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TEMPERATURE DISTRIBUTION IN A SPINNING SPHERICAL SHELL IN THE SOLAR FLUX

UNIFORMLY VALID PERTURBATION EXPANSION FOR A THIN SHELL

by W. P. Brown, Jr.

Prepared under Contract No. NASw-1056 *by*
HUGHES AIRCRAFT COMPANY
Malibu, Calif.
for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MAY 1965

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I. INTRODUCTION

The exact calculation of the temperature distribution in a radiating body involves the solution of a differential equation with a nonlinear boundary condition. Since there are no general mathematical techniques available for the exact analytical solution of such nonlinear equations, it is necessary to employ approximate techniques of solution. Considerable attention has recently been given to the calculation of the temperature distribution in thin-walled objects subject to solar radiation.[1-3] The approximations employed in these calculations are based on the hypothesis that the temperature distribution in a thin shell differs very little from that in a shell of zero thickness. The validity of the hypothesis is unquestionable for sufficiently thin shells, but there is an open question as to how thin is sufficiently thin. In other words, it would be useful to have an estimate of the error incurred by replacing a thin shell by a shell of zero thickness.

The work reported here not only provides an answer to this question, but also presents a systematic procedure for the generation of a uniformly valid asymptotic expansion of the temperature distribution in powers of a thickness parameter. The first term in the asymptotic expansion is the conventional thin shell approximation, and succeeding terms represent the corrections necessitated by the nonzero thickness of the shell. The effect of spinning the shell about an axis at an angle $\beta (0 < \beta < \pi/2)$ to the direction of the solar flux is also considered. Although the analysis is performed for the specific case of a spherical shell, it probably can be generalized to include any shell that has azimuthal symmetry about the spin axis (body of revolution).

II. FORMULATION OF THE PROBLEM

A. Differential Equation

The temperature distribution in a spinning spherical shell composed of homogeneous, isotropic material satisfies the conventional diffusion equation modified by the addition of a term that originates from the convection of heat introduced by the motion[4]

$$\left[\nabla^2 - \frac{1}{\kappa}\left(\frac{\partial}{\partial t} + \underline{v}(\underline{r}) \cdot \nabla\right)\right] T(\underline{r}, t) = 0, \quad \underline{r} \text{ in } D \ . \qquad \text{(II. 1)}$$

κ is the diffusivity of the material (assumed to be independent of temperature and position), t denotes time, $\underline{v}(\underline{r})$ is the velocity of the sphere at the point $\underline{r}$, and D denotes the domain occupied by the spherical shell. In the problem to be considered here, the sphere is in the vacuum of outer space and it is assumed that the region interior to the shell is also evacuated. Therefore, there is no convective transfer of heat at either boundary. The source of heat is a uniform solar flux F_0 which is assumed to be incident along a direction which makes an angle β with the spin axis of the sphere (see Fig. 1).

Since the transient behavior of the temperature distribution is not of particular interest in the practical applications envisaged for the results to be derived here (principally the calculation of satellite temperature distribution), it will be assumed that the steady state has been reached. This implies that the temperature distribution is not a function of time. The partial differential equation satisfied by the steady state temperature distribution is

$$\left(\nabla^2 - \frac{\omega}{\kappa}\frac{\partial}{\partial \phi}\right) T(\underline{r}) = 0 \qquad \underline{r} \text{ in } D \ . \qquad \text{(II. 2)}$$

Equation (II. 2) is obtained from (II. 1) by replacing $\partial/\partial t$ with zero and $\underline{v}(\underline{r})$ with $\underline{a}_\phi[\omega r] \sin\theta$ ($\omega \equiv$ rate of rotation, $\underline{a}_\phi \equiv$ unit vector in the ϕ direction).

B. Boundary Conditions

The boundary conditions at the surface of the shell can be derived from the physical condition that the net heat transfer across a boundary must be continuous. Within the shell, heat is transferred by conduction. The component of flux normal to the boundaries of the shell is $\pm\ K\ \partial T/\partial r$ where K denotes the thermal conductivity

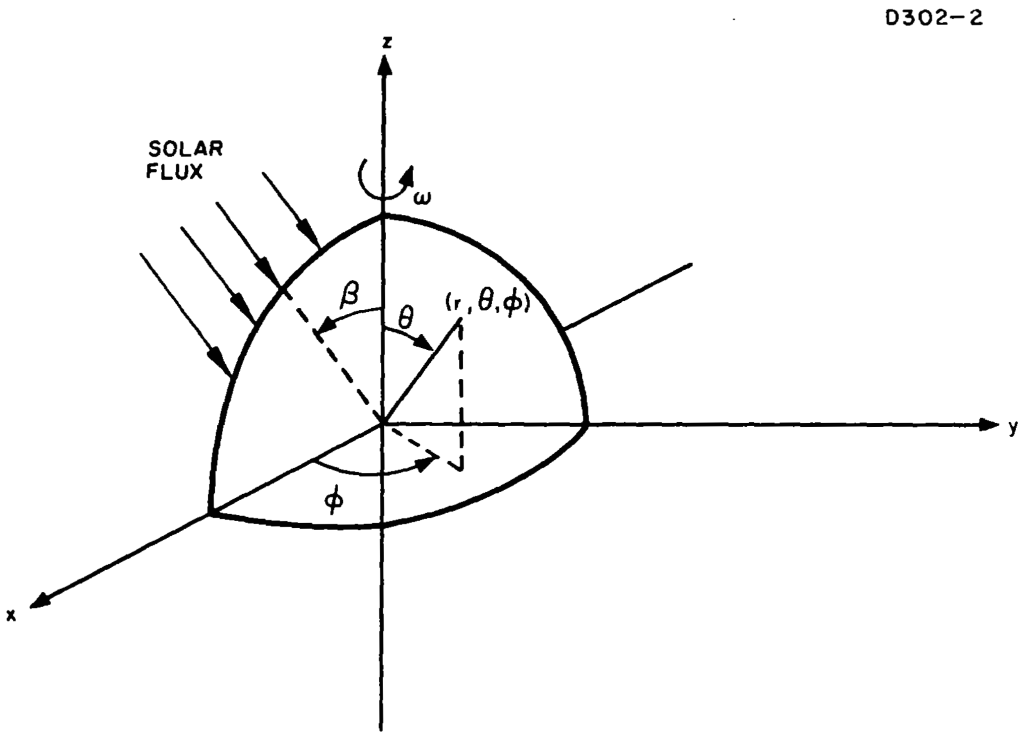

Fig. 1. Coordinate system and orientation of the solar flux with respect to the spin axis.

of the shell material. The minus sign is taken at the outer boundary and the plus sign at the inner. Outside of the shell, heat is transferred by emission and absorption of radiation. The magnitude of the radiated flux is given by the Stefan-Boltzmann law

$$\text{radiated flux} = \sigma\epsilon\ T^4(\underline{r}), \qquad \underline{r} \text{ on } S \tag{II. 3}$$

where σ is Stefan's constant (1.37×10^{-12} cal/cm^2-sec-deg^4), and ϵ is the emissivity of the surface. The magnitude of the flux absorbed will be calculated under the assumption that each surface element emits and absorbs radiation in accordance with Lambert's law. When the heat radiated and absorbed at the outer surface of the spherical shell is equated to the normal component of heat flux conducted at the boundary, the following condition is obtained:

$$-K \frac{\partial T(\underline{r})}{\partial r} = \sigma\epsilon\ T^4(\underline{r}) - \epsilon\ F_o G_S(\theta, \phi, \beta), \quad \underline{r} \text{ on } S_o \tag{II. 4}$$

where $-\epsilon\ F_o G_S(\theta, \phi, \beta)$ denotes the solar flux absorbed by the shell at θ, ϕ.

$$G_S(\theta, \phi, \beta) = \cos\beta\cos\theta + \sin\beta\sin\theta\cos\phi\ , \tag{II. 5}$$

for the values of θ and ϕ where the quantity on the right hand side is greater than zero and G_S is zero for all other θ and ϕ.

On the inner surface of the spherical shell the heat absorbed at any given surface element comes from the heat radiated and reflected from all other points on the surface. A calculation of the absorbed component of heat flux is given in Appendix A. This calculation reveals that the heat absorbed at a point on the inner surface of a spherical shell is a constant independent of the coordinates of the point.*

$$\begin{bmatrix} \text{flux absorbed on} \\ \text{the inner surface} \\ \text{of a spherical} \\ \text{shell} \end{bmatrix} = Q = \frac{\sigma\epsilon}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta(\sin\theta\ T^4(\underline{r})), \quad \underline{r} \text{ on } S_i\ . \tag{II. 6}$$

*In the more general case of a body of revolution, the heat absorbed at a point is a function of the latitudinal coordinate but not of the azimuthal coordinate.

Hence the boundary condition at the inner surface of the spherical shell is

$$K \frac{\partial T(\underline{r})}{\partial r} = \sigma\epsilon\, T^4(\underline{r}) - Q, \qquad \underline{r} \text{ on } S_i \quad . \tag{II. 7}$$

The remainder of this report is devoted to the approximate solution of the boundary value problem posed in eqs. (II. 2), (II. 4), and (II. 7).

III. PERTURBATION CALCULATION FOR A THIN SHELL

A. Thickness Parameter

Sometimes the solution of a complicated problem is closely approximated by the solution of a much simpler problem. In essence, this is the basis of perturbation theory. The characteristic feature of all problems solvable by perturbation theory is the presence of a small parameter. Typically, the parameter is the ratio of two lengths, times, or magnitudes (mass, temperature, force, etc.). The first term in a perturbation series is the solution of the original problem for the limiting case in which the parameter is zero. Subsequent terms in the series are the corrections to the zero order solution necessitated by the nonzero value of the parameter. In the case of the temperature distribution problem formulated in Section II, the small parameter is the ratio of the half thickness δ of the spherical shell to the mean radius a. Hence the zero order term in the perturbation solution of the temperature distribution problem corresponds to the temperature distribution in a shell of zero thickness, and the higher order terms are corrections necessitated by the nonzero thickness of the shell.

When the differential (II. 2) is written in terms of the conventional spherical coordinates r, θ, and ϕ, the thickness parameter δ/a does not appear explicitly. Thus, it is necessary to express (II. 2) in terms of other variables that make explicitly evident the dependence of $T(\underline{r})$ on δ/a. This can be accomplished by replacing the radial variable r by a new variable ζ

$$\zeta \equiv \frac{r - a}{\delta} \quad . \tag{III. 1}$$

This change of variable is suggested by the success of a similar change employed in the approximate theory of the elastic deformation of thin plates and shells. Another change that is not essential, but which simplifies the equation, is the representation of $T(\zeta, \theta, \phi)$ in terms of a dimensionless variable $U(\zeta, \theta, \phi)$

$$T(\zeta, \theta, \phi) = \left(\frac{F_o}{\sigma}\right)^{1/4} U(\zeta, \theta, \phi) \quad . \tag{III. 2}$$

When written in terms of the variables U, ζ, θ, and ϕ, the problem posed in (II. 2), (II. 4), and (II. 7) becomes

$$\left\{ \frac{\partial^2}{\partial \zeta^2} + \frac{\delta}{a} \left[2 \frac{\partial}{\partial \zeta} \left(\zeta \frac{\partial}{\partial \zeta} \right) \right] \right.$$

$$+ \left(\frac{\delta}{a}\right)^2 \left[\frac{\partial}{\partial \zeta} \left(\zeta^2 \frac{\partial}{\partial \zeta} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} - \frac{\omega a^2}{\kappa} \frac{\partial}{\partial \phi} \right]$$

$$+ \left(\frac{\delta}{a}\right)^3 \left[- \frac{2\omega a^2}{\kappa} \zeta \frac{\partial}{\partial \phi} \right]$$

$$\left. + \left(\frac{\delta}{a}\right)^4 \left[- \frac{\omega a^2}{\kappa} \zeta^2 \frac{\partial}{\partial \phi} \right] \right\} U(\zeta, \theta, \phi) = 0, \quad \begin{array}{l} -1 < \zeta < 1 \\ 0 < \theta < \pi \\ 0 < \phi < 2\pi \end{array} \qquad \text{(III. 3)}$$

$$- \frac{\partial U}{\partial \zeta} = \frac{\delta}{a} \alpha \left[U^4 - G_S (\theta, \phi, \beta) \right], \qquad \delta = 1 \qquad \text{(III. 4)}$$

$$\frac{\partial U}{\partial \zeta} = \frac{\delta}{a} \alpha \left[U^4 - q \right], \qquad \delta = -1 \qquad \text{(III. 5)}$$

where

$$\alpha = \frac{a \epsilon \sigma^{1/4} F_o^{3/4}}{K} . \qquad \text{(III. 6)}$$

$$q = \frac{1}{4\pi} \int_o^{2\pi} d\phi \int_o^{\pi} d\theta (\sin \theta \, U^4(-1, \theta, \phi)) . \qquad \text{(III. 7)}$$

B. Generation of an Asymptotic Expansion by the Repeated Application of a Limit Process

Before proceeding with the solution of the problem posed in (III. 3) to (III. 5), we will discuss a formal procedure that can be used to generate asymptotic expansions in such problems. Lagerstrom and Cole[5] have shown that the asymptotic solution of a partial differential equation containing a small parameter can be obtained by repeatedly applying a limit process to the equations. The appropriate limit process in a given problem depends on the nature of the exact solution of the problem. In regular perturbation problems the limit process simply involves taking the limit as the small parameter goes to zero with the independent variables fixed. This procedure yields results that are not uniformly valid in singular perturbation problems. In such cases it is generally necessary to stretch the independent and dependent variables by factors proportional to the small parameter and take a limit as the small parameter goes to zero with the stretched variables fixed. The details of the formal procedure for generating asymptotic expansions in the case of singular perturbation problems are discussed further in Section IV-B. In this discussion we are mainly interested in demonstrating a formal procedure for generating asymptotic expansions, regardless of the particular type of limit process involved.

