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ABSTRACT

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The construction and nonlinear aspects of phase-locked-loop and automatic frequency control systems are discussed. The nonlinear systems are approximated by linear systems, and a time lag is introduced into the system. Several methods of designing a digital controller for a second-order system with a large time lag are investigated. The time lag is assumed to be 10 to 20 times larger than the sampling period. The principal criterion used is the deadbeat response to a ramp input, although other criteria are considered. Several illustrations of the type of system response attainable with these digital controllers are presented to enable the reader to judge the merits of this controller for his particular application.

I. INTRODUCTION

The linear and nonlinear behavior of continuous automatic frequency controls (AFC) and phase-locked loops (APC) is well known (Ref. 1-5). A linear analysis of a high-speed digital (sampled-data and quantized) phase-locked loop has been investigated (Ref. 6). The present problem presents two new considerations for an AFC system. The proposed system will contain a large pure time delay and will be partially digital with a slow sampling rate.

An investigation of the nonlinear characteristics of the error-detecting devices used in AFC and phase-locked loops shows that the AFC system can be made linear over its useful operating range, or it can be made nonlinear if such a characteristic is desired. A phase-locked loop used as an AFC system will be inherently nonlinear except for very small error signal magnitudes. Thus a nonlinear analysis may be necessary for two reasons. Under operating conditions of continuous disturbances, the system error will often build up to a large magnitude, since the system will not immediately take corrective action because of the large time delay.

II. GENERAL SYSTEMS

Automatic frequency control systems and phase-locked loops differ mainly in the method of error detection, the input and output variables, and an integration in the phase-locked loop. The phase-locked loop can be used as an AFC system. We will consider digital AFC and digital APC systems with two pure time delays and two samplers within the control loop. Such systems can be visualized if one considers a receiver located in a planetary satellite that is to be tuned (controlled) from Earth. The samplers and delays correspond to the sending of error signals (coded pulses) from the vehicle to Earth and the transmitting of control commands (coded pulses) from Earth to the satellite. Figure 1 is a block diagram of two such systems. Both systems can be inherently nonlinear, with the major contributions lumped in the error detecting mechanisms. The output of the VCO is usually limited so that the VCO is essentially a limiter. However, its range usually exceeds the input range. Both error-detecting elements are usually linear for small error signal magnitudes and are useless when the error signal magnitude exceeds some limit.

As a first step herein, the nonlinear aspects of each device are investigated and its linear approximation is derived. Secondly, if a phase-locked loop is to be used, the nonlinear behavior of the system dictates the "pull-in" range of the system (Ref. 5). The pull-in characteristic is used to set the sweeping characteristic of the system when a sweep mode is added to the system.

Although no actual nonlinear analysis is presented herein, the nonlinear characteristics of the error detectors are discussed. Several different methods of analysis for sampled-data nonlinear systems are also discussed, along with a special problem created by the long time delay. The long time delay appears to raise the order of the difference equation describing the system, which would indicate that a second-order system with a time delay (larger than the sampling period) could not be adequately described by a phase plane representation (or an incremental phase plane).

Fortunately, some AFC systems can be described by linear methods. A major portion of this work is devoted to the design of a digital controller for a linear AFC system with a long time lag. This design will be based on "optimum control through digital compensation" (Ref. 7) or "deadbeat response" (Ref. 8). Both single-rate and multirate systems will be investigated. Minimization of the sum of squared-error samples is used to achieve an optimum compromise between the ramp and step response of the system. The large time lag increases the complexity of the optimum digital controller, especially in the case of the multirate controller.

A subrate controller, which receives n error signals for each command (corrective) signal it sends to the plant to be controlled, is also investigated. With this additional information about the system performance (more error information), it may be possible to design a digital controller to yield optimum dead beat response for more than one type of input. The problems of designing a subrate controller are presented.

III. NONLINEAR ASPECTS OF ERROR DETECTION

A. Frequency Detector

The function of the frequency-sensing error detector is to provide a signal (dc voltage) that is proportional to the difference between the input and output frequencies ($\omega_e = \omega_i - \omega_o$). The general scheme for attaining this goal is shown in Fig. 2. The frequency of the feedback signal ω_o is shifted an amount ω_c by the mixer. The input signal with a frequency ω_i and the shifted feedback signal are both fed into a second mixer. The output of this mixer is passed through a high-gain limiter, a narrow-band amplifier, and a discriminator to produce the error signal voltage ω_e.

To begin a mathematical description of the frequency detector, let the input signal $R(t)$ and the feedback signal $C(t)$ be given by

$$R(t) = R \sin \omega_i t \tag{1}$$

$$C(t) = C \sin (\omega_o t + \phi_c) \tag{2}$$

The output $v(t)$ of the ω_c mixer is given by

$$v(t) = V(C) \sin [(\omega_o - \omega_c) t + \phi_v] \tag{3}$$

where the reference input $Q(t)$ to the mixer is given by

$$Q(t) = Q \sin (\omega_c t + \phi_q) \tag{4}$$

Thus the total input $I(t)$ to the nonlinear element section of the second mixer is given by

$$I(t) = R(t) + v(t) \tag{5}$$

$$I(t) = R \sin \omega_i t + V(C) \sin (\omega_v t + \phi_v) \tag{6}$$

where

$$\omega_v = \omega_o - \omega_c \tag{7}$$

It has been shown that the output $m(t)$ of a nonlinear device with a double sinusoidal input is of the form

$$m(t) = \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} M_{lj}^{*} \sin\left[(l\omega_i + j\omega_v)t + \psi_{lj}\right] + M_{lj} \sin\left[(l\omega_i - j\omega_v)t + \phi_{lj}\right] \tag{8}$$

$$\begin{aligned} &= M_{00}^{*} + M_{00} + M_{10}^{*} \sin(\omega_i t + \psi_{10}) + M_{10} \sin(\omega_i + \phi_{10}) \\ &+ M_{01}^{*} \sin(\omega_v t + \psi_{01}) - M_{01} \sin(\omega_v t - \phi_{01}) \\ &+ M_{11}^{*} \sin\left[(\omega_i + \omega_o - \omega_c)t + \psi_{11}\right] + M_{11} \sin\left[(\omega_i - \omega_o + \omega_c)t + \phi_{11}\right] \end{aligned} \tag{9}$$

Now, if the narrow-band amplifier is designed to pass only the component of the output of the nonlinear element of interest $-M_{11} \sin\left[(\omega_i - \omega_o + \omega_c)t + \phi_{11}\right]$ – the output $h(t)$ of the narrow-band amplifier will be given by

$$h(t) = H \sin(\omega_i - \omega_o + \omega_c)t + \phi_h \tag{10}$$

The amplifier should have a center frequency of ω_c radians and a half bandwidth $\Delta\omega_b$ on each side of ω_c such that $\omega_c - \Delta\omega_b \leq \omega_i - \omega_o + \omega_c \leq \omega_c + \Delta\omega_b$ within the expected range of signal deviations (error magnitudes). Of greater importance, the bandwidth must be narrow enough to reject all other components of the nonlinear device.

The output of the narrow-band amplifier $h(t)$ is fed into a symmetrical high-gain limiter (essentially a relay), whose output $k(t)$ can be expressed as

$$K(t) = \sum_{n=1}^{\infty} K_n \sin\left[(2n-1)(\omega_i - \omega_o + \omega_c)t + \phi_n\right] \tag{11}$$

where $K_1 = 1.273$.

The second narrow-band filter also has a center frequency of ω_c radians and is essentially flat over the narrow bandwidth $2\Delta\omega_b$ so the gain $K(j\omega)$ of the filter is given by

$$K(j\omega) = 1 \qquad \omega_c - \Delta\omega_b \le \omega_i - \omega_o + \omega_c \le \omega_c + \Delta\omega_b$$
$$\tag{12}$$
$$K(j\omega) = 0 \qquad \text{all other frequencies}$$

Then the output of the second narrow-band filter is given by

$$g(t) = 1.273 \sin \left[(\omega_i - \omega_o + \omega_c)\, t + \phi_g\right] \tag{13}$$

A typical discriminator is shown in Fig. 3 (Ref. 9). It consists of a primary $L_1 C_1$ tuned to the center frequency ω_c, and two secondary circuits $L_2 C_2$ and $L_3 C_3$ tuned at $\omega_c + \Delta\omega_c$ and $\omega_c - \Delta\omega_c$. Each diode develops across its load impedance R_2 a voltage that varies with frequency as shown in Fig. 4a – assuming a fixed input amplitude and suitable smoothing (dc filtering) after the diode. The output is the sum of these two diode outputs, which are of opposite sign as shown in Fig. 4b.[1] If the discriminator is operating in the linear region between $\omega_c - \Delta\omega_c$ and $\omega_c + \Delta\omega_c$ the output $e(t)$ of the discriminator is given by

$$e(t) = K_d \cdot (\omega_i - \omega_o) \tag{14}$$

where K_d is the discriminator constant.

Of course the discriminator curve (S-curve) of Fig. 4b can be made to assume almost any arbitrary shape through manipulation of the tuned circuits $R_2 C_2$ and $R_3 C_3$ and substitution of more complicated circuits in their place. In this case the characteristic of the discriminator may never be linear, and its output becomes a function of the frequency difference as given by

$$e(t) = E_d\, (\omega_i - \omega_o) \tag{15}$$

Thus the frequency detector can be represented as shown in Fig. 5, in which both the linear and nonlinear representations are given.

[1] See Ref. 9 for details.

B. Phase Detection

The function of a phase detector is to produce a signal (dc voltage) proportional to the difference between the input and output phases. In this case the input and output frequencies appear as a linearly increasing (with time) phase. When the input and output phase difference is a constant, the output frequency must equal the input frequency (except where phase changes continually perfectly offset the frequency difference). Thus the phase-locked loop and the phase detector can also be used as frequency control system and frequency measuring device, respectively. There are two suggested basic methods of phase detection – multiplication and diode detection (Ref. 1).

The simplest method of phase detection is to multiply the two sinusoidal signals and filter the output to remove the undesirable component. This method will yield either a sum or difference of the two frequencies. Let the inputs to the multiplier be given by

$$\begin{aligned} R_1(t) &= r_1 \sin(\omega_1 t + \psi_1) \\ R_2(t) &= r_2 \cos \omega_2 t \end{aligned} \tag{16}$$

The output $E_m(t)$ of the multiplier is given by

$$\begin{aligned} E_m(t) &= r_1 r_2 \left[\cos \omega_2 t\right] \sin(\omega_1 t + \psi_1) \\ &= \frac{r_1 r_2}{2} \left\{ \sin\left[(\omega_1 + \omega_2) t + \psi_1\right] + \sin\left[(\omega_1 - \omega_2) t + \psi_1\right\} \right. \end{aligned} \tag{17}$$

Now let

$$\frac{r_1 r_2}{2} = E_m \qquad \omega_1 t + \psi_1 = \theta_1 \qquad \omega_2 t = \theta_2 \tag{18}$$

Then

$$E_m(t) = E_m \sin(\theta_1 + \theta_2) + E_m \sin(\theta_1 - \theta_2) \tag{19}$$

If this multiplier output is passed through a low-pass filter, the filter output is

$$e_o(t) = E_o \sin(\theta_1 - \theta_2) \tag{20}$$

If the phase difference $\theta_1 - \theta_2$ is small, the linear approximation

$$e_o(t) = E_o (\theta_1 - \theta_2) \tag{21}$$

is valid. However, it should be noted that the output $e_o(t)$ is proportional not only to the frequency difference but also to the input sinusoidal amplitudes r_1 and r_2. This condition might be eliminated by some amplitude correction scheme.

