots + A_N\bar{\varphi}_N \cdot \bar{\varphi}_1 &= \bar{X} \cdot \bar{\varphi}_1 \\
&\vdots \\
A_0\bar{\varphi}_0 \cdot \bar{\varphi}_N + A_1\bar{\varphi}_1 \cdot \bar{\varphi}_N + \ldots + A_N\bar{\varphi}_N \cdot \bar{\varphi}_N &= \bar{X} \cdot \bar{\varphi}_N
\end{aligned}
$$

Obviously, if $\varphi_0(\beta)$, $\varphi_1(\beta), \ldots, \varphi_N(\beta)$ are chosen so that $\bar{\varphi}_i \cdot \bar{\varphi}_j = \delta_{ij}$, then the problem is greatly simplified.

Now let us define the vectors $\overline{\varphi'_0}$, $\overline{\varphi'_1}, \ldots, \overline{\varphi'_N}$ as follows:

$$\overline{\varphi'_j} = \bar{\varphi}_j - (\bar{\varphi}_j, \bar{e}_0)\bar{e}_0 - (\bar{\varphi}_j, \bar{e}_1)\bar{e}_1 - \ldots - (\bar{\varphi}_j, \bar{e}_{j-1})\bar{e}_{j-1}, \quad (j = 0, 1, \ldots, N)$$

where $\bar{e}_j = \dfrac{\overline{\varphi'_j}}{\|\overline{\varphi'_j}\|}$ and $(\bar{\varphi}_j, \bar{e}_k) = \bar{\varphi}_j \cdot \bar{e}_k$.

The vectors $\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_N$ form an orthonormal collection and

$$\bar{e}_j = \frac{1}{\|\overline{\varphi'_j}\|} \left\{ \bar{\varphi}_j - (\bar{\varphi}_j,\bar{e}_0)\bar{e}_0 - (\bar{\varphi}_j,\bar{e}_1)\bar{e}_1 - \ldots - (\bar{\varphi}_j,\bar{e}_{j-1})\bar{e}_{j-1} \right\}.$$

In addition, define $A_\gamma(-1) = \dfrac{1}{\|\overline{\varphi'_\gamma}\|}$ and $A_\gamma(k) = \dfrac{(\bar{\varphi}_\gamma,\bar{e}_k)}{\|\overline{\varphi'_\gamma}\|}$,

$k = 0,1,\ldots,\gamma-1.$

<u>Theorem</u>: If $A_\gamma(k) = (\bar{\varphi}_\gamma,\bar{e}_k)\Big/\|\overline{\varphi'_\gamma}\|$, $k = 0,1,\ldots,\gamma-1$, then

$$A_\gamma(k) = \{A_\gamma(-1)A_k(-1)(\bar{\varphi}_\gamma,\bar{\varphi}_k) - A_\gamma(0)A_k(0) - A_\gamma(1)A_k(1) - \ldots$$

$$- A_\gamma(k-1)A_k(k-1)\}.$$

<u>Proof</u>:

$$A_\gamma(k) = (\bar{\varphi}_\gamma,\bar{e}_k)\Big/\|\overline{\varphi'_\gamma}\|$$

$$= \frac{1}{\|\overline{\varphi'_\gamma}\|}\left\{\bar{\varphi}_\gamma,\ \bar{\varphi}_k\Big/\|\overline{\varphi'_k}\| - (\bar{\varphi}_k,\bar{e}_0)\bar{e}_0\Big/\|\overline{\varphi'_k}\|\right.$$

$$\left.- \ldots - (\bar{\varphi}_k,\bar{e}_{k-1})\bar{e}_{k-1}\Big/\|\overline{\varphi'_k}\|\right\}$$

$$= (\bar{\varphi}'_\gamma,\bar{\varphi}_k)\Big/\{\|\overline{\varphi'_\gamma}\|\,\|\overline{\varphi'_k}\|\} - (\bar{\varphi}_k,\bar{e}_0)(\bar{\varphi}_\gamma,\bar{e}_0)\Big/\{\|\bar{\varphi}_\gamma\|\,\|\overline{\varphi'_k}\|\}$$

$$- \ldots - (\bar{\varphi}_k,\bar{e}_{k-1})(\bar{\varphi}_\gamma,\bar{e}_{k-1})\Big/\{\|\overline{\varphi'_\gamma}\|\,\|\overline{\varphi'_k}\|\}$$

$$= A_\gamma(-1)A_k(-1)(\bar{\varphi}_\gamma,\bar{\varphi}_k) - A_\gamma(0)A_k(0) - \ldots - A_\gamma(k-1)A_k(k-1).$$

