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ELECTRICAL CONDUCTIVITY OF A MEDIUM CONTAINING AN ARRAY OF WIDELY SEPARATED, PERFECTLY CONDUCTING, CYLINDERS

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Information concerning the structure of an array of particles with a certain electrical conductivity, which are embedded in a medium of arbitrary conductivity, can be obtained by a measurement of the effective conductivity of the system. In order to relate the electrical measurement to the structure of the system, several workers have calculated the effective conductivity of conducting media containing arrays of perfectly conducting and nonconducting spheres and cylinders.[1-5]

The effective conductivity between rows in an array of widely separated, perfectly conducting cylinders will be considered herein by means of the same conformal mapping that is used to calculate the interelectrode capacitances of an idealized planar triode.

An array of perfectly conducting, equal-sized cylinders embedded in a medium with conductivity σ_0 and dielectric constant ϵ is shown in Fig. 1. In order to find the electrical conductivity in the x-direction, assume that an electric field $\vec{E} = -\hat{a}_x E_0$ is applied to the system. It is advantageous to note that the surfaces $[(n + 1)d_1 + n\, d_2, y, z]$, where $n = 0, \pm 1, \pm 2, \cdot \cdot \cdot$ and $d_{1,2}$ are the x-direction separation distances between cylinders, are equipotential planes. For example, the $n = +1$ surface is shown by asterisks in Fig. 1.

Thus, it is merely necessary to consider the geometry shown in Fig. 2 in order to find the conductivity of the system shown in Fig. 1. The equipotential planes denoted by the numbers 1 and 2 may be assumed to be perfectly conducting. The geometry shown in Fig. 2 resembles that of an idealized triode for the case in which the cylinders are widely separated ($b, d_{1,2} \gg a$). The planes represent the cathode and anode; the cylinders represent the grid.

Ramo and Whinnery[6] have solved for the interelectrode capacitances per unit length C_{nm} of this structure using the complex mapping

$$z' = \exp(-\pi z/b)$$

by which the triode geometry in Fig. 2 is mapped into that of Fig. 3 for the condition $b, d_{1,2} \gg a$. The interelectrode conductance per unit length G_{nm} is related to C_{nm} by $C_{nm} = (\epsilon/\sigma_0)G_{nm}$. Thus

$$G_{12} \equiv \frac{\sigma_0}{\epsilon} C_{12} = \frac{\sigma_0 \ln\left(\frac{1}{2} \csc \frac{\pi a}{2b}\right)}{2\pi\ \Delta}; \quad G_{13} \equiv \frac{\sigma_0}{\epsilon} C_{13} = \frac{\sigma_0\ d_2}{2b\ \Delta}; \quad G_{23} \equiv \frac{\sigma_0}{\epsilon} C_{23} = \frac{\sigma_0\ d_1}{2b\ \Delta}$$

$$\Delta = \frac{(d_1 + d_2)}{2b} \left[\frac{d_2}{2b} + \frac{1}{2\pi} \ln\left(\frac{1}{2} \csc \frac{\pi a}{2b}\right)\right] - \left(\frac{d_2}{2b}\right)^2 \tag{1}$$

where the values for C_{nm} are those given in reference 6.

An equivalent circuit for Fig. 2 is given in Fig. 4. Thus

$$\left.\begin{array}{c} G_{\text{eff}}^{(1,2)} = G_{12} + \dfrac{G_{13}G_{23}}{G_{13} + G_{23}} \qquad G_{\text{eff}}^{(1,3)} = G_{13} + \dfrac{G_{12}G_{23}}{G_{12} + G_{23}} \\[2ex] G_{\text{eff}}^{(2,3)} = G_{23} + \dfrac{G_{12}G_{13}}{G_{12} + G_{23}} \end{array}\right\} \tag{2}$$

It is easily shown that

$$\left[G_{\text{eff}}^{(1,2)}\right]^{-1} = \frac{\left[G_{\text{eff}}^{(1,3)}\right]^{-1}}{(1 + e)} + \frac{\left[G_{\text{eff}}^{(2,3)}\right]^{-1}}{(1 + e)} \tag{3}$$

where $e = 2C_{12}/(C_{13} + C_{23})$. Thus $\left[G_{\text{eff}}^{(1,3)}\ (1 + e)\right]$ and $\left[G_{\text{eff}}^{(2,3)}\ (1 + e)\right]$ are the effective conductances per unit length between adjacent rows of cylinders, d_1 and d_2 apart, respectively, where a plane through the axes of the cylinders in each row is virtually equipotential.