A second method of phase detection, known as a "balanced phase detector" (Ref. 1),is composed of peak detecting diodes. Let the inputs to the detector be given by

$$\begin{aligned} e_1(t) &= E_1 \cos \phi_1 \\ e_2(t) &= E_2 \sin \phi_2 \end{aligned} \tag{22}$$

One diode is fed the sum of $e_1(t)$ and ½ $e_2(t)$ while the other is fed the difference between $e_1(t)$ and ½ $e_2(t)$. It has been shown that the difference E_d between the output of the summing diode and the difference diode is given by

$$E_d = \frac{2E_1 E_2}{E_{d_{sum}} + E_{d_{diff}}} \sin(\phi_1 - \phi_2) \tag{23}$$

Now, if $E_1 >> E_2$

Then

$$E_{d_{sum}} + E_{d_{diff}} \approx 2E_1$$

and thus

$$E_d = E_2 \sin(\phi_1 - \phi_2) \tag{24}$$

Actually a more accurate analysis of this detector shows that there is really a detector constant caused by the half-wave rectification of the diodes and a necessary smoothing filter. If we lump this constant and the input amplitude E_2 into one constant K_p, the output of the detector becomes

$$E_d = K_p \sin(\phi_1 - \phi_2) \tag{25}$$

Again, for small error signal amplitudes ($\phi_1 - \phi_2 << \pi$), the linear approximation

$$E_d = K_p \cdot (\phi_1 - \phi_2) \tag{26}$$

is valid.

Thus the phase detector can be represented as shown in Fig. 6, in which both the linear and nonlinear representations are given.

IV. BLOCK DIAGRAMS OF AFC AND APC SYSTEMS

Block diagram representation of the elements (other than the detector) in an AFC or APC system is straightforward. Using the conventional Laplace transform notation and the sampled-data starred-function notation (Ref. 7, pp. 98 – 102) to represent digital systems, the following transfer functions are defined.

$$\text{time delay} = \epsilon^{-T_d s} = \exp(-T_d s)$$

$$\text{digital controller} = D^*(s)$$

$$\text{control function and VCO (plant)} = G(s)$$

where T_d is the communication time in seconds between the vehicle and the Earth.

The digital controller is the portion of the loop located on the Earth and is completely flexible within the limits of physical reliability. We will assume that the control function and VCO contains a zero-order hold, a simple lag network for output smoothing, and an integration with an APC system. Assuming these components and the detector transfer functions derived earlier, the linear and nonlinear block diagrams for the AFC system are as shown in Fig. 7. Similarly, Fig. 8 shows the nonlinear and linear block diagrams for an APC system.

V. THE NONLINEAR AFC PROBLEM

The basic mathematical tool for the study of continuous nonlinear systems is the nonlinear differential equation describing the system. Similarly, the nonlinear difference equation is the basic tool for analysis of the nonlinear sampled-data system (Ref. 10 – 12).

We shall consider the nonlinear AFC system shown in Fig. 7a with several simplifying assumptions. The two samplers are syncronous and have the same sampling rate. The total dead time $2\,T_d$ is an integral multiple of the sampling period $T\,(2T_d = vT)$. This allows the entire delay to be lumped in the plant function $G(s)$, which now has the form

$$G(s) = \frac{(1 - \epsilon^{-Ts})\,K\,\epsilon^{-vTs}}{s(1 + T_1\,s)} \tag{27}$$

We shall assume that the nonlinear element (S-curve) is frequency-insensitive. (If we consider a sinusoidal variation in the input frequency, the characteristic of the nonlinear element does not depend on the frequency of the sinusoidal variation.) Since the nonlinear element is only amplitude-sensitive and the sampler is only frequency-sensitive, the error sampler and the S-curve can be interchanged. We shall assume that the digital controller transfer function $D^*(s)$, when transformed into the z-domain (z-transformation), can be expressed as the simple ratio of two polinomial in powers of z^{-1} so that

$$D(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \cdots}{1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \cdots} \tag{28}$$

Upon z-transforming the plant function $G(s)$ we obtain

$$G(z) = \frac{K\,z^{-v-1}\,(1 - \epsilon^{-T/T_1})}{1 - \epsilon^{-T/T_1}\,z^{-1}} \tag{29}$$

After considering the above assumptions and transformations, the block diagram of Fig. 9 is derived.

Now we shall derive the difference equation that describes the digital AFC system shown in Fig. 9. Let $a(kT)$ denote the value of the variable $a(t)$ at the kth sampling instant ($t = kT$). Consider only the value of the variables at the sampling instants. Then

$$e(kT) = \omega_i(kT) - \omega_o(kT)$$

$$\omega_o(kT) - \epsilon^{-T/T_1}\,\omega_o[(k-1)T] = K(1 - \epsilon^{-T/T_1})\,q[(k-v-1)T] \tag{30}$$

and

$$\begin{aligned} q(kT) &+ b_1 q[(k-1)T] + b_2 q[(k-2)T] + b_3 q[(k-3)T] + \cdots \\ &= a_0 m(kT) + a_1 m[(k-1)T] + a_2 m[(k-2)T] + a_3 m[(k-3)T] + \cdots \end{aligned} \tag{31}$$

Thus

$$\begin{aligned} q[(k+v-1)T] &= a_0 m[(k-v-1)T] + a_1 m[(k-v-2)T] + a_2 m[(k-v-3)T] + a_3 m[(k-v-4)T] + \cdots \\ &\quad - b_1 q[(k-v-2)T] - b_2 q[(k-v-3)T] - b_3 q[(k-v-4)T] - \cdots \end{aligned} \tag{32}$$

Therefore,

$$\begin{aligned} e(kT) = \omega_i(kT) &- \epsilon^{-T/T_1}\,\omega_o(kT-T) - K(1-\epsilon^{-T/T_1})\;\{a_0 m(kT-vT-T) \\ &+ a_1 m(kT-vT-2T) + a_2 m(kT-vT-3T) + \cdots \\ &- b_1 q(kT-vT-2T) - b_2 q(kT-vT-3T) - \cdots\} \end{aligned} \tag{33}$$

We shall assume that the output of the nonlinear element (S-curve) is related to the input by

$$m(kT) = F[e(kT)]$$

Now

$$\omega_o(kT) = \omega_i(kT) - e(kT)$$

$$q[(k-v-1)T) = \frac{\omega_o(kT)}{K(1-\epsilon^{-T/T_1})} - \frac{\epsilon^{-T/T_1}\omega_o(kT-T)}{K(1-\epsilon^{-T/T_1})} \tag{34}$$

Let

$$\alpha = 1/K(1-\epsilon^{-T/T_1})$$

$$\beta = \epsilon^{-T/T_1} \tag{35}$$

Thus

$$\begin{aligned}\omega_o(kT) = \frac{1}{\alpha}\{&a_0 F[e(kT-vT-T)] + a_1 F[e(kT-vT-2T)] + a_2 F[e(kT-vT-3T)] \\ &+ a_3 F[e(kT-vT-4T)] + \cdots\} - b_1\omega_o(kT-vT-2T) \\ &+ (\beta b_1 - b_2)\omega_o(kT-vT-3T) + (\beta b_2 - b_3)\omega_o(kT-vT-4T) \\ &+ (\beta b_3 - b_4)\omega_o(kT-vT-5T) + (\beta b_4 - b_5)\omega_o(kT-vT-6T) + \cdots\end{aligned} \tag{36}$$

But

$$e(kT) = \omega_i(kT) - \omega_o(kT) \tag{37}$$

Therefore,

$$\omega_o(kT) = \frac{1}{K(1-\epsilon^{-T/T_1})} \{a_0 F[\omega_i(kT-vT-T) - \omega_o(kT-vT-T)]$$

$$+ a_1 F[\omega_i(kT-vT-2T) - \omega_o(kT-vT-2T)] + a_2 F[\omega_i(kT-vT-3T) - \omega_o(kT-vT-3T)]$$

$$+ \cdots \} - b_1\, \omega_o(kT-vT-2T) + (b_1 \epsilon^{-T/T_1} b_2)\, \omega_o(kT-vT-3T)$$

$$+ (b_2 \epsilon^{-T/T_1} - b_3)\, \omega_o(kT-vT-4T) + \cdots \tag{38}$$

Thus we can express the present value at the kth sampling instant of the output in terms of past values of the input and the output. A similar expression can be obtained for the system error $e(kt)$ by substituting from Eq. (38) into Eq. (37).

Example 1. Simple nonlinear AFC System

To simulate a simple AFC system as shown in Fig. 9,

Assume

$$D(z) = \frac{a_0 + a_1 z^{-1}}{1 + b_1 z^{-1}}$$

$$v = 14 \tag{39}$$

Thus

$$\omega_o(kT) = \frac{1}{K(1-\epsilon^{-T/T_1})} \{a_0 F[\omega_i(kT - 15T)$$

$$- \omega_o(kT - 16T)] + a_1 F[\omega_i(kT-16T) - \omega_o(kT-16T)]\}$$

$$- b_1\, \omega_o(kT-16T) + \epsilon^{-T/T_1} b_1 \omega_o(kT-17T) \tag{40}$$

or since

$$e(kT) = \omega_i(kT) - \omega_o(kT) \tag{41}$$

$$e(kT) = \omega_i(kT) + b_1\,\omega_i(kT - 16T) + \epsilon^{-T/T_1}\,b_1\,\omega_i(kT - 17T)$$

$$- \frac{1}{K(1 - \epsilon^{-T/T_1})}\,\{a_0 F\,[e(kT - 15T)] + a_1 F\,[e(kT - 16T)]\}$$

$$- b_1\,e(kT - 16T) - \epsilon^{-T/T_1}\,b_1\,e(kT - 17T) \tag{42}$$

Following the general assumption often used in continuous nonlinear systems, we will assume that the S-curve can be approximated by a piecewise linear equivalent nonlinear element. We will consider the approximation shown in Fig. 10.