Although the norms of the vectors $\overline{\varphi'_0}, \overline{\varphi'_1},\ldots,\overline{\varphi'_N}$ are utilized in defining the $A_\gamma(k)$ of the previous theorem, it is not necessary to actually construct this collection of vectors in order to calculate these norms. This is shown in the following theorem:

<u>Theorem</u>: If $\overline{\varphi'_k} = \overline{\varphi}_k - (\overline{\varphi}_k,\overline{e}_0)\overline{e}_0 - (\overline{\varphi}_k,\overline{e}_1)\overline{e}_1 - \dots - (\overline{\varphi}_k,\overline{e}_{k-1})\overline{e}_{k-1}$,

$$= \overline{\varphi}_k - \sum_{j=0}^{k-1} (\overline{\varphi}_k,\overline{e}_j)\overline{e}_j,$$

then $||\overline{\varphi}'_k|| = \left\{ (\overline{\varphi}_k, \overline{\varphi}_k) - \sum_{j=0}^{k-1} (\overline{\varphi}_k,\overline{e}_j)(\overline{\varphi}_k,\overline{e}_j) \right\}^{1/2}.$

<u>Proof</u>: $||\overline{\varphi}'_k|| = \left\{ (\overline{\varphi}_k - \sum_{j=0}^{k-1} (\overline{\varphi}_k,\overline{e}_j)\overline{e}_j) \cdot (\overline{\varphi}_k - \sum_{j=0}^{k-1} (\overline{\varphi}_k,\overline{e}_j)\overline{e}_j \right\}^{1/2}$

$$= \Big\{ (\overline{\varphi}_k, \overline{\varphi}_k) - \overline{\varphi}_k \cdot \sum_{j=0}^{k-1} (\overline{\varphi}_k,\overline{e}_j)\overline{e}_j - \overline{\varphi}_k \cdot \sum_{j=0}^{k-1} (\overline{\varphi}_k,\overline{e}_j)\overline{e}_j$$

$$+ \{(\overline{\varphi}_k,\overline{e}_0)\overline{e}_0 \cdot \sum_{j=0}^{k-1} (\overline{\varphi}_k,\overline{e}_j)\overline{e}_j + (\overline{\varphi}_k,\overline{e}_1)\overline{e}_1 \cdot \sum_{j=0}^{k-1} (\overline{\varphi}_k,\overline{e}_j)\overline{e}_j +$$

$$\dots + (\overline{\varphi}_k,\overline{e}_{k-1})\overline{e}_{k-1} \cdot \sum_{j=0}^{k-1} (\overline{\varphi}_k,\overline{e}_j)\overline{e}_j\} \Big\}^{1/2}$$

$$= \Big\{ (\overline{\varphi}_k, \overline{\varphi}_k) - \sum_{j=0}^{k-1} (\overline{\varphi}_k,\overline{e}_j)(\overline{\varphi}_k,\overline{e}_j) - \sum_{j=0}^{k-1} (\overline{\varphi}_k,\overline{e}_j)(\overline{\varphi}_k,\overline{e}_j)$$

$$+ \{ (\overline{\varphi}_k,\overline{e}_0)(\overline{\varphi}_k,\overline{e}_0) + (\overline{\varphi}_k,\overline{e}_1)(\overline{\varphi}_k,\overline{e}_1) +$$

$$\dots + (\overline{\varphi}_k,\overline{e}_{k-1})(\overline{\varphi}_k,\overline{e}_{k-1}) \} \Big\}^{1/2}$$

$$= \left\{ (\overline{\varphi}_k, \overline{\varphi}_k) - \sum_{j=0}^{k-1} (\overline{\varphi}_k,\overline{e}_j)(\overline{\varphi}_k,\overline{e}_j) \right\}^{1/2}.$$

