VON KARMAN CENTER

ADVANCED PROGRAMS DEPARTMENT

THERMAL STRAIN ANALYSIS OF ADVANCED MANNED SPACECRAFT HEAT SHIELDS

Second Quarterly Status Report
to the

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AEROJET
GENERAL TIRE
GENERAL

CONTRACT FULFILLMENT STATEMENT

This is the second of three quarterly progress reports submitted in partial fulfillment of the National Aeronautics and Space Administration Contract No. NAS 9-1986. This report covers the period from 1 December 1963 to 29 February 1964.

Approved:

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I. SUMMARY OF PROGRESS TO DATE

During the reporting period the derivation of all equations and boundary conditions based on "thick-shell" and "thin-shell" theory in spherical and toroidal coordinates was completed. The full set of equations for the "thin-shell" theory and the sequence for their solution is included in this report, as is the finite-difference formulation for this case.

A re-evaluation of the equations near the singularity on the geometric axis-of-symmetry was made during the reporting period. This re-evaluation resulted in a useable set of equations for the singular-point for non-axisymmetric case. These equations are also presented in this report. As stated in the previous quarterly report, since this point is common to all meridian planes, using it as a common node would considerably reduce the total number of nodes.

Efforts were initiated during the reporting period in the areas of overrelaxation and convergence criteria. Initial results will be reported in the next monthly report.

The effort in the programming area continued on the axisymmetric case. It is expected that this case will be completed by 1 April.

Details of accomplishments since the last report are below, based on the revised schedule presented in the first quarterly report.

A. INVESTIGATION OF ENGINEERING MODELS (A" in Schedule)

The full set of displacement equations of the thin-shell theory in tori-spherical coordinates and the proper sequence to be employed in their solution is presented in Appendix A. The "zero-order" and "first-order" parts of the equations are clearly indicated. A comparison of this set of equations with the equations generated using the "standard" thin-shell formulation reveals plainly the inconsistencies in the "standard" formulation. These inconsistencies are manifested by missing terms in the "first-order" equations. Thus, the standard formulation is at best a "zero-order" formulation. However, the consistent zero-order formulation is considerably simpler than the "standard" formulation.

As mentioned in the fourth monthly report, an attempt was made to avoid fourth-order terms in the derivation. This was accomplished at the cost of increasing the number of equations from two to six as shown in Appendix A.

The finite-difference equivalent for the "thin-shell" equations is presented in Appendix B.

(This completes Task A").

B. THE SINGULAR POINT

A re-examination of the equations for the non-axisymmetric case at the singularity on the axis-of-symmetry was made during the month of February. This effort was prompted by the progress made in the programming of the non-axisymmetric case which requires special treatment near the singularity. This effort resulted in the derivation of a programmable relatively simple set of equations for this point for the general non-axisymmetric case. These equations are presented in Appendix C.

C. DEVELOP CONVERGENCE CRITERIA

An effort in this area was initiated during the reporting period. It is expected that initial results will be presented in the next reporting period.

D. OVERRELAXATION METHOD

A considerable effort was expended in this area during the reporting period. It was established that "line-relaxation" offers considerable advantages over point-relaxation for the present problem. Various line-relaxation approaches were investigated. Presently the "lines" used are those of constant R (and appropriate constant r) because they assure the proper effect of the boundary conditions on the solution and enable "starting" the solution in a relatively easy fashion.

E. PROGRAMMING

The effort in the programming area was concentrated on the completion of the axisymmetric case. It is expected that this effort will be completed by 1 April 1964.

The major problems encountered are in satisfying all the boundary conditions by the "first-guess" solution without generating discontinuities at nearby mesh-points. The difficulty stems from the fact that the finite-difference formulation utilizing the mesh-size imposed by the problem does not model the (continuous) derivatives satisfactorily near the boundary when standard finite-difference formulations are utilized. A considerable number of non-standard formulations have already been tried providing insight into the behavior of the equations and the formulations that offer most promise.

With the receipt of the information on ablator-thickness and base-temperature data, the programming of the final three-dimensional problem has resumed.

II. PLANNED ACTIVITIES FOR THE NEXT REPORTING PERIOD

A. DEVELOP CONVERGENCE CRITERIA

Effort in this area will continue during the next reporting period with initial results expected to be available for reporting in the next monthly report.

B. OVERRELAXATION METHOD

With the expected completion of the axisymmetric case during the next reporting period (by 1 April), details of the results of this effort should also be available for reporting in the next monthly report.

C. SANDWICH CORE PROPERTIES ANALYSIS

In order to enable fulfillment of the requirement for a "five-layer" solution of the final non-axisymmetric case, the proper sandwich-core properties will have to be developed. This effort will be initiated during the next reporting period. An unisotropic analysis of the sandwich core will be performed. Based on this analysis the "equivalent-isotropic" values of the material properties utilized in the equations of thin and thick shell analysis will be developed.

D. PROGRAMMING

The axisymmetric test case will be completed by 1 April. The programming effort on the non-axisymmetric case will continue.

III. PROBLEM AREAS

No new technical problem areas have developed since the last report.

IV. PROGRAM CHANGES

During a meeting on 2 March 1964 between NASA/MSC and Aerojet, tentative agreement was reached on the following changes in personnel for this program.

Program Manager: W. T. Cox - replacing Dr. A. Zukerman

Principal Investigator: W. T. Cox - replacing B. Mazelsky

Consultant: Dr. H. H. Hilton - replacing Dr. A. J. A. Morgan

These changes will not affect the performance and schedule of this contract.

APPENDIX A

Detailed Stress Equations for the Thin-Shell Solution

The zero-order and first-order stress equations for the thin-shell solution are derived in the toroidal system since the spherical case is readily attained from the toroidal case by letting a = 0.

The first-order expansion of the tangential stresses are

$$\sigma_{\varphi\varphi} = \sigma_{\varphi\varphi_0} + \left(\frac{z}{\rho}\right)\sigma_{\varphi\varphi_1} \tag{1}$$

$$\sigma_{\theta\theta} = \sigma_{\theta\theta_0} + \left(\frac{z}{\rho}\right)\sigma_{\theta\theta_1} \tag{2}$$

$$\sigma_{\varphi\theta} = \sigma_{\varphi\theta_0} + \left(\frac{z}{\rho}\right)\sigma_{\varphi\theta_1} \tag{3}$$

where $\sigma_{\varphi\varphi_0}$, $\sigma_{\varphi\varphi_1}$, $\sigma_{\theta\theta_0}$, $\sigma_{\theta\theta_1}$, $\sigma_{\varphi\theta_0}$, and $\sigma_{\varphi\theta_1}$ have been derived in Reference 2, Equations (48) through (53) of Appendix A. Substituting these equations into equations (1), (2) and (3) and simplifying the stress equations in toroidal coordinates are obtained in terms of u_0, v_0, w_0 and σ_{rr_2}.