Thus

$m(kT) = K_1\,e(kT)$	for	$\lvert e(kT)\rvert \leq e_1$	Range 1	
$m(kT) = K_2\,e(kT) + (K_1 - K_2)\,e_1$	for	$e_1 \leq e(kT) \leq e_2$	Range 2	
$m(kT) = K_2\,e_2 + (K_1 - K_2)\,e_1$	for	$e(kT) \geq e_2$	Range 3	
$m(kT) = K_2\,e(kT) - (K_1 - K_2)\,e_1$	for	$-e_1 \geq e(kT) \geq -e_2$	Range 4	
$m(kT) = -[K_2\,e(kT) + (K_1 - K_2)\,e_1]$	for	$e(kT) \leq -e_2$	Range 5	(43)

Thus, for Range 1 $[\,\lvert e(kT)\rvert \leq e_1]$, the linear difference equation becomes

$$e(kT) = \omega_i(kT) + b_1\,\omega_i(kT - 16T) + \epsilon^{-T/T_1}\,b_2\,\omega_i(kT - 17T)$$

$$- \frac{1}{K(1 - \epsilon^{-T/T_1})}\,[a_0 K_1\,e(kT - 15T) + a_1 K_1\,e(kT - 16T)]$$

$$- b_1\,e(kT - 16T) - \epsilon^{-T/T_1}\,b_1\,e(kT - 17T) \tag{44}$$

Or, in general,

$$e(kT) = \alpha_1 e(kT - 15T) + \alpha_2 e(kT - 16T) + \alpha_3 e(kT - 17T)$$

$$+ \omega_i(kT) + b_1 \omega_i(kT - 16T) + \epsilon^{-T/T_1} b_2 \omega_i(kT - 17T) \tag{45}$$

Thus, even in this simple example, we will obtain a difference equation of order $(v + 2)$, where v is the ratio of the time delay to the sampling period (Ref. 13). By analogy to a phase plane study of continuous systems (without time delays) described by an $(v + 2)$th-order nonlinear differential equation, one might conclude that a phase plane study – or incremental phase plane (Ref. 14) – for this digital AFC system will not be sufficient to describe the system performance. An $(v + 2)$th-order phase space may be required for an accurate system analysis (Ref. 14, p. 613). Unfortunately, our analogy is weak without a knowledge of rigorous phase plane treatment of continuous (or sampled-data) systems with pure time delays.[2]

A common method of handling the pure time delay is to approximate the ϵ^{-Ts} term (or other notations for the time delay) with some type of finite series approximation such as the first few terms of a Taylor series or a Padé approximation (Ref. 15, pp. 546 – 553). This would lead directly to the higher-order differential equation which would indicate the use of a higher-order phase space solution. Actually, an infinite series is needed to describe the delay. Thus, it seems that an infinite-order phase space is required to study the system performance, but each successive dimension added to the phase space is of decreasing importance in the solution.

If direct solution of linear differential equations with time delays is attempted in the Laplace domain by retaining the ϵ^{-Ts} term, the solution will have an infinite number of roots in the s-domain. This, too, indicates the necessity of a higher-order phase space for the study of systems with pure time delays. However, as with the series approximation, it is often possible to obtain a valid solution from the first few terms (roots) of the approximate solution.

Continuous systems with time delays are mathematically described by differential – difference equations (Ref. 18). A differential – difference equation can be considered an infinite-order differential equation, and its solution may have many of the properties of an infinite-order differential equation (such as an infinite number of possible oscillations). If the time delay is small, the differential – differential equation can usually be approximated by a finite-order differential equation.

[2] References 16 and 17 contain extensive bibliographies on the time delay problem.

However, a second-order differential – difference equation can usually be handled in the phase plane. Initial values for the time interval $T_d \leq t \leq 0$, where T_d is the time delay, are required for the solution of the differential – difference equation as opposed to initial conditions for the differential equation. The phase portrait is handled in a conventional manner except that one must keep track of time in the phase plane to correct for the time delay.

The phase plane for a differential – difference equation has many unusual properties. The phase portrait may cross itself. This occurs because an infinite number of solutions exist at any one point in the phase plane. The correct solution depends upon the initial values, which are really an infinite collection of initial conditions.

Thus, unless the initial values are restricted, a complete investigation of the system performance requires an infinite number of phase portraits, since each set of initial values will usually yield a different phase portrait. This might be considered analogous to the infinite phase space requirement for the infinite-order differential equation.

The above discussion indicates that a phase plane analysis of the sampled-data system with a pure time lag is sufficient to describe the system performance. However, such an analysis is hopeless unless the initial values at the sampling instants can be restricted by a knowledge of the operating conditions.

VI. LINEAR ANALYSIS

It has been shown that the phase-locked loop operates in a linear manner only when the difference between the input and output phases is small. A difference between the input and output frequencies appears as a difference in phase that increases linearly with time. Since no initial or sudden change in the input frequency can be corrected at the output in a time less than the pure time delay, a large difference between the input and output phase will exist when a sudden change in the input frequency occurs. A linear analysis will be invalid during this important portion of the system operation. Because of the long time lag, even very small "unpredictable" frequency errors will cause nonlinear operation of the system. For these reasons, a linear analysis of a digital phase-locked loop with large time delays seems meaningless.

However, it was also shown that the AFC can be designed to operate in a linear fashion over most of its useful range of automatic frequency control. When the operating conditions leave the linear range, the system often switches to a sweeping search mode to return the operating point to the linear mode. Thus a linear analysis of the AFC system is the logical starting point in the investigation of digital AFC systems.

The design of an AFC system with a large time delay requires some kind of "predictor" to achieve satisfactory control if we wish to compensate for the time delay. A common type of "predictor" controller is a digital controller based on "deadbeat" response (Ref. 7). Variations of this basic technique are ripple free (Ref. 7) and multirate deadbeat controllers (Ref. 19). Each of these controllers is designed for a specific input, and the response to other inputs is often very poor (especially when the ratio of the time delay to the sampling period is large). Design on the principle of minimization of the sum of the squared-error-samples is sometimes used to achieve a compromise between the designs for two or more different inputs (Ref. 7). Statistical considerations are also sometimes used as a basis of design of a digital controller when statistically described inputs or disturbances (or both) are expected under operating conditions (Ref. 7).

Each method of design will be applied to the digital AFC system described previously. The results and expected responses can be compared and evaluated in the light of proposed applications. In most cases, these methods will not provide adequate control and some other method should be sought.

One such method is the subrate controller. The sampler at the output of the subrate controller operates n times slower than the error sampler, as opposed to the multirate controller, whose sampler operates n times faster than the error sampler. Although the design procedures for the subrate controller are not completed, the philosophy of the subrate controller is covered herein.

A. Deadbeat Digital Controller for an AFC System

First, the design of a digital controller for the system in Fig. 11a is considered on the basis of "minimum settling time" or deadbeat[3] response. To achieve deadbeat response, the system error must be identically zero at every sampling instant after a specified interval of time following the application of the test input. Note that this does not specify the system error between the sampling instants. Minimum settling time is achieved by holding the interval of nonzero error to a minimum within the limits of system stability and a physically realizable digital controller. Later we will sacrifice the minimum settling time to reach a compromise be tween satisfactory ramp deadbeat response and satisfactory step deadbeat response.

The following assumptions are made to simplify the design of a digital controller for the system shown in Fig. 11a. The two samplers have the same sampling period T and operate syncronously. The ratio of the pure time delays T_d to the sampling period is an integer. Thus the time delays can be combined into one time delay appearing in the plant for simplicity. The entire loop gain $K_d K_g$ can be lumped into a plant gain K. The plant also contains the zero-order hold and time lag. Thus the transfer function $G(s)$ of the plant is given by

$$G(s) = \frac{K \epsilon^{-2T_d s} (1 - \epsilon^{-Ts})}{1 + T_1 s} \tag{46}$$

The z-transformation of the plant transfer function yields

$$G(z) = \frac{K z^{-(v+1)} (1 - \epsilon^{-T/T_1})}{1 - \epsilon^{-T/T_1} z^{-1}} \tag{47}$$

where

$$v = 2T_d/T$$

Let $D(z)$ represent the z-transform of the digital controller pulse network. Then the z-transformation representation of the system shown in Fig. 11b.

[3] We shall consider deadbeat response to specify the system only at the sampling instants. Others (Ref. 8) require zero error for all time, once deadbeat response has been achieved.

By conventional block-diagram reduction we obtain

$$E(z) = \frac{1}{1 + G(z)D(z)} R(z) \tag{48}$$

$$C(z) = \frac{G(z)D(z)}{1 + G(z)D(z)} R(z) \tag{49}$$

Let

$$W_e(z) = \frac{1}{1 + G(z)D(z)} \tag{50}$$

Therefore

$$E(z) = W_e(z)R(z) \tag{51}$$

$$C(z) = [1 - W_e(z)]\ R(z) \tag{52}$$

and

$$D(z) = \frac{[1 - W_e(z)]}{G(z) \cdot W_e(z)} \tag{53}$$

Now, $E(z)$ can be expanded to yield

$$E(z) = e_0 + e_1 z^{-1} + e_2 z^{-2} + e_3 z^{-3} + \cdots \tag{54}$$

However, $E(z)$ is defined as

$$E(z) = \sum_{k=0}^{\infty} e(kT)\, z^{-k} \tag{55}$$

Thus

$$
\begin{aligned}
e_0 &= e(t)\big|_{t=0} \\
e_1 &= e(t)\big|_{t=T} \\
e_2 &= e(t)\big|_{t=2T}
\end{aligned}
\tag{56}
$$

If $E(z)$ can be expressed in a finite series in z^{-1} ($e_k = 0$ for all k greater than some value K_m), the system error will be zero at each sampling instant after the K_mth sampling instant. Now consider inputs $R(s)$ of the form

$$R(s) = 1/s^n \tag{57}$$

Then

$$R(z) = A(z^{-1})/(1 - z^{-1})^n \tag{58}$$

Let $W_e(z)$ be of the form

$$W_e(z) = (1 - z^{-1})^n F(z^{-1}) \tag{59}$$

Then the system error will be a finite series in z^{-1} given by

$$E(z) = A(z^{-1})\, F(z^{-1}) \tag{60}$$

and the requirement of zero error at the sampling instants after a finite time will be satisfied.

To satisfy the requirement of a physically realizable digital controller and reduce system stability problems, the following restrictions must be imposed upon the functions $W_e(z)$ and $1 - W_e(z)$.

1. $W_e(z)$ should contain as its zeros all the poles of $G(z)$ which lie on or outside the unit circle of the z plane.

2. $1 - W_e(z)$ should contain as its zeros all the zeros of $G(z)$ which lie on or outside the unit circle of the z plane.

3. $1 - W_e(z)$ should contain z^{-n} as a factor when $G(z)$ contains z^{-n} as a factor (n will usually be unity if $G(z)$ does not have a time delay, and n will usually be greater than unity if $G(z)$ has a time delay).