Consider the following partial differential equation and boundary conditions

$$\mathcal{L}W(\underline{x}, \epsilon) = 0 \qquad \underline{x} \text{ in } D \tag{III. 8}$$

$$\mathcal{L}_B W(\underline{x}, \epsilon) = F(W(\underline{x}, \epsilon), \underline{x}, \epsilon) \qquad \underline{x} \text{ on } S \tag{III. 9}$$

where $\underline{x}$ denotes the unstretched independent variables in the case of a regular perturbation problem and the stretched independent variables in a singular perturbation problem, and ϵ is a small parameter in terms of which the operators $\mathcal{L}$ and $\mathcal{L}_B$ can be written

$$\mathcal{L} = \sum_{m=0,1,2,\ldots} \epsilon^m \mathcal{L}_m \quad , \tag{III. 10}$$

$$\mathcal{L}_B = \sum_{m=0,1,2,\ldots} \epsilon^m \mathcal{L}_{Bm} \quad . \tag{III. 11}$$

The function F in (III. 9) is a known function of $W, \underline{x},$ and ϵ. Define the limit process

$$\operatorname{Lim}_{\underline{x}} \equiv \text{limit as } \epsilon \to 0 \text{ with } \underline{x} \text{ fixed,} \tag{III. 12}$$

and denote the result of applying $\mathrm{Lim}_{\underline{x}}\ \epsilon^{-n}$ to a function by a subscript n. According to the procedure outlined by Lagerstrom and Cole, the first term in the asymptotic expansion of $W(\underline{x}, \epsilon)$ satifies the equations obtained when $\mathrm{Lim}_{\underline{x}}$ is applied to (III. 8) and (III. 9)

$$\mathcal{L}_o W_o = 0 \qquad \underline{x} \text{ in } D, \tag{III. 13}$$

$$\mathcal{L}_{Bo} W_o = F_o \qquad \underline{x} \text{ on } S \tag{III. 14}$$

$$W_o(\underline{x}) = \mathrm{Lim}_{\underline{x}}\ W(\underline{x}, \epsilon) \quad . \tag{III. 15}$$

The second term in the asymptotic expansion of $W(\underline{x}, \epsilon)$ satisfies the equations obtained when $\mathrm{Lim}_{\underline{x}}\ \epsilon^{-1}$ is applied to the result obtained when (III. 13) and (III. 14) are substracted from (III. 8) and (III. 9).

$$\mathcal{L}_o W_1 + \mathcal{L}_1 W_o = 0 \qquad \underline{x} \text{ in } D, \tag{III. 16}$$

$$\mathcal{L}_{Bo} W_1 + \mathcal{L}_{B1} W_o = F_1 \qquad \underline{x} \text{ on } S \tag{III. 17}$$

$$W_1(\underline{x}) = \mathrm{Lim}_{\underline{x}}\ \epsilon^{-1}(W(\underline{x}, \epsilon) - W_o(\underline{x})) \quad . \tag{III. 18}$$

The third term in the asymptotic expansion of $W(\underline{x}, \epsilon)$ satisfies the equations obtained when $\mathrm{Lim}_{\underline{x}}\ \epsilon^{-2}$ is applied to the difference between (III. 5), ((III. 6)) and the sum of (III. 10), ((III. 11)) and ϵ times (III. 13), ((III. 14)).

$$\mathcal{L}_o W_2 + \mathcal{L}_1 W_1 + \mathcal{L}_2 W_o = 0 \qquad \underline{x} \text{ in } D, \tag{III. 19}$$

$$\mathcal{L}_{Bo} W_2 + \mathcal{L}_{B1} W_1 + \mathcal{L}_{B2} W_o = F_2 \qquad \underline{x} \text{ on } S \tag{III. 20}$$

$$W_2(\underline{x}) = \mathrm{Lim}_{\underline{x}}\ \epsilon^{-2}(W(\underline{x}) - W_o(\underline{x}) - \epsilon W_1(\underline{x})) \quad . \tag{III. 21}$$

Higher order terms in the asymptotic expansion satisfy equations obtained in a similar fashion. The n^{th} term satisfies the equations obtained when $\mathrm{Lim}_{\underline{x}}\, \epsilon^{-n}$ is applied to the difference between (III. 8) ((III. 9)) and the sum of ϵ^m times the equations satisfied by $W_m(\underline{x})$ for all m between zero and (n - 1).

$$\sum_{m=0}^{n} \mathcal{L}_m W_{n-m}(\underline{x}) = 0 \qquad \underline{x} \text{ in } D \ , \tag{III. 22}$$

$$\sum_{m=0}^{n} \mathcal{L}_{Bm} W_{n-m}(\underline{x}) = F_n(\underline{x}) \qquad \underline{x} \text{ on } S \ . \tag{III. 23}$$

$$W_n(\underline{x}) = \mathrm{Lim}_{\underline{x}}\, \epsilon^{-n} \left[W(\underline{x}, \epsilon) - \sum_{m=0}^{n-1} \epsilon^m W_m(\underline{x}) \right] \tag{III. 24}$$

The procedure outlined in the preceding paragraph does not yield results that could not have been obtained in another fashion. For instance, the same equations are obtained when $W(\underline{x}, \epsilon)$ is replaced in (III. 8) and (III. 9) by a power series expansion in ϵ and like powers of ϵ are equated to zero. The procedure developed by Lagerstrom and Cole, however, is somewhat more convenient to apply in singular perturbation problems. Furthermore, it clearly places in evidence the nature of the approximation involved in replacing $W(\underline{x}, \epsilon)$ by the expansion so derived. The difference between $W(\underline{x}, \epsilon)$ and an N term expansion is

$$R_N(\underline{x}, \epsilon) = W(\underline{x}, \epsilon) - \sum_{m=0}^{N} \epsilon^m W_m(\underline{x}) \ . \tag{III. 25}$$

Application of the limit process $\mathrm{Lim}_{\underline{x}}\, \epsilon^{-(N+1)}$ to $R_N(\underline{x}, \epsilon)$ yields a relation between the limiting behavior of $R_N(\underline{x}, \epsilon)$ and the $(N + 1)^{st}$ term in the perturbation expansion

$$\mathrm{Lim}_{\underline{x}}\, \epsilon^{-(N+1)} R_N(\underline{x}, \epsilon) = W_{N+1}(\underline{x}) \ . \tag{III. 26}$$

The result given in (III. 26) implies that $R_N(\underline{x}, \epsilon)$ is of the order

$$R_N(\underline{x}, \epsilon) = O(\epsilon^{N+1} W_{N+1}(\underline{x})) \quad , \tag{III. 27}$$

and thus the error incurred in approximating $W(\underline{x}, \epsilon)$ by an N term expansion is of the order of the first term neglected.

C. Prefatory Comments on the Perturbation Calculation of $U(\zeta, \theta, \phi)$

The problem posed in (III. 3) to (III. 5) is somewhat more complicated than that discussed in the preceding section. Instead of being a simple one parameter perturbation problem, the calculation of $U(\zeta, \theta, \phi)$ involves three parameters: δ/a, $\alpha = (a\epsilon\, \sigma^{1/4} F_o^3/K)$, and $\omega a^2/\kappa$. The significance of the thickness parameter δ/a has already been discussed. In this section, the effect of the remaining two parameters will be examined.

The parameter α appears in the boundary condition equations (III. 4) and (III. 5), and is a measure of the relative magnitude of radiated and conducted flux. The mathematical significance of α is evinced by a consideration of its effect on the boundary conditions obtained from (III. 4) and (III. 5) in the limit of small δ/a. A crucial step in the solution of (III. 3) to (III. 5) by perturbation theory is the linearization of the boundary conditions when $\delta/a \to 0$. Application of the limit process

$$\text{Lim}\,\frac{\delta}{a} \to 0; \quad \zeta, \theta, \phi \text{ fixed} \equiv \text{Lim}_\zeta \tag{III. 28}$$

to (III. 4) and (III. 5) yields

$$-\frac{\partial U_o}{\partial \zeta} = \text{Lim}_\zeta\, \frac{\delta}{a}\, \alpha \left[U^4 - G_S\right] , \quad \zeta = 1 \tag{III. 29}$$

$$\frac{\partial U_o}{\partial \zeta} = \text{Lim}_\zeta\, \frac{\delta}{a}\, \alpha \left[U^4 - q\right] , \quad \zeta = -1 \ . \tag{III. 30}$$

Thus, a necessary condition for the linearization of the boundary conditions is

$$\alpha << \frac{a}{\delta} \quad , \qquad \text{(III. 31)}$$

because when α is of the same order as a/δ, the righthand sides of (III. 29) and (III. 30) do not vanish in the limit of small δ/a.

Very small values of α are also troublesome. For α's of the order $O(\delta/a)$ the zero order temperature distribution satisfies a nonlinear partial differential equation that is not any easier to solve than the exact equations from which it is obtained (see Appendix B for a derivation of this equation).

In most of the applications for which these calculations are intended, α is neither too large nor too small. It is well to remember, however, that the perturbation calculations given in this report yield a valid approximation only in those problems for which

$$\delta/a << \alpha << a/\delta \quad . \qquad \text{(III. 32)}$$

The parameter $\omega a^2/\kappa$ is a measure of the relative magnitude of the rotation rate ω and a circumferential equilibration rate κ/a^2. When $\omega a^2/\kappa$ is small, the temperature distribution is close to that of a stationary shell. Likewise, when $\omega a^2/\kappa$ is large, the temperature distribution approximates that for the case $\omega \rightarrow \infty$. The form of (III. 3) indicates that the transition of the temperature distribution from that which exists when $\omega = 0$ to that which exists when $\omega \rightarrow \infty$ is encompassed by the following cases:

1. $\underline{\frac{\omega a^2}{\kappa} = O(1)}$ — The terms in (III. 3) that originate from the rotation of the shell are all of second order or smaller in the thickness parameter δ/a. Consequently, the zero order temperature distribution is unaffected by the rotation. The temperature distribution in this case is obviously quite close to that which exists when $\omega = 0$ (see Section III-D-1).

2. $\underline{\frac{\omega a^2}{\kappa} = O(a/\delta)}$ — Although the effect of rotation enters (III. 3) in terms of the order δ/a or smaller, the zero order temperature distribution is significantly influenced by rotation when $\omega a^2/\kappa = O(a/\delta)$. Calculations given in Section V indicate that $U_o(\zeta, \theta, \phi)$ satisfies a nonlinear first order ordinary differential equation. The transition from a temperature distribution characteristic of $\omega \sim 0$ to that for $\omega \rightarrow \infty$ is described by the solutions of this nonlinear equation.

3. $\frac{\omega a^2}{\kappa} = O(a/\delta)^2$ — The effect of rotation enters the differential equation for $U(\zeta, \theta, \phi)$ as a zero order term in δ/a. Calculations given in Section III-D-2 reveal that the zero order temperature distribution in this case corresponds to that which would exist if $\omega \to \infty$. Thus, the temperature distribution is only slightly perturbed from the distribution for $\omega \to \infty$.

When the magnitude of $\omega a^2/\kappa$ is outside the range covered in items 1, 2, and 3, the zero order temperature distribution is identical with that which exists for $\omega = 0$ $(\omega a^2/\kappa < O(1))$ or $\omega \to \infty$ $(\omega a^2/\kappa > O(a/\delta)^2)$. The departures of the temperature distributions from those obtained in item 1 or 3 appear in the higher order correction terms.

D. Regular Perturbation Calculations

The procedure outlined in Section III-B will now be applied to the problem posed in (III. 3) to (III. 5). At the outset it will be assumed that this is a regular perturbation problem, so that the appropriate dependent and independent variables are U, ζ, θ, and ϕ, as written in (III. 3) to (III. 5). This assumption proves to be valid if portions of the domain D are excluded. The domains that must be excluded are boundary layer regions within which the angular derivatives of the temperature distribution become quite large. The solution of (III. 3) to (III. 5) in these regions is deferred until Section IV. Calculations will be given here for the temperature distribution when $(\omega a^2/\kappa) = O(1)$ and $O(a/\delta)^2$. The transitional case $(\omega a^2/\kappa) = O(a/\delta)$ will be considered separately in Section V.

1. $\frac{\omega a^2}{\kappa} = O(1)$ (Slow Rotation)

The first two terms in the regular perturbation expansion of $U(\zeta, \theta, \phi)$ will be derived in this section. Subsequent terms can be obtained in a similar fashion. Application of the limit process Lim_ζ to (III. 3) to (III. 5) yields the following equations for the zero order temperature distribution.

$$\frac{\partial^2 U_o}{\partial \zeta^2} = 0 \quad , \qquad \text{(III. 33)}$$

$$\frac{\partial U_o}{\partial \zeta} = 0, \qquad \zeta = \pm 1 \quad . \qquad \text{(III. 34)}$$

The solution of (III. 33) and (III. 34) is a function independent of ζ

$$U_o = A_o(\theta, \phi) \quad . \tag{III. 35}$$

Equation (III. 35) verifies an assumption often made in the approximate calculation of the temperature distribution in thin shells; viz., that the zero order temperature distribution is independent of the shell thickness.