Detailed Stress Equations for the Thin-Shell Solution (cont.)

$$
\begin{aligned}
\sigma_{\varphi\varphi} = {} & C_1 \left[\frac{1}{\rho}\right]\frac{\partial v_o}{\partial \varphi} - C_1\left[\frac{\nu_o}{s}\right]\frac{\partial w_o}{\partial \theta} + C_1\left[\frac{\nu_o \cos\varphi}{s}\right] v_o \\
& + C_1\left[\frac{1}{\rho} + \frac{\nu_o \sin\varphi}{s}\right] u_o - \frac{E_o \alpha_o T_o}{1-\nu_o} - \left[\frac{\nu_o}{1-\nu_o}\right]\left[\left(\frac{p_1+p_2}{2}\right) + \left(\frac{h}{2\rho}\right)^2 \sigma_{rr_2}\right] \qquad \text{zero order terms} \\
& + C_2\left[\frac{z}{\rho^2}\right]\frac{\partial v_o}{\partial \varphi} - C_1\left[\frac{z}{\rho^2}\right]\left[u_o + \frac{\partial^2 u_o}{\partial \varphi^2} - \frac{\rho}{G_o}\frac{\partial \sigma_{r\varphi_o}}{\partial \varphi}\right] - C_3\left[\frac{z}{\rho s}\right]\frac{\partial w_o}{\partial \theta} \\
& - C_1\left[\frac{\nu_o z}{s}\right]\left[\frac{\sin\varphi \cos\varphi}{s} v_o + \frac{\sin^2\varphi \, u_o}{s} + \frac{1}{s}\frac{\partial^2 u_o}{\partial \theta^2} + \frac{1}{G_o}\frac{\partial \sigma_{r\theta_o}}{\partial \theta}\right] \\
& + C_3\left[\frac{z}{\rho}\frac{\cos\varphi}{s}\right] v_o + C_1\left[\frac{\nu_o z}{\rho}\frac{\cos\varphi}{s}\right]\left[v_o - \frac{\partial u_o}{\partial \varphi} + \frac{\rho}{G_o}\sigma_{r\varphi_o}\right] \\
& + \left[C_2\left(\frac{z}{\rho^2}\right) + C_3\left(\frac{z}{\rho}\frac{\sin\varphi}{s}\right)\right] u_o \qquad \text{first order terms} \qquad (4) \\
& + C_1\left[\frac{z}{\rho^2} + \frac{\nu_o z}{\rho}\frac{\sin\varphi}{s}\right]\left\{\frac{-\nu_o}{1-\nu_o}\left[\frac{\partial v_o}{\partial \varphi} + u_o + \frac{\rho}{s}\left(u_o \sin\varphi + v_o \cos\varphi - \frac{\partial w_o}{\partial \theta}\right)\right]\right. \\
& \qquad \left. + \left[\frac{\rho}{E_o}\frac{(1-2\nu_o)(1+\nu_o)}{1-\nu_o}\right]\sigma_{rr_o} + \left(\frac{1+\nu_o}{1-\nu_o}\right)\rho \, \alpha_o \, T_o\right\} \\
& - \left(\frac{z}{\rho}\right) C_4 + \left(\frac{z}{h}\right)\left[\frac{\nu_o}{1-\nu_o}\right]\left[(p_1 - p_2)\right] - \frac{z}{\rho}\left[\frac{\nu_1}{(1-\nu_o)^2}\right]\left[\left(\frac{p_1+p_2}{2}\right) + \left(\frac{h}{2\rho}\right)^2 \sigma_{rr_2}\right]
\end{aligned}
$$

Detailed Stress Equations for the Thin-Shell Solution (cont.)

$$\sigma_{\theta\theta} = C_1\left[\frac{\nu_o}{\rho}\right]\frac{\partial v_o}{\partial\varphi} - C_1\left[\frac{1}{s}\right]\frac{\partial w_o}{\partial\theta} + C_1\left[\frac{\cos\varphi}{s}\right] v_o$$

$$+ C_1\left[\frac{\sin\varphi}{s} + \frac{\nu_o}{\rho}\right] u_o - \frac{E_o\alpha_o T_o}{1-\nu_o} - \left[\frac{\nu_o}{1-\nu_o}\right]\left[\left(\frac{p_1+p_2}{2}\right) + \left(\frac{h}{2\rho}\right)^2 \sigma_{rr_2}\right]$$

zero order terms

$$+ C_3\left[\frac{z}{\rho^2}\right]\frac{\partial v_o}{\partial\varphi} - C_1\left[\frac{\nu_o z}{\rho^2}\right]\left[u_o + \frac{\partial^2 u_o}{\partial\varphi^2} - \frac{\rho}{G_o}\frac{\partial\sigma_{r\varphi o}}{\partial\varphi}\right] - C_2\left[\frac{z}{\rho s}\right]\frac{\partial w_o}{\partial\theta}$$

$$- C_1\left[\frac{z}{s}\right]\left[\frac{\sin^2\varphi}{s} u_o + \frac{\rho}{s}\frac{\partial^2 u_o}{\partial\theta^2} + \frac{\rho}{G_o}\frac{\partial\sigma_{r\theta_o}}{\partial\theta}\right] + C_1\left[\frac{z\cos\varphi}{\rho s}\right]\cdot$$

$$\left[(1-\frac{\rho}{s})v_o - \frac{\partial u_o}{\partial\varphi} + \frac{\rho}{G_o}\sigma_{r\varphi_o}\right]$$

$$+ C_2\left[\frac{z\cos\varphi}{\rho s}\right] v_o + C_1\left[\frac{z\sin\varphi}{\rho s} + \frac{\nu_o z}{\rho^2}\right]\left\{\frac{-\nu_o}{1-\nu_o}\left[\frac{\partial v_o}{\partial\varphi} + u_o + \frac{\rho}{s}\left(u_o\sin\varphi\right.\right.\right.$$

