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TWO MODELS FOR TIME CONSTRAINED MAINTENANCE

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The RAND Corporation
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PREFACE

This RAND Memorandum derives methods for computing probabilities associated with certain maintenance policies. Theoretical in nature, it is part of RAND's continuing interest in mathematical properties of maintenance policies. The focus is on systems whose operating life is restricted to a limited time span. This contrasts with the situation, often considered, of an infinite time horizon.

This Memorandum is addressed to statisticians, operations researchers, and others concerned with properties of maintenance policies. The investigation is an outgrowth of work on checkout and maintenance operations performed as a part of the Apollo Checkout System Study which RAND is conducting for Headquarters, National Aeronautics and Space Administration, under Contract NASr-21(08).

SUMMARY

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This Memorandum examines time constrained maintenance policies. It derives the probability that all of a number of initially inoperative systems will function at a fixed time after repair begins. Two maintenance disciplines are considered: in the first, no system undergoing repair is turned on until all systems have been fixed, then all are turned on simultaneously ("deferred turn-on"); in the second, a system is turned on as soon as it is repaired ("immediate turn-on"). In both disciplines only one system can be in repair at a time. It is assumed that failure times and repair times are exponentially distributed with known parameters and that these times are totally independent. Each system is repaired only once.

An explicit expression for the optimum time to begin repair is obtained for only the simplest case of one system; however, the probability of all systems functioning at a certain time is given in closed form, so that in a more general case the optimum time can be numerically obtained.

The focus of this Memorandum is on systems whose operating life is restricted to a limited time span. This contrasts with the situation, often considered, of an infinite time horizon.

AUTHOR

CONTENTS

I. INTRODUCTION

Maintenance policies for systems whose time-to-failure and repair time are random variables have been intensively studied in recent years. (See Ref. [1] for a survey.) Among the topics treated in this area are the timing of checkout, repair, and/or replacement activities. Often such times are chosen to optimize some property asymptotically. Clearly, results that hold for systems operated indefinitely need not hold for systems operated only a finite time or systems that need to operate only upon demand. In this Memorandum we study the maintenance of systems whose operating life is restricted to a limited time span. We will pay particular attention to the probability that our systems are working at the end of the period of interest.

In Sec. II we present some mathematical details needed as a background for our investigation.

In Sec. III we assume it known that a certain number of systems need repair, that only one system can be worked on at a time, that failure times and repair times are exponentially distributed with known parameters, and that these times are totally independent. We derive the probability that these failed systems are all working at a fixed future time for two situations: in the first, no system undergoing repair is turned on until all systems have been fixed, then all are turned on simultaneously ("deferred turn-on"); in the second, a system is turned on as soon as it is repaired ("immediate turn-on"). Each system is repaired only once.

II. SOME MATHEMATICAL PRELIMINARIES

In this section we record some formulas and results that will be used in Section III.

A. The exponential distribution with parameter λ ($\lambda > 0$) is given by

$$f(t) = \begin{cases} \lambda e^{-\lambda t} , & t \geq 0 \\ 0 , & t < 0 \end{cases} \quad . \tag{1}$$

If a lifetime (repair time) has the exponential distribution with parameter λ, the probability that the length of life (time to effect the repair) is at least t is $e^{-\lambda t}$.

B. If $X_1, \ldots, X_n$ are independent, non-negative random variables with density functions $f_1, \ldots, f_n$, respectively, and if $S = X_1 + \ldots + X_n$, then the density, f, of S is given by

$$f = f_1 * \cdots * f_n \tag{2}$$

where $f_1 * f_2$ is the <u>convolution</u> of f_1 and f_2 defined by

$$(f_1 * f_2)(t) = \int_0^t f_1(u) f_2(t - u)\, du \quad . \tag{3}$$

The convolution of more than two densities is defined recursively by

$$f_1 * \cdots * f_n = (f_1 * \cdots * f_{n-1}) * f_n \quad . \tag{4}$$

Equation (2) becomes, explicitly,

$$f(t) = \int_0^t \int_0^{u_n} \cdots \int_0^{u_2} f_1(u_1) f_2(u_2 - u_1) \cdots f_{n-1}(u_n - u_{n-1}) f_n(t - u_n)\, du_1 \cdots du_{n-1}\, du_n . \tag{5}$$

C. If $X_1, X_2, \ldots, X_n$ are independent and have exponential distributions with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, the density f of their sum $S = X_1 + \ldots + X_n$ depends on the relations among the parameters λ_i.