Thus, $\bar{e}_j = A_j(-1)\,\bar{\varphi}_j - A_j(0)\bar{e}_0 - \dots - A_j(j-1)\bar{e}_{j-1}$,

where the coefficients in this representation can be obtained from the following triangular array:

$$
\begin{array}{llll}
A_0\,(-1) & & & \\
A_1\,(-1) & A_1\,(0) & & \\
A_2\,(-1) & A_2\,(0) & A_2\,(1) & \\
A_3\,(-1) & A_3\,(0) & A_3\,(1) & A_3\,(2) \\
\multicolumn{4}{l}{- -}
\end{array}
$$

Notice that only the elements in the first column require any new calculations, since all other elements in the array can be written recursively using these elements and the two theorems above.

To show how these coefficients are to be utilized, let us define

$$f_0(\beta) = A_0(-1)\,\varphi_0(\beta), \text{ and}$$

$$f_j(\beta) = A_j(-1)\varphi_j(\beta) - A_j(0)f_0(\beta) - A_j(1)f_1(\beta) - \dots$$

$$- A_j(j-1)f_{j-1}(\beta), \text{ for } j = 1,2,\dots,N.$$

Notice that each $f_j(\beta)$ is a linear combination of the N + 1 independent functions $\varphi_0(\beta), \varphi_1(\beta), \dots, \varphi_N(\beta)$.

<u>Theorem</u>: If $\bar{f}_j = \{f_j(\beta_0), f_j(\beta_1), \dots, f_j(\beta_n)\}$, $j = 0,1,\dots,N$, then $\bar{f}_j = \bar{e}_j$.

<u>Proof</u>: $\bar{f}_0 = \{f_0(\beta_0), f_0(\beta_1), \dots, f_0(\beta_n)\} = \{A_0(-1)\varphi_0(\beta_0), A_0(-1)\varphi_0(\beta_1),$

$$\dots, A_0(-1)\varphi_0(\beta_n)\} = A_0(-1)\bar{\varphi}_0 = \bar{e}_0$$

Now assume that $\bar{f}_{k-1} = \bar{e}_{k-1}$, for $1 \leq k-1 \leq N-1$. Then

$$\bar{f}_k = \{f_k(\beta_0),\ f_k(\beta_1),\ldots,f_k(\beta_n)\}$$

$$= \Big\{A_k(-1)\varphi_k(\beta_0) - A_k(0)f_0(\beta_0) - A_k(1)f_1(\beta_0) - \ldots$$

$$- A_k(k-1)f_{k-1}(\beta_0),\ A_k(-1)\varphi_k(\beta_1) - A_k(0)f_0(\beta_1) -$$

$$A_k(1)f_1(\beta_1) - \ldots - A_k(k-1)f_{k-1}(\beta_1),$$

$$- ,$$

$$A_k(-1)\varphi_k(\beta_n) - A_k(0)f_0(\beta_n) - A_k(1)f_1(\beta_n) - \ldots$$

$$- A_k(k-1)f_{k-1}(\beta_n)\Big\}$$

$$= A_k(-1)\ \{\varphi_k(\beta_0),\ \varphi_k(\beta_1),\ldots,\varphi_k(\beta_n)\}$$

$$- A_k(0)\ \{f_0(\beta_0),\ f_0(\beta_1),\ldots,f_0(\beta_n)\}$$

$$- A_k(1)\ \{f_1(\beta_0),\ f_1(\beta_1),\ldots,f_1(\beta_n)\} - \ldots$$

$$- A_k(k-1)\ \{f_{k-1}(\beta_0),\ f_{k-1}(\beta_1),\ldots,f_{k-1}(\beta_n)\}$$

$$= A_k(-1)\ \bar{\varphi}_k - A_k(0)\bar{e}_0 - A_k(1)\bar{e}_1, - \ldots - A_k(k-1)\bar{e}_{k-1} = \bar{e}_k.$$