$$\left.\left.\left. + v_o\cos\varphi - \frac{\partial w_o}{\partial\theta}\right)\right] + \left[\frac{\rho}{E_o}\frac{(1-2\nu_o)(1+\nu_o)}{1-\nu_o}\right]\sigma_{rr_o} + \left(\frac{1+\nu_o}{1-\nu_o}\right)\rho\,\alpha_o T_o\right\} \qquad (5)$$

$$+ C_2\left[\frac{\sin\varphi}{s} + C_3\frac{1}{\rho}\right]\left[\frac{z}{\rho}\right] u_o - \left[\frac{z}{\rho}\right] C_4 + \frac{z}{h}\left[\frac{\nu_o}{1-\nu_o}\right]\left[(p_1-p_2)\right]$$

$$- \frac{z}{\rho}\left[\frac{\nu_1}{(1-\nu_o)^2}\right]\left[\left(\frac{p_1+p_2}{2}\right) + \left(\frac{h}{2\rho}\right)^2 \sigma_{rr_2}\right]$$

first order terms

Detailed Stress Equations for the Thin-Shell Solution (cont.)

$$\sigma_{\varphi\theta} = - C_5\left[\frac{1}{s}\right]\frac{\partial v_o}{\partial\theta} + C_5\left[\frac{1}{\rho}\right]\frac{\partial w_o}{\partial\varphi} - C_5\left[\frac{\cos\varphi}{s}\right] w_o \quad \Big\} \text{ zero order terms}$$

$$\left.\begin{aligned}
& - C_6\left[\frac{z}{\rho s}\right]\frac{\partial v_o}{\partial\theta} - C_5\left[\frac{z}{\rho s}\right]\left[\left(1 - \frac{\rho}{s}\sin\varphi\right)\frac{\partial v_o}{\partial\theta} - \frac{\partial^2 u_o}{\partial\varphi\partial\theta} + \frac{\rho}{G_o}\frac{\partial\sigma_{r\varphi_o}}{\partial\theta}\right] \\
& + C_6\left[\frac{z}{\rho^2}\right]\frac{\partial w_o}{\partial\varphi} + C_5\left[\frac{z}{\rho}\right]\left[\left(\frac{s\cos\varphi + (\rho-r)\sin\varphi\cos\varphi}{s^2}\right) w_o\right. \\
& \left. + \left(\frac{\sin\varphi}{s} - \frac{1}{\rho}\right)\frac{\partial w_o}{\partial\varphi} + \frac{1}{s}\frac{\partial^2 u_o}{\partial\theta\partial\varphi} - \frac{r\cos\varphi}{s^2}\frac{\partial u_o}{\partial\theta} + \frac{1}{G_o}\frac{\partial\sigma_{r\theta_o}}{\partial\varphi}\right] \\
& - C_6\left[\frac{z}{\rho s}\cos\varphi\right] w_o - C_5\left[\frac{z}{s}\cos\varphi\right]\left[\frac{\sin\varphi}{s} w_o + \frac{1}{s}\frac{\partial u_o}{\partial\theta} + \frac{1}{G_o}\sigma_{r\theta_o}\right]
\end{aligned}\right\} \text{ first order terms} \qquad (6)$$

where

$$s = a + r\sin\varphi \qquad (7)$$

$$C_1 = \frac{E_o}{1-\nu_o^2} \qquad (8)$$

$$C_2 = \frac{E_1}{1-\nu_o^2} + \frac{2E_o\nu_o\nu_1}{(1-\nu_o^2)^2} \qquad (9)$$

$$C_3 = \frac{E_1\nu_o}{1-\nu_o^2} + \frac{E_o\nu_1(1+\nu_o^2)}{(1-\nu_o^2)^2} \qquad (10)$$

$$C_4 = \frac{E_o\alpha_o T_1}{1-\nu_o} + \frac{E_o\alpha_1 T_o}{1-\nu_o} + \frac{E_1\alpha_o T_o}{1-\nu_o} + \frac{E_o\alpha_o T_o\nu_1}{(1-\nu_o)^2} \qquad (11)$$

$$C_5 = \frac{E_o}{1+\nu_o} \qquad (12)$$

$$C_6 = \frac{E_1}{1+\nu_o} - \frac{E_o\nu_1}{(1+\nu_o)^2} \qquad (13)$$

Detailed Stress Equations for the Thin-Shell Solution (cont.)

Equations (4), (5) and (6), as previously stated, are in terms of u_o, v_o, w_o and σ_{rr_2} which are obtained by solving the six equilibrium equations as given in Reference 2, Equations (33) through (38) of Appendix A.

APPENDIX B

Thin Shell Solution: Finite Difference Formulation

The generalized finite difference analogue formulation was developed in Reference 1 and is directly applicable to the thin-shell solution given in Reference 2 with the following changes: coordinate α_1 will be eliminated since it corresponds to the radial direction*, subscript i will also be dropped to conform with the above statement.

The finite difference solution of the equilibrium equations will first be obtained for the general case where the grid spacing is assumed to be irregular. Then a solution will also be obtained for regular grid spacing. In both cases the first, second and mixed derivatives are needed for the central, forward and backward grid combinations. A typical general grid spacing is shown in Figure 1 below.