(i) If $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \lambda$ (say), then it is well known (e.g., [2, p. 11]) that S has the gamma distribution with scale parameter λ and shape parameter n. That is

$$f(t) = \begin{cases} \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} & , \quad t \geq 0 \\ 0 & , \quad t < 0 \ . \end{cases} \tag{6}$$

(ii) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct, then [2, p. 17]

$$f(t) = \begin{cases} \sum_{i=1}^{n} \lambda_i e^{-\lambda_i t} \sum_{\substack{j=1 \\ j \neq i}}^{n} \lambda_j / (\lambda_j - \lambda_i) & , \quad t \geq 0 \\ 0 & , \quad t < 0 \ . \end{cases} \tag{7}$$

(iii) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are such that

$$\begin{aligned} \lambda_1 = \ldots = \lambda_{r_1} &= \alpha_1 \\ \lambda_{r_1+1} = \ldots = \lambda_{r_1+r_2} &= \alpha_2 \\ &\vdots \\ \lambda_{r_1+\ldots+r_{m-1}+1} = \ldots = \lambda_{r_1+\ldots+r_m} &= \alpha_m \end{aligned}$$

with the α_i distinct, each $r_i \geq 1$, and $r_1 + \ldots + r_m = n$, then

$$f(t) = \begin{cases} \prod_{j=1}^{m} \alpha_j^{r_j} \sum_{i=1}^{m} \sum_{\ell=1}^{r_i} \frac{\Phi_{i\ell}(-\alpha_i)}{(i-\ell)!(\ell-1)!} t^{r_i-\ell} e^{-\alpha_i t} & , t \geq 0 \\ 0 & , t < 0 \end{cases} \tag{8}$$

with

$$\Phi_{i\ell}(x) = \frac{d^{\ell-1}}{dx^{\ell-1}} \prod_{\substack{j=1 \\ j\neq i}}^{m} (\alpha_j + x)^{-r_j} . \tag{9}$$

There seems to be no convenient closed form for $\Phi_{i\ell}(-\alpha_i)$ in general. If each $r_i = 1$, however, equation (8) reduces to equation (7), as is to be expected.

Equation (8) was obtained by inverting the Laplace transform of the density of the sum,* not from equation (5).

D. If $X_1, X_2, \ldots, X_n$ are independent and have exponential distributions with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, then $X = \min (X_1, X_2, \ldots, X_n)$ has the exponential distribution with parameter $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

E. If f is the probability density function of system repair time and $\overline{G}$ is the system survival probability function (that is, $\overline{G}(x)$ is the probability that time-to-failure exceeds x), then the probability P(t) that a system which is inoperative at time zero has been repaired and is operating at time t, is given by

$$P(t) = \int_0^t f(u) \, \overline{G} \, (t - u) \, du . \tag{10}$$

This is assuming that the repair starts at time zero and the system is turned on immediately when repaired, and also assuming that repair and failure times are independent.

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*I thank Albert Madansky for this result.

III. TWO MAINTENANCE DISCIPLINES

INTRODUCTION

We consider the following problem in this section. It is known that a certain number, k, of systems are inoperative; that the repair time for the i-th system is a random variable having exponential distribution with parameter γ_i; that the failure time for the i-th system is a random variable having exponential distribution with parameter λ_i; that all failure and repair times are independent; and that the standing failure rate and the turn-on stress are zero.

We consider two maintenance disciplines. In the first, no repaired system is turned on until all systems have been repaired and then all are turned on simultaneously. In the second, a system is turned on as soon as it is repaired. We call these disciplines "deferred turn-on" and "immediate turn-on," respectively. Of course, they coincide for the case of only one failed system. In both situations, we suppose that only one system can be undergoing repair at a time. Each system is repaired only once.

For the models and under the assumptions described above, we derive the probability P(t) that all the systems are operating at a time t after the repairs have begun. This is equivalent to the probability that all systems will be working at a fixed time T if repairs are begun at time T - t. It would be desirable to give the time, say $\hat{t}$, which maximizes P(t) as this would determine the time, $T - \hat{t}$, when repairs should begin so as to maximize the probability of all systems being operative at time T. We are able to give an explicit formula for $\hat{t}$ only in the case of one system. However, P(t) is given in closed form for both maintenance disciplines considered, so that $\hat{t}$ can be obtained numerically for any given situation to any desired accuracy.*

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*A closely related problem in which optimum times are obtained is discussed by Port [3].

