Technical Report No. 32-650 (Part III)

The Rotating Superconductor
Part III: The Superelectrons as an Incompressible Charged Fluid

M. M. Saffren

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

May 31, 1965

Technical Report No. 32-650 (Part III)

The Rotating Superconductor Part III: The Superelectrons as an Incompressible Charged Fluid

M. M. Saffren

R. J. Mackin, Jr., Manager
Physics Section

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

May 31, 1965

Prepared Under Contract No. NAS 7-100
National Aeronautics & Space Administration

CONTENTS

24923

ABSTRACT

In this Report, we explore the theory of the superfluid without the London constraint. For brevity, we call this theory BSH theory after its original promulgators. We note its similarity to and differences from the London theory. In particular, we examine the BSH theory as it applies to a hollow cylinder, at rest and rotating, and with and without an applied field.

author

I. INTRODUCTION

In this Report, we explore the theory of the superfluid without the London constraint. For brevity, we call this theory BSH theory after its original promulgators, Becker, Sauter, and Heller (Ref. 1). We note its similarity to and differences from the London theory. In particular, we examine the BSH theory as it applies to a hollow cylinder, at rest and rotating, and with and without an applied field.

The London equations as constitutive equations for superelectrons are derivable with the aid of the equations of motion of an inviscid, incompressible, charged fluid (Ref. 2). The equations of motion show that if London's first equation,

$$\nabla \times \mathbf{v}_s + \frac{e}{mc}\mathbf{B} = 0$$

holds at any instant, it holds thereafter; they also serve to yield London's second equation once the first equation is assumed.

In terms of the general solutions of the equations of motion, however, London's first equation may be regarded simply as a constraint. Perhaps the easiest way to see this is to notice that the equations of motion require (see Section II, and Ref. 3) that at a point moving with the fluid

$$\nabla \times \mathbf{v} + \frac{e}{mc}\mathbf{B} \equiv 2\boldsymbol{\Omega} = \mathbf{const} \tag{1}$$

Since $\nabla \times \mathbf{v}$ is twice the local angular velocity, we can write this as Larmor's theorem (Ref. 3):

$$\triangle \boldsymbol{\omega} + \frac{e}{2\,mc}\triangle \mathbf{B} = 0 \tag{2}$$

where the deltas refer to differences at two successive moments. The London constraint now amounts to the dropping of the deltas. This step can be justified by the uniqueness of the superconducting ground state. However, were the superconducting ground state degenerate,* the deltas would have to remain. The fluid, although superconducting, would then fail to display a Meissner effect. Qualitatively, such a superconductor would behave much the same as a superconductor of the second kind in a field greater than its first critical field.

*This would be the case if, for example, the Cooper pairs of the superconducting ground state could exist in angular momentum states of $l = 1$ or $l = 2$ that were degenerate in energy with the $l = 0$ state. Such Cooper pair states have been examined by, for example, S. V. Vonsovskii and M. S. Svirskii as presented in "Superconductivity of an Electron System with Singlet or Triplet Pairs," *Soviet Physics, JETP*, Vol. 19, p. 1095, 1964, and references given there.

II. THE EQUATIONS OF BSH THEORY

The equation of motion for an inviscid and incompressible charged fluid of electrons (charge e, mass m, velocity field $\mathbf{v}$, pressure p, number density ρ_0) is

$$m\frac{\partial \mathbf{v}}{\partial t} - m\mathbf{v} \times \nabla \times \mathbf{v} + \frac{m}{2}\nabla \mathbf{v}^2 = e\mathbf{E} + \frac{1}{\rho_0}\nabla p + \frac{e\mathbf{v}}{c} \times \mathbf{B} \tag{3}$$

When $\mathbf{E}$ and $\mathbf{B}$ are expressed in terms of the potentials $\mathbf{A}$ and ϕ, Eq. (3) becomes

$$\frac{\partial}{\partial t}\left(\mathbf{v} + \frac{e}{mc}\mathbf{A}\right) - \mathbf{v} \times \nabla \times \left(m\mathbf{v} + \frac{e}{c}\mathbf{A}\right) = \nabla\left(\frac{mv^2}{2} + e\phi + \frac{p}{\rho_0}\right) \tag{4}$$

By taking the curl of this equation, we obtain (Ref. 2)

$$\frac{\partial}{\partial t}\left(\nabla \times \mathbf{v} + \frac{e}{mc}\mathbf{B}\right) = \nabla \times \mathbf{v} \times \left(\nabla \times \mathbf{v} + \frac{e}{mc}\mathbf{B}\right) \tag{5}$$

Since

$$\nabla \cdot \left(\nabla \times \mathbf{v} + \frac{e}{mc}\mathbf{B}\right) = 0$$

we can deduce (Ref. 4) Eq. (1).