Now rather than finding the function $\sum_{j=0}^{N} A_j\varphi_j(\beta)$ such that $\sum_{i=0}^{n} \{X(\beta_i) - \sum_{j=0}^{N} A_j\varphi_j(\beta_i)\}^2$ is minimized, we now attempt to find the function $\sum_{j=0}^{N} A_j' f_j(\beta)$ such that $\sum_{i=0}^{n} \{X(\beta_i) - \sum_{j=0}^{N} A_j' f_j(\beta_i)\}^2$ is minimized. The conditions necessary for this are the normal equations

$$
\begin{aligned}
A_0' &= \overline{X} \cdot \overline{e}_0 \\
A_1' &= \overline{X} \cdot \overline{e}_1 \\
&\ \ \vdots \\
A_N' &= \overline{X} \cdot \overline{e}_N .
\end{aligned}
$$

This yields the function $\sum_{j=0}^{N} A_j' f_j(\beta) = \sum_{j=0}^{N} A_j \varphi_j(\beta)$ such that

$\sum_{i=0}^{N} \{X(\beta_i) - \sum_{j=0}^{N} A_j \varphi_j(\beta)\}^2$ is minimum.

V. BOUNDS FOR THE ERROR FUNCTION

A recursion procedure has been developed for obtaining the coefficients $A_0, A_1, \ldots, A_N$ of the function $A_0\varphi_0(\beta) + A_1\varphi_1(\beta) + \cdots + A_N\varphi_N(\beta)$ such that

$$E = \sum_{i=0}^{n} \left\{ X(\beta_i) - \sum_{j=0}^{N} A_j\varphi_j(\beta_i) \right\}^2$$

is minimum. This scheme yields the coefficients of the approximating function without having to solve the normal equations. Of course, the least squares procedure minimizes the sum of the squared errors, yet we have no assurance of the relative size of this error. In this report, we will develop a process for choosing the approximating function in such a fashion that the error will not exceed a given tolerance.

Before doing this, let us examine more closely the error E incurred by using the function $\sum_{j=0}^{N} A_j\varphi_j(\beta)$ as an approximating function. If the vectors $\bar{\varphi}_0, \bar{\varphi}_1, \ldots, \bar{\varphi}_N$, $N < n$, are used to obtain the collection $\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_N$ of orthonormal vectors, then the error E can be written as follows:

$$E = \sum_{i=0}^{n} \left[X(\beta_i) - \sum_{j=0}^{N} A_j\varphi_j(\beta_i) \right]^2 = \| \bar{X} - A_0\bar{\varphi}_0 - A_1\bar{\varphi}_1 - \cdots - A_N\bar{\varphi}_N \|^2 = \| \bar{X} - \sum_{j=0}^{N} (\bar{X}, \bar{e}_j)\, \bar{e}_j \|^2 =$$

$$\| \bar{X} \|^2 - \left[\bar{X}, \sum_{j=0}^{N} (\bar{X}, \bar{e}_j)\, \bar{e}_j \right] - \left[\bar{X}, \sum_{j=0}^{N} (\bar{X}, \bar{e}_j)\, \bar{e}_j \right] + \| \sum_{j=0}^{N} (\bar{X}, \bar{e}_j)\, \bar{e}_j \|^2 = \| \bar{X} \|^2 - \sum_{j=0}^{N} (\bar{X}, \bar{e}_j)^2 .$$

From this representation of E, we are able to observe the following:

1) $\| \overline{X} \|^2$ is an upper bound for E.

2) A sum of any k of the N + 1 terms $(\overline{X}, \overline{e}_j)^2$, $0 < k < N$, will yield an error $E' > E$.

3) If $\overline{e}_{N+1}$ is any other non-zero vector orthogonal to each of $\overline{e}_0, \overline{e}_1, \ldots, \overline{e}_N$, then $\| \overline{X} \|^2 - \sum_{j=0}^{N+1} (\overline{X}, \overline{e}_j)^2 < E$.

VI. SELECTION OF THE FUNCTION

After evaluating $\| \overline{X} \|^2 - \sum_{j=0}^{N} (\overline{X}, \overline{e}_j)^2$, we may find that this value still exceeds a given error tolerance δ. Then we wish to find $\overline{\varphi}_{N+1}$ such that

$$\| \overline{X} \|^2 - \sum_{j=0}^{N} (\overline{X}, \overline{e}_j)^2 - (\overline{X}, \overline{e}_{N+1})^2 \leq \delta;$$

i.e., find $\overline{\varphi}_{N+1}$ such that

$$(\overline{X}, \overline{e}_{N+1})^2 \geq \| \overline{X} \|^2 - \sum_{j=0}^{N} (\overline{X}, \overline{e}_j)^2 - \delta,$$

where $\overline{e}_{N+1}$ is the vector associated with $\overline{\varphi}_{N+1}$ that is orthonormal to $\overline{e}_0, \overline{e}_1, \ldots, \overline{e}_N$.