The first order term in the regular perturbation expansion of $U(\zeta, \theta, \phi)$ satisfies the equations obtained when Lim_ζ (a/δ) is applied to the difference between (III. 3) ((III. 4), (III. 5)) and (III. 33) ((III. 34), (III. 34))*:

$$\frac{\partial^2 U_1}{\partial \zeta^2} + 2 \frac{\partial}{\partial \zeta}\left(\zeta \frac{\partial U_o}{\partial \zeta}\right) = 0 \quad , \tag{III. 36}$$

$$- \frac{\partial U_1}{\partial \zeta} = \alpha \left[U_o^4 - G_S\right] , \qquad \zeta = 1 \tag{III. 37}$$

$$\frac{\partial U_1}{\partial \zeta} = \alpha \left[U_o^4 - q_o\right] , \qquad \zeta = -1 \tag{III. 38}$$

where

$$q_o = \mathrm{Lim}_\zeta \; q = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta (\sin \theta \; U_o^4(-1, \theta, \phi)) \quad . \tag{III. 39}$$

The solution of (III. 36) is of the form (note that $(\partial U_o / \partial \zeta) = 0$)

$$U_1(\zeta, \theta, \phi) = A_1(\theta, \phi) + \zeta B_1(\theta, \phi) \quad . \tag{III. 40}$$

Substitution of (III. 40) in the boundary conditions (III. 37) and (III. 38) yields the equations

*The notation employed here is to be interpreted as meaning (III. 33) is subtracted from (III. 3), (III. 34) from (III. 4), and (III. 34) from (III. 5).

$$-B_1 = a\left[A_o^4 - G_S\right] , \tag{III.41}$$

$$B_1 = a\left[A_o^4 - q_o\right] , \tag{III.42}$$

from which it is a simple matter to obtain expressions for A_o and B_1.

$$A_o(\theta, \phi) = \left(\frac{G_S + q_o}{2}\right)^{1/4} , \tag{III.43}$$

$$B_1(\theta, \phi) = \frac{a}{2}\left(G_S - q_o\right) . \tag{III.44}$$

The zero order temperature distribution expressed in (III.43) is identical with the result obtained when one equates to zero the net outward flux from a shell of zero thickness. This can be proved by noting that heat is not conducted in the θ and ϕ directions in a shell of zero thickness. Thus, the flux balance can be written

$$\text{flux radiated at outer surface} + \text{flux radiated at inner surface} = 0 \tag{III.45}$$

and in terms of the dimensionless variable U_o this becomes

$$\left(U_o^4 - G_S\right) + \left(U_o^4 - q_o\right) = 0 . \tag{III.46}$$

Solution of (III.46) for U_o yields a result identical to that obtained by the perturbation calculation. In anticipation of the nonuniformity of the regular perturbation expansion in the boundary layer region near the shadow boundary, the calculation of q_o will be deferred until Section IV. This is necessary since q_o depends on the zero order temperature distribution over the entire inside surface of the shell.

In order to complete the determination of $U_1(\zeta, \theta, \phi)$, it is necessary to consider the equations satisfied by $U_2(\zeta, \theta, \phi)$. Those equations are obtained when $\text{Lim}\ (a/\delta)^2$ is applied to the difference between (III.3) ((III.4), (III.5)) and the sum of (III.33) ((III.34), (III.34)) and δ/a times (III.36) ((III.37), (III.38)).

$$\frac{\partial^2 U_2}{\partial \zeta^2} + 2 \frac{\partial}{\partial \zeta} \left(\zeta \frac{\partial U_1}{\partial \zeta} \right) + \mathcal{L}_{\theta, \phi, \omega} U_o = 0 \quad , \tag{III. 47}$$

$$- \frac{\partial U_2}{\partial \zeta} = 4 \, a \, U_o^3 \, U_1 \qquad \zeta = 1 \tag{III. 48}$$

$$\frac{\partial U_2}{\partial \zeta} = 4 \, a \, U_o^3 \, U_1 - a \, q_1 \, , \qquad \zeta = -1 \tag{III. 49}$$

where

$$q_1 = \operatorname{Lim}_{\zeta} \frac{a}{\delta} q = \frac{1}{\pi} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta (\sin\theta \, U_o^3(-1, \theta, \phi) \, U_1(-1, \theta, \phi)) \tag{III. 50}$$

and $\mathcal{L}_{\theta, \phi, \omega}$ denotes the operator

$$\mathcal{L}_{\theta, \phi, \omega} = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} - \frac{\omega a^2}{\kappa} \frac{\partial}{\partial\phi} \, . \tag{III. 51}$$

Equation (III. 47) can be integrated to obtain the following solution for $U_2(\zeta, \theta, \phi)$.

$$U_2(\zeta, \theta, \phi) = A_2(\theta, \phi) + \zeta B_2(\theta, \phi) - \frac{\zeta^2}{2} \left[2B_1(\theta, \phi) + \mathcal{L}_{\theta, \phi, \omega} A_o(\theta, \phi) \right] . \tag{III. 52}$$

Substitution of (III. 52) in the boundary conditions (III. 48) and (III. 49) then yields two equations that can be solved for the undetermined function $A_1(\theta, \phi)$ (and also $B_2(\theta, \phi)$, if desired). The result is

$$A_1(\theta,\phi) = A_o^{-3}(\theta,\phi) \left[q_1 + \frac{B_1(\theta,\phi)}{2a} + \frac{1}{4a} \mathcal{L}_{\theta,\phi,\omega} A_o(\theta,\phi) \right] . \tag{III. 53}$$

Examination of the results obtained for the first two terms in the regular perturbation expansion of $U(\zeta,\theta,\phi)$ reveals that $U_o(\zeta,\theta,\phi)$ is a bounded continuous function of θ and ϕ, whereas $U_1(\zeta,\theta,\phi)$ is bounded and continuous only if the vicinity of the shadow boundary is excluded.* The singular behavior of $U_1(\zeta,\theta,\phi)$ originates in the term $\mathcal{L}_{\theta,\phi,\omega} A_o$ which appears in the expression (III. 53) for $A_1(\theta,\phi)$. As a consequence of this singular behavior, the perturbation expansion derived in this section is not uniformly valid over the entire sphere. There is a small region in the vicinity of the shadow boundary within which singular perturbation theory must be applied. This will be done in Section IV.

2. $\frac{\omega a^2}{\kappa} = O(a/\delta)^2$ (Rapid Rotation)

The statement that $\omega a^2/\kappa$ is of the order $(a/\delta)^2$ implies that $\mathrm{Lim}\,(\delta/a)^2\,\omega a^2/\kappa$ is a bounded, nonzero number. Thus, application of the limit process Lim_ζ to (III. 3) to (III. 5) yields the following equations for the zero order temperature distribution in a rapidly rotating spherical shell.

$$\left(\frac{\partial^2}{\partial \zeta^2} - \frac{\omega\delta^2}{\kappa} \frac{\partial}{\partial \phi} \right) U_o = 0, \qquad -1 \le \zeta \le 1, \quad 0 \le \phi \le 2\pi \tag{III. 54}$$

$$\frac{\partial U_o}{\partial \zeta} = 0, \qquad \zeta = \pm 1 \quad . \tag{III. 55}$$

The only solution of (III. 54) that satisfies (III. 55) is the trivial solution

$$U_o(\zeta,\theta,\phi) = C_o(\theta) \quad . \tag{III. 56}$$

*The term "shadow boundary" denotes the boundary between the region in which $G_S(\theta,\phi,\beta) > 0$ and that where $G_S(\theta,\phi,\beta) = 0$.

Thus, the zero order temperature distribution in the case of a rapidly rotating thin spherical shell is independent of both the ζ and ϕ coordinates.

The equations satisfied by the first order terms in the regular perturbation expansion of $U(\zeta, \theta, \phi)$ are obtained when $\mathrm{Lim}_{\epsilon}\, a/\delta$ is applied to the difference between (III. 3) ((III. 4), (III. 5)) and (III. 54) ((III. 55), (III. 55)).

$$\left(\frac{\partial^2}{\partial \zeta^2} - \frac{\omega\delta^2}{\kappa}\frac{\partial}{\partial \phi}\right) U_1 = 0, \qquad -1 \le \zeta \le 1, 0 \le \phi \le 2\pi \qquad \text{(III. 57)}$$

$$-\frac{\partial U_1}{\partial \zeta} = a\left[U_o^4 - G_S\right], \qquad \zeta = 1 \qquad \text{(III. 58)}$$

$$\frac{\partial U_1}{\partial \zeta} = a\left[U_o^4 - q_o\right], \qquad \zeta = 1 \quad . \qquad \text{(III. 59)}$$

A solution of the problem posed in (III. 57) to (III. 59) can be obtained by separation of variables. For the sake of brevity, the present discourse will be confined to a skeletal outline of the crucial steps in the solution and a quotation of the results. Since U_1 is a real periodic function of ϕ, the solution of (III. 57) is expressed in the form

$$U_1(\zeta, \theta, \phi) = \mathrm{Re} \sum_{m=o}^{\infty} Z_{1m}(\zeta, \theta)\, e^{im\phi} \qquad \text{(III. 60)}$$

where Re denotes that the real part of the sum is to be taken. The $Z_{1m}(\zeta, \theta)$ are solutions of the equation

$$\frac{\partial^2 Z_{1m}}{\partial \zeta^2} - i\frac{\omega\, \delta^2 m}{\kappa} Z_{1m} = 0 \quad . \qquad \text{(III. 61)}$$

Solution of (III.61) for Z_{lm} then yields the following expression for $U_1(\zeta, \theta, \phi)$:

$$U_1(\zeta, \theta, \phi) = C_1(\theta) + \zeta D_1(\theta)$$

$$+ \sum_{m=1}^{\infty} \left\{ e^{\beta_m \zeta} \left[A_{rm}(\theta) \cos(\beta_m \zeta + m\phi) - A_{im}(\theta) \sin(\beta_m \zeta + m\phi) \right] \right.$$

$$\left. + e^{-\beta_m \zeta} \left[B_{rm}(\theta) \cos(\beta_m \zeta - m\phi) - B_{im}(\theta) \sin(\beta_m \zeta - m\phi) \right] \right\} , \qquad \text{(III. 62)}$$

$$\beta_m = \left(\frac{\omega \delta^2 m}{2} \right)^{1/2} . \qquad \text{(III. 63)}$$

Substitution of (III. 62) in the boundary conditions (III. 58) and (III. 59), multiplication of the results by $\cos m\phi d\phi$ or $\sin m\phi d\phi$, and integration over the interval $0 \leq \phi \leq 2\pi$ gives equations that can be solved for $C_o(\theta)$, $D_1(\theta)$, $A_{rm}(\theta)$, $A_{im}(\theta)$, $B_{rm}(\theta)$, and $B_{im}(\theta)$. The results of this calculation are

$$C_o(\theta) = \left(\frac{\frac{1}{2\pi} \int_o^{2\pi} G_S(\theta, \phi, \beta) d\phi + q_o}{2} \right)^{1/4} = \left(\frac{\overline{G_S(\theta, \beta)} + q_o}{2} \right)^{1/4} , \qquad \text{(III. 64)}$$

$$D_1(\theta) = \frac{a}{2} \left(\overline{G_S(\theta, \beta)} - q_o \right) , \qquad \text{(III. 65)}$$

$$A_{rm}(\theta) = \frac{M_{11}}{\det M} \Gamma_m(\theta, \beta) , \qquad \text{(III. 66)}$$

$$A_{im}(\theta) = \frac{M_{12}}{\det M}\, \Gamma_m(\theta, \beta) \quad , \qquad \text{(III. 67)}$$

$$B_{rm}(\theta) = \frac{M_{13}}{\det M}\, \Gamma_m(\theta, \beta) \quad , \qquad \text{(III. 68)}$$

$$B'_{im}(\theta) = \frac{M_{14}}{\det M}\, \Gamma_m(\theta, \beta) \quad , \qquad \text{(III. 69)}$$

where

$$\Gamma_m(\theta, \beta) = \frac{a}{\pi \beta_m} \int_0^{2\pi} G_S(\theta, \phi, \beta) \cos m\phi \, d\phi, \quad m = 1, 2, 3, \ldots \; .$$

The M_{ij} $(j = 1, 2, 3, 4)$ are cofactors and $\det M$ the determinant of the matrix

$$M = \begin{bmatrix} e^{\beta_m}(\cos\beta_m - \sin\beta_m) & -e^{-\beta_m}(\cos\beta_m + \sin\beta_m) & -e^{\beta_m}(\cos\beta_m + \sin\beta_m) & -e^{-\beta_m}(\cos\beta_m - \sin\beta_m) \\ e^{-\beta_m}(\cos\beta_m + \sin\beta_m) & -e^{\beta_m}(\cos\beta_m - \sin\beta_m) & -e^{-\beta_m}(\cos\beta_m - \sin\beta_m) & -e^{\beta_m}(\cos\beta_m + \sin\beta_m) \\ -e^{\beta_m}(\cos\beta_m + \sin\beta_m) & e^{-\beta_m}(\cos\beta_m - \sin\beta_m) & -e^{\beta_m}(\cos\beta_m - \sin\beta_m) & -e^{-\beta_m}(\cos\beta_m + \sin\beta_m) \\ -e^{-\beta_m}(\cos\beta_m - \sin\beta_m) & e^{\beta_m}(\cos\beta_m + \sin\beta_m) & -e^{-\beta_m}(\cos\beta_m + \sin\beta_m) & -e^{\beta_m}(\cos\beta_m - \sin\beta_m) \end{bmatrix}$$

(III. 70)

After considerable mathematical manipulation, the $M_{ij}(j = 1, 2, 3, 4)$ and $\det M$ obtained from (III. 70) can be written in the surprisingly simple form

$$M_{11} = 2^{3/2} \cos\left(\beta_m + \frac{\pi}{4}\right) \left[-e^{3\beta_m} + e^{-\beta_m}(1 + 2 \sin 2\beta_m)\right] \quad ,$$

(III. 71)

$$M_{12} = 2^{3/2} \cos\left(\beta_m - \frac{\pi}{4}\right) \left[-e^{\beta_m}(1 - 2 \sin 2\beta_m) + e^{-3\beta_m}\right] \quad ,$$

(III. 72)

$$M_{13} = 2^{3/2} \cos\left(\beta_m - \frac{\pi}{4}\right) \left[e^{3\beta_m} - e^{-\beta_m} (1 - 2 \sin 2\beta_m) \right] , \quad \text{(III. 73)}$$

$$M_{14} = 2^{3/2} \cos\left(\beta_m + \frac{\pi}{4}\right) \left[-e^{\beta_m} (1 + 2 \sin 2\beta_m) + e^{-3\beta_m} \right] , \quad \text{(III. 74)}$$

$$\det M = 8(\cos 4\beta_m - \cosh 4\beta_m) \quad . \quad \text{(III. 75)}$$

Note that the zero order temperature distribution $C_o(\theta)$ expressed in (III. 64) is the same as would be derived by the methods of the preceding section for a source having the averaged solar flux distribution $\overline{G_S(\theta, \beta)}$. A sphere rotating at an infinite rate would be irradiated by such a flux distribution. Thus, the zero temperature distribution in a spherical shell spinning at a rate of the order $(a/\delta)^2$ is identical to that when $\omega \to \infty$. The difference in the temperature distribution for the two cases appears in the first and higher order terms in the perturbation expansions.