We can also deduce from Eq. (4) that

$$\oint\left(m\mathbf{v} + \frac{e}{c}\mathbf{A}\right) \cdot \hat{\boldsymbol{\lambda}}\, d\lambda \tag{6}$$

is conserved as long as the contour is a closed streamline. This theorem is a lot weaker than is the corresponding theorem of London's theory — the conservation of the fluxoid, as we now show.

If the contour encloses a hole, we can write this as

$$\int_{hole}\left(m\mathbf{v} + \frac{e}{c}\mathbf{A}\right) \cdot \hat{\boldsymbol{\lambda}}\, d\lambda + 2\oint_{\substack{streamline \\ minus\ hole}} \boldsymbol{\Omega} \cdot \hat{\mathbf{n}}\, d\sigma$$

The first integral is the fluxoid, $e/c\ \Phi$, associated with the hole, and it is the same as in London theory. The second integral, a surface integral, is absent in London theory. In BSH theory, then, we no longer have conservation of the fluxoid. Instead, we have the following theorem: *The rate of change of a fluxoid associated with a hole is given by the negative rate of change of the flux* $2 \oint \boldsymbol{\Omega} \cdot \hat{\mathbf{n}}\, d\sigma$ *enclosed by any streamline encircling the hole.* Of course, this flux is defined only in the superconductor. While in general, the fluxoid is not conserved in BSH theory, it is conserved when the superelectron velocity field is sufficiently symmetrical, as we now show.

III. THE HOLLOW CYLINDER

In this discussion, we examine the fluxoid theorem (6) as it applies to the hollow cylinder. We assume that for the infinitely long hollow cylinder, the streamlines are always cylindrically symmetric, which here means that the velocity field and magnetic field depend solely on the radial coordinate ρ. The velocity field lies along the tangential unit vector $\hat{\boldsymbol{\Phi}}$, and $\nabla \times \mathbf{v}$ and $\mathbf{B}$ are along $\hat{\mathbf{Z}}$. We then can write Eq. (1) as

$$\frac{m}{\rho}\frac{\partial}{\partial \rho}\rho v + \frac{e}{c}B = f(\rho) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho u \tag{7}$$

where u is as yet undetermined. In fact, from (6) we have that

$$\Phi(\rho) = 2\pi m \rho v + \frac{2\pi e}{c}\int_{\rho_i}^{\rho} \rho B d\rho + B_i \pi \rho_i^2 \tag{8}$$

is conserved (ρ_i is the inner radius of the cylinder, B_i is the field in the bore). However, unlike the fluxoid of London's theory, this quantity is a function of ρ. From comparison of Eq. (7) with Eq. (8), it follows that

$$\Phi(\rho') - \Phi(\rho) = 2\pi\left[\rho' u(\rho') - \rho u(\rho)\right] \tag{9}$$

The quantity u, which is zero in London theory, is the distinguishing characteristic of BSH theory. Evidently, it is another quantity determined when the cylinder becomes superconducting, just as is the constant $\Phi(\rho_i)$.

IV. THE FREE ENERGY

Just as in the London theory, we determined Φ at the time of transition by finding the value of Φ which made the free energy a minimum, we now determine both u and $\Phi(\rho_i)$ in BSH theory in the same way.

The variation of the free energy of a superconductor having a rigid lattice and rotating at a constant angular velocity in a uniform magnetic field is given by (Ref. 5)

$$\delta \int_{sup} \left\{ \frac{m\rho_0}{2} (\mathbf{v}_s - \mathbf{v}_l)^2 + \frac{\mathbf{B}^2}{8\pi} \right\} d\tau - \oint_{surf} \hat{\mathbf{n}} \cdot \delta \mathbf{A} \times \mathbf{B} d\sigma + \delta \int_{holes} \frac{\mathbf{B}^2}{8\pi} d\tau \tag{10}$$

where $\mathbf{v}_l$ is the velocity field of the lattice. We can write (10) as

$$\int_{sup} \rho_0 (\mathbf{v}_s - \mathbf{v}_l)\, \delta \left(m\mathbf{v}_s + \frac{e}{c} \mathbf{A} \right) d\tau = \frac{1}{c} \sum_{cuts} J_{cut}\, \delta\Phi_{cut} + m \int_{sup} \rho_0 (\mathbf{v}_s - \mathbf{v}_l) \cdot \delta \mathbf{u} d\tau \tag{11}$$