Suppose we let

$$\overline{\varphi}_{N+1} = (\lambda_0, \lambda_1, \ldots, \lambda_n).$$

Then

$$\bar{\varphi}'_{N+1} = \bar{\varphi}_{N+1} - (\bar{\varphi}_{N+1}, \bar{e}_0)\,\bar{e}_0 - \cdots - (\bar{\varphi}_{N+1}, \bar{e}_N)\,\bar{e}_N$$

$$= (\lambda_0,\lambda_1,\ldots,\lambda_n) - \Big(\sum_{i=0}^{n} \lambda_i\, e_{0i}\Big)\,\bar{e}_0 - \cdots$$

$$- \Big(\sum_{i=0}^{n} \lambda_i\, e_{Ni}\Big)\,\bar{e}_N,$$

if $\bar{e}_j = (e_{j0}, e_{j1}, \ldots, e_{jn})$, $j = 0,1,\ldots,N$.

Therefore,

$$\bar{e}_{N+1} = \frac{(\lambda_0,\lambda_1,\ldots,\lambda_n) - \Big(\sum_{i=0}^{n} \lambda_i\, e_{0i}\Big)\,\bar{e}_0 - \cdots - \Big(\sum_{i=0}^{n} \lambda_i\, e_{Ni}\Big)\,\bar{e}_N}{\left\{\sum_{i=0}^{n} \lambda_i^2 - \Big(\sum_{i=0}^{n} \lambda_i\, e_{0i}\Big)^2 - \cdots - \Big(\sum_{i=0}^{n} \lambda_i\, e_{Ni}\Big)^2\right\}^{1/2}},$$

and if $\bar{X} = (t_0,t_1,\ldots,t_n)$, then

$$(\bar{X}, \bar{e}_{N+1})^2 = \frac{\left[\sum_{i=0}^{n} \lambda_i\, t_i - \Big(\sum_{i=0}^{n} \lambda_i\, e_{0i}\Big)(\bar{X}, \bar{e}_0) - \cdots - \Big(\sum_{i=0}^{n} \lambda_i\, e_{Ni}\Big)(\bar{X}, \bar{e}_N)\right]^2}{\sum_{i=0}^{n} \lambda_i^2 - \Big(\sum_{i=0}^{n} \lambda_i\, e_{0i}\Big)^2 - \cdots - \Big(\sum_{i=0}^{n} \lambda_i\, e_{Ni}\Big)^2}.$$

Thus, to have

$$(\bar{X}, \bar{e}_{N+1})^2 \geq \|\bar{X}\|^2 - \sum_{j=0}^{N} (\bar{X}, \bar{e}_j)^2 - \delta,$$

we must have

$$\left[\sum_{i=0}^{n} \lambda_i\, t_i - \Big(\sum_{i=0}^{n} \lambda_i\, e_{0i}\Big)(\bar{X}, \bar{e}_0) - \cdots - \Big(\sum_{i=0}^{n} \lambda_i\, e_{Ni}\Big)(\bar{X}, \bar{e}_N)\right]^2 \geq$$

$$\left[\sum_{i=0}^{n} \lambda_i^2 - \Big(\sum_{i=0}^{n} \lambda_i\, e_{0i}\Big)^2 - \cdots - \Big(\sum_{i=0}^{n} \lambda_i\, e_{Ni}\Big)^2\right]\left[\|\bar{X}\|^2 - \sum_{j=0}^{N} (\bar{X}, \bar{e}_j)^2 - \delta\right],$$

or

$$\left[\sum_{i=0}^{n} \lambda_i \{t_i - (\overline{X}, \overline{e}_0) e_{0i} - \ldots - (\overline{X}, \overline{e}_N) e_{Ni}\}\right]^2 \geq$$