To complete the determination of $U_1(\zeta, \theta, \phi)$, note that $C_1(\theta)$ can be obtained from the equations for $U_2(\zeta, \theta, \phi)$ in exactly the same manner as $C_o(\theta)$ was obtained from those for $U_1(\zeta, \theta, \phi)$. Since this calculation is quite straightforward, we simply quote the result

$$C_1(\theta) = C_o^{-3}\,\theta \left[q_1 + \frac{D_1(\theta)}{2a} + \frac{1}{4a} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial C_o(\theta)}{\partial \theta} \right) \right] . \quad \text{(III. 76)}$$

As in the case of slow rotation, the regular perturbation expansion of the temperature distribution in a rapidly rotating sphere is nonuniform in certain domains. These troublesome domains are located in the vicinity of the circles $\theta = \cos^{-1}(\pm \sin \beta)$ where the second derivatives with respect to θ of the averaged flux distribution $\overline{G_S(\theta, \beta)}$ (and thus, also $C_o(\theta)$) are singular.

IV. APPLICATION OF SINGULAR PERTURBATION THEORY TO OBTAIN UNIFORMLY VALID PERTURBATION EXPANSIONS

A. Nonuniformity of the Regular Perturbation Expansions in Certain Domains (Boundary Layers)

The singular behavior of the first order term in the regular perturbation expansion of $U(\zeta, \theta, \phi)$ has already been noted in Sections III-D-1 and III-D-2. Calculation of additional terms in this expansion only serves to worsen the situation, since the singularity becomes progressively stronger in each successive term. The error incurred in an asymptotic approximation is of the order of the last term omitted. Consequently, although the zero order term in the regular perturbation expansion of $U(\zeta, \theta, \phi)$ is well behaved everywhere, it is not a uniformly valid approximation since the error (that is, $\delta/a\ U_1(\zeta, \theta, \phi)$) is unbounded in the domains where $U_1(\zeta, \theta, \phi)$ is singular. In order to obtain a uniformly valid approximation, the regular perturbation expansion must be replaced in the vicinity of its singularities by an expansion that is well behaved in these regions.

Perturbation problems in which the straightforward application of regular perturbation theory fails to yield uniformly valid results are referred to as singular perturbation problems. The source of the singular behavior of the regular perturbation expansion for such problems can usually be traced to the improper treatment of a particular term, or group of terms, in the exact equation. In regular perturbation theory, the terms in a differential equation are ordered according to the order of their coefficients. For instance, the m^{th} term in the differential equation ($\mathcal{L}_m$ is a differential operator)

$$\sum_{m=0}^{N} \epsilon^m \mathcal{L}_m W(\underline{x}) = 0 \qquad \text{(IV.1)}$$

is taken to be $O(\epsilon^m)$. This assumes, however, that $\mathcal{L}_m W(\underline{x})$ is of the order unity. Singular perturbation problems are characterized by the existence of vanishingly small domains within which this condition is violated. These domains are located in regions where the nature of the solution changes rapidly and drastically.

The most famous example of a singular perturbation problem is the fluid flow past an obstacle in the limit of small viscosity. At a sufficiently great distance from the surface of the obstacle, regular perturbation theory yields a valid approximation, the first term of which corresponds to the inviscid flow field past the obstacle. The regular perturbation expansion, however, fails to describe the flow field in a thin layer near the surface of the obstacle (boundary layer). In this region, there is a rapid change in the field from that characteristic of inviscid flow to that characteristic of viscous flow. Prandtl[6] recognized that the difficulties associated with the regular perturbation expansion in the boundary layer could be remedied by correctly appraising the order of magnitude of the various terms in the governing differential equations. Subsequent work on singular perturbation theory has also been largely directed toward the solution of fluid flow problems. As a consequence, the domains within which regular perturbation theory fails are commonly referred to as boundary layer regions even in problems that have nothing to do with fluid flow.

The singular behavior of the perturbation calculation of $U(\zeta, \theta, \phi)$ is caused by a lack of sufficient smoothness in the incident solar flux distribution $G_S(\theta, \phi, \beta)$. Although G_S is continuous everywhere, its derivatives are discontinuous across the shadow boundary. The resultant singular behavior of the second and higher order derivatives produces a rapid variation of $U(\zeta, \theta, \phi)$ in the vicinity of the shadow boundary. Thus the assumption that

$$\left(\frac{\delta}{a}\right)^2 \left(\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2}\right) U(\zeta, \theta, \phi)$$

is of the order $(\delta/a)^2$ is not legitimate in this domain. A systematic procedure that can be utilized to obtain a valid perturbation expansion in the boundary layer domains will be discussed in the next section.

B. Method of Matched Asymptotic Expansions

In the time that has elapsed since Prandtl introduced his boundary layer theory, the ideas embodied in that work have been developed and extended by others into a systematic procedure for the solution of singular perturbation problems (e.g., Friedrichs,[7,8] Kaplun,[9,10] and Lagerstrom and Kaplun[11]). For the sake of brevity, the discussion given here will be limited to the essential steps in the procedure. Those interested in a more detailed account of the method are referred to a recent book by Van Dyke.[12]

The first step in the derivation of a perturbation expansion which is to be valid in a boundary layer region is to identify the source of nonuniformity in the regular perturbation expansion. This can be accomplished in a number of interrelated ways. An examination of the nature of the singularity will usually reveal which terms in the governing differential equations have been improperly accounted for in the conventional perturbation procedure. Moreover, in physical problems, a qualitative understanding of the nature of the solution assists in the identification of these terms. Generally speaking, careful scrutiny of all terms involving second or higher order derivatives is recommended.

After the source of the nonuniformity has been identified, the next step involves a stretching of the independent variables by factors that depend on the smallness parameter.* The degree of stretching required is usually suggested again either by the behavior of the regular perturbation expansion in the boundary layer domain or by physical intuition. When both of these approaches fail, the requisite stretching can be determined by trial and error. An incorrect choice for the stretching factor will yield a perturbation expansion that cannot satisfy the matching condition (to be discussed below). Thus, although rather costly in terms of time, the trial and error approach will eventually lead to the appropriate stretching factor.

The equations satisfied by the various terms in the boundary layer perturbation expansion are obtained by a repeated application of a limit process that involves taking a limit as the smallness parameter goes to zero with the stretched variables held fixed. That is, the equations are obtained in the same manner as those in the regular perturbation theory except that the stretched variables are held fixed instead of the original variables. The domain within which these new equations apply is a stretched copy of the original boundary layer domain. In contrast to the original domain, which is vanishingly small, the stretched domain is usually of infinite extent.

The final step in the procedure is to match the boundary layer expansion obtained from the stretched equations with the regular perturbation expansion in a domain of common validity. Matching to the m^{th} order requires that

$$\lim_{\substack{\epsilon \to 0 \\ \text{Intermediate variables fixed}}} \epsilon^{-n} \left[\text{Regular expansion - boundary layer expansion} \right] = 0 \qquad \text{(IV.2)}$$

*In some problems it is also necessary to stretch the dependent variables.

where ϵ is the smallness parameter and both expansions are expressed in variables (intermediate variables) appropriate to the domain of common validity. The intermediate variables are stretched but not as much as the boundary layer variables. The existence of a domain within which (IV. 2) is satisfied ensures the uniformity of the perturbation expansion in the transition from the region exterior to that interior to the boundary layer domain. As mentioned above, the nonexistence of a domain within which (IV. 2) is satisfied indicates that the stretching factor employed in the second step is incorrect.

C. Singular Perturbation Calculations

Singular perturbation theory will be applied in this section to the calculation of the temperature distribution in the vicinity of the singularities of the regular perturbation expansions. Preliminary to this, however, an alternative derivation of the zero order equations satisfied by the temperature distribution will be given. The results obtained provide a clear insight into the rationale behind the selection of the stretching factor utilized in the singular perturbation calculations.

The alternative derivation starts with an integration of the exact equation (II. 2) over a small volume element ΔV

$$\Delta V = \int_{\theta}^{\theta+d\theta} \int_{\phi}^{\phi+d\phi} \int_{a-\delta}^{a+\delta} r^2 \sin\theta \, d\theta \, d\phi \, dr \quad . \tag{IV. 3}$$

Application of Gauss's theorem to the resulting equation then yields the relation

$$\int_S \frac{\partial T}{\partial n} \, dS - \frac{\omega}{\kappa} \int_{\Delta V} \frac{\partial T}{\partial \phi} \, dV = 0 \quad , \tag{IV. 4}$$

where S denotes the surface defined by the boundaries of ΔV. The integrals involving the angular coordinates θ and ϕ can easily be evaluated because of the incremental nature of the volume element with respect to these variables. Substitution of the boundary conditions (II. 4) and (II. 7) in the resultant expression then yields

$$\left\{ \frac{\sigma\epsilon}{K} \left[-(a+\delta)^2 T^4(a+\delta, \theta, \phi) - a^2 T^4(a-\delta, \theta, \phi) + \frac{\epsilon F_o G_S + Q}{\sigma\epsilon} \right] \right.$$

$$+ \int_{a-\delta}^{a+\delta} \frac{1}{\sin \theta d\theta} \left[\sin(\theta + d\theta) \frac{\partial T(r, \theta + d\theta, \phi)}{\partial \theta} - \sin \theta \frac{\partial T(r, \theta, \phi)}{\partial \theta} \right] rdr$$

$$\left. + \int_{a-\delta}^{a+\delta} \frac{1}{\sin^2 \theta d\phi} \left[\frac{\partial T(r, \theta, \phi + d\phi)}{\partial \phi} - \frac{\partial T(r, \theta, \phi)}{\partial \phi} \right] rdr - \frac{\omega}{K} \int_{a-\delta}^{a+\delta} \frac{\partial T(r, \theta, \phi)}{\partial \theta} r^2 dr \right\} \quad 0 \ .$$

(IV. 5)

Now expand (IV. 5) in the limit of small δ (much less than a) and take the limit as the angular increments $d\theta$ and $d\phi$ go to zero. The following relation is obtained when terms of zero and first order in δ are retained.

$$\frac{\sigma\epsilon}{K} \left[-2 T^4(a, \theta, \phi) + \frac{\epsilon F_o G_S + Q}{\sigma\epsilon} \right]$$

$$+ \frac{\delta}{a} 2 \left[\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) \left(\sin \theta \frac{\partial}{\partial \theta} \right) \right] + \frac{\partial^2}{\partial \phi} \quad T(a, \theta, \phi) - 2 \frac{\omega\delta}{K} \frac{\partial T(a, \theta, \phi)}{\partial \theta} \ .$$

(IV. 6)

The conventional zero order approximations to $T(\underline{r})$ are obtained when the limit process Lim_{ζ} is applied to (IV. 6) (the form of the equations obtained depends on the relative magnitudes of ω and δ). As mentioned previously, however, the angular derivatives of $T(\underline{r})$ are very large in the boundary layer domains. Even though the second term in (IV. 6) is multiplied by the small parameter δ/a, it should not be neglected in these domains. To negate the factor δ/a multiplying the second term, the angular variables must be stretched by a factor $(a/\delta)^{1/2}$. Thus, the stretching factor $(a/\delta)^{1/2}$ will be employed in the boundary layer calculations.

1. $\underline{\frac{\omega a^2}{K} = O(1) \text{ (Slow Rotation)}}$

The first two terms in the perturbation expansion of $U(\zeta, \theta, \phi)$ valid in the vicinity of the shadow boundary will be derived in this section. Additional terms can be obtained in the same manner.