The sum stems from the longitudinal part of $\mathbf{v}_s - \mathbf{v}_l$ (see Ref. 5) and the integral from the transverse part. We now see that in BSH theory, the presence of the integral shows that the minimum free energy is achieved when $\mathbf{v}_s - \mathbf{v}_l = 0$—no current anywhere in the superconductor. In London theory, the integral is absent. The minimum is therefore given when the *average* current J_{cut}, passing through a cut in a multiply connected superconductor, vanishes. The London theory further requires that in a simply connected superconductor, rotating or stationary, in a fixed field, there must in general be net currents present if the state is an equilibrium state. In BSH theory there is no such requirement. This clearly shows the effect of setting $\mathbf{u} = 0$ (the London constraint). We now apply what we have just deduced from the general expression (10) to the determination of the fields associated with a hollow cylinder.

Since now we see that $\mathbf{v}_s = \mathbf{v}_l$ at the time of transition, we can write

$$\nabla \times \mathbf{v}_s + \frac{e}{mc} \mathbf{B} = \nabla \times (\mathbf{v}_l)_0 + \frac{e}{mc} (\mathbf{B})_0 = \frac{e}{mc} \left(\mathbf{B} + \frac{2mc\omega_l}{e} \right)_0 \tag{12}$$

which holds for all subsequent times, as long as the cylinder remains superconducting. (The subscript zero on a quantity indicates its value at the time of transition.) Furthermore, in the holes of a superconductor, the field is always $\mathbf{B}_0$, since $\mathbf{B}$ penetrates a superconductor completely both before and after the transition. We therefore obtain for the fluxoid of the cylinder

$$\oint \left(m\mathbf{v} + \frac{e}{c} \mathbf{A} \right) \cdot \hat{\lambda} d\lambda = \frac{\pi \rho^2 e}{c} \left[\frac{2mc}{e} \omega_l + B \right]_0 \tag{13}$$

We then obtain from the Maxwell equation

$$\nabla \times \mathbf{B} = \frac{4\pi e \rho_0}{c} [\mathbf{v}_s - \mathbf{v}_l] \tag{14}$$

the equation

$$\frac{c}{4\pi\rho_0 e} \nabla \times \nabla \times \mathbf{B} + 2\boldsymbol{\omega}_l + \frac{e}{mc} \mathbf{B} = \frac{e}{mc} \left(\mathbf{B}_0 + \frac{2mc\boldsymbol{\omega}_{l0}}{e} \right) \tag{15}$$

or, with

$$\mathbf{b} = \mathbf{B} + \frac{2mc}{e} \boldsymbol{\omega}_l \qquad \text{and} \qquad \lambda^2 = \frac{mc^2}{4\pi\rho_0 e^2}$$

we have

$$\lambda^2 \nabla \times \nabla \times (\mathbf{b} - \mathbf{b}_0) + (\mathbf{b} - \mathbf{b}_0) = 0 \tag{16}$$

and

$$mv + \frac{e}{c} A = \frac{e}{2c} \rho b_0 \tag{17}$$

so at $\rho = \rho_i$, the inner radius

$$\frac{m}{\rho_i} [-\rho_i \omega_l + v(\rho_i)] = \frac{e}{2c} [b_0 - b(\rho_i)] \tag{18}$$

We introduce this into (14) and obtain

$$- \left. \frac{\partial b}{\partial \rho} \right|_{\rho = \rho_i} = - \frac{2\pi e^2 \rho_0}{mc^2} \rho_i (b_0 - b_i)$$

so that

$$\left. \frac{\partial (b - b_0)}{\partial \rho} \right|_{\rho = \rho_i} = \frac{\rho_i}{2\lambda^2} (b - b_0) \bigg|_{\rho = \rho_i} \tag{19}$$

is the boundary condition to be imposed on the solution of Eq. (16) at the inner surface. In a simply connected superconductor, we merely require that v be finite at $\rho = 0$. The solution of (16) and (19) is (see Ref. 6)

$$b - b_0 = [b(\rho_{ext}) - b_0] \times \frac{\left[I_2\left(\frac{\rho_i}{\lambda}\right) K_0\left(\frac{\rho}{\lambda}\right) - K_2\left(\frac{\rho_i}{\lambda}\right) I_0\left(\frac{\rho}{\lambda}\right)\right]}{D_{20}}$$

$$D_{20} = I_2\left(\frac{\rho_i}{\lambda}\right) K_0\left(\frac{\rho_{ext}}{\lambda}\right) - K_2\left(\frac{\rho_i}{\lambda}\right) I_0\left(\frac{\rho_{ext}}{\lambda}\right) \tag{20}$$