$$\left[\sum_{i=0}^{n} \lambda_i^2 - (\sum_{i=0}^{n} \lambda_i e_{0i})^2 - \ldots - (\sum_{i=0}^{n} \lambda_i e_{Ni})^2\right]\left[\| \overline{X} \|^2 - \sum_{j=0}^{N} (\overline{X}, \overline{e}_j)^2 - \delta\right],$$

or

$$\sum_{i=0}^{n} \left[\lambda_i^2 \{t_i - (\overline{X}, \overline{e}_0) e_{0i} - \ldots - (\overline{X}, \overline{e}_N) e_{Ni}\}^2\right.$$

$$+ 2\lambda_i \sum_{\substack{k=0\\k>i}}^{n} \lambda_k \{t_i - (\overline{X}, \overline{e}_0) e_{0i} - \ldots - (\overline{X}, \overline{e}_N) e_{Ni}\}\{t_k - (\overline{X}, \overline{e}_0) e_{0k} -$$

$$\left.\ldots - (\overline{X}, \overline{e}_N) e_{Nk}\}\right] \geq \sum_{i=0}^{n} \left[\lambda_i^2 - \lambda_i^2 e_{0i}^2 - 2\lambda_i \sum_{\substack{k=0\\k>i}}^{n} \lambda_k e_{0i} e_{0k} - \ldots\right.$$

$$\left.- \lambda_i^2 e_{Ni}^2 - 2\lambda_i \sum_{\substack{k=0\\k>i}}^{n} \lambda_k e_{Ni} e_{Nk}\right]\left[\| \overline{X} \|^2 - \sum_{j=0}^{N} (\overline{X}, \overline{e}_j)^2 - \delta\right],$$

or

$$\sum_{i=0}^{n} \left[\lambda_i^2 \left\{ \{t_i - (\overline{X}, \overline{e}_0) e_{0i} - \ldots - (\overline{X}, \overline{e}_N) e_{Ni}\}^2 - \{ \| \overline{X} \|^2 - \right.\right.$$

$$\sum_{j=0}^{N} (\overline{X}, \overline{e}_j)^2 - \delta\} + e_{0i}^2 \{ \| \overline{X} \|^2 - \sum_{j=0}^{N} (\overline{X}, \overline{e}_j)^2 - \delta\} + \ldots +$$

$$e_{Ni}^{2} \{ \| \overline{X} \|^{2} - \sum_{j=0}^{N} (\overline{X}, \overline{e}_j)^{2} - \delta\} \Big\} + \lambda_i \Big\{ 2 \sum_{\substack{k=0\\k>i}}^{n} \lambda_k \{t_i - (\overline{X}, \overline{e}_0)\, e_{0i}$$

$$- \dots - (\overline{X}, \overline{e}_N)\, e_{Ni}\}\{t_k - (\overline{X}, \overline{e}_0)\, e_{0k} - \dots - (\overline{X}, \overline{e}_N)\, e_{Nk}\} +$$

$$2 \{ \| \overline{X} \|^{2} - \sum_{j=0}^{N} (\overline{X}, \overline{e}_j)^{2} - \delta\}\{ \sum_{\substack{k=0\\k>i}}^{n} \lambda_k\, e_{0i}\, e_{0k} + \dots + \sum_{\substack{k=0\\k>i}}^{n} \lambda_k\, e_{Ni}\, e_{Nk}\} \Big\} \Big]$$

$$\geq 0.$$

If we let

$$A_i = \Big\{\{t_i - (\overline{X}, \overline{e}_0)\, e_{0i} - \dots - (\overline{X}, \overline{e}_N)\, e_{Ni}\}^{2} - \{ \| \overline{X} \|^{2} - \sum_{j=0}^{N} (\overline{X}, \overline{e}_j)^{2} - \delta\}$$

$$+ e_{0i}^{2} \{ \| \overline{X} \|^{2} - \sum_{j=0}^{N} (\overline{X}, \overline{e}_j)^{2} - \delta\} + \dots + e_{Ni}^{2} \{ \| \overline{X} \|^{2} - \sum_{j=0}^{N} (\overline{X}, \overline{e}_j)^{2} - \delta\} \Big\},$$