ONE SYSTEM*

The time-to-failure distribution is exponential with parameter λ, the repair time distribution is exponential with parameter γ, and these times are independent. We denote by P(t) the probability that a failed system has been repaired and is still operating at a time t after repair begins. From equation (10)

$$P(t) = \int_0^t \gamma\, e^{-\gamma u}\, e^{-\lambda(t-u)}\, du$$

$$= \frac{\gamma}{\gamma-\lambda} \left(e^{-\lambda t} - e^{-\gamma t}\right) \quad . \qquad (11)$$

The value $\hat{t}$ which maximizes P(t) is easily obtained by differentiation of P(t) and is

$$\hat{t} = \frac{\ln \gamma - \ln \lambda}{\gamma - \lambda} \quad . \qquad (12)$$

Substituting (12) into (11) yields for the maximum probability

$$P(t)_{max} = P(\hat{t}) = \left(\frac{\lambda}{\gamma}\right)^{\lambda/(\gamma-\lambda)} \quad . \qquad (13)$$

k SYSTEMS: DEFERRED TURN-ON

We assume k systems are inoperative, that the repair time distribution of the i-th system is exponential with parameter γ_i, that the time-to-failure distribution of the i-th system is exponential with parameter λ_i,

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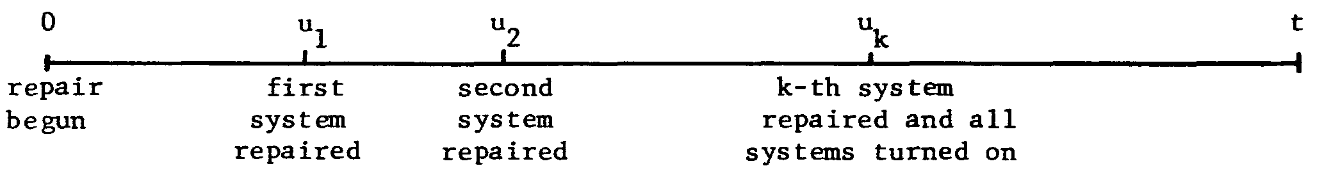
*Truelove [4] has discussed the case for a single system where precisely one checkout is to be performed during a fixed interval of time prior to a critical event. He assumes exponentially distributed failure times, and log-normally distributed repair times, and gives tables and graphs for the optimum checkout timing when all parameters are known, and an iterative min-max procedure when some of the parameters are uncertain.

and that these times are totally independent. Only one system can be in repair at any given time. The turn-on of a repaired system is deferred until each system has been repaired, then all systems are turned on simultaneously. Let P(t) denote the probability that all the systems have been repaired and are all still working at a time t after the repairs have begun.

The sketch below illustrates this situation.



Note that we can write $u_k = (u_k - u_{k-1}) + \ldots + (u_2 - u_1) + u_1$, and that $u_i - u_{i-1}$ is the repair time of the i-th system, i = 1, ..., k, with the convention that $u_0 = 0$. Thus, by our various distribution and independence assumptions, the density, f, of u_k is given by $f = f_1 * \cdots * f_k$. Depending on whether the parameters γ_i are all equal, distinct, or neither all equal nor distinct, the form of the density f of u_k is given by equation (6), (7), or (8) with n = k and $\lambda_i = \gamma_i$.

Now denote by λ the sum $\lambda_1 + \ldots + \lambda_k$. Then by the considerations of Secs. II.D and II.E, we have:

(i) if all the γ_i are equal to γ (say)

$$P(t) = \frac{\gamma^k}{(k-1)!} \int_0^t x^{k-1} e^{-\gamma x} e^{-\lambda(t-x)} dx$$

$$= \left(\frac{\gamma}{\gamma-\lambda}\right)^k \frac{e^{-\lambda t}}{(k-1)!} \int_0^{(\gamma-\lambda)t} y^{k-1} e^{-y} dy ; \qquad (14)$$