Example 2. General Deadbeat Design for a Ramp Input

Assume that a ramp input $[r(t) = t]$ is applied to the system shown in Fig. 11. Then

$$R(z) = \frac{T z^{-1}}{(1 - z^{-1})^2} \tag{61}$$

Also assume that $G(z)$ is stable ($\epsilon^{-T/T_1} < 1$). According to the deadbeat criterion stated earlier

$$W_e(z) = (1 - z^{-1})^2 (1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \cdots) \tag{62}$$

$$1 - W_e(z) = z^{-v-1} (a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \cdots) \tag{63}$$

Equation (62) can be rewritten as

$$W_e(z) = 1 - a_0 z^{-v-1} - a_1 z^{-v-2} - a_2 z^{-v-3} - a_3 z^{-v-4} - a_4 z^{-v-5} - \cdots \tag{64}$$

Expanding Eq. (62) and equating coefficients of like powers of z^{-n} in Eq. (62) and (64) yield the following set of equations

$$
\begin{aligned}
b_1 - 2 &= 0 \\
b_2 - 2b_1 + 1 &= 0 \\
b_3 - 2b_2 + b_1 &= 0 \\
&\vdots \\
b_v - 2b_{v-1} + b_{v-2} &= 0 \\
b_{v+1} - 2b_v + b_{v-1} &= -a_0 \\
b_{v+2} - 2b_{v+1} + b_v &= -a_1 \\
b_{v+3} - 2b_{v+2} + b_{v+1} &= -a_2 \\
b_{v+4} - 2b_{v+3} + b_{v+2} &= -a_3 \\
&\vdots \\
b_{v+k} - 2b_{v+k-1} + b_{v+k-2} &= -a_k
\end{aligned}
\tag{65}
$$

The solution to these equations is given by

$$
\begin{aligned}
b_1 &= 2 \\
b_2 &= 3 \\
b_3 &= 4 \\
b_4 &= 5 \\
&\vdots \\
b_v &= v + 1 \\
b_{v+1} &= -a_0 + v + 2 \\
b_{v+2} &= -a_1 - 2a_0 + v + 3 \\
b_{v+3} &= -a_2 - 2a_1 - 3a_0 + v + 4 \\
b_{v+4} &= -a_3 - 2a_2 - 3a_1 - 4a_0 + v + 5 \\
&\vdots \\
b_{v+k} &= -a_{k-1} - 2a_{k-2} = \cdots - k\,a_0 + v + k + 1 \\
&\vdots
\end{aligned}
\tag{66}
$$

If the system is to achieve zero error at the sampling instants in the shortest possible time, the finite series in z^{-1} for $W_e(z)$ should be of the lowest possible order consistent with Eq. (66). This is achieved if

$$\begin{aligned} a_0 &= +\,(v+2) \\ a_1 &= -\,(v+1) \\ a_2 &= 0 \\ a_3 &= 0 \\ a_4 &= 0 \\ &\vdots \end{aligned} \tag{67}$$

Thus

$$1 - W_e(z) = (v+2)\, z^{-v-1} - (v+1)\, z^{-v-2} \tag{68}$$

$$W_e(z) = 1 - (v+z)\, z^{-v-1} + (v+1)\, z^{-v-2} \tag{69}$$

$$D(z) = \frac{(1-\epsilon^{-T/T_1} z^{-1})\left[(v+2) - (v+1)\, z^{-1}\right]}{K(1-\epsilon^{-T/T_1})\left[1 - (v+2)\, z^{-v-1} + (v+1)\, z^{-v-2}\right]} \tag{70}$$

and the z-transform of the ramp response is given by

$$C(z)_{ramp} = (v+2)\, z^{-(v+2)} + (v+3)\, z^{-(v+3)} + (v+4)\, z^{-(v+4)} + \ldots \tag{71}$$

From the definition of the z-transform we obtain the system output at the sampling instants to be

$$\begin{aligned} C(kT) &= 0 & \qquad 0 \le k < v+2 \\ C(kT) &= k & \qquad k \ge v+2 \end{aligned} \tag{72}$$

Interpretation of Eq. (71) shows that the system output will follow the input with zero error at the sampling instants $2T_d + 2T$ sec after the application of the ramp input. Thus a large decrease in the sampling period will cause only a slight increase in the speed of response of the system to a ramp input when the sampling period is a small fraction of the delay.

However, the size of the sampling period has a significant effect on two other aspects of the system performance — the system output between the sampling instants and the overshoot when the system is subjected to stop inputs.

First we will consider the step response of the AFC system with a digital controller designed for a ramp input. The unit step input $[r(t) = u(t)]$ is defined as

$$u(t) = \begin{cases} 0 & \text{for} \quad t > 0 \\ 1 & \text{for} \quad t \geq 0 \end{cases} \tag{73}$$

The z-transform of the unit step is given by

$$\mathfrak{z}\{u(t)\} = \frac{1}{1 - z^{-1}} \tag{74}$$

Thus

$$R(z)_{step} = \frac{1}{1 - z^{-1}} \tag{75}$$

The z-transform of the system step response is given by

$$\begin{aligned} C(z) &= [1 - W_e(z)]\, R(z)_{step} \\ &= (v + 2)\, z^{-v-1} + z^{-v-2} + z^{-v-3} + z^{-v-4} + \cdots \end{aligned} \tag{76}$$

Thus the step response at the sampling instants is given by

$$c(kT) = \begin{array}{lll} 0 & \text{for} & k \leq v \\ v + 2 & \text{for} & k = v + 1 \\ 1 & \text{for} & k \geq v + 2 \end{array} \tag{77}$$

It should be noted that the step response also has zero error at the sampling instants after $2T_d + 2T$ sec.

From Eq. (72) it is obvious that the overshoot is at least $v + 1$ units and may be greater between the sampling instants. Thus T should be large to decrease the value of v and thus reduce the step overshoot magnitude. However, T should be small to provide adequate speed of response to ramp and step inputs. An optimum compromise between these two conflicting requirements is presented on page 35.

Determination of the system output between the sampling instants requires the use of the modified z-transformation, which is defined by

$$\mathfrak{z}_m \{C(t)\} = C(z,m)$$

$$= z^{-1} \sum_{k=0}^{\infty} C(kT + mT)\, z^{-k} \tag{78}$$

for

$$0 < m \leq 1$$

Thus, if the modified z-transform is expanded into a series such as

$$C(z,m) = z^{-1} \left[C_0(m) + C_1(m)\, z^{-1} + C_2(m)\, z^{-2} + C_3(m)\, z^{-3} + \cdots\right] \tag{79}$$

then

$$\begin{aligned}
C_0(m) &= C(t)\,|_{t=mT} \\
C_1(m) &= C(t)\,|_{t=T+mT} \\
C_2(m) &= C(t)\,|_{t=2T+mT} \\
&\vdots \\
C_n(m) &= C(t)\,|_{t=nT+mT} \\
&\vdots
\end{aligned}$$

where $0 < m \leq 1$.

The modified z-transform of the output $C(z,m)$ of the system shown in Fig. 11 is given by

$$C(z,m) = \frac{D(z)\,G(z,m)}{1 + D(z)\,G(z)}\,R(z)$$

$$= \frac{G(z,m)}{G(z)}\,[1 - W_e(z)]\,R(z) \qquad (81)$$

where $G(z,m)$ is the modified z-transform associated with the plant transfer function $G(s)$ so that

$$G(z,m) = \mathfrak{z}_m \{G(s)\} \qquad (82)$$

which can be obtained from several different tables or direction application of the definition (78). For the system under study

$$G(s) = \frac{K\,\epsilon^{-vTs}\,(1 - \epsilon^{-Ts})}{(1 + T_1 s)} \qquad (83)$$

so

$$G(z,m) = \frac{K\,[(1 - \epsilon^{-mT/T_1}) - (\epsilon^{-T/T_1} - \epsilon^{-mT/T_1})\,z^{-1}]}{1 - \epsilon^{-T/T_1}\,z^{-1}} \qquad (84)$$

and

$$\frac{G(z,m)}{G(z)} = \left[\frac{1 - \epsilon^{-mT/T_1}}{1 - \epsilon^{-T/T_1}} - \frac{\epsilon^{-T/T_1} - \epsilon^{-mT/T_1}}{1 - \epsilon^{-T/T_1}}\,z^{-1}\right] \qquad (85)$$

Thus the modified z-transform of the system output is given by

$$C(z,m) = \frac{[(1 - \epsilon^{-mT/T_1}) - (\epsilon^{-T/T_1} - \epsilon^{-mT/T_1})\,z^{-1}]\,[(v+2) - (v+1)\,z^{-1}]\,z^{-v-1}}{(1 - \epsilon^{-T/T_1})}\,R(z) \qquad (86)$$

The modified z-transform of the system ramp response is given by

$$C(z,m)_{ramp} = \frac{T(v+2)z^{-(v+2)}\left[1 - \dfrac{\epsilon^{-T/T_1} - \epsilon^{-mT/T_1}}{1-\epsilon^{-mT/T_1}}\right]\left[1 - \dfrac{v+1}{v+2}z^{-1}\right]}{(1-\epsilon^{-T/T_1})(1-\epsilon^{-mT/T_1})(1-z^{-1})^2}$$

$$= T\left\{\left(\frac{1-\epsilon^{-mT/T_1}}{1-\epsilon^{-T/T_1}}\right)(v+2)\,z^{-(v+2)}\right.$$

$$+\left[v+3-\left(\frac{\epsilon^{-mT/T_1}-\epsilon^{-T/T_1}}{1-\epsilon^{-T/T_1}}\right)\right]z^{-(v+3)}$$

$$\left. + \cdots + \left[v+k-\left(\frac{\epsilon^{-mT/T_1}-\epsilon^{-T/T_1}}{1-\epsilon^{-T/T_1}}\right)\right]z^{-(v+k)} + \cdots \right\} \tag{87}$$

Thus the ramp response is given by

$$c(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t \leq (v+1)\,T \\[2ex] \dfrac{1-\epsilon^{-mT/T_1}}{1-\epsilon^{-T/T_1}}\,(v+2) & \text{for} \quad t = (v+1+m)\,T \\[2ex] v+k+1-\dfrac{\epsilon^{-mT/T_1}-\epsilon^{-T/T_1}}{1-\epsilon^{-mT/T_1}} & \text{for} \quad t = (v+k+m)\,T \end{cases}$$

$$\text{where} \qquad 0 \leq m \leq 1 \qquad\qquad k \geq 2 \tag{88}$$

Similarly, the modified z-transform of the system step response is given by

$$C(z,m)_{step} = \frac{(v+2)\,z^{-v-1}}{(1-\epsilon^{-T/T_1})(1-\epsilon^{-mT/T_1})} \; \frac{\left[1-\dfrac{\epsilon^{-T/T_1}-\epsilon^{-mT/T_1}}{1-\epsilon^{-mT/T_1}}\,z^{-1}\right]\left[1-\dfrac{v+1}{v+2}\,z^{-1}\right]}{(1-z^{-1})}$$

$$= \left[\frac{1-\epsilon^{-mT/T_1}}{1-\epsilon^{-T/T_1}}\right](v+2)\,z^{-v-1} + \left\{(v+2)-(v+1)\left[\frac{1-\epsilon^{-mT/T_1}}{1-\epsilon^{-T/T_1}}\right]\right\} z^{-v-2}$$

$$+ z^{-v-3} + z^{-v-4} + \cdots \tag{89}$$

Thus the step response is given by

$$c(t) = \begin{cases} 0 & \text{for} \quad 0 \le t \le vT \\[2ex] \dfrac{1-\epsilon^{-mT/T_1}}{1-\epsilon^{-T/T_1}}\,(v+2) & \text{for} \quad t = v + mT \\ & \qquad\;\; 0 \le m \le 1 \\[2ex] (v+2)-(v+1)\,\dfrac{1-\epsilon^{-mT/T_1}}{1-\epsilon^{-T/T_1}} & \text{for} \quad t = (v+1+m)\,T \\ & \qquad\;\; 0 \le m \le 1 \\[2ex] 1 & \text{for} \quad t \le (v+2)\,T \end{cases} \tag{90}$$

Example 3. Design of Remote-Controlled Receiver for Earth-to-Venus Operation

Let us consider the remote tuning of a receiver located in a vehicle near Venus with a digital controller on the Earth. With the digital controller on the Earth there is complete freedom to change the control characteristics to compensate for unexpected operating conditions in space travel. In this case the one-way communications time delay will be 140 sec ($2T_d$ = 280 sec). In the initial design there are two parameters in addition to the digital controller which must be chosen. They are the sampling period T and the plant time constant T_1. It has already been shown that the step overshoot and the speed of response put conflicting requirements on the ratio of the time delay to the sampling period. In addition, the ratio of the plant time constant to the sampling period also determines the response characteristics between the sampling instants.