The source of the nonuniformity of the regular perturbation expansion of $U(\zeta, \theta, \phi)$ is the rapid variation of the temperature distribution in the vicinity of the shadow boundary. According to the stretching principle the variables along the direction of rapid change should be stretched. To eliminate the necessity of stretching more than one variable, eq. (III. 3) will be rewritten in terms of variables defined with respect to a coordinate system oriented in the direction of the incident flux (see Fig. 2). The distance from the shadow boundary is then a function only of the polar angle θ'. When written in terms of r, θ', and ϕ', eq. (III. 3) assumes the form

$$\left\{ \frac{\partial^2}{\partial \zeta^2} + \frac{\delta}{a}\left[2 \frac{\partial}{\partial \zeta}\left(\zeta \frac{\partial}{\partial \zeta}\right)\right] + \left(\frac{\delta}{a}\right)^2 \left[\frac{\partial}{\partial \zeta}\left(\zeta^2 \frac{\partial}{\partial \zeta}\right) + \frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'}\left(\sin \theta' \frac{\partial}{\partial \theta'}\right) + \frac{1}{\sin^2 \theta'} \frac{\partial^2}{\partial \phi'^2} - \frac{\omega a^2}{\kappa} \frac{\partial}{\partial \phi}\right] + \left(\frac{\delta}{a}\right)^3 \left[- \frac{2\omega a^2}{\kappa} \zeta \frac{\partial}{\partial \phi}\right] + \left(\frac{\delta}{a}\right)^4 \left[- \frac{\omega a^2}{\kappa} \zeta^2 \frac{\partial}{\partial \phi}\right]\right\} U(\zeta, \theta', \phi') = 0 \ , \tag{IV. 7}$$

where the unprimed ϕ derivative in (IV. 7) is

$$\frac{\partial}{\partial \phi} = \sin \beta \sin \phi' \frac{\partial}{\partial \theta'} + (\cos \beta + \cot \theta' \cos \phi' \sin \beta) \frac{\partial}{\partial \phi'} \ . \tag{IV. 8}$$

Calculations given in the introductory portion of this section indicate that the appropriate stretching factor is $(a/\delta)^{1/2}$. To achieve this stretching, the variables θ' will be replaced by a new variable

$$\xi = \left(\frac{a}{\delta}\right)^{1/2} \cos \theta' \tag{IV. 9}$$

in terms of which the equations satisfied by $U(\zeta, \theta, \phi)$ become

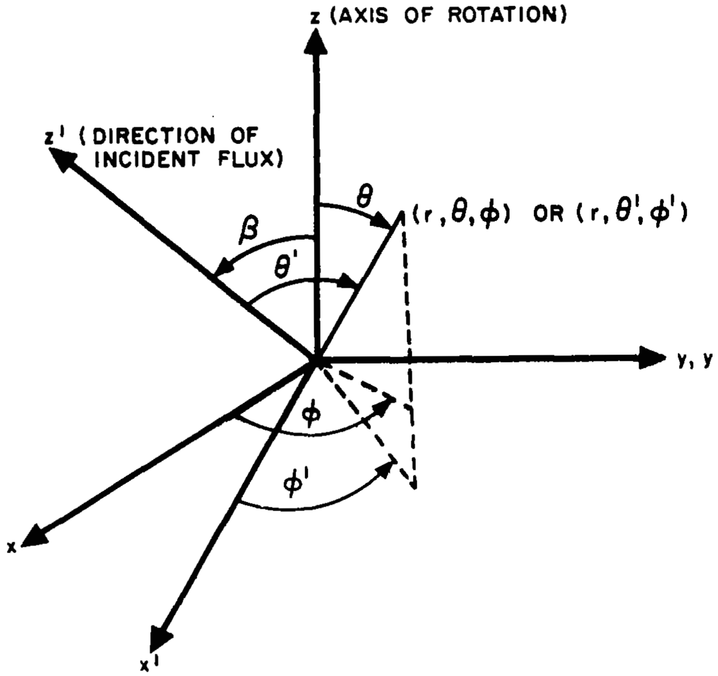

Fig. 2. A new coordinate system.

$$\left\{ \frac{\partial^2}{\partial \zeta^2} + \frac{\delta}{a} \left[2 \frac{\partial}{\partial \zeta} \left(\zeta \frac{\partial}{\partial \zeta} \right) + \frac{\partial^2}{\partial \xi^2} \right] + \left(\frac{\delta}{a} \right)^{3/2} \left[\frac{\omega a^2}{\kappa} \sin \beta \sin \phi' \frac{\partial}{\partial \xi} \right] \right.$$

$$+ \left(\frac{\delta}{a} \right)^2 \left[\frac{\partial}{\partial \zeta} \left(\zeta^2 \frac{\partial}{\partial \zeta} \right) - \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial}{\partial \xi} \right) + \frac{\partial^2}{\partial \phi'^2} - \frac{\omega a^2}{\kappa} \cos \beta \frac{\partial}{\partial \phi'} \right]$$

$$+ \left(\frac{\delta}{a} \right)^{5/2} \left[- \frac{\omega a^2}{\kappa} \sin \beta \left(\sin \phi' \frac{\xi^2}{2} \frac{\partial}{\partial \xi} - \cos \phi' \, \xi \frac{\partial}{\partial \phi'} \right) \right]$$

$$\left. + \, 0 \left(\frac{\delta}{a} \right)^3 \right\} \cdot U(\zeta, \xi, \phi) = 0 \qquad \begin{aligned} -1 &< \zeta < 1 \\ -\infty &< \xi < \infty \\ 0 &< \phi < 2\pi \end{aligned} \qquad \text{(IV. 10)}$$

$$- \frac{\partial U}{\partial \zeta} = \frac{\delta}{a} \alpha \left[U^4 - \left(\frac{\delta}{a} \right)^{1/2} \xi \, H(\xi) \right. \quad = 1 \; , \qquad \text{(IV. 11)}$$

$$\frac{\partial U}{\partial \zeta} = \frac{\delta}{a} \alpha \; [U^4 - q] \qquad \zeta = -1 \; . \qquad \text{(IV. 12)}$$

where $H(\xi)$ is a unit step function.

The equations satisfied by the zero order term in the boundary layer perturbation expansion can be obtained from (IV. 10) to (IV. 12) by applying the limit process

$$\text{Lim } \delta/a \rightarrow 0; \quad \zeta, \xi, \phi' \text{ fixed} \equiv \text{Lim}_{\zeta, \xi} \; . \qquad \text{(IV. 13)}$$

It is found that

$$\frac{\partial^2 U_{oi}}{\partial \zeta^2} = 0 \quad , \tag{IV.14}$$

$$\frac{\partial U_{oi}}{\partial \zeta} = 0 \quad , \quad \zeta = \pm 1 \tag{IV.15}$$

where the subscript i denotes that U_{oi} is a boundary layer term. The solution of (IV.14) and (IV.15) is the trivial solution

$$U_{oi}(\zeta, \xi, \phi') = A_{oi}(\xi, \phi') \quad . \tag{IV.16}$$

The thickness parameter appears in (IV.10) to (IV.12) in powers of $(\delta/a)^{1/2}$. This indicates that the perturbation expansion of $U(\zeta, \xi, \phi')$ should be sought in powers of $(\delta/a)^{1/2}$. The equations for the second term in the boundary layer expansion are obtained when $\mathrm{Lim}_{\zeta,\xi}(a/\delta)^{1/2}$ is applied to the difference between (IV.10) ((IV.11), (IV.12)) and (IV.14) ((IV.15), (IV.15)). These equations are identical to (IV.14) and (IV.15). Therefore $U_{1i}(\zeta, \xi, \phi')$ can be written

$$U_{1i}(\zeta, \xi, \phi') = A_{1i}(\xi, \phi') \quad . \tag{IV.17}$$

To determine A_{oi} and A_{1i}, the equations satisfied by U_{2i} and U_{3i} must be considered. Because the method of deriving such equations has been amply demonstrated in the preceding portions of this report, we will henceforth dispense with an account of the details of these derivations. U_{2i} is found to satisfy the equations

$$\frac{\partial^2 U_{2i}}{\partial \zeta^2} + \frac{\partial^2 A_{oi}}{\partial \xi^2} = 0 \quad , \tag{IV.18}$$

$$-\frac{\partial U_{2i}}{\partial \zeta} = \alpha A_{oi}^4 \quad , \quad \zeta = 1 \tag{IV.19}$$

$$\frac{\partial U_{2i}}{\partial \zeta} = \alpha \left[A_{oi}^4 - q_o \right] \quad , \qquad \zeta = -1 \quad . \tag{IV.20}$$

and U_{3i} the equations

$$\frac{\partial^2 U_{3i}}{\partial \zeta^2} + \frac{\partial^2 A_{1i}}{\partial \xi^2} = 0 \quad , \tag{IV.21}$$

$$- \frac{\partial U_{3i}}{\partial \zeta} = 4\alpha \, A_{oi}^3 \, A_{1i} - \alpha \xi \, H(\xi) \quad , \qquad \zeta = 1 \quad . \tag{IV.22}$$

$$\frac{\partial U_{3i}}{\partial \zeta} = 4\alpha \, A_{oi}^3 \, A_{1i} \quad , \qquad \zeta = -1 \quad . \tag{IV.23}$$

Consider first the determination of A_{oi}. Since A_{oi} is independent of ζ, integration of (IV.18) yields the following expression for $U_{2i}(\zeta, \xi, \phi')$

$$U_{2i}(\zeta, \xi, \phi') = A_{2i}(\xi, \phi') + \zeta B_{2i}(\xi, \phi') - \frac{\zeta^2}{2} \frac{\partial^2 A_{oi}}{\partial \xi^2} \quad . \tag{IV.24}$$

Substitution of (IV.24) in the boundary conditions (IV.19) and (IV.20) gives equations that can be solved directly for $B_{2i}(\xi, \phi')$ and from which an equation satisfied by $A_{oi}(\xi, \phi')$ can be obtained. As a result we find that

$$B_{2i}(\xi, \phi') = - \frac{\alpha}{2} q_o \quad , \tag{IV.25}$$

$$\frac{\partial^2 A_{oi}}{\partial \xi^2} = \alpha \left[A_{oi}^4 - \frac{q_o}{2} \right] \quad , \qquad - \infty < \xi < \infty \quad . \tag{IV.26}$$

The desired solution of (IV. 26) is uniquely determined by the condition that U_{oi} and U_o match in their domains of common validity. Assuming that there exists an $\eta, 0 < \eta < 1/2$, and a bounded number γ such that the domains of common validity are given by

$$\xi = \left(\frac{a}{\delta}\right)^{\eta} \gamma \quad , \qquad \text{(IV. 27)}$$

$$\cos \theta' = \left(\frac{\delta}{a}\right)^{(1/2)-\eta} \gamma \ , \qquad \text{(IV. 28)}$$

it can be shown that the matching condition requires that

$$U_{oi}(\pm \infty) = U_o(0) = \left(\frac{q_o}{2}\right)^{1/4} \quad , \qquad \text{(IV. 29)}$$

$$\frac{\partial^m U_{oi}}{\partial \xi^m} = 0, \qquad \xi = \pm \infty, \qquad m = 1, 2, 3, \ldots \quad . \qquad \text{(IV. 30)}$$

The only solution of (IV. 22) that satisfies (IV. 29) and (IV. 30) is

$$A_{oi}(\xi, \phi') = \left(\frac{q_o}{2}\right)^{1/4} \quad . \qquad \text{(IV. 31)}$$

The behavior of the zero order term in the perturbation expansion of $U(\zeta, \theta, \phi)$ is now known everywhere. Substitution of the results expressed in (III. 43) and (IV. 31) in (III. 39) yields an equation that determines the hitherto unspecified constant q_o. The result is

$$q_o = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \; G_S(\theta, \phi, \beta) = \frac{1}{4} \quad . \qquad \text{(IV. 32)}$$

An equation for $A_{1i}(\xi, \phi')$ can be derived from (IV.21) to (IV.23) in exactly the same manner as was that for A_{oi} from (IV.18) to (IV.20). We find that

$$\frac{\partial^2 A_{1i}}{\partial \xi^2} = (4\alpha A_{oi}^3)\, A_{1i} - \frac{\alpha}{2}\, \xi\, H(\xi), \qquad -\infty < \xi < \infty \quad . \qquad \text{(IV.33)}$$

$A_{1i}(\xi, \phi')$ is uniquely determined by the conditions that must be imposed to ensure matching. The matching condition is

$$\lim_{\frac{\delta}{a} \to 0,\ \gamma \text{ fixed}} \left(\frac{a}{\delta}\right)^{1/2} \left[U_o\left(\zeta, \cos^{-1} \frac{\delta}{a}^{(1/2)-\eta} \gamma, \phi\right) - U_{oi}\left(\zeta, \left(\frac{a}{\delta}\right)^{\eta} \gamma, \phi'\right) - \left(\frac{\delta}{a}\right)^{1/2} U_{1i}\left(\zeta, \left(\frac{a}{\delta}\right)^{\eta} \gamma, \phi'\right) \right] = 0$$

(IV.34)

and reference to results previously obtained then indicates that

$$\lim_{\xi \to \infty} A_{1i}(\xi, \phi') \propto \xi \quad , \qquad \text{(IV.35)}$$

$$\lim_{\xi \to -\infty} A_{1i}(\xi, \phi') = 0 \quad . \qquad \text{(IV.36)}$$

Solution of (IV.33) by variation of parameters with the conditions (IV.35) and (IV.36) imposed at $\xi = \pm\infty$ yields

$$A_{1i}(\xi, \phi') = \frac{1}{8A_{oi}^3} \left[\left(\xi + \frac{1}{2b} e^{-b\xi} \right) H(\xi) + \frac{1}{2b} e^{b\xi} (1 - H(\xi)) \right], \tag{IV.37}$$

where

$$b = \left[4\alpha \left(\frac{q_o}{2} \right)^{3/4} \right]^{1/2} . \tag{IV.38}$$

2. $\underline{\frac{\omega a^2}{K} = O(a/\delta)^2 \text{ (Rapid Rotation)}}$

The regular perturbation expansion of the temperature distribution in a rapidly rotating shell is nonuniform in the vicinity of the circles $\theta = \cos^{-1}(\pm \sin \beta)$. As seen from Fig. 3, these circles are the boundary lines between the regions on a rotating sphere where the incident solar flux at a point is (1) always nonzero, (2) nonzero at some instants and zero at others, and (3) always zero. Regular perturbation theory does not properly account for the rapid variation of the temperature distribution in the transition from one of these regions to another. In this section singular perturbation theory will be utilized to derive the first three terms in the perturbation expansion of $U(\zeta, \theta, \phi)$ valid in the vicinity of $\theta = \cos^{-1}(\sin \beta)$. Additional terms in this expansion and an expansion valid near $\theta = \cos^{-1}(-\sin \beta)$ can be obtained in a similar manner.