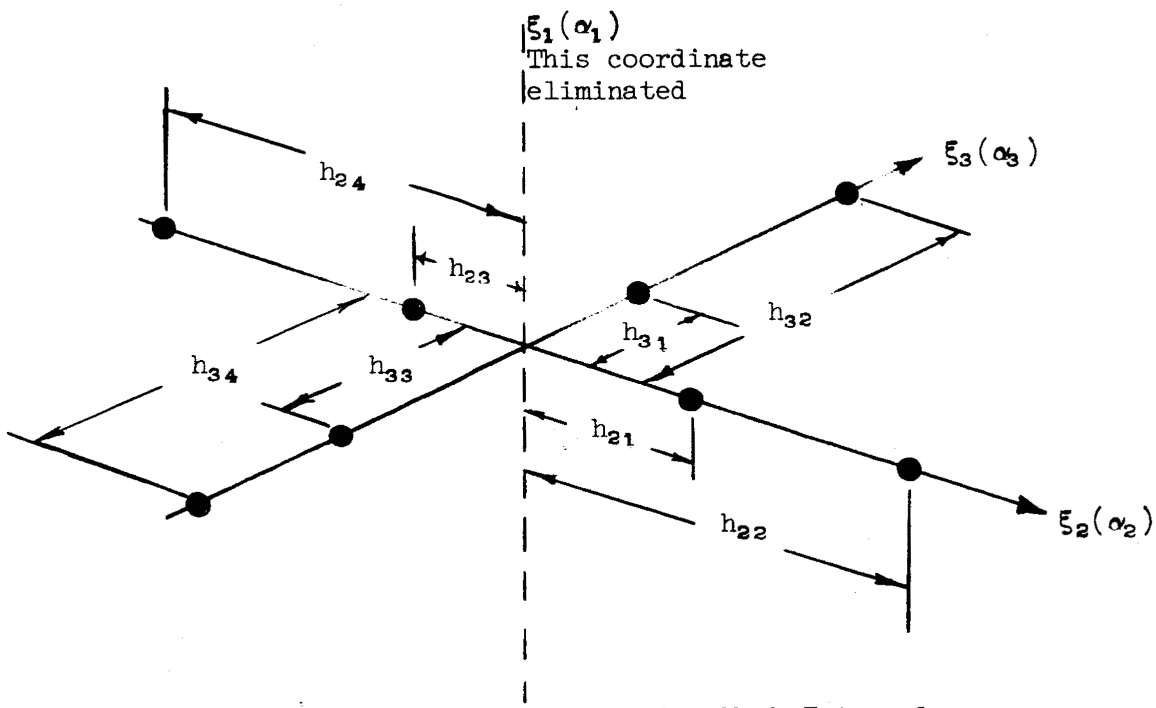

Figure 1 - Coordinates of Irregular Mesh Intervals

* The partial derivatives in the equilibrium equations given in Reference 2 are taken with respect to α_2 and α_3 (φ and θ respectively).

The increments in the vicinity of a node will be designated by the following, in accordance to Figure 1.

$$\begin{aligned}
h_{21} &= (\alpha_2)_{i+1} - (\alpha_2)_i & h_{31} &= (\alpha_3)_{i+1} - (\alpha_3)_i \\
h_{22} &= (\alpha_2)_{i+2} - (\alpha_2)_i & h_{32} &= (\alpha_3)_{i+2} - (\alpha_3)_i \\
h_{23} &= (\alpha_2)_i - (\alpha_2)_{i-1} & h_{33} &= (\alpha_3)_i - (\alpha_3)_{i-1} \\
h_{24} &= (\alpha_2)_i - (\alpha_2)_{i-2} & h_{34} &= (\alpha_3)_i - (\alpha_3)_{i-2}
\end{aligned} \tag{1}$$

From Reference 2 $f(\xi_1, \xi_2, \xi_3)$, the function of the coordinates with the origin at i, j, k is:

$$\begin{aligned}
f(\xi_1, \xi_2, \xi_3) = f_{i,j,k} &+ B_1 \xi_1 + B_2 \xi_2 + B_3 \xi_3 + B_4 \xi_1 \xi_2 \\
&+ B_5 \xi_2 \xi_3 + B_6 \xi_3 \xi_1 + B_7 \xi_1^2 + B_8 \xi_2^2 + B_9 \xi_3^2 \\
&+ B_{10} \xi_1 \xi_2 \xi_3 + B_{11} \xi_1 \xi_2^2 + B_{12} \xi_1 \xi_3^2 + B_{13} \xi_1^2 \xi_2 \\
&+ B_{14} \xi_2 \xi_3^2 + \ldots
\end{aligned} \tag{2}$$

The first and second derivatives of $f(\alpha_1, \alpha_2, \alpha_3)$ with respect to α_2 and α_3 are obtained from Equation (2).

$$\begin{aligned}
&\left.\frac{\partial f}{\partial \alpha_2}\right|_{i,j,k} = \left.\frac{\partial f}{\partial \xi_2}\right|_{0,0,0} = B_2 , \quad \left.\frac{\partial f}{\partial \alpha_3}\right|_{i,j,k} = \left.\frac{\partial f}{\partial \xi_3}\right|_{0,0,0} = B_3 \\
&\left.\frac{\partial^2 f}{\partial \alpha_2 \, \partial \alpha_3}\right|_{i,j,k} = \left.\frac{\partial^2 f}{\partial \xi_2 \, \partial \xi_3}\right|_{0,0,0} = B_5 \\
&\left.\frac{\partial^2 f}{\partial \alpha_2^2}\right|_{i,j,k} = \left.\frac{\partial^2 f}{\partial \xi_2^2}\right|_{0,0,0} = 2B_8 , \quad \left.\frac{\partial^2 f}{\partial \alpha_3^2}\right|_{i,j,k} = \left.\frac{\partial^2 f}{\partial \xi_3^2}\right|_{0,0,0} = 2B_9
\end{aligned} \tag{3}$$

The constants B_i are evaluated in terms of the function at these nodes and the grid spacing as shown in Figure 1 by considering the values of $f(\xi_1, \xi_2, \xi_3)$ at the eight nodes adjacent to j,k, and proceeding in a manner similar to that outlined in Reference 1.