Figures 12 – 16 show the ramp and step response of the digital AFC with a digital controller designed for a ramp input for several different ratios v of the pure time delay $2T_d$ to the sampling period $T\,[v = 2T_d/T]$. In each figure the intersample ripples are shown for the case when the ratio q of the time constant of the plant T_1 to the sampling period T was 0.25 ($q = T_1 T = 0.25$). From these response curves it is obvious that the ratio $2T_d/T = v$ also influences the intersample ripples.

The effect of the ratio T_1/T upon the output ripples is shown more clearly in the exploded view of the system output (Fig. 17). The same ratio of time delay to sampling period is used in both response curves of Fig. 17 (v = 14). However, two different values of T_1/T (T_1/T = 0.125 and T_1/T = 0.25) were used for the exploded views showing the ramp response when deadbeat response is first attained. The ripples are reduced by increasing the ratio T_1/T in this case. This is to be expected since a low T_1/T ratio indicates little filtering, and the output should approach the clamped condition between samples because of the zero-order hold. As T_1/T increases, the filter approaches a pure integration, which will allow perfect ramp response (no ripples). The problems of output ripples will be elaborated in the following section.

B. Ripple-Free Design

It has been proposed (Ref. 7) that the intersampling can be removed at the expense of slower system response. The following is a summary of the proposed method. From Eq. (81) it is known that

$$C(z,m) = \frac{D(z)\,G(z,m)}{1 + D(z)\,G(z)}\,R(z) \tag{91}$$

Let

$$C(z,m) = G_e(z)\,G(z,m)\,R(z) \tag{92}$$

where

$$G_e(z) = \frac{D(z)}{1 + D(z)\,G(z)}$$

$$= \frac{1}{G(z)}\,[1 - W_e(z)] \tag{93}$$

"It can be shown that the pulse-transfer function $G(z)\,G(z,m)$ as a polynomial in z^{-1} is the necessary condition for the system response to a basic input (step, ramp, parabolic, etc.) to be free from ripples after a reasonably short transient period has elapsed.[4]" Thus, observing the above requirement for $G_e(z)$ and restating the previous restrictions for deadbeat response design, the following criterion is established foı the design of a digital controller to provide ripple-free deadbeat response. The transfer function of the digital controller is given by

$$D(z) = \frac{1}{G(z)}\,\frac{1 - W_e(z)}{W_e(z)} \tag{94}$$

for basic input of the form $R(s) = 1/s^n$

$$R(z) = \frac{F(z^{-1})}{(1 - z^{-1})^n} \tag{95}$$

Following the deadbeat design pattern, the functions $W_e(z)$ and $1 - W_e(z)$ must be of the form

$$W_e(z) = (1 - z^{-1})^n\,(1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \cdots) \tag{96}$$

$$1 - W_e(z) = z^{-k}\,(b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \cdots) \tag{97}$$

[4] This statement (quoted from Ref. 7, p. 513) has not been verified, and a following example will show that the above condition may be necessary but is not sufficient.

where

1. $W_e(z)$ must contain in its zeros all the poles of $G(z,m)$ which lie on or outside the unit circle of the z-plane.

2. $1 - W_e(z)$ must contain in its zeros all the zeros of $G(z)$.

3. $1 - W_e(z)$ must contain z^{-k} as a factor if $G(z)$ contains z^{-k} as a factor (k usually equals unity unless $G(z)$ contains a pure time delay).

Thus, the only addition to the deadbeat response criterion is that $W_e(z)$ must contain all the zeros of $G(z)$. Previously, only the zeros on or outside the unit circle in the z-plane needed to be included in the $W_e(z)$.

Immediate application of the ripple-free criterion to the system proposed in Examples 1 and 2 will show that this criterion does not always yield a ripple-free design. Furthermore, it will be shown that it is impossible to make the system output ripple-free for a ramp input without changing the plant portion of the system. This fact is hinted at by Jury (Ref. 8, p. 196) when he states, "However, the possibility of a true deadbeat (ripple-free) response is dependent on the plant transfer function . . . "

Example 4. Ripple-Free Design

Consider the system of Examples 2 and 3, where

$$G(z) = \frac{K(1 - \epsilon^{-T/T_1})\, z^{-(v+1)}}{1 + \epsilon^{-T/T_1}\, z^{-1}} \tag{98}$$

and

$$G(z,m) = \frac{K[(1 - \epsilon^{-mT/T_1}) + (\epsilon^{-mT/T_1} - \epsilon^{-T/T_1})\, z^{-1}]\, z^{-(v+1)}}{1 + \epsilon^{-T/T_1}\, z^{-1}} \tag{99}$$

Examination of Eq. (98) reveals that $G(z)$ has no zeros. Thus the criterion for ripple-free deadbeat controller design is identical to the previous criterion for a deadbeat controller, which yielded a system with output ripples for a ramp input. Thus, the proposed criterion will not yield a ripple-free response for a ramp input when the plant consists of zero-order hold, pure time delay, and a first-order lag.

Now it will be shown that it is impossible to design a digital controller to provide ripple-free deadbeat response to a ramp input for the above-described system.

From Eq. (92) and (93) we have

$$C(z,m) = \frac{1 - W_e(z)}{G(z)} G(z,m) R(z) \tag{100}$$

for a ramp input

$$R(z) = \frac{T z^{-1}}{(1 - z^{-1})^2} \tag{101}$$

so

$$C(z,m) = \frac{T z^{-1}}{(1 - z^{-1})^2} [1 - W_e(z)] \frac{G(z,m)}{G(z)} \tag{102}$$

where $1 - W_e(z)$ should be a finite polynomial in z^{-1} to satisfy the deadbeat response requirements. For a deadbeat ripple-free response, the terms of an expansion of $C(z,m)$ in z^{-1} must be of the form $T(m+k)z^{-(k-1)}$ for all k greater than some finite value.

For our system

$$C(z,m) = \frac{[1 - W_e(z)]\, T\, [(1 - \epsilon^{-mT/T_1}) - (\epsilon^{-T/T_1} - \epsilon^{-mT/T_1})\, z^{-1}]\, z^{-1}}{(1 - z^{-1})^2 (1 - \epsilon^{-T/T_1})} \tag{103}$$

Now, to satisfy the requirement for deadbeat response and physical realizability of the digital controller, $1 - W_e(z)$ should be of the form

$$1 - W_e(z) = z^{-v-1} (a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \cdots) \tag{104}$$

Thus

$$C(z,m) = \frac{T z^{-(v+2)} (a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots) [1 - \epsilon^{-mT/T_1} - (\epsilon^{-T/T_1} - \epsilon^{-mT/T_1}) z^{-1}]}{(1 - z^{-1})^2 (1 - \epsilon^{-T/T_1})} \tag{105}$$

Let

$$\beta = \frac{\epsilon^{-T/T_1} - \epsilon^{-mT/T_1}}{1 - \epsilon^{-T/T_1}} \tag{106}$$

Then

$$\begin{aligned} C(z,m) = T \{ & a_0 (1 + \beta) z^{-(v+2)} + [a_1 + 2a_0 + (a_0 + a_1) \beta] z^{-(v+3)} \\ & + \cdots + [ak + 2a_{k-1} + \cdots + (k+1) a_0 + (a_0 + a_1 + \cdots + ak) \beta z^{-(v+2+k)} + \cdots \} \end{aligned} \tag{107}$$

Thus, to satisfy the restriction on the coefficients of the $C(z,m)$ expansion mentioned above,

$$\begin{aligned} a_0 + a_0 \beta &= v + m + 1 \\ a_1 + 2a_0 + (a_0 + a_1) \beta &= v + m + 2 \\ a_k + 2a_{k-1} + \cdots + (k+1) a_0 + (a_0 + a_1 + \cdots + a_k) \beta &= v + m + k + 1 \end{aligned} \tag{108}$$

If the ripples are to disappear after the kth sampling period

$$a_n + 2a_{n-1} + \cdot \cdot \cdot + (n+1)\, a_0 + (a_0 + a_1 + a_2 + \cdot \cdot \cdot + a_n) \quad \frac{\epsilon^{-T/T_1} - \epsilon^{-mT/T_1}}{1 - \epsilon^{-T/T_1}} = k + m \tag{109}$$

Since the a's cannot be a function of m, the various combinations of a's and $1 - \epsilon^{-T/T_1}$ can be lumped into constants M, N, and K to yield

$$M + N\epsilon^{-mT/T_1} = K + m$$

or

$$\epsilon^{-mT/T_1} = \alpha + \gamma m \tag{110}$$

However, Eq. (110) has no solution when α and γ are not functions of m.

Thus, it is impossible to design a digital controller to yield a ripple-free deadbeat ramp response when the open-loop plant consists of a zero-order hold, pure time delay, and a first-order lag. This statement can also be proven using the design approach of Jury (Ref. 8) or Schroeder (Ref. 19).

The ripples can be removed by changing the system plant and applying the above-mentioned criterion. Replacing the zero-order hold with a first-order hold is one obvious method. The addition of an integration in the plant (other than the hold) would be a second method.

C. Minimization of the System Error

It was noted that a least-settling-time, deadbeat, digital-controlled system, when designed for a ramp input, yielded a very large step response overshoot when the system contained a large time lag. This overshoot can be reduced at the sacrifice of the deadbeat response by introducing a "staleness weighting factor" (Ref. 7, p. 519) in the digital controller design. A second method of reducing the step function overshoot is to retain the deadbeat response but increase the settling time. Both of these methods will be attempted.

A mathematically useful criterion for optimization of system parameters for sampled-data systems is the minimization of the sum of the squared-error samples of the error pulse sequence. It has been shown that the sum of the squared-error samples can be written as

$$\sum_{k=0}^{\infty} [e(kT)]^2 = \frac{1}{2\pi j} \oint_{\Gamma} E(z)\, E(z^{-1})\, z^{-1}\, dz \tag{111}$$

Where the z-transform of the error pulse sequence is given by

$$E(z) = \sum_{k=0}^{\infty} e(kT)\, z^{-k} \tag{112}$$

and the Γ contour is the unit circle in the z-plane.

Application of this criterion to the design of a digital controller is straightforward and, with a small modification, follows directly from the design of a deadbeat digital controller. The problem is to choose the parameters in $W_e(z)$ and $1 - W_e(z)$ in such a manner that the above error criterion is minimized.

The first modification to the deadbeat response requirements will be the addition of the "staleness weighting factor" to the design function $W_e(z)$ and $1 - W_e(z)$. The following design procedure is taken from Ref. 7. It will be shown that the introduction of the "staleness weighting factor" is not mathematically practical when the system contains a time delay that is much greater than the sampling period.