The rapid variation of $U(\zeta, \theta, \phi)$ near $\theta = \cos^{-1}(\sin \beta)$ will be accounted for by replacing the variable θ with a new stretched variable μ*

$$\mu = \left(\frac{a}{\delta} \right)^{1/2} (\cos \theta - \sin \beta) . \tag{IV.39}$$

When expressed in terms of the variables ζ, μ, and ϕ, the equations satisfied by a rapidly rotating spherical shell assume the form

*As is customary, the stretched variable is defined such that it goes to zero in the boundary layer.

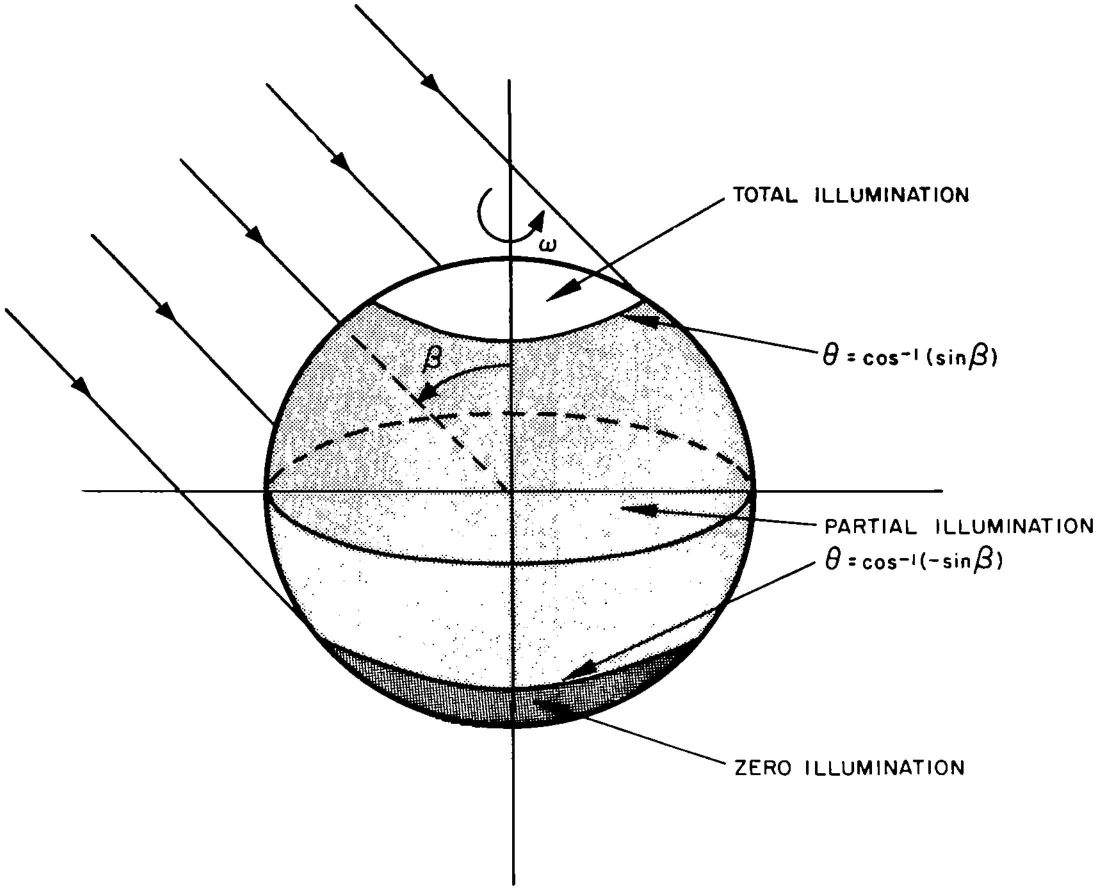

Fig. 3. Flux distribution on a rapidly rotating shell.

$$\left\{\left[\frac{\partial^2}{\partial\zeta^2} - \frac{\omega\delta^2}{\kappa}\frac{\partial}{\partial\phi}\right] + \frac{\delta}{a}\left[2\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial}{\partial\zeta}\right) + \frac{\partial^2}{\partial\mu^2} - 2\frac{\omega\delta^2}{\kappa}\zeta\frac{\partial}{\partial\phi}\right]\right.$$

$$+ \left(\frac{\delta}{a}\right)^2\left[\frac{\partial}{\partial\zeta}\left(\zeta^2\frac{\partial}{\partial\zeta}\right) - \sin^2\beta\frac{\partial^2}{\partial\mu^2} + \frac{1}{\cos^2\beta}\frac{\partial^2}{\partial\phi^2} - \frac{\omega\delta^2}{\kappa}\zeta^2\frac{\partial}{\partial\phi}\right]$$

$$\left. + \left(\frac{\delta}{a}\right)^{5/2}\left[-2\sin\phi\frac{\partial}{\partial\mu}\left(\mu\frac{\partial}{\partial\mu}\right) + \frac{2}{\cos^4\beta}\mu\frac{\partial^2}{\partial\phi^2}\right] + O\left(\frac{\delta}{a}\right)^3\right\}\cdot$$

$$U(\zeta,\mu,\phi) = 0 \quad , \qquad \text{(IV.40)}$$

$$-\frac{\partial U}{\partial\zeta} = \frac{\delta}{a}\,\alpha\left[U^4 - G_S(\mu,\phi,\beta,\delta/a)\right] , \qquad \zeta = 1 \qquad \text{(IV.41)}$$

$$\frac{\partial U}{\partial\zeta} = \frac{\delta}{a}\,\alpha\left[U^4 - q\right] , \qquad \zeta = -1 \quad . \qquad \text{(IV.42)}$$

Application of the limit process

$$\text{Lim } \delta/a \to 0;\ \zeta,\mu,\phi \text{ fixed} \equiv \text{Lim}_{\zeta,\mu} \quad , \qquad \text{(IV.43)}$$

to (IV.40) to (IV.42) yields the equations satisfied by the zero order term in the boundary layer expansion.

$$\left(\frac{\partial^2}{\partial\zeta^2} - \frac{\omega\delta^2}{\kappa}\frac{\partial}{\partial\phi}\right)U_{oi}(\zeta,\mu,\phi) = 0 \quad , \qquad \text{(IV.44)}$$

$$\frac{\partial U_{oi}}{\partial \zeta} = 0, \qquad \zeta = \pm 1 \quad . \tag{IV.45}$$

The solution of (IV.44) and (IV.45) is simply

$$U_{oi}(\zeta, \mu, \phi) = C_{oi}(\mu) \tag{IV.46}$$

where C_{oi} is a function independent of ζ and ϕ.

The form of the regular perturbation expansion in the vicinity of the singularity at $\theta = \cos^{-1}(\sin \beta)$ indicates that the boundary layer perturbation expansion should be sought in powers of $(\delta/a)^{1/4}$. It can easily be shown that the second, third, and fourth terms in such an expansion also satisfy (IV.44) and (IV.45). Therefore, these terms are also independent of ζ and ϕ.

$$U_{ji}(\zeta, \mu, \phi) = C_{ji}(\mu), \qquad (j = 0, 1, 2, 3) \tag{IV.47}$$

To determine the functional forms of the $C_{ji}(\mu)$ $(j = 0, 1, 2, 3)$, the equations satisfied by $U_{ki}(\zeta, \mu, \phi)$ $(k = 4, 5, 6, 7)$ must be considered. These equations are easily obtained from (IV.40) to (IV.42) by repeatedly applying $\mathrm{Lim}_{\zeta,\mu}$ in the manner illustrated in the calculations given in the preceding portions of this report. The resulting equations can be expressed in the form

a. $\underline{U_{4i}(\zeta, \mu, \phi)}$

$$\left(\frac{\partial^2}{\partial \zeta^2} - \frac{\omega \delta^2}{\kappa} \frac{\partial}{\partial \phi}\right) U_{4i} + \frac{\partial^2 C_{oi}}{\partial \mu^2} = 0 \quad , \tag{IV.48}$$

$$- \frac{\partial U_{4i}}{\partial \zeta} = a \left[C_{oi}^4 - G_{So}\right] , \qquad \zeta = 1 \tag{IV.49}$$

$$\frac{\partial U_{4i}}{\partial \zeta} = a \left[C_{oi}^4 - q_o\right] , \qquad \zeta = -1 \quad . \tag{IV.50}$$

b. $\underline{U_{5i}(\zeta, \mu, \phi)}$

$$\left(\frac{\partial^2}{\partial \zeta^2} - \frac{\omega \delta^2}{\kappa} \frac{\partial}{\partial \phi}\right) U_{5i} + \frac{\partial^2 C_{1i}}{\partial \mu^2} = 0 \quad , \tag{IV.51}$$

$$- \frac{\partial U_{5i}}{\partial \zeta} = a \left[4\, C_{oi}^3\, C_{1i} - G_{S1} \right] , \qquad \zeta = 1 \tag{IV.52}$$

$$\frac{\partial U_{5i}}{\partial \zeta} = a\, 4\, C_{oi}^3\, C_{1i}, \qquad \zeta = -1 \quad . \tag{IV.53}$$

c. $\underline{U_{6i}(\zeta, \mu, \phi)}$

$$\left(\frac{\partial^2}{\partial \zeta^2} - \frac{\omega \delta^2}{\kappa} \frac{\partial}{\partial \phi}\right) U_{6i} + \frac{\partial^2 C_{2i}}{\partial \mu^2} = 0 \quad , \tag{IV.54}$$

$$- \frac{\partial U_{6i}}{\partial \zeta} = a \left[4 C_{oi}^3\, C_{2i} + 6 C_{oi}^2\, C_{1i}^2 - G_{S2} \right] \tag{IV.55}$$

$$\frac{\partial U_{6i}}{\partial \zeta} = a \left[4 C_{oi}^3\, C_{2i} + 6 C_{oi}^2\, C_{1i}^2 \right] , \qquad \zeta = -1 \quad . \tag{IV.56}$$

d. $\underline{U_{7i}(\zeta, \mu, \phi)}$

$$\left(\frac{\partial^2}{\partial \zeta^2} - \frac{\omega \delta^2}{\kappa} \frac{\partial}{\partial \phi}\right) U_{7i} + \frac{\partial^2 C_{3i}}{\partial \mu^2} = 0 \quad , \tag{IV.57}$$

$$- \frac{\partial U_{7i}}{\partial \zeta} = a \left[4 (C_{oi}^3\, C_{3i} + C_{oi}\, C_{1i}^3 + C_{oi}^2\, C_{1i}\, C_{2i}) - G_{S3} \right] , \qquad \zeta = 1 \tag{IV.58}$$

$$\frac{\partial U_{7i}}{\partial \zeta} = a4\left[C_{oi}^3 C_{3i} + C_{oi} C_{1i}^3 + C_{oi}^2 C_{1i} C_{2i}\right], \qquad \zeta = -1, \qquad \text{(IV.59)}$$

where

$$G_{Sm}(\mu, \phi, \beta) = \mathrm{Lim}_{\zeta,\mu}\left(\frac{a}{\delta}\right)^{m/4}\left[G_S\left(\mu, \phi, \beta; \frac{\delta}{a}\right) - \sum_{n=0}^{m-1}\left(\frac{\delta}{a}\right)^{m-1} G_{Sn}(\mu, \phi, \beta)\right]. \qquad \text{(IV.60)}$$

To obtain an equation for $C_{oi}(\mu)$, integrate (IV.48) to (IV.50) with respect to ϕ over the interval $0 < \phi < 2\pi$ and divide the result by 2π. Since $U_{4i}(\zeta, \mu, \phi)$ is periodic in ϕ with period 2π and $C_{oi}(\mu)$ in independent of ϕ, the following equations are obtained (quantities averaged over the interval $0 \leq \phi \leq 2\pi$ are denoted by a bar)

$$\frac{\partial^2 \overline{U}_{4i}}{\partial \zeta^2} + \frac{\partial^2 C_{oi}}{\partial \mu^2} = 0, \qquad \text{(IV.61)}$$

$$-\frac{\partial \overline{U}_{4i}}{\partial \zeta} = a\left[C_{oi}^4 - \overline{G}_{So}\right], \qquad \zeta = 1 \qquad \text{(IV.62)}$$

$$\frac{\partial U_{4i}}{\partial \zeta} = a\left[C_{oi}^4 - q_o\right], \qquad \zeta = -1 \qquad \text{(IV.63)}$$

Integration of (IV.61) and substitution of the result in (IV.62) and (IV.63) yields two equations from which an equation for $C_{oi}(\mu)$ can be obtained. The result is

$$\frac{d^2 C_{oi}}{d\omega^2} = a\, C_{oi}^4 - \frac{a}{2}\left[\overline{G}_{So} + q_o\right], \qquad -\infty < \mu < \infty \qquad \text{(IV.64)}$$