I. GENERAL CASE - IRREGULAR GRID SPACING

A. CENTRAL DERIVATIVES

$$\left.\frac{\partial f}{\partial \alpha_2}\right|_{j,k} = \frac{h_{23}^2\, f_{j+1,k} + (h_{21}^2 - h_{23}^2)\, f_{j,k} - h_{21}^2\, f_{j-1,k}}{h_{21}\, h_{23}\, (h_{21} + h_{23})} \tag{4}$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right|_{j,k} = \frac{h_{33}^2\, f_{j,k+1} + (h_{31}^2 - h_{33}^2)\, f_{j,k} - h_{31}^2\, f_{j,k-1}}{h_{31}\, h_{33}\, (h_{31} + h_{33})} \tag{5}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2\, \partial \alpha_3}\right|_{j,k} = \frac{1}{h_{31}\, h_{33}\, (h_{21} + h_{23})\, (h_{31} + h_{33})} \Big[h_{33}^2 \left(f_{j+1,k+1} - f_{j-1,k+1}\right) - (h_{33}^2 - h_{31}^2) \left(f_{j+1,k} - f_{j-1,k}\right) - h_{31}^2 \left(f_{j+1,k-1} - f_{j-1,k-1}\right) \Big] \tag{6}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2\, \partial \alpha_3}\right|_{j,k} = \frac{1}{h_{21}\, h_{23}\, (h_{21} + h_{23})(h_{31} + h_{33})} \Big[h_{23}^2 \left(f_{j+1,k+1} - f_{j+1,k-1}\right) - (h_{23}^2 - h_{21}^2) \left(f_{j,k+1} - f_{j,k-1}\right) - h_{21}^2 \left(f_{j-1,k+1} - f_{j-1,k-1}\right) \Big] \tag{7}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2^2}\right|_{j,k} = \frac{2\left[h_{23}\, f_{j+1,k} - (h_{21} + h_{23})\, f_{j,k} + h_{21}\, f_{j-1,k}\right]}{h_{21}\, h_{23}\, (h_{21} + h_{23})} \tag{8}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3^2}\right|_{j,k} = \frac{2\left[h_{33}\, f_{j,k+1} - (h_{31} + h_{33})\, f_{j,k} + h_{31}\, f_{j,k-1}\right]}{h_{31}\, h_{33}\, (h_{31} + h_{33})} \tag{9}$$

B. FORWARD DERIVATIVES

$$\left.\frac{\partial f}{\partial \alpha_2}\right|_{j,k} = \frac{h_{22}^2 \, f_{j+1,k} - (h_{22}^2 - h_{21}^2)\, f_{j,k} - h_{21}^2 \, f_{j+2,k}}{h_{21}\, h_{22}\, (h_{22} - h_{21})} \tag{10}$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right|_{j,k} = \frac{h_{32}^2 \, f_{j,k+1} - (h_{32}^2 - h_{31}^2)\, f_{j,k} - h_{31}^2 \, f_{j,k+2}}{h_{31}\, h_{32}\, (h_{32} - h_{31})} \tag{11}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2\, \partial \alpha_3}\right|_{j,k} = \frac{1}{h_{31}\, h_{33}\, (h_{21} - h_{22})(h_{31} + h_{33})} \Big[h_{33}^2 \big(f_{j+1,k+1} - f_{j+1,k} - f_{j+2,k+1} + f_{j+2,k} \big) - h_{31}^2 \big(f_{j+1,k-1} - f_{j+1,k} - f_{j+2,k-1} + f_{j+2,k} \big) \Big] \tag{12}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2\, \partial \alpha_3}\right|_{j,k} = \frac{1}{h_{21}\, h_{23}\, (h_{31} - h_{32})(h_{21} + h_{23})} \Big[h_{23}^2 \big(f_{j+1,k+1} - f_{j,k+1} - f_{j+1,k+2} + f_{j,k+2} \big) - h_{21}^2 \big(f_{j-1,k+1} - f_{j,k+1} - f_{j-1,k+2} + f_{j,k+2} \big) \Big] \tag{13}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2^2}\right|_{j,k} = \frac{2 \left[- h_{22}\, f_{j+1,k} + (h_{22} - h_{21})\, f_{j,k} + h_{21}\, f_{j+2,k} \right]}{h_{21}\, h_{22}\, (h_{22} - h_{21})} \tag{14}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3^2}\right|_{j,k} = \frac{2 \left[- h_{32}\, f_{j,k+1} + (h_{32} - h_{31})\, f_{j,k} + h_{31}\, f_{j,k+2} \right]}{h_{31}\, h_{32}\, (h_{32} - h_{31})} \tag{15}$$

C. BACKWARD DERIVATIVES

$$\left.\frac{\partial f}{\partial \alpha_2}\right|_{j,k} = \frac{h_{23}\, f_{j-2,k} + (h_{24}^2 - h_{23}^2)\, f_{j,k} - h_{24}^2\, f_{j-1,k}}{h_{23}\, h_{24}\, (h_{24} - h_{23})} \tag{16}$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right|_{j,k} = \frac{h_{33}^2\, f_{j,k-2} + (h_{34}^2 - h_{33}^2)\, f_{j,k} - h_{34}^2\, f_{j,k-1}}{h_{33}\, h_{34}\, (h_{34} - h_{33})} \tag{17}$$

$$\left.\frac{\partial^2 f}{\partial\alpha_2\,\partial\alpha_3}\right|_{j,k} = \frac{1}{h_{31}\,h_{33}\,(h_{23}-h_{24})(h_{31}+h_{33})}\left[h_{33}^2\left(f_{j-1,k+1} - f_{j-2,k+1} + f_{j-2,k} - f_{j-1,k}\right) - h_{31}^2\left(f_{j-1,k-1} - f_{j-1,k} - f_{j-2,k-1} + f_{j-2,k}\right)\right] \tag{18}$$

$$\left.\frac{\partial^2 f}{\partial\alpha_2\,\partial\alpha_3}\right|_{j,k} = \frac{1}{h_{21}\,h_{23}\,(h_{33}-h_{34})(h_{21}+h_{23})}\left[h_{23}^2\left(f_{j+1,k-1} - f_{j+1,k-2} + f_{j,k-2} - f_{j,k-1}\right) - h_{21}^2\left(f_{j-1,k-1} - f_{j,k-1} - f_{j-1,k-2} + f_{j,k-2}\right)\right] \tag{19}$$

$$\left.\frac{\partial^2 f}{\partial\alpha_2^2}\right|_{j,k} = \frac{2\left[h_{23}\,f_{j-2,k} + (h_{24}-h_{23})\,f_{j,k} - h_{24}\,f_{j-1,k}\right]}{h_{23}\,h_{24}\,(h_{24}-h_{23})} \tag{20}$$

$$\left.\frac{\partial^2 f}{\partial\alpha_3^2}\right|_{j,k} = \frac{2\left[h_{33}\,f_{j,k-2} + (h_{34}-h_{33})\,f_{j,k} - h_{34}\,f_{j,k-1}\right]}{h_{33}\,h_{34}\,(h_{34}-h_{33})} \tag{21}$$

II. GENERAL CASE - REGULAR GRID SPACING

When the grid spacing is regular, then

$$h_2 = h_{21} = h_{23} = \frac{1}{2}h_{22} = \frac{1}{2}h_{24}$$
$$h_3 = h_{31} = h_{33} = \frac{1}{2}h_{32} = \frac{1}{2}h_{34} \tag{22}$$

Substituting the conditions of Equation (22) into Equations (4) through (21) the first, second and mixed derivatives for the central, forward, and backward regular grid spacing combinations are obtained.