The "staleness weighting factor" α is introduced so that

$$W_e(z) = \frac{(1 - z^{-1})^n\, (1 + b_1 z^{-1} + b_2 z^{-2} + \cdots)}{1 - \alpha z^{-1}} \tag{113}$$

$$1 - W_e(z) = \frac{z^{-(v+1)}\, (a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots)}{1 - \alpha z^{-1}} \tag{114}$$

where the numerators of $W_e(z)$ and $1 - W_e(z)$ satisfy the requirements for deadbeat response, physically realizable digital controller, and system stability

Now

$$E(z) = W_e(z)\, R(z) \tag{115}$$

In deriving (Eq. 113 – 115), we have assumed an input of the form

$$R(z) = \frac{F(z^{-1})}{(1 - z^{-1})^n} \tag{116}$$

so

$$E(z) = \frac{[F(z^{-1})]\,[1 + b_1 z^{-1} + b_2 z^{-2} + \cdots]}{1 - \alpha z^{-1}} \tag{117}$$

Now it is known that $F(z^{-1})$ and $W_e(z)$ are finite series in z^{-1} so that

$$E(z) = \frac{\alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_{m+n} z^{(m+n)}}{z^{m+n-1}(z - \alpha)} \tag{118}$$

Thus

$$E(z^{-1}) = \frac{\alpha_0 z^{m+n} + \alpha_1 z^{m+n-1} + \cdots + \alpha_{m+n}}{(1 - \alpha z)} \tag{119}$$

where $n \geq v + 1$.

Thus

$$\sum_{k=0}^{\infty} [e(kT)]^2 = \frac{1}{2\pi j} \oint_{\Gamma} \frac{(\alpha_0 + \alpha_1 z + \cdots + \alpha_{m+n} z^{m+n})(\alpha_0 z^{m+n} + \cdots + \alpha_{m+n})}{z^{m+n}(z - \alpha)(1 - \alpha z)}\, dz \tag{120}$$

where $|\alpha| < 1$ and the contour Γ is the unit circle in the z plane. This integral is readily evaluated by determining the residues of the integrand at its poles inside the contour (the poles appear at $z = \alpha$ and $z = 0$). However, the pole at the origin is of order $m + n$, which is at least of order $v + 1$. The residue, assuming an order of $v + 1$, is given by

$$\mathscr{R} = \frac{1}{v!} \frac{d^v}{dz^v} \left[\frac{(a_0 + a_1 z + \cdots a_{m+n} z^{m+n})(a_0 z^{m+n} + a_1 z^{m+n-1} + \cdots + a_{m+n})}{(z - \alpha)(1 - \alpha z)} \right] \tag{121}$$

which, from an engineering point of view, is mathematically impractical to evaluate.

Thus, the "staleness weighting factor" is not a logical approach to reduce the system error when long time delays exist in the system.

Assume that the system should have deadbeat response to a ramp input but that the minimum settling time requirement of the previous examples is not necessary. Then extra terms can be added to the digital controller transfer function $D(z)$ to reduce the step overshoot. This method will be optimized by employing the "minimum sum of the squared-error samples" of the error pulse sequence criterion on the selection of the extra terms.

The basic requirements of deadbeat response are retained so that

$$W_e(z) = \frac{1}{1 + D(z)\, G(z)} \tag{122}$$

$$E(z) = W_e(z)\, R(z) \tag{123}$$

$$C(z) = [1 - W_e(z)]\ R(z) \tag{124}$$

$$D(z) = \frac{1}{G(z)} \frac{1 - W_e(z)}{W_e(z)} \tag{125}$$

$$R(z)_{ramp} = \frac{T z^{-1}}{(1 - z^{-1})^2} \tag{126}$$

$$R(z)_{step} = \frac{1}{1 - z^{-1}} \tag{127}$$

For deadbeat response to a ramp input, $W_e(z)$ and $1 - W_e(z)$ should be of the form

$$W_e(z) = (1 - z^{-1})^2 (1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \cdots) \tag{128}$$

$$1 - W_e(z) = z^{-(v+1)} (a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \cdots) \tag{129}$$

and must satisfy the conditions:

1. $W_e(z)$ should contain as its zeros all the poles of $G(z)$ which lie on or outside the unit circle of the z plane.
2. $1 - W_e(z)$ should contain as its zeros all the zeros of $G(z)$ which lie on or outside the unit circle of the z plane.
3. $1 - W_e(z)$ should contain z^{-n} as a factor when $G(z)$ contains z^{-n} as a factor. (n will usually be unity if $G(z)$ does not contain a pure delay.)

Under these conditions, it was shown that

$$\begin{aligned}
b_1 &= 2 \\
b_2 &= 3 \\
b_3 &= 4 \\
&\vdots \\
b_v &= v + 1 \\
b_{v+1} &= -a_0 + v + 2 \\
b_{v+2} &= -a_1 - 2a_0 + v + 3 \\
&\vdots \\
b_{v+m} &= -a_{m-1} - 2a_{m-2} - 3a_{m-3} - \cdots - ma_0 + v + m + 1 \\
&\vdots
\end{aligned} \tag{130}$$

To obtain a minimum settling time response, it was shown that $b_k = 0$ for all $k \geq v + 1$. By relaxing the minimum settling time, one can arbitrarily choose values of b_k's for $v + 1 \leq k \leq v + m$. In this case, the system ramp response will have a settling time of $(m + 2)\,T + T_d$ sec.

However, if the b_k's are chosen arbitrarily, there is no assurance that the step overshoot will be a minimum for the amount of settling time sacrificed. Although not necessarily the same criterion as minimum overshoot, minimization of the sum of the squared-error samples of the error pulse sequence is a convenient and logical method of determining an optimum set of values for the b_k's. Pursuing this approach yields

$$E(z)_{ramp} = W_e(z)\,R(z)_{ramp} \tag{131}$$

for step and ramp inputs

$$E(z)_{ramp} = T z^{-1}\,[1 + 2z^{-1} + 3z^{-2} + \cdots + (v+1)\,z^{-v} + b_{v+1}\,z^{-(v+1} + b_{v+2}\,z^{-(v+2)} + \cdots + b_{v+m}\,z^{-(v+m)}] \tag{132}$$

similarly

$$E(z)_{step} = 1 + z^{-1} + z^{-2} + \cdots + z^{-v} + (b_{v-1} - v - 1)\,z^{-(v+1)}$$

$$+ (b_{v+2} - b_{v+1})\,z^{-(v+2)} + \cdots + (b_{v+m} - b_{v+m-1})\,z^{-(v+m)} - b_{v+m}\,z^{-(v+m+1)} \tag{133}$$

For a ramp input

$$E(z)\,E(z^{-1})\,z^{-1} = T^2\,z^{-(m+v+1)}\,\{b_{v+m}$$

$$+ (b_{v+m-1} + 2b_{v+m})\,z + (b_{v+m-2} + 2b_{v+m-1} + 3b_{v+m})\,z^2 + \cdots + e_{v+m}\,z^{v+m}$$

$$+ e_{v+m+1}\,z^{v+m+1} + \cdots + b_{v+m}\,z^{2v+2m}\} \tag{134}$$

where

$$e_{v+m} = 1^2 + 2^2 + 3^2 + \cdots + v^2 + (v+1)^2 + (b_{v+1})^2 + (b_{v+2})^2 + \cdots + (b_{v+m})^2 \tag{135}$$

It has already been shown that the sum of the squared-error samples can be expressed as

$$\sum_{k=0}^{\infty} [e(kT)]^2 = \frac{1}{2\pi j} \oint_{\Gamma} E(z)\, E(z^{-1})\, z^{-1}\, dz \tag{136}$$

Substitution of Eq. (134) into (136) and evaluation of the contour integration by residue methods yields

$$\sum_{k=0}^{\infty} [e(kT)]^2 = \frac{T^2}{(v+m)!} \frac{d^{v+m}}{dz^{v+m}} \left[b_{v+m} + (b_{v+m-1} + 2b_{v+m})\, z + \cdots + e_{v+m}\, z^{v+m} \right.$$

$$\left. + \, e_{v+m+1}\, z^{v+m+1} + \cdots + b_{v+m}\, z^{2v+2m} \right] \Big|_{z=0} \tag{137}$$

$$= T^2\, e_{v+m}$$

$$= T^2 \{ 1^2 + 2^2 + 3^2 + 4^2 + \cdots + v^2 + (v+1)^2$$

$$+ (b_{v+1})^2 + (b_{v+2})^2 + (b_{v+3})^2 + \cdots + (b_{v+m})^2 \} \tag{138}$$

Minimization of this sum by conventional calculus techniques $(\partial \Sigma / \partial b_k = 0)$ yields

$$\frac{\partial}{\partial b_{v+l}} \left\{ \sum_{k=0}^{\infty} [e(kT)]^2 \right\} = 0 \qquad l = 1, 2, \cdots, m$$

$$\begin{aligned} b_{v+1} &= 0 \\ b_{v+2} &= 0 \\ &\vdots \\ b_{v+m} &= 0 \end{aligned} \tag{139}$$

which should be expected since this is the minimum settling time solution. Expression (138) will be useful as a measure of how much each extra term in the digital controller increases the ramp error.

The next step is to evaluate the sum of the squared-error samples for a step input. From (133) we have

$$E(z)E(z^{-1})\,z^{-1} = \frac{1}{z^{v+m+2}}\,(-b_{v+m} - b_{v+m-1}\,z - b_{v+m-2}\,z^2 - \cdots$$

$$+ e_{v+m+1}\,z^{v+m+1} + \cdots - b_{v+m-1} - b_{v+m}) \tag{140}$$

where

$$e_{v+m+1} = (v+1) + (b_{v+1} - v - 1)^2 + (b_{v+2} - b_{v+1})^2 + \cdots + (b_{v+m} - b_{v+m-1})^2 + (b_{v+m})^2 \tag{141}$$

Thus

$$\sum_{k=0}^{\infty} [e(kT)]^2 = \frac{1}{(v+m+1)!}\,\frac{d^{v+m+1}}{dz^{v+m+1}}\,[-b_{v+m} - b_{v+m-1}\,z - \cdots$$

$$+ e_{v+m+1}\,z^{v+m+1} + \cdots - b_{v+m-1} - b_{v+m}]\Big|_{z=0} \tag{142}$$

$$= e_{v+m+1}$$

$$= (v+1) + (b_{v+1} - v - 1)^2 + (b_{v+2} - b_{v+1})^2$$

$$+ (b_{v+3} - b_{v+2})^2 + \cdots + (b_{v+m} - b_{v+m-1})^2 + (b_{v+m})^2 \tag{143}$$

Minimization of the step sum of the squared-error samples by conventional calculus techniques ($\partial\Sigma/\partial b_k = 0$) yields

$$\frac{\partial}{\partial b_{v+l}} \left\{ \sum_{k=0}^{\infty} [e(kT)]^2 \right\} = 0 \qquad l = 1, 2, \cdots, m$$

$$\begin{aligned}
-(v+1) + 2b_{v+1} - b_{v+2} &= 0 \\
-b_{v+1} + 2b_{v+2} - b_{v+3} &= 0 \\
&\vdots \\
-b_{v+m-2} + 2b_{v+m-1} - b_{v+m} &= 0 \\
-b_{v+m-1} + 2b_{v+m} &= 0
\end{aligned} \tag{144}$$

or

$$\begin{aligned}
b_{v+1} &= \frac{m}{m+1}(v+1) \\
b_{v+2} &= \frac{m-1}{m+1}(v+1) \\
b_{v+3} &= \frac{m-2}{m+1}(v+1) \\
&\vdots \\
b_{v+m-1} &= \frac{2}{m+1}(v+1) \\
b_{v+m} &= \frac{1}{m+1}(v+1)
\end{aligned} \tag{145}$$

Thus for a step input

$$\sum_{k=0}^{\infty} [e(kT)]^2 = (v+1) + \left(\frac{v+1}{m+1}\right)^2 \tag{146}$$

and for a ramp input

$$\sum_{k=0}^{\infty} [e(kT)]^2 = \sum_{q=1}^{v+1} q^2 + \left(\frac{v+1}{m+1}\right)^2 \sum_{j=1}^{m} j^2 \tag{147}$$

Actually the first terms in (146) and (148) above represent the error for the first $v + 1$ samples and cannot be reduced by adding extra terms to the controller.