Equations for $C_{1i}(\mu)$, $C_{2i}(\mu)$ and $C_{3i}(\mu)$ can be obtained in exactly the same manner. The results are

$$\frac{d^2 C_{1i}}{d\mu^2} = 4a\, C_{oi}^3 C_{1i} - \frac{a}{2}\overline{G}_{S1}, \qquad \text{(IV.65)}$$

$$\frac{d^2 C_{2i}}{d\mu^2} = 4a\, C_{oi}^3\, C_{2i} + a \left[6C_{oi}^2\, C_{1i}^2 - \frac{1}{2} \overline{G}_{S2} \right], \tag{IV.66}$$

$$\frac{d^2 C_{3i}}{d\mu^2} = 4a\, C_{oi}^3\, C_{2i} + a \left[4C_{oi}^2\, C_{1i}\, C_{2i} + 4C_{oi}\, C_{1i}^3 - \frac{1}{2} \overline{G}_{S3} \right], \tag{IV.67}$$

where $-\infty < \mu < \infty$. The functional forms of the $\overline{G_{S_m}(\mu)}$ are determined by averaging (IV.60) over ϕ and expanding the average of $G_S(\mu, \phi, \beta; \delta/a)$ about $\mu = 0$. This yields

$$\overline{G_{So}(\mu)} = \sin\beta \cos\beta \quad , \tag{IV.68}$$

$$\overline{G_{S1}(\mu)} = 0 \quad , \tag{IV.69}$$

$$\overline{G_{S2}(\mu)} = \mu \cos\beta \quad , \tag{IV.70}$$

$$\overline{G_{S3}(\mu)} = (-\mu)^{3/2}\, H(-\mu)\, M(\beta) \quad , \tag{IV.71}$$

where

$$M(\beta) = (2 \sin\beta \cos^2\beta)^{-1/2}$$

$$\left[\frac{5}{2} \sec\beta + \frac{3}{2} \sin\beta \tan\beta + \frac{3\tan\beta}{\cos^2\beta} - \frac{1}{3\sin\beta\cos^3\beta} \right] \quad \beta \neq 0,\ \pi/2 \ , \tag{IV.72}$$

As in the case of slow rotation, the solutions of the boundary layer equations are uniquely determined by the matching condition. In general, matching to the order $(\delta/a)^{m/4}$ requires that

$$\lim_{\delta/a \to 0,\ \nu\ \text{fixed}} \left(\frac{a}{\delta}\right)^{m/4} \left[\sum_{n=0}^{N} \left(\frac{\delta}{a}\right)^{n} U_n(\zeta, \theta, \phi) - \sum_{n=0}^{P} \left(\frac{\delta}{a}\right)^{n/4} U_{ni}(\zeta, \mu, \phi) \right] = 0$$

(IV.73)

where ν is an intermediate variable appropriate in the domain of common validity. In this domain the variables θ and μ assume the form

$$\mu = \left(\frac{a}{\delta}\right)^{\eta} \nu \quad , \tag{IV.74}$$

$$\theta = \cos^{-1}\left(\sin\beta + \left(\frac{\delta}{a}\right)^{1/2-\eta} \nu\right) , \tag{IV.75}$$

where $0 < \eta < 1/2$ and ν is bounded. Expansion of the regular and boundary layer perturbation series in terms of the intermediate variable defined in (IV.74) and (IV.75), substitution of the result in (IV.73), and application of the limit indicated there for $m = 0, 1, 2, 3$ yields the following conditions on the $C_{ji}(\mu)$ $(j = 0, 1, 2, 3)$.

$$C_{oi}(\pm\infty) = U_o(\zeta, \cos^{-1}\sin\beta, \phi) = \left(\frac{\sin\beta\cos\beta + q_o}{2}\right)^{1/4} ,$$

(IV.76)

$$C_{1i}(\pm\infty) = 0 \quad , \tag{IV.77}$$

$$\lim_{|\mu| >> 1} C_{2i}(\mu) \propto \mu \quad , \tag{IV.78}$$

$$\lim_{|\mu| >> 1} C_{3i}(\mu) \propto (-\mu)^{3/2}\, H(-\mu) \quad . \tag{IV.79}$$

When the conditions at infinity are taken into account, solution of (IV.64) to (IV.67) gives

$$C_{oi}(\mu) = \left(\frac{\sin\beta\cos\beta + q_o}{2}\right)^{1/4} , \tag{IV.80}$$

$$C_{1i}(\mu) = 0 , \tag{IV.81}$$

$$C_{2i}(\mu) = \left(\frac{\cos\beta}{8C_{oi}^3}\right)\mu , \tag{IV.82}$$

$$C_{3i}(\mu) = \frac{M(\beta)}{16\,C_{oi}^3}\left\{\frac{3\sqrt{\pi}}{4h^{3/2}}\,e^{-h\mu} + \left[2(-\mu)^{3/2} + \frac{3}{2h}\left(e^{h\mu}\int_o^{-\mu} e^{hz}\,d(z^{1/2}) - e^{-h\mu}\int_o^{-\mu} e^{-hz}\,d(z^{1/2})\right)\right]H(-\mu)\right\}$$

(IV.83)

where $h = (4\alpha\, C_{oi}^3)^{1/2}$. As in the case of slow rotation, it is a simple matter now to calculate the constant q_o. We find

$$q_o = \frac{1}{2}\int_o^{\pi} d\theta\,\sin\theta\,\overline{G_S(\theta,\beta)} = \frac{1}{4} . \tag{IV.84}$$

V. TRANSITION FROM SLOWLY TO RAPIDLY ROTATING SPHERES — ZERO ORDER TEMPERATURE DISTRIBUTION

A. Introduction

Quantitative estimates of the effect of rotation on the temperature distribution in a spinning spherical shell are of considerable importance. Physically, it is obvious that the temperature extremes on a rapidly rotating shell are less severe than those on a stationary one. A question of practical importance, however, is how fast the shell must rotate to keep the extremes within specified limits. To obtain an answer to this question the mathematical implications of rotation must be investigated. The results obtained in the preceding sections have revealed that the temperature distribution in a spinning shell is affected relatively little when $(\omega a^2/\kappa) = O(1)$ and very significantly when $(\omega a^2/\kappa) = O(a/\delta)^2$. In the first case the temperature distribution differs only slightly from that for a stationary shell and in the second it approximates that for a shell rotating at an infinite rate (i. e., maximum smoothing occurs). Calculations will now be given for the transition of the temperature distribution from that characteristic of slow rotation to that characteristic of rapid rotation. As discussed earlier, this transition occurs when $(\omega a^2/\kappa) = O(a/\delta)$.

B. Differential Equation for Zero Order Temperature Distribution

The differential equation satisfied by the temperature distribution in a thin spherical shell rotating at a rate that is neither slow nor fast can be derived from the zero and first order regular perturbation equations. These equations are obtained from (III. 3) to (III. 5) by applying Lim_ζ in the manner illustrated in the preceding sections. U_o is found to satisfy the equations

$$\frac{\partial^2 U_o}{\partial \zeta^2} = 0 \quad , \tag{V.1}$$

$$\frac{\partial U_o}{\partial \zeta} = 0, \qquad \zeta = \pm 1 \quad , \tag{V.2}$$

and U_1 the equations

$$\frac{\partial^2 U_1}{\partial \zeta^2} + \left[2 \frac{\partial}{\partial \zeta}\left(\zeta \frac{\partial}{\partial \zeta}\right) - \Omega \frac{\partial}{\partial \phi}\right] U_o = 0 \tag{V.3}$$

$$-\frac{\partial U_1}{\partial \zeta} = a\left[U_o^4 - G_s\right], \qquad \zeta = 1 \tag{V.4}$$

$$\frac{\partial U_1}{\partial \zeta} = a\left[U_o^4 - q_o\right], \qquad \zeta = -1 \tag{V.5}$$

where

$$\Omega = \frac{\omega a \delta}{\kappa} \quad . \tag{V.6}$$

Note that the nonzero value of $\lim_{\zeta} \omega a\delta/\kappa$ has been taken into account in the derivation of (V. 3). Integration of (V. 1) indicates that U_o is independent of ζ . Hence, when (V. 3) is integrated we obtain

$$U_1(\zeta, \theta, \phi) = A_1(\theta, \phi) + \zeta B_1(\theta, \phi) + \frac{\zeta^2}{2} \Omega \frac{\partial U_o(\theta, \phi)}{\partial \phi} \quad . \tag{V.7}$$

Substitution of (V.7) in the boundary conditions (V.4) and (V.5) and elimination of $B_1(\theta, \phi)$ from the resultant equations yields the following nonlinear differential equation for $U_o(\theta, \phi)$.

$$\frac{\partial U_o}{\partial \phi} = \frac{\gamma}{2\pi}\left[U_o^4 - \left(\frac{G_s + q_o}{2}\right)\right], \quad 0 \le \phi \le 2\pi \quad . \tag{V.8}$$

where

$$\gamma = \frac{2\pi a}{\Omega} \quad . \tag{V.9}$$

As in the case of slow and fast rotation, the constant q_o can be shown to have the value $1/4$.

Equation (V.8) cannot be solved by exact analytical techniques. Furthermore, perturbation techniques are of little use since the solutions of most interest are those for $\gamma \sim O(1)$.* Solutions for $U_o(\theta, \phi; \beta, \gamma)$, however, can easily be obtained numerically on a digital computer. A number of solutions for various parameter values have been computed. The results of these computations will be discussed next.

C. Zero Order Temperature Distribution

The solution of (V.8) has been numerically computed on a digital computer for fifty-two different parameter combinations. In each case, the solar flux was assumed to be incident at an angle $\beta = (\pi/4)$. This value was selected because the temperature distribution so obtained is typical of what can be expected in the general case (i.e., a rather hot zone near the upper pole, a zone of intermediate temperature in the vicinity of the equator, and a relatively cool zone at the lower pole). Four values of $\theta(\pi/6, \pi/3, \pi/2, 2\pi/3)$ and thirteen values of $\gamma(0.10, 0.50, 0.75, 1.00, 1.25, 1.50, 2.00, 2.50, 3.00, 3.50, 4.00, 4.50, 5.00)$ were considered.**, *** The values of γ's selected cover the range within which perturbation theory fails to yield a good approximation to U_o. When γ is outside this range, perturbation theory can be employed (for small γ perturb about $\gamma = 0$ and for large γ perturb about $\gamma \rightarrow \infty$).

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*Nichols[1] and Hrycak[2] linearize (V.8) by expressing U_o as the sum of a term $U_{o\,\infty}$ that corresponds to the temperature on a sphere rotating at an infinite rate and a remainder term $U_o - U_{o\,\infty}$ which is assumed to be much smaller than $U_{o\,\infty}$. They then obtain solutions of the linearized equation satisfied by $(U_o - U_{o\,\infty})$. This approach is basically a perturbation calculation valid for sufficiently small γ.

**Note that (V.8) can be solved by inspection when $\theta > \cos^{-1}(-\sin\beta)$. The result is $U_o(\theta, \phi) = (q_o/2)^{1/4} = 0.595$. Likewise, when $\theta = 0$, (V.8) is easily solved to yield $U_o(\theta, \phi) = 0.832$.

***The relation between γ and the rate of rotation in a typical case is $\gamma = 0.132/\omega$ with ω expressed in rpm ($F_o = 0.0324$ cal/cm^2-sec, typical of solar flux; $\epsilon = 0.9$, $K = 0.48$, $\kappa = 0.86$, aluminum shell assumed; $a = 60$ cm, $\delta/a = 10^{-3}$).

The maximum and minimum temperatures for the fifty-two cases computed are given in Table I. Note that although the temperature distribution is nicely smoothed in the ϕ direction at high rates of rotation, the temperature variation is still rather large in the θ direction. A detailed plot of the temperature distribution at $\theta = (\pi/2)$ as a function of ϕ for six values of γ is shown in Fig. 4. As expected, the temperature distribution for $\gamma = 0.1$ very closely approximates that for $\gamma = 0$, and the distribution for $\gamma = 5.0$ is close to that for $\gamma = \infty$. The transition from a distribution characteristic of large ω to that characteristic of small ω occurs when $\gamma \sim 1$.