A. CENTRAL DERIVATIVES

$$\left.\frac{\partial f}{\partial \alpha_2}\right|_{j,k} = \frac{f_{j+1,k} - f_{j-1,k}}{2h_2} \qquad (23)$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right|_{j,k} = \frac{f_{j,k+1} - f_{j,k-1}}{2h_3} \qquad (24)$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2\, \partial \alpha_3}\right|_{j,k} = \frac{1}{4h_2h_3} \left[f_{j+1,k+1} - f_{j+1,k-1} - f_{j-1,k+1} + f_{j-1,k-1}\right] \qquad (25)$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2^{\,2}}\right|_{j,k} = \frac{1}{h_2^{\,2}} \left[f_{j+1,k} - 2\, f_{j,k} + f_{j-1,k}\right] \qquad (26)$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3^{\,2}}\right|_{j,k} = \frac{1}{h_3^{\,2}} \left[f_{j,k+1} - 2\, f_{j,k} + f_{j,k-1}\right] \qquad (27)$$

B. FORWARD DERIVATIVES

$$\left.\frac{\partial f}{\partial \alpha_2}\right|_{j,k} = \frac{1}{2h_2} \left[4\, f_{j+1,k} - 3\, f_{j,k} - f_{j+2,k}\right] \qquad (28)$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right|_{j,k} = \frac{1}{2h_3} \left[4\, f_{j,k+1} - 3\, f_{j,k} - f_{j,k+2}\right] \qquad (29)$$

$$\left.\frac{\partial^2 f}{\partial\alpha_2\,\partial\alpha_3}\right|_{j,k} = \frac{-1}{2h_2h_3}\left[f_{j+1,k+1} - f_{j+2,k+1} - f_{j+1,k-1} + f_{j+2,k-1}\right] \quad (30)$$

$$\left.\frac{\partial^2 f}{\partial\alpha_2\,\partial\alpha_3}\right|_{j,k} = \frac{-1}{2h_2h_3}\left[f_{j+1,k+1} - f_{j+1,k+2} - f_{j-1,k+1} + f_{j-1,k+2}\right] \quad (31)$$

$$\left.\frac{\partial^2 f}{\partial\alpha_2^2}\right|_{j,k} = \frac{1}{h_2^2}\left[-2\,f_{j+1,k} + f_{j,k} + f_{j+2,k}\right] \quad (32)$$

$$\left.\frac{\partial^2 f}{\partial\alpha_3^2}\right|_{j,k} = \frac{1}{h_3^2}\left[-2\,f_{j,k+1} + f_{j,k} + f_{j,k+2}\right] \quad (33)$$

C. BACKWARD DERIVATIVES

$$\left.\frac{\partial f}{\partial\alpha_2}\right|_{j,k} = \frac{1}{2h_2}\left[f_{j-2,k} + 3\,f_{j,k} - 4\,f_{j-1,k}\right] \quad (34)$$

$$\left.\frac{\partial f}{\partial\alpha_3}\right|_{j,k} = \frac{1}{2h_3}\left[f_{j,k-2} + 3\,f_{j,k} - 4\,f_{j,k-1}\right] \quad (35)$$

$$\left.\frac{\partial^2 f}{\partial\alpha_2\,\partial\alpha_3}\right|_{j,k} = \frac{-1}{2h_2h_3}\left[f_{j-1,k+1} - f_{j-2,k+1} - f_{j-1,k-1} + f_{j-2,k-1}\right] \quad (36)$$

$$\left.\frac{\partial^2 f}{\partial\alpha_2\,\partial\alpha_3}\right|_{j,k} = \frac{-1}{2h_2h_3}\left[f_{j+1,k-1} - f_{j+1,k-2} - f_{j-1,k-1} + f_{j-1,k-2}\right] \quad (37)$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2^2}\right|_{j,k} = \frac{1}{h_2^2}\left[f_{j-2,k} + f_{j,k} - 2\, f_{j-1,k}\right] \quad (38)$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3^2}\right|_{j,k} = \frac{1}{h_3^2}\left[f_{j,k-2} + f_{j,k} - 2\, f_{j,k-1}\right] \quad (39)$$

APPENDIX C

EQUATIONS FOR THE SINGULAR POINT FOR THE NON-AXISYMMETRIC CASE

A. CONDITIONS AT THE POINT, $\varphi = 0$

The components of displacement u, v, w in the R, φ, θ directions respectively have the following properties at $\varphi = 0$,

$$
\begin{aligned}
\frac{\partial u}{\partial \theta} &= 0 & \frac{\partial^3 u}{\partial \theta^2 \partial \varphi} &= -\frac{\partial u}{\partial \varphi} \\
\frac{\partial w}{\partial \theta} &= v & \frac{\partial^3 w}{\partial \theta^2 \partial \varphi} &= -2\frac{\partial^2 v}{\partial \theta \partial \varphi} \\
\frac{\partial v}{\partial \theta} &= w & \frac{\partial^3 v}{\partial \theta^2 \partial \varphi} &= 2\frac{\partial^2 w}{\partial \theta \partial \varphi} \\
\frac{\partial \beta}{\partial \theta} &= 0 & \beta &= \int_{T_o}^{T} \alpha(T)\, \partial T
\end{aligned}
\qquad (1)
$$

These results are derived in Section B of this Appendix.

Evaluation of Strains

From Equation (1) the singular terms in the strain-displacement equations can be evaluated. Thus we find

$$e_{\theta\theta} = \frac{1}{R\sin\varphi}\frac{\partial w}{\partial \theta} + \frac{u}{R} + \frac{v\cot\varphi}{R}$$

$$\xrightarrow[\varphi \to 0]{} \frac{1}{R}\left(\frac{\frac{\partial w}{\partial \theta} + v}{\sin\varphi}\right) + \frac{u}{R}$$

and from Equation (1), we note that the bracketed part is $\left(\frac{0}{0}\right)$ and thus we have the result,

$$e_{\theta\theta} = \frac{1}{R}\left(\frac{\partial^2 w}{\partial \theta \partial \varphi} + \frac{\partial v}{\partial \varphi} + u\right) \quad (\varphi = 0) \qquad (2)$$

Equations for the Singular Point for the Non-Axisymmetric Case (cont.)