If σ_x^∞ is the x-direction effective conductivity between two adjacent, actually equipotential, planes and $\sigma_{x_1}^\infty(\sigma_{x_2}^\infty)$ is the x-direction effective conductivity between two rows of cylinders that are $d_1(d_2)$ apart, Eq. (3) can be written as

$$\frac{d_1 + d_2}{\sigma_x^\infty 2b} = \frac{d_1}{\sigma_{x_1}^\infty 2b} + \frac{d_2}{\sigma_{x_2}^\infty 2b} \tag{4}$$

where

$$\frac{\sigma_x^\infty}{\sigma_0} = \frac{d_1 + d_2}{2b} G_{eff}^{(1,2)} = 1 \tag{5a}$$

and

$$\frac{\sigma_{x\binom{1}{2}}^\infty}{\sigma_0} = \frac{d_{\binom{1}{2}}}{2b} G_{eff}^{\binom{1}{2},3} = \left[\frac{d_{\binom{1}{2}}}{d_1 + d_2}\right] \left[\frac{\dfrac{d_1 + d_2}{2b} + \dfrac{\ln\left(\frac{1}{2}\csc\frac{\pi a}{2b}\right)}{\pi}}{\dfrac{d_{\binom{1}{2}}}{2b} + \dfrac{\ln\left(\frac{1}{2}\csc\frac{\pi a}{2b}\right)}{2\pi}}\right] \tag{5b}$$

$\sigma_{x_1}^\infty(\sigma_{x_2}^\infty)$ is also the x-direction effective conductivity between a row of cylinders and the equipotential surface that is a distance $d_1(d_2)$ from it.

When Eq. (5a) is compared with the finite difference calculation of Ref. 5, it is seen that, for $d_1 = d_2 = b$ and $(a/b) < 0.1$, the error in the above mapping technique is 1.5 percent. Since the error in the mapping decreases as d_1 and d_2 increase, Eq. (5b) is in error by less than 1.5 percent for $d_1, d_2 > b$ and $(a/b) < 0.1$. When $d_1 = d_2$, $\sigma_{x\binom{1}{2}}^\infty/\sigma_0 = 1$, which indicates that a plane through the axes of cylinders in a row is equipotential.

The above mapping technique can be used to calculate y-direction effective conductivities only for the case of $d_1 = d_2$. Thus, in a manner identical to the above techniques, it can be shown that the y-direction effective conductivity between two rows of cylinders or between two equipotential surfaces is σ_0 (for $d_1 = d_2$).

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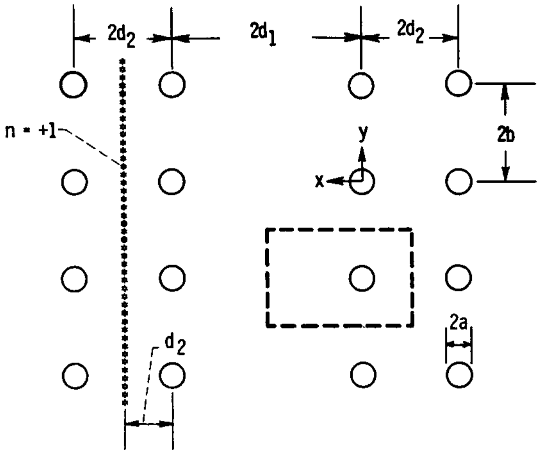

Fig. 1. Array of cylinders.

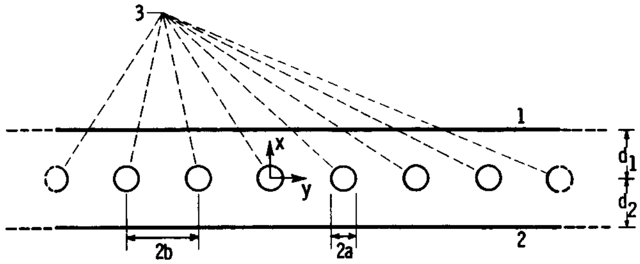

Fig. 2. Equivalent array of cylinders.

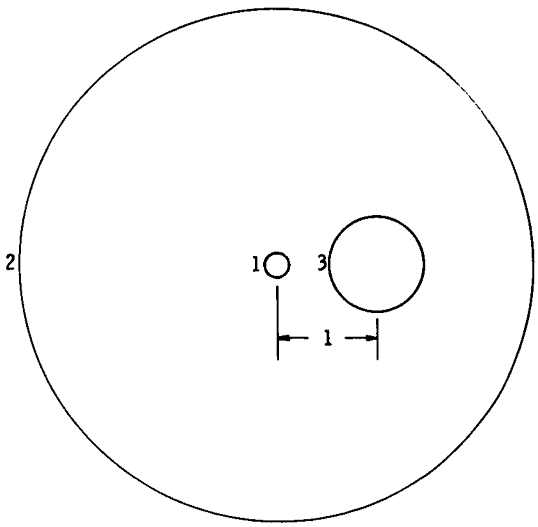

Fig. 3. The z' plane (not to scale).

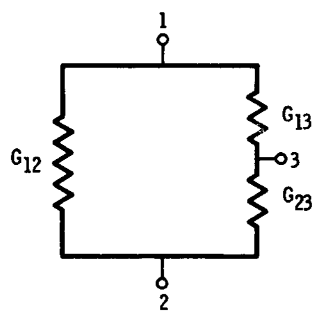

Fig. 4. Equivalent circuit of array of cylinders.