TABLE I

Maximum and Minimum Temperatures; Intermediate Rotation Rates

γ \ θ	$\frac{\pi}{6}$		$\frac{\pi}{3}$		$\frac{\pi}{2}$		$\frac{2\pi}{3}$	
	$U_{o_{max}}$	$U_{o_{min}}$	$U_{o_{max}}$	$U_{o_{min}}$	$U_{o_{max}}$	$U_{o_{min}}$	$U_{o_{max}}$	$U_{o_{min}}$
0.10	0.813	0.808	0.761	0.753	0.702	0.695	0.625	0.623
0.50	0.824	0.796	0.777	0.735	0.714	0.683	0.628	0.619
0.75	0.830	0.790	0.787	0.725	0.721	0.676	0.630	0.617
1.00	0.836	0.783	0.796	0.715	0.729	0.669	0.633	0.615
1.25	0.842	0.777	0.805	0.705	0.736	0.662	0.635	0.613
1.50	0.846	0.771	0.813	0.696	0.743	0.656	0.638	0.611
2.00	0.855	0.762	0.827	0.680	0.756	0.644	0.643	0.608
2.50	0.861	0.754	0.838	0.667	0.767	0.635	0.648	0.605
3.00	0.866	0.747	0.847	0.656	0.777	0.627	0.652	0.603
3.50	0.869	0.742	0.854	0.647	0.785	0.620	0.657	0.601
4.00	0.872	0.738	0.860	0.636	0.792	0.615	0.661	0.600
4.50	0.874	0.734	0.864	0.630	0.798	0.611	0.665	0.598
5.00	0.876	0.731	0.867	0.625	0.803	0.608	0.668	0.598

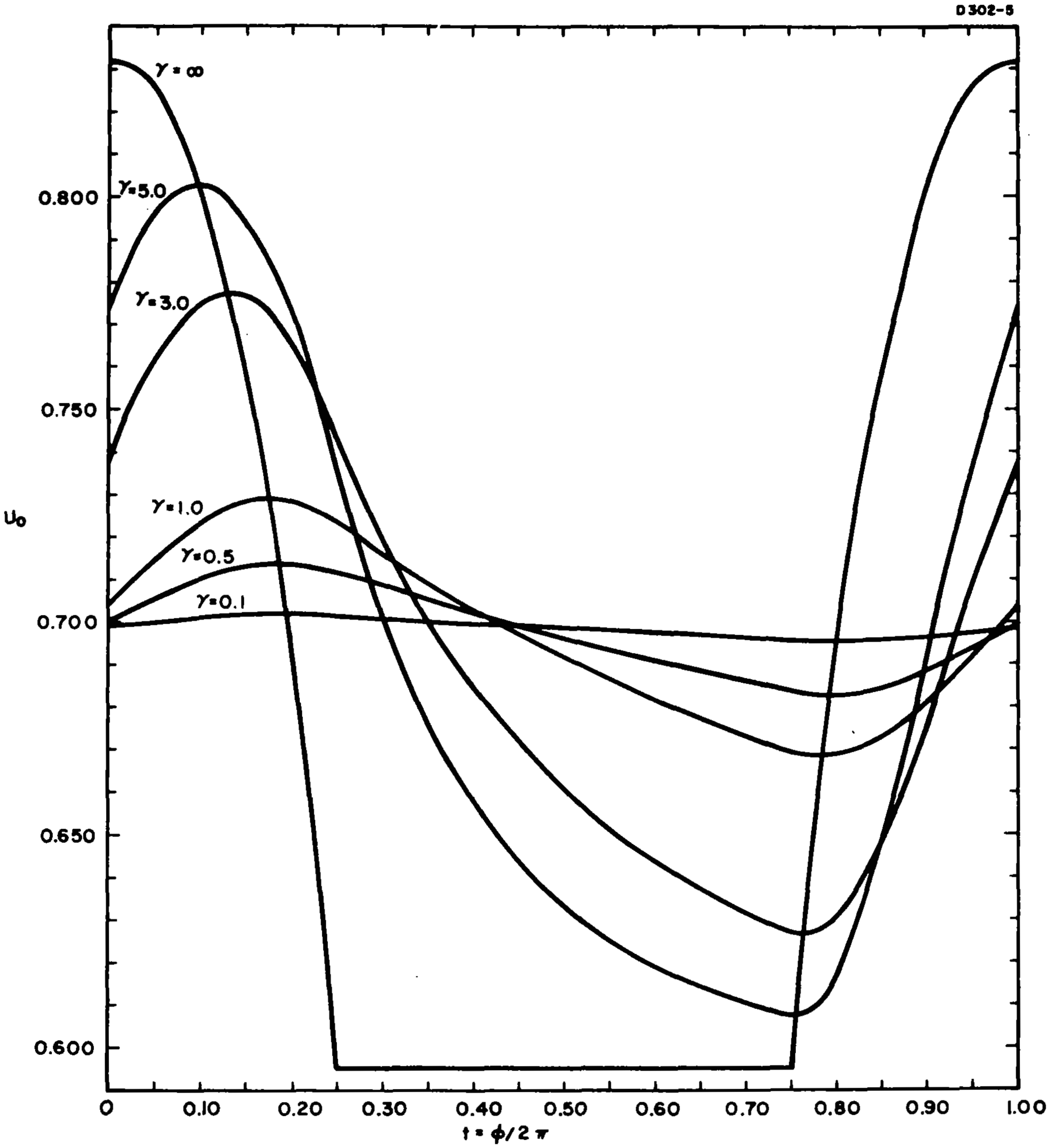

Fig. 4. Temperature distribution — intermediate rotation rates ($\beta = \frac{\pi}{4}$, $\theta = \frac{\pi}{2}$).

VI. SUMMARY

A method for calculating a uniformly valid perturbation expansion of the temperature distribution in a thin spherical shell has been discussed. Examination of the results obtained reveals that the conventional perturbation approach (i.e., regular perturbation calculation) yields a valid zero order approximation although it provides no means of estimating the error incurred in the approximation. Information regarding the magnitude of the error and the form of the correction terms can be obtained only if singular perturbation techniques are employed. To illustrate the perturbation method, we have derived the first few terms in the uniformly valid perturbation expansions of the temperature distribution in spinning shells for the cases of slow $((\omega a^2/\kappa) = O(1))$ and rapid $((\omega a^2/\kappa) = O(a/\delta)^2)$ rotation. Additional terms can be derived in the same manner. This may be desirable when the shell thickness is not really very thin, since the error incurred in approximating the temperature distribution by the uniformly valid perturbation expansion is always less than the order of the last term retained.

The analysis has shown that the zero order temperature distribution in a shell rotating at a rate such that $(\omega a^2/\kappa) = O(1)$ is identical to that in a stationary shell. Likewise, when $(\omega a^2/\kappa) = O(a/\delta)^2$, the zero order temperature distribution is identical to that in the case of an infinite rate of rotation. These two cases are referred to in this report as slow and rapid rotation. The transition from a distribution characteristic of slow rotation to one characteristic of rapid rotation occurs when $(\omega a^2/\kappa) = O(a/\delta)$. We derived the differential equation satisfied by the zero order temperature distribution in this case. Since the equation is nonlinear, numerical solution techniques were employed. The results of the numerical computation reveal that the transition in the temperature distribution occurs when the parameter

$$\gamma = \frac{2\pi a \epsilon \sigma^{1/4} F_o^{3/4}}{K} \cdot \left(\frac{\omega a \delta}{\kappa}\right)^{-1}$$

is in the range $0.1 < \gamma < 5.0$. For γ's outside this range the temperature distribution closely approximates that for $\gamma = 0 (\omega \to \infty)$ or $\gamma \to \infty$ $(\omega = 0)$.

In all of the calculations, the angle between the direction of the solar flux vector and the axis of rotation was taken to be an arbitrary parameter β with values in the range $0 \leq \beta \leq (\pi/2)$. In the case of slow rotation $(\omega a^2/\kappa = O(1))$, the temperature distribution is centered about an axis defined by the direction of the solar flux vector. In other cases the temperature distribution generally consists of a hot zone near the upper pole defined by two axis of rotation, a zone of intermediate temperature in the vicinity of the equator, and a relatively cool zone near the lower pole. The boundaries of these regions are the circles $\theta = \cos^{-1}(\pm \sin \beta)$.

APPENDIX A – HEAT TRANSFER WITHIN AN EVACUATED SPHERE

The calculation of heat transfer within a closed surface is complicated by the fact that the heat radiated from each point on the surface is eventually absorbed or reflected at some other point. Heat is radiated in accordance with the Stefan-Boltzmann law

$$\text{radiated heat flux} = \sigma\epsilon\, T^4\,(\underline{r}) \quad . \qquad (A.1)$$

The rate at which energy radiated from the interior of a spherical surface is absorbed on a surface element dA can be written (see Fig. A-1)

$$\begin{bmatrix} \text{rate of absorption} \\ \text{at dA of heat radiated} \\ \text{from other surface} \\ \text{elements} \end{bmatrix} = Q'\, dA$$

$$= \epsilon \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta'\, r^2 \sin\theta'\, \sigma\epsilon\, T^4(r, \theta', \phi') \cos\frac{\theta'}{2}\; \frac{dA\cos\frac{\theta'}{2}}{4\pi r^2 \cos^2\frac{\theta'}{2}}$$

$$\left[= \frac{\sigma\epsilon^2}{4\pi} \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta'\, T^4(r, \theta', \phi') \right] dA \qquad (A.2)$$

where the scattering is assumed to be Lambertian.

Note that the heat radiated from other surface elements and absorbed at dA is independent of the coordinates of dA. Likewise, the radiated heat that is scattered from dA and all other surface elements is independent of position. Thus, the rate at which heat scattered from other surface elements is absorbed on dA can be written

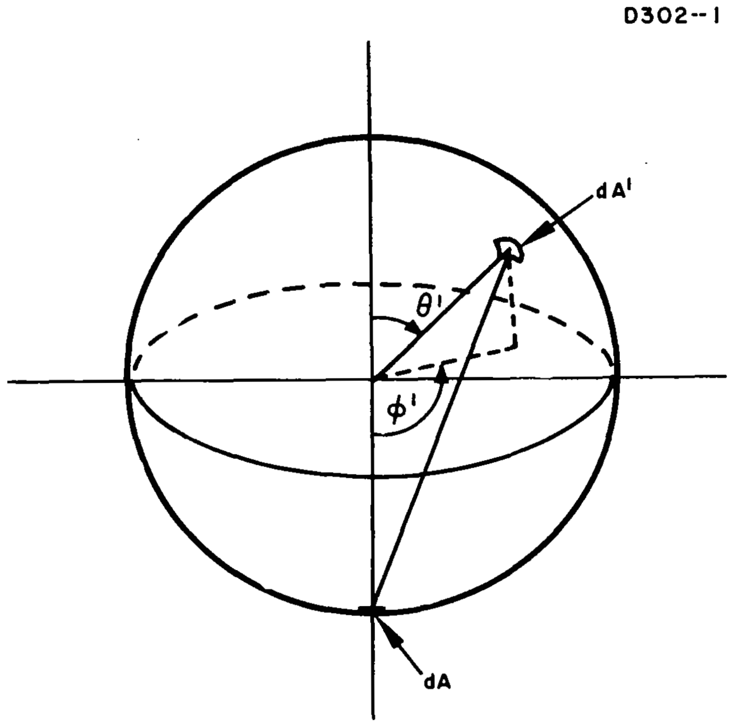

Fig. A-1. Coordinate system.

$$\begin{bmatrix} \text{rate of absorption at } dA \\ \text{of heat scattered from} \\ \text{other surface elements} \end{bmatrix} = Q'' \, dA = (1 - \epsilon) \, [\text{total flux absorbed}] \, dA$$

$$= (1 - \epsilon)(Q' + Q'') \, dA \quad . \qquad (A.3)$$

Solution of (A. 3) for $Q' + Q''$ in terms of Q' yields

$$Q = [\text{total flux absorbed}] = Q' + Q'' = \frac{1}{\epsilon} Q'$$

$$= \frac{\sigma\epsilon}{4\pi} \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta' \, T^4(r, \theta', \phi') \quad . \qquad (A.4)$$

APPENDIX B — DIFFERENTIAL EQUATION SATISFIED BY U_o WHEN $\alpha = O(\delta/a)$

When $\alpha = O(\delta/a)$, the zero, first, and second order equations obtained from (III. 3) to (III. 5) are

$$\frac{\partial^2 U_o}{\partial \zeta^2} = 0 \tag{B.1}$$

$$\frac{\partial U_o}{\partial \zeta} = 0, \qquad \zeta = \pm 1 \tag{B.2}$$

$$\frac{\partial^2 U_1}{\partial \zeta^2} + 2 \frac{\partial}{\partial \zeta}\left(\zeta \frac{\partial U_o}{\partial \zeta}\right) = 0 \tag{B.3}$$

$$\frac{\partial U_1}{\partial \zeta} = 0 , \qquad \zeta = \pm 1 \tag{B.4}$$

$$\frac{\partial^2 U_2}{\partial \zeta^2} + 2 \frac{\partial}{\partial \zeta}\left(\zeta \frac{\partial U_1}{\partial \zeta}\right) + \frac{\partial}{\partial \zeta}\left(\zeta^2 \frac{\partial U_o}{\partial \zeta}\right) + \mathcal{L}_{\theta, \phi, \omega} U_o = 0 \tag{B.5}$$

$$\frac{\partial U_2}{\partial \zeta} = \frac{a\alpha}{\delta}\left[U_o^4 - G_s\right], \qquad \zeta = 1 \tag{B.6}$$

$$\frac{\partial U_2}{\partial \zeta} = \frac{a\alpha}{\delta}\left[U_o^4 - q_o\right], \qquad \zeta = -1 \tag{B.7}$$

where

$$\mathscr{L}_{\theta,\phi,\omega} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - \frac{\omega a^2}{\kappa}\frac{\partial^2}{\partial\phi^2}. \quad \text{(B.8)}$$

Equations (B.1) to (B.4) indicate that U_o and U_1 are independent of ζ. Thus, integration of (B.5) yields

$$U_2(\zeta,\theta,\phi) = A_2(\theta,\phi) + \zeta\, B_2(\theta,\phi) - \frac{\zeta^2}{2}\mathscr{L}_{\theta,\phi,\omega}\, U_o(\theta,\phi). \quad \text{(B.9)}$$

Substitution of (B.9) in the boundary conditions (B.6) and (B.7) and elimination of B_2 from the resulting equations then yields the following equation for U_o.

$$\mathscr{L}_{\theta,\phi,\omega}\, U_o = \frac{a\alpha}{\delta}\left[U_o^4 - \left(\frac{G_s + q_o}{2}\right)\right] \quad . \quad \text{(B.10)}$$

Note that $U_o(\theta,\phi)$ satisfies a nonlinear partial differential equation.

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