Similarly we find:

$$e_{\varphi\theta} = \frac{1}{2R}\left(\frac{\partial w}{\partial \varphi} - w \cot\varphi + \frac{1}{\sin\varphi}\frac{\partial v}{\partial \theta}\right)$$

and again through Equation (1) we obtain

$$e_{\varphi\theta} = \frac{1}{2R}\left(\frac{\partial w}{\partial \varphi} - \frac{\partial w}{\partial \varphi} + \frac{\partial^2 v}{\partial \theta \partial \varphi}\right) \quad (\varphi = 0)$$

$$e_{\varphi\theta} = \frac{1}{2R}\frac{\partial^2 v}{\partial \theta \partial \varphi} \quad (\varphi = 0) \tag{3}$$

Finally,

$$e_{R\theta} = \frac{1}{2R}\left(\frac{1}{\sin\varphi}\frac{\partial u}{\partial \theta} - w + R\frac{\partial u}{\partial R}\right)$$

$$e_{R\theta} = \frac{1}{2R}\left(\frac{\partial^2 u}{\partial \theta \partial \varphi} - w + R\frac{\partial u}{\partial R}\right) \quad (\varphi = 0) \tag{4}$$

<u>Equations of Equilibrium</u>

The equations of equilibrium

$$\frac{\partial \tau_{RR}}{\partial R} + \frac{1}{R}\frac{\partial \tau_{R\varphi}}{\partial \varphi} + \frac{1}{R\sin\varphi}\frac{\partial \tau_{R\theta}}{\partial \theta} + \frac{2\tau_{RR} - \tau_{\varphi\varphi} - \tau_{\theta\theta} + \tau_{R\varphi}\cot\varphi}{R} = 0 \tag{5}$$

$$\frac{\partial \tau_{R\varphi}}{\partial R} + \frac{1}{R}\frac{\partial \tau_{\varphi\varphi}}{\partial \varphi} + \frac{1}{R\sin\varphi}\frac{\partial \tau_{\varphi\varphi}}{\partial \theta} + \frac{3\tau_{R\varphi} + (\tau_{\varphi\varphi} - \tau_{\theta\theta})\cot\varphi}{R} = 0 \tag{6}$$

$$\frac{\partial \tau_{R\theta}}{\partial R} + \frac{1}{R}\frac{\partial \tau_{\varphi\theta}}{\partial \varphi} + \frac{1}{R\sin\varphi}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{3\tau_{R\theta} + 2\tau_{\varphi\theta}\cot\varphi}{R} = 0 \tag{7}$$

have the following indeterminate parts:

From (5), we have

$$\frac{1}{R\sin\varphi}\,\frac{\partial\tau_{R\theta}}{\partial\theta} + \frac{\tau_{R\varphi}\cot\varphi}{R} \tag{8}$$

Putting in the values of $\tau_{R\theta}$ and $\tau_{R\varphi}$ in terms of displacements, we get

$$\frac{\partial\tau_{R\theta}}{\partial\theta} + \tau_{R\varphi} = \frac{\mu}{R}\left[\frac{1}{\sin\varphi}\,\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial u}{\partial\varphi} - \left(\frac{\partial w}{\partial\theta} + v\right) + \frac{\partial}{\partial R}\left(\frac{\partial w}{\partial\theta} + v\right)\right] \tag{9}$$

From Equation (1) we find that the expression (9) is zero at $\varphi = 0$. Thus we may write, for expression (8),

$$\frac{1}{R}\left(\frac{\partial^2\tau_{R\theta}}{\partial\theta\partial\varphi} + \frac{\partial\tau_{R\varphi}}{\partial\varphi}\right)$$

which is the evaluation of the indeterminate portion of Equation (5).

From Equation (6), the indeterminate part is

$$\frac{1}{R\sin\varphi}\,\frac{\partial\tau_{\varphi\theta}}{\partial\theta} + \frac{\tau_{\theta\varphi} - \tau_{\theta\theta}}{R}\cot\varphi \tag{10}$$

In terms of the displacements

$$\frac{\partial\tau_{\theta\varphi}}{\partial\theta} + \tau_{\varphi\varphi} - \tau_{\theta\theta} =$$

$$\frac{\mu}{R}\left[\frac{\partial^2 w}{\partial\theta\partial\varphi} + 2\frac{\partial v}{\partial\varphi} + \frac{1}{\sin\varphi}\,\frac{\partial}{\partial\theta}\left(\frac{\partial v}{\partial\theta} - w\right) - \frac{2}{\sin\varphi}\left(\frac{\partial w}{\partial\theta} + v\right)\right]$$

Equations for the Singular Point
for the Non-Axisymmetric Case (cont.)

From Equation (1) the terms in parenthesis are zero. Thus we have

$$\frac{\mu}{R}\left[\frac{\partial^2 w}{\partial\theta\partial\varphi} + \frac{\partial^3 v}{\partial\theta^2\partial\varphi} - \frac{\partial^2 w}{\partial\varphi\partial\theta} - 2\frac{\partial^2 w}{\partial v\partial\varphi}\right]$$

$$= \frac{\mu}{R}\left(\frac{\partial^3 v}{\partial\theta^2\partial\varphi} - 2\frac{\partial^2 w}{\partial\theta\partial\varphi}\right)$$

which is zero by Equation (1).

Therefore we may write expression (10) as

$$\frac{1}{R}\frac{\partial^2\tau_{\varphi\theta}}{\partial\theta\partial\varphi} + \frac{1}{R}\left(\frac{\partial\tau_{\varphi\varphi}}{\partial\varphi} - \frac{\partial\tau_{\theta\theta}}{\partial\varphi}\right)$$

which is the evaluation of the indeterminate part of Equation (6).