Thus

$$\sum_{k=v+1}^{\infty} [e(kT)]^2_{step} = \frac{(v+1)^2}{m+1} \tag{148}$$

$$\sum_{k=v+2}^{\infty} [e(kT)]^2_{ramp} = \left(\frac{v+1}{m+1}\right)^2 (1 + 2^2 + 3^2 + \cdots + m^2) \tag{149}$$

represent the sum of the squared-error samples after the controller has begun correcting the system performance for step and ramp input respectively. These sums are plotted vs m in Fig. 18 for $v = 14$.

It is obvious that reducing the step error leads to an increase in the ramp error and that some compromise value of m should be chosen. A plot such as Fig. 18 for a given system will aid in the selection of such a compromise. Figures 19 to 21 show the ramp and step responses of the system of Example 2 with a time constant ratio T_1/T_1 of 0.25, a time delay ratio v of 14, and three different values of m ($m = 1$, 2, and 3, respectively).

VII. MULTIRATE CONTROLLER

It has been suggested that a multirate controller will often yield a faster response with less ripple content than its single-rate counterpart (Ref. 20). A multirate controller is a digital controller whose output sampler operates at a sampling rate n times as fast as the system error sampler.

We shall consider the design of a multirate controller for the simple unity-feedback error-sampled system shown in Fig. 22. The system error is the input to the digital controller $D^*(s)$. The output of the digital controller (sampled control commands) becomes the input to the plant $G(s)$. The plane output is the system output.

The conventional method of analyzing multirate sampled data systems is to replace the multirate samplers with n delays, advances, and samplers of the basic sampling period T (Ref. 7, p. 281–304). This substitution yields the equivalent system shown in Fig. 23. From Fig. 23 it can easily be shown that

$$E(z) = \frac{R(z)}{1 + D(z)G(z) + \sum_{p=1}^{n-1} \mathcal{Z}\{\epsilon^{sTp/n} D(s)\} \mathcal{Z}\{\epsilon^{-sTp/n} G(s)\}} \tag{150}$$

$$C(z) = \frac{R(z) \sum_{p=0}^{n-1} \mathcal{Z}\{\epsilon^{sTp/n} D(s)\} \mathcal{Z}\{\epsilon^{-sTp/n} G(s)\}}{1 + D(z)G(z) + \sum_{p=1}^{n-1} \mathcal{Z}\{\epsilon^{sTp/n} D(s)\} \mathcal{Z}\{\epsilon^{-sTp/n} G(s)\}} \tag{151}$$

Although reasonable equations for analysis, Eq. (150) and (151) are not convenient design equations. Thus, another approach is required. The z_n-transform provides a design approach.

The z-transform is defined by

$$C(z) = \mathfrak{z}\{c(t)\}$$

$$= \sum_{k=0}^{\infty} c(kT)\, z^{-k} \tag{152}$$

Similarly, we shall define the z_n-transform as

$$C(z_n) = \mathfrak{z}_n\, [c(t)]$$

$$= \sum_{k=0}^{\infty} c\left(k\, \frac{T}{n}\right) z^{-k} \tag{153}$$

The z-transform can be considered the z-transform with respect to the basic sampling period T. The z_n-transform is the z-transform with respect to the higher rate sampling period T/n. By conventional definition we also have

$$z = \epsilon^{sT}$$

Similarly

$$z_n = \epsilon^{sT/n}$$

Thus

$$z = (z_n)^n \tag{154}$$

The following three equations can be obtained by inspection from z_n- and z-transform block diagram of Fig. 24.

$$E(z) = R(z) - C(z) \tag{155}$$

$$C(z) = E(z)\, \mathfrak{z}\, \{D(z_n)\, G(z_n)\} \tag{156}$$

$$C(z_n) = E(z)\, D(z_n)\, G(z_n) \tag{157}$$

Thus

$$E(z) = \frac{R(z)}{1 + \mathcal{Z}\,\{D(z_n)\, G(z_n)\}} \tag{158}$$

$$C(z_n) = \frac{R(z)\, D(z_n)\, G(z_n)}{1 + \mathcal{Z}\,\{D(z_n)\, G(z_n)\}} \tag{159}$$

now let

$$K(z_n) = \frac{D(z_n)\, G(z_n)}{1 + \mathcal{Z}\,\{D(z_n)\, G(z_n)\}} \tag{160}$$

Then it can be shown that

$$D(z_n) = \frac{K(z_n)}{G(z_n)\, [1 - K(z)]} \tag{161}$$

where $K(z) = \mathcal{Z}\,[K(z_n)]$ is the z-transform of $K(z_n)$ with respect to the basic sampling period T. Now assume that the system input $R(z)$ is of the form

$$R(z) = \frac{F(z^{-1})}{(1 - z^{-1})^r} \tag{162}$$

Then it can be shown that $K(z_n)$ and $1 - K(z)$ should be of the form

$$K(z_n) = [1 + z_n^{-1} + z_n^{-2} + \cdots + z_n^{-(n-1)}]^r\, B(z_n) \tag{163}$$

$$1 - K(z) = (1 - z^{-1})^r A(z) \tag{164}$$

If $G(z_n)$ has poles $\prod_q (1 - b_q z_n)^k$ on or outside the unit circle in the z_n plane or zeros $\prod_p (1 - a_p z_n^{-1})^i$ on or outside the unit circle in the z_n plane so that $G(z_n)$ is of the form

$$G(z_n) = \frac{\prod_p (1 - a_p z_n^{-1})^i P(z_n) z_n^{-(v+1)}}{\prod_q (1 - b_q z_n^{-1})^k Q(z_n)} \tag{165}$$

then it has been shown that $K(z_n)$ and $1 - K(z)$ should also satisfy the conditions

$$K(z_n) = \prod_p (1 - a_p z_n^{-1})^i M(z_n) \prod_q [1 + b_q z_n^{-1} + \cdots + b_q^{n-1} z_n^{-(n-1)}]^k z_n^{-(v+1)} \tag{166}$$

$$1 - K(z) = \prod_q (1 - b_q^n z_n^{-1})^k N(z) \tag{167}$$

Solutions for $K(z_n)$, $1 - K(z)$, and $D(z_n) G(z_n)$ are given in Table 1 (reproduced from Ref. 21) for step and ramp inputs when $G(z_n)$ is stable with a small pure time delay T_d (T_d $2T + 2T/n$, where T is the high rate sampler period). However, a more straightforward approach is possible.

The following expressions can be obtained from Fig. 24.

$$E(z_n) = R(z_n) - C(z_n) \tag{168}$$

$$C(z_n) = E(z) G(z_n) D(z_n) \tag{169}$$

$$C(z_n) = \frac{R(z) D(z_n) G(z_n)}{1 + \mathcal{Z}\{D(z_n) G(z_n)\}} \tag{170}$$

Now let

$$K(z_n) = \frac{D(z_n)\,G(z_n)}{1 + \mathfrak{z}\,\{D(z_n)\,G(z_n)\}} \tag{171}$$

$$D(z_n)G(z_n) = \frac{K(z_n)}{1 - K(z)} \tag{172}$$

so

$$E(z_n) = R(z_n) - R(z_n^n)\,K(z_n) \tag{173}$$

Now let the plant transfer function $G(s)$ be of the form

$$G(s) = \frac{K_g(1 + T_1 s)(1 + T_2 s)\cdots(1 + T_b s)\,\epsilon^{-vTs}}{s^p(1 + a_1 s)(1 + a_2 s)\cdots(1 + a_b s)} \tag{174}$$

Then, in general

$$G(z) = \frac{K_g\,P(z)\,z^{-m-1}}{Q(z)} \tag{175}$$

$$G(z_n) = \frac{K_g\,P(z_n)\,z_n^{-m_1-1}}{Q(z_n)} \tag{176}$$

where

$$\begin{aligned} &(m-1)\,T < vT \le mT \\ &(m_1-1)\,\frac{T}{n} < vT \le m_1\frac{T}{n} \end{aligned} \tag{177}$$

Again, for physical realizability of $D(z_n)$ and to improve the system stability:

1. $K(z_n)$ should contain $z_n^{-m_1 - 1}$ as a factor.

2. $K(z_n)$ should contain in its zeros the zeros of $G(z_n)$ that lie on or outside the unit circle of the z_n plane.

3. $1 - K(z)$ should contain in its zeros all the poles of $G(z_n)$ that lie on or outside the unit circle.

Now consider inputs of the form $R(s) = 1/s^r$ so that

$$R(z) = \lim_{a \to 0} \frac{(-1)^{r-1}}{(r-1)!} \frac{\partial^{r-1}}{\partial a^{r-1}} \left\{ \frac{z}{z - \epsilon^{-aT}} \right\}$$

$$= \frac{A(z)}{(z-1)^r} \frac{T^{r-1}}{(r-1)!}$$

$$= \frac{B_r(z)}{(1-z)^r} \tag{178}$$

$$R(z_n) = \frac{B_{r1}(z_n)}{(1 - z_n^{-1})^r} \tag{179}$$

$$R(z_n^n) = \frac{B_{r2}(z_n)}{(1 - z_n^{-n})^r} \tag{180}$$

Thus the system error is given by

$$E(z_n) = \frac{B_{r1}(z_n)}{(1 - z_n^{-1})^r} - \frac{B_{r2}(z_n)}{(1 - z_n^{-n})^r} K(z_n) \tag{181}$$

Let $K(z_n)$ be of the form

$$K(z_n) = \frac{(1 - z_n^{-n})^r}{(1 - z_n^{-1})^r} (a_0 + a_1 z_n^{-1} + \cdots) Z_0(z_n) z_n^{-m-1} \tag{182}$$

where $Z_0(z_n)$ contains the zeros of $G(z_n)$ that lie on or outside the unit circle of the z_n plane.

Thus

$$E(z_n) = \frac{B_{r1}(z_n) - B_{r2}(z_n) Z_0(z_n) z^{-m_1 -1} (a_0 + a_1 z_n^{-1} + \cdots)}{(1 - z_n^{-1})^r} \tag{183}$$

$$= b_0 + b_1 z_n^{-1} + b_2 z_n^{-2} + b_3 z_n^{-3} + \cdots + b_q z_n^{-q} \tag{184}$$

It is possible to choose the a_k's and b_q's in (183) and (184) so that the minimum settling time is achieved (q attains the smallest possible value) or the sum of the error-squared samples is minimized. Both of these criteria are illustrated in the following examples.Designs for both are illustrated for a ramp input in the following examples.