For a thick-walled superconductor, this reduces to

$$b - b_0 = [b(\rho_{ext}) - b_0]\sqrt{\frac{\rho_{ext}}{\rho}} \sinh\left[\frac{\rho_i - \rho}{\lambda}\right] \tag{21}$$

We now see that while the expression for the field b_i is the same as it is in London theory, namely,

$$b_i - b_0 = 0$$

the field in the wall in BSH theory is also given by $b = b_0$, whereas in the London theory, the field in the wall is $b = 0$.

V. DISCUSSION

As we have seen, for the hollow cylinder, one of the main differences between BSH theory and London theory is that in BSH theory, the fluxoid is in general not conserved. However, under the assumption that the field of a hollow cylinder is itself cylindrically symmetric, we found that the field trapped inside the bore of the cylinder, as given by BSH theory, was the same as that given by London theory. The complete trapping of the flux in the bore followed from fluxoid conservation, which (under the assumption of cylindrical symmetry) was shown to hold in both theories. Also, the value of the field trapped—the field that is required to be in the bore when the system is in thermal equilibrium—is seen to be the same in both theories. This latter result follows just from the requirement that in equilibrium there be no magnetic pressure drop across a superconductor.

Thus, the theories differ only in the field present inside the superconducting material itself. In London theory, the constraint—London's first equation—requires a magnetic pressure drop from the outside to the inside of the superconductor. In BSH theory, however, even such a drop is absent. This difference expresses itself in the fact that while in London theory it is only the average current through the cross section of wall that must be zero, in the BSH theory the current must be zero point by point.

The reason for the difference in the equilibrium current distribution for the two theories, at least as seen from Eq. (1), is that in London theory, $m\mathbf{v}_s + e/c\mathbf{A} = \nabla\chi$, while in BSH theory, $m\mathbf{v}_s + e/c\mathbf{A} = \nabla\chi + \mathbf{u}$, where* $\mathbf{u} = \nabla \times \mathbf{w}$, so that the London constraint as it appears here is $\mathbf{u} = 0$. The extra degree of freedom represented by $\mathbf{u}$ always allows us to reach a lower minimum of the free energy. The states represented by this minimum are states in which the magnetic field is not excluded from the superconductor.

Perhaps a word of explanation is in order regarding the lower free energy of BSH theory and the "constraint" that converts BSH theory into London theory. As a constraint, London's first equation raises the free energy (or at least never lowers it) beyond the value it has in BSH theory. It is important to realize that this is only an apparent raising, however. What we have done is not to include the condensation energy (the pairing energy of BCS theory) in the free energy; once we do this, the London constraint ceases to be a constraint and becomes merely a property of the state having the lowest free energy.

*We express $\mathbf{u}$ as $\nabla \times \mathbf{w}$ to make $\nabla\chi$ unique. Then, in either theory, χ must vanish in a simply connected superconductor, and in either theory, χ need not be single-valued in a multiply connected superconductor.

ACKNOWLEDGMENT

The author would like to thank Dr. A. H. Hildebrandt for his constant encouragement, stimulation and interest in the work presented in this Report.

REFERENCES

1. Becker, F., Sauter, F., and Heller, C., "Über die Stromverteilung in einer supraleitenden Kugel," *Zeitschrift für Physik*, Vol. 85, 1933, p. 772.
2. London, F., *Superfluids*, John Wiley and Sons, Inc., New York, 1950.*
3. Saffren, M. M., "London's Equations as an Expression of Larmor's Theorem," *Space Programs Summary No. 37-31*, Jet Propulsion Laboratory, Pasadena, California, January 31, 1965.
4. Panofsky, W. K. H., and Phillips, M., *Classical Electricity and Magnetism*, Addison-Wesley Publishing Co., Inc., Reading, Massachusetts, 1955.
5. Saffren, M. M., *The Rotating Superconductor, Part II: The Free Energy*, Technical Report No. 32-650 (Part II), Jet Propulsion Laboratory, Pasadena, California, May 14, 1965.
6. Saffren, M. M., *The Rotating Superconductor, Part I: The Fluxoid*, Technical Report No. 32-650 (Part I), Jet Propulsion Laboratory, Pasadena, California, March 15, 1965.

*This reference was inadvertently omitted from Part I of Technical Report No. 32-650. It should have been cited there as Ref. 2 in place of the second Hildebrandt reference.