(ii) if the γ_i are distinct

$$P(t) = \sum_{i=1}^{k} [\gamma_i/(\gamma_i - \lambda)] \prod_{\substack{j=1 \\ j\neq i}}^{k} [\gamma_j/(\gamma_j - \gamma_i)] \left(e^{-\lambda t} - e^{-\gamma_i t}\right); \quad (15)$$

(iii) if $\gamma_1, \ldots, \gamma_k$ are such that

$$\begin{aligned} \gamma_1 = \cdots = \gamma_{r_1} &= \alpha_1 \\ \gamma_{r_1+1} = \cdots = \gamma_{r_1+r_2} &= \alpha_2 \\ &\vdots \\ \gamma_{r_1+\ldots+r_{m-1}+1} = \cdots = \gamma_{r_1+\ldots+r_m} &= \alpha_m \end{aligned}$$

with the α_i distinct, each $r_i \geq 1$, and $r_1 + \ldots + r_m = k$, then

$$P(t) = \prod_{j=1}^{m} \alpha_j^{r_j} \sum_{i=1}^{m} \sum_{\ell=1}^{r_i} \frac{\Phi_{i\ell}(-\alpha_i)\, e^{-\lambda t}}{(i-\ell)!(\ell-1)!(\alpha_i-\lambda)^{r_i-\ell+1}} \int_0^{(\alpha_i-\lambda)u} y^{r_i-\ell} e^{-y}\, dy \, . \quad (16)$$

The integrals in Formulas (14) and (16) can, for some parameter combinations, be evaluated from Pearson's tables [5], or generally by using the fact that for positive integers h

$$\int_0^A e^{-x} x^h \, dx = h!\left[1 - e^{-A} \sum_{i=0}^{h} A^i/i!\right] . \quad (17)$$

Depending on the combination of parameters appearing in equations (14), (15), or (16), P(t) may have one or several relative maxima. In the latter case, one would want to determine the value, $\hat{t}$, of t which yields the largest of these relative maxima. However, since no system is turned on until all systems have been repaired, it is clearly immaterial in which order the systems are repaired.

k SYSTEMS: IMMEDIATE TURN-ON

We assume k systems are inoperative, that the repair time distribution of the i-th system is exponential with parameter γ_i, that the time-to-failure distribution of the i-th system is exponential with parameter λ_i, and that these times are totally independent. Only one system can be in repair at any given time. Each system is turned on as soon as it is repaired. Let P(t) denote the probability that all systems have been repaired and are working at a time t after the repairs have begun.

As before, we will exploit the results of Sec. II to determine P(t). Writing $u_0 = 0$,

$$t - u_i = (t - u_k) + (u_k - u_{k-1}) + \dots + (u_{i+1} - u_i)$$

and

$$\begin{aligned} \alpha_1 &= \gamma_1 \\ \alpha_i &= \gamma_i + \lambda_1 + \dots + \lambda_{i-1}, \qquad i = 2, \dots, k \qquad (18) \\ \alpha_{k+1} &= \lambda_1 + \dots + \lambda_k, \end{aligned}$$

we have

$$P(t) = \int_0^t \int_0^{u_k} \dots \int_0^{u_3} \int_0^{u_2} e^{-\alpha_{k+1}(t-u_k)} \prod_{i=1}^{k} \gamma_i \, e^{-\alpha_i(u_i-u_{i-1})} \, du_i$$

or

$$P(t) = \left(\prod_{i=1}^{k} \gamma_i \Big/ \prod_{i=1}^{k+1} \alpha_i \right) \left[(f_1 * \dots * f_{k+1})(t) \right], \qquad (19)$$

where

$$f_i(t) = \alpha_i e^{-\alpha_i t} . \qquad (20)$$

Thus P(t) is obtained (mutatis mutandis) from equation (6), (7), or (8) according as the α_i are all equal, distinct, or neither all equal nor distinct.

Unlike the deferred turn-on maintenance discipline, here the order in which systems are repaired matters. So any optimization of P(t) by appropriate choice of t must be for a given repair schedule. We are not able to solve, in general, the problem of determining the best repair schedule. Common sense suggests that if $\gamma_1 \le \gamma_2 \le \ldots \le \gamma_k$ and $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_k$, repair the systems in order 1, 2, ..., k. That is, if system 1 has the largest mean repair time and the largest time-to-failure among the k systems, repair it first, etc. This statement, however, remains a conjecture.

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