and

$$B_i = 2 \Big\{ \sum_{\substack{k=0\\k>i}}^{n} \lambda_k \{t_i - (\overline{X}, \overline{e}_0)\, e_{0i} - \dots - (\overline{X}, \overline{e}_N)\, e_{Ni}\}\{t_k - (\overline{X}, \overline{e}_0)\, e_{0k} -$$

$$\dots - (\overline{X}, \overline{e}_N)\, e_{Nk}\} + \{ \| \overline{X} \|^{2} - \sum_{j=0}^{N} (\overline{X}, \overline{e}_j)^{2} - \delta\}\{ \sum_{\substack{k=0\\k>i}}^{n} \lambda_k\, e_{0i}\, e_{0k} +$$

$\dots + \sum_{\substack{k=0\\k>i}}^{n} \lambda_k\, e_{Ni}\, e_{Nk}\} \Big\}$, we can write this inequality as

$\sum_{i=0}^{n} (A_i \lambda_i^2 + B_i \lambda_i) \geq 0$, and this inequality is satisfied if

$A_i \lambda_i^2 + B_i \lambda_i \geq 0$, for $i = 0, 1, \ldots, n$. Notice that these conditions are much stronger than are necessary and we will need to examine some cases that might arise.

<u>Case 1</u>: If $A_i \geq 0$ for some i, $0 \leq i \leq n$, choose $\lambda_t = 0$, $t = n, n-1, \ldots, i_0 + 1$, where i_0 is the largest value of i such that $A_i \geq 0$, $\lambda_{i_0} = 1$, and $\lambda_k = -\dfrac{B_k}{A_k}$, $k = 0, 1, \ldots, i_0 - 1$, provided $A_k \neq 0$, or $\lambda_k = B_k$, $k = 0, 1, \ldots, i_0 - 1$, for $A_k = 0$.

<u>Case 2</u>: If $A_i < 0$, for all i, tentatively choose $\lambda_k = 1$ and examine

(1) $B_{k-1}^2 - 4A_k A_{k-1} \geq 0$, $k = n, n-1, \ldots, 2, 1, 0$.

If (1) is not true, choose $\lambda_k = 0$ and proceed to examine,

(2) $B_{k-2}^2 - 4A_{k-1} A_{k-2} \geq 0$ for $\lambda_{k-1} = 1$.

If (2) is false, choose $\lambda_{k-1} = 0$ and proceed as before.

If (1) is satisfied for some value of k, let i be the first such positive integer in the sequence n, n-1, ... , 1, 0. Then $(B_{i-1})^2 - 4A_{i-1} A_i \lambda_i^2 \geq 0$, and we are assured of a solution λ_{i-1} to the equation

$$A_{i-1} \lambda_{i-1}^2 + B_{i-1} \lambda_{i-1} + A_i \lambda_i^2 = 0.$$

Notice that the left side of this equation is just the sum of the ith and (i-1)st terms of the sum $\sum_{i=0}^{n} (A_i \lambda_i^2 + B_i \lambda_i)$. Thus, let λ_{i-1} be either solution of the equation

$$A_{i-1} \lambda_{i-1}^2 + B_{i-1} \lambda_{i-1} + A_i \lambda_i^2 = 0.$$

Then $\lambda_j = -\frac{B_j}{A_j}$, $j = i-1, \ldots , 1, 0,$

will assure the satisfaction of the suceeding inequalities.

In the newly computed vector $\bar{\varphi}_{N+1} = (\lambda_0, \lambda_1, \ldots , \lambda_n)$, suppose we let λ_i be the value of some ideal function $\varphi_{N+1}(\beta)$ at β_i; i.e., $\varphi_{N+1}(\beta_i) = \lambda_i$. Then this ideal function assures us that the error E, where

$$E = \sum_{i=0}^{n} \left[X(\beta_i) - \sum_{j=0}^{N} A_j \varphi_j(\beta_i) - A_{N+1} \varphi_{N+1}(\beta_i)\right]^2,$$

is less than the imposed tolerance δ. Since we know the values of this ideal function at the tabular values β_i, our next objective is to develop a technique for computing $\varphi_{N+1}(\beta')$, for some value $\beta' \neq \beta_i$, $i = 0,1,\ldots,n$, such that the error obtained by using $\sum_{j=0}^{N+1} A_j \varphi_j(\beta')$ to approximate $X(\beta')$, in the sense of least squares, is as small, if not smaller, than the error obtained by approximating $X(\beta')$ with $\sum_{j=0}^{N} A_j \varphi_j(\beta')$. We obtain this value $\varphi_{N+1}(\beta')$ in the following manner.