From Equation (7), the indeterminate parts are

$$\frac{1}{R\sin\varphi}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + 2\frac{\tau_{\varphi\theta}}{R}\cot\varphi \tag{11}$$

and

$$\frac{1}{R}\frac{\partial\tau_{\theta\theta}}{\partial\theta} \tag{12}$$

For (11) we have (at $\varphi \rightarrow 0$)

$$\frac{\partial\tau_{\theta\theta}}{\partial\theta} + 2\,\tau_{\theta\varphi} =$$

$$= \frac{2\mu}{R}\frac{\partial w}{\partial \varphi} + \frac{2(\lambda + \mu)}{R}\frac{\partial u}{\partial \theta} + \lambda \frac{\partial^2 u}{\partial R \partial \theta}$$

$$+ \frac{\lambda}{R}\frac{\partial^2 v}{\partial \theta \partial \varphi} - \frac{\partial \beta(T)}{\partial \theta}$$

$$+ \frac{2\mu}{R\sin\varphi}\left(\frac{\partial v}{\partial \theta} - w\right) - \frac{(\lambda + 2\mu)}{R\sin\varphi}\frac{\partial}{\partial \theta}\left(\frac{\partial w}{\partial \theta} + v\right)$$

It can be seen from Equation (1) that the terms with $\frac{1}{\sin\varphi}$ have the form $\frac{0}{0}$ and that the terms in u and β are zero. Thus we can write

$$\frac{\partial \tau_{\theta\theta}}{\partial \theta} + 2\tau_{\theta\varphi} = \frac{1}{R}\left(2\mu \frac{\partial^2 v}{\partial \theta \partial \varphi} + (\lambda + 2\mu)\frac{\partial^3 w}{\partial \theta^2 \partial \varphi} + (\lambda + 2\mu)\frac{\partial^2 v}{\partial \theta \partial \varphi} + \lambda \frac{\partial^2 v}{\partial \theta \partial \varphi}\right)$$

$$= \frac{\lambda + 2\mu}{R}\left(\frac{\partial^3 w}{\partial \theta^2 \partial \varphi} + 2\frac{\partial^2 v}{\partial \theta \partial \varphi}\right)$$

$$= 0 \quad \text{by Equation (1).}$$

Thus we may write expression (10) as

$$\frac{1}{R}\frac{\partial^2 \tau_{\theta\theta}}{\partial \theta \partial \varphi} + 2\frac{\partial \tau_{\theta\varphi}}{\partial \varphi},$$

For (12) we expand the functions in powers of φ thus:

$$w = w_o + w_1 \varphi + w_2 \varphi^2 + \cdots$$

etc

Then we may write (for $\varphi \to 0$)

$$\frac{\partial \tau_{\theta\varphi}}{\partial \varphi} = \frac{\partial}{\partial \varphi}\left[w_1 + 2w_2 \varphi - \frac{w_o + w_1 \varphi + w_2 \varphi^2}{\varphi} + \left(\frac{\partial v_o}{\partial \theta} + \frac{\partial v_1}{\partial \theta} \varphi + \frac{\partial v_2}{\partial \theta} \varphi^2 \right) / \varphi \right]$$

$$= 2w_2 + \frac{w_o - \frac{\partial v_o}{\partial \theta}}{\varphi^2} - w_2 + \frac{\partial v_2}{\partial \theta}$$

And since $w_o = \frac{\partial v_o}{\partial \theta}$ from (1), we have

$$\frac{\partial \tau_{\theta\varphi}}{\partial \varphi} = w_2 + \frac{\partial v_2}{\partial \theta} \qquad (\varphi = 0)$$

$$= \frac{1}{2}\left(\frac{\partial^2 w}{\partial \varphi^2} + \frac{\partial^3 v}{\partial \theta \partial \varphi^2} \right) \qquad (\varphi = 0) \tag{13}$$

Thus, the one term, $\frac{\partial \tau_{\theta\varphi}}{\partial \varphi}$, in the equilibrium equations is not evaluated by simple differentiation with respect to φ.•

B. RELATIONS BETWEEN DISPLACEMENTS AT $\varphi = 0$

To obtain the equations given in (1) we take the gradient and the Laplacian of the displacement vector, i.e.,

$$\nabla \vec{\xi} = \nabla (\hat{R}u + \hat{\varphi}v + \hat{\theta}w)$$

and

$$\nabla^2 \vec{\xi} = \nabla^2 (\hat{R}u + \hat{\varphi}v + \hat{\theta}w)$$

• See note at end of Appendix C.

Expanding in powers of φ, viz,

$$u = u_o + u_1 \varphi + u_2 \varphi^2 + \dots$$

etc

and letting $\varphi = \rightarrow 0$ after obtaining $\nabla \xi$ and $\nabla^2 \xi$, we find the following terms involving $\frac{1}{\varphi}$ as a factor

$$\hat{\theta}\hat{\theta}\left(\frac{\partial w_o}{\partial \theta} + v_o\right)$$

$$\hat{\theta}\hat{\varphi}\left(\frac{\partial v_o}{\partial \theta} - w_o\right)$$

$$\hat{\theta}\hat{R}\left(\frac{\partial u_o}{\partial \theta}\right)$$

$$\hat{R}\left(\frac{\partial^2 u_1}{\partial \theta^2} + u_1\right)$$

$$\hat{\varphi}\left(\frac{\partial^2 v_1}{\partial \theta^2} - 2\frac{\partial w_1}{\partial \theta}\right)$$

$$\hat{\theta}\left(\frac{\partial^2 w_1}{\partial \theta^2} + 2\frac{\partial v_1}{\partial \theta}\right)$$

Since $\nabla \vec{\xi}$ and $\nabla^2 \vec{\xi}$ must be finite when $\varphi = 0$, the above expressions (having $\frac{1}{\varphi}$ as a multiplier) must all vanish. Thus we have

$$\frac{\partial w_o}{\partial \theta} = - v_o$$

$$\frac{\partial v_o}{\partial \theta} = w_o$$

$$\frac{\partial u_o}{\partial \theta} = 0$$

$$\frac{\partial^2 w_1}{\partial \theta^2} = -2 \frac{\partial v_1}{\partial \theta}$$

$$\frac{\partial^2 v_1}{\partial \theta^2} = 2 \frac{\partial w_1}{\partial \theta}$$

$$\frac{\partial^2 u_1}{\partial \theta^2} = - u_1$$

The equations in (1) are equivalent to these when $\varphi = 0$.

The equation

$$\frac{\partial \beta}{\partial \theta} = 0 \qquad (\varphi = 0)$$

where

$$\beta = \int_{T_o}^{T} \alpha(T)dT$$

can be seen from the fact that at $\varphi = 0$ there is only one point for all θ, so that β cannot vary with θ. (The same argument could have been applied to the quantity u. It could not be applied to w or v since these are functions which have extension in θ.)

NOTE: The further evaluation of (13) may reduce the expression to a simple derivation process, but such a possibility will not be investigated since the expression has already been made determinate.

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