First, we compute $A_{N+1}(k)$, $k = -1, 0, 1, \ldots, N$, $\bar{e}_{N+1}$ and A'_{N+1} as follows:

$$A_{N+1}(-1) = \frac{1}{\left\| \bar{\varphi}_{N+1} - \sum_{j=0}^{N} (\bar{\varphi}_{N+1}, \bar{e}_j)\, \bar{e}_j \right\|}$$

$$A_{N+1}(0) = A_{N+1}(-1)\, A_0(-1)(\bar{\varphi}_{N+1}, \bar{\varphi}_0)$$

$$\vdots$$

$$A_{N+1}(N) = A_{N+1}(-1)\, A_N(-1)(\bar{\varphi}_{N+1}, \bar{\varphi}_N) - \sum_{j=0}^{N-1} A_{N+1}(j)\, A_N(j).$$

$$\bar{e}_{N+1} = A_{N+1}(-1)\, \bar{\varphi}_{N+1} - \sum_{j=0}^{N} A_{N+1}(j)\, \bar{e}_j$$

$$A'_{N+1} = (\bar{X}, \bar{e}_{N+1}).$$

Finally, compute the (N+2) A_j's, $j = 0, 1, \ldots, N+1$, as follows:

$$A_{N+1} = A'_{N+1}\, A_{N+1}(-1)$$

$$A_N = A_N(-1)\left[A'_N - A'_{N+1} A_{N+1}(N) \right]$$

$$A_{N-1} = A_{N-1}(-1)\left\{A'_{N-1} - A'_N A_N(N-1) + A'_{N+1}\left[- A_{N+1}(N-1) + A_{N+1}(N)A_N(N-1) \right]\right\},$$

$$\vdots$$

Now let $\beta_{i'}$ be a β_i such that $\| \beta_{i'} - \beta' \| = \min\limits_{0 \le i \le n} \left\{ \| \beta_i - \beta' \| \right\}$,

and let us define the following function:

$$\sum_{j=0}^{N} A_j \varphi_j (\beta') + A_{N+1} M(\beta'),$$

$$\text{where } M(\beta') = \lambda_{i'} \left[\frac{L(\beta_{i'}) - 2 \| \beta_{i'} - \beta' \|}{L(\beta_{i'})} \right],$$

$$\text{for } 2 \| \beta_{i'} - \beta' \| < L (\beta_{i'}),$$

$$= 0, \text{ otherwise},$$

$$\text{where } L(\beta_{i'}) = \min_{\substack{0 \le i \le n \\ i \ne i'}} \left\{ \| \beta_i - \beta_{i'} \| \right\}.$$

Thus, when β' is chosen, we are able to use the function above to approximate $X(\beta')$, being assured that the approximation obtained here is no worse that the value $\sum\limits_{j=0}^{N} A_j \varphi_j (\beta')$ obtained by using the initial least squares approximating function.

Writing this multiple of λ_i as

$$\frac{\frac{1}{2} L(\beta_{i'}) - \| \beta_{i'} - \beta' \|}{\frac{1}{2} L(\beta_{i'})},$$

we see that we have a factor which varies from zero to one as β' varies from a position on the boundary to a position at the center of the ball

$$\left\{ \beta \,\middle|\, \| \beta_{i'} - \beta' \| \leq \frac{1}{2} L(\beta_{i'}) \right\} .$$

Thus, the factor $\lambda_{i'}$ which was derived in association with the vector $\beta_{i'}$ is weighted depending on the nearness of β' to $\beta_{i'}$.

VII. RECOMMENDATIONS

The contractor recommends that the computation technique developed in this report be utilized as soon as feasible in constructing least squares approximating functions for

1) The development of steering programs needed by space vehicles at each instant of time in order to satisfy given cut-off equations in an optimum manner, and

2) The development of optimum trajectories satisfying given cut-off equations.

VIII. BIBLIOGRAPHY

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