Example 5. Multirate Minimum Settling Time Digital Controller

Consider the multirate, error-sampled, unity feedback system shown in Fig. 24 in which the controller sampler operates twice as fast as the error sampler ($n = 2$). The plant transfer function is given by

$$G(z) = \frac{K_g (1 - \epsilon^{-T/T_1}) z^{-v-1}}{1 - \epsilon^{-T/T_1} z^{-1}} \tag{185}$$

and

$$G(z_n) = \frac{K_g (1 - \epsilon^{-T/nT_1}) z_n^{-2v+1}}{1 - \epsilon^{-T/nT_1} z_n^{-1}} \tag{186}$$

Assume a ramp input so that

$$R(z) = \frac{T z^{-1}}{(1 - z^{-1})^2} \tag{187}$$

$$R(z_n) = \frac{\frac{T}{n} z_n^{-1}}{(1 - z_n^{-1})^2} \tag{188}$$

$$R(z_n^n) = \frac{T z_n^{-2}}{(1 - z_n^{-2})^2} \tag{189}$$

Substitution of (188) and (189) into (183) yields

$$E(z_n) = \frac{T}{n} z_n^{-1} - T z_n^{-2} z_n^{-2v-1} (a_0 + a_1 z_n^{-1} + a_2 z_n^{-2} + \cdots)$$

$$= T \left[\frac{z_n^{-1}}{2} - a_0 z_n^{-2v-3} - a_1 z_n^{-2v-4} - a_2 z_n^{-2v-5} - \cdots \right]$$

$$= T (1 - z_n^{-1})^2 [b_0 + b_1 z_n^{-1} + b_2 z_n^{-2} + \cdots + b_2 z_n^{-q}] \tag{190}$$

$$E(z_n) = T\,\{b_0 + (b_1 - 2b_0)\,z_n^{-1} + (b_2 - 2b_1 + b_0)\,z_n^{-2}$$
$$+ (b_3 - 2b_2 + b_1)\,z_n^{-3} + \cdots + (b_q - 2b_{q-1} + b_{q-2})\,z_n^{-q}$$
$$+ (-2b_q + b_{q-1})\,z_n^{-(q+1)} + b_q\,z_n^{-(q+2)}\} \tag{191}$$

Equating like coefficients yields

$$\begin{aligned}
b_0 &= 0 \\
b_1 - 2b_0 &= \frac{1}{2} \\
b_2 - 2b_1 + b_0 &= 0 \\
b_3 - 2b_2 + b_1 &= 0 \\
&\vdots \\
b_{2v+2} - 2b_{2v+1} + b_{2v} &= 0 \\
b_{2v+3} - 2b_{v+2} + b_{2v+1} &= -a_0 \\
b_{2v+4} - 2b_{v+3} + b_{2v+2} &= -a_1 \\
&\vdots \\
b_q - 2b_{q-1} + b_{q-2} &= -a_{q-2v-3} \\
-2b_q + b_{q-1} &= -a_{q-2v-2} \\
b_q &= -a_{q-2v-1}
\end{aligned} \tag{192}$$

Solving for the b_q's yields

$$
\begin{aligned}
b_0 &= 0 \\
b_1 &= \frac{1}{2} \\
b_2 &= 1 \\
b_3 &= \frac{3}{2} \\
b_4 &= 2 \\
&\vdots \\
b_{2v+2} &= v + 1 \\
b_{2v+3} &= -a_0 + v + \frac{3}{2} \\
b_{2v+4} &= -a_1 - 2a_0 + v + 2 \\
b_{2v+5} &= -a_2 - 2a_1 - 3a_0 + v \frac{5}{2} \\
&\vdots
\end{aligned}
\tag{193}
$$

As in the case of the minimum settling time single-rate controller, minimum settling time is achieved in the above design by letting $b_q = 0$ for $q \geq 2v + 3$. Thus

$$
\begin{aligned}
a_0 &= v + \frac{3}{2} \\
a_1 &= -(v + 1) \\
a_m &= 0 \qquad m \geq 2
\end{aligned}
\tag{194}
$$

Thus

$$K(z_n) = (1 + z_n^{-1})^2 \left[\left(v + \frac{3}{2} \right) - (v + 1)\, z_n^{-1} \right] z_n^{-2v-1} \tag{195}$$

It is shown in Appendix A that

$$K(z) = (2v + 3)\, z^{-v} - (v + 1)\, z^{-(v+1)} - (v + 1)\, z^{-(v+2)} \tag{196}$$

Now

$$D(z_n) = \frac{K(z_n)}{G(z_n)\ [1 - K(z)} \tag{197}$$

$$D(z_n) = \frac{(1 - \epsilon^{-T/nT_1}\, z_n^{-1})\,(1 - z_n^{-1})^2 \left[\left(v + \frac{3}{2} \right) - (v + 1)\, z_n^{-1} \right]}{K_g\ (1 - \epsilon^{-T/nT_1})\ [1 - (2v + 3)\, z_n^{-2v} + (v + 1)\, z_n^{-2v-2}\ (v + 1)\, z_n^{-2v-4}]} \tag{198}$$

The system output can be obtained from

$$C(z_n) = R(z_n^n)\, K(z_n) \tag{199}$$

For a ramp input

$$R(z_n^n) = \frac{T\, z_n^{-2}}{(1 - z_n^{-2})^2} \tag{200}$$

and

$$C(z_n) = T \left[15.5\, z_n^{-31} + 16\, z_n^{-32} + 16.5\, z_n^{-33} + \cdots\ \frac{N}{2}\ z_n^{-N} + \cdots \right] \tag{201}$$

For a step input

$$R(z_n^n) = \frac{1}{1 - z_n^{-2}} \tag{202}$$

and

$$C(z_n) = 15.5\, z_n^{-29} + 16\, z_n^{-30} + z_n^{-31} + z_n^{-32} + \cdots \tag{203}$$

The ramp and step response of the above system is plotted in Fig. 25 for the following parameters; $v = 14$, $T_1/T_1 = 0.25$, and $T = 20$ sec. It should be noted that the response is slightly faster than the response of the single-rate system with the same basic sampling period and parameters (Fig. 13). Deadbeat response is reached in 310 sec vs 320 sec for the single-rate system. The ripples have also been "reduced" in the multirate system output. An examination of the multirate controller transfer function (198) reveals that the multirate controller usually will be much more complicated than the single-rate controller, with a small gain in overall system performance. As in the single-rate case, the minimum settling time design for a ramp input yields very poor step response. The addition of extra terms to the digital controller will reduce the step overshoot when properly chosen. The design for minimum sum of the squared-error samples, which is discussed next, will provide a method of properly choosing the extra terms.

Example 6. Minimization of the Sum of the Squared-Error Samples for a Multirate System

Minimization of the sum of the squared-error samples of the error pulse sequence for a multirate controlled system can be obtained in the same manner as in the case of a single-rate system. This would require the evaluation of

$$\sum_{k=0}^{\infty} [e(kT)]^2 = \frac{1}{2\pi j} \oint_{\Gamma} E(z_n)\, E(z_n^{-1})\, z_n^{-1}\, dz_n \tag{204}$$

which becomes very tedious and conducive to calculation error. Equation (204) is especially convenient when the error pulse sequence is infinite or extremely long. However, with deadbeat response, the error sequence is finite. Since $e(kT)$ can be obtained from a finite expansion of $E(z_n)$, it may be desirable to find the individual $e(kT)$'s and form the desired sum from the individual terms. If

$$E(z_n) = b_0 + b_1 z_n^{-1} + b_2 z_n^{-2} + b_3 z_n^{-3} + \cdots + b_q z_n^{-q} \tag{205}$$

where the b_q's are a function of the "extra" terms in $K(z_n)$. Then

$$\sum_{k=0}^{\infty} [e(kT)]^2 = \sum_{i=0}^{q} [b_i]^2 \tag{206}$$

Conventional calculus techniques can be used to determine the b_q's so that the sum is minimized.

This approach will now be applied to the system of Example 3. The digital controller sampler operates twice as fast as the error sampler ($n = 2$). The plant transfer function $G(z_n)$ is stable with a z_n^{-2v-1} factor.

It has been shown that

$$E(z_n) = R(z_n) - R(z_n^n)\, K(z_n) \tag{207}$$

For step inputs

$$R(z_n) = \frac{1}{1 - z_n^{-1}}$$

$$R(z_n^n) = \frac{1}{1 - z_n^{-2}} \tag{208}$$

It has been shown that $K(z_n)$ should be of the form

$$K(z_n) = \frac{(1 - z_n^{-2})^2}{(1 - z_n^{-1})^2} (a_0 + a_1 z_n^{-1} + a_2 z_n^{-2} + \cdots + a_n z^{-n})\, z_n^{-(2v+1)} \tag{209}$$

to obtain deadbeat response for a ramp input.

Thus the step error is given by

$$E(z_n) = \frac{1}{(1 - z_n^{-1})} - \frac{(1 - z_n^{-2})\, z_n^{-(2v+1)}}{(1 - z_n^{-1})^2} \left[a_0 + a_1 z_n^{-1} + \cdots + a_m z_n^{-m}\right]$$

$$= 1 + z_n^{-1} + \cdots + z_n^{-2v} + (1 - a_0)\, z_n^{-(2v+1)}$$

$$+ (1 - 2a_0 - a_1)\, z_n^{-(2v+2)} + (1 - 2a_0 - 2a_1 - a_2)\, z_n^{-(2v+3)}$$

$$+ \cdots + (1 - 2a_0 - 2a_1 - \cdots - 2a_{m-1} - a_m)\, z_n^{-(2v+m+1)}$$

$$+ (1 - 2a_0 - 2a_1 - \cdots - 2a_m)\, z_n^{-(2v+m+2)}$$

$$+ (1 - 2a_0 - 2a_1 - \cdots - 2a_m)\, z_n^{-(2v+m+3)} + \cdots \tag{210}$$

From Eq. (210) it appears that the error pulse sequence is an infinite sequence. However, the following substitutions will yield a finite sequence.

Since the a_k's must satisfy the deadbeat ramp response requirements, Eq. (193) must still be valid. Thus

$$\begin{aligned}
a_0 &= -d_0 + v + \frac{3}{2} \\
a_1 &= -d_1 + 2d_0 - (v + 1) \\
a_2 &= -d_2 + 2d_1 - d_0 \\
a_3 &= -d_3 + 2d_2 - d_1 \\
&\ \ \vdots \\
a_{m-2} &= -d_{m-2} + 2d_{m-3} - d_{m-4} \\
a_{m-3} &= +2d_{m-2} - d_{m-3} \\
a_m &= -d_{m-2}
\end{aligned} \tag{211}$$

Substituting from Eq. (211) into (210) yields

$$E(z_n) = 1 + z_n^{-1} + z_n^{-2} + z_n^{-3} + \cdots + z_n^{-2v}$$

$$+ \left(d_0 - v - \frac{1}{2}\right) z_n^{-(2v+1)} + (d_1 - v - 1)\, z_n^{-(2v+2)} + (d_2 - d_0)\, z_n^{-(2v+3)}$$

$$+ (d_3 - d_1)\, z_n^{-(2v+4)} + \cdots + (d_{m-4} - d_{m-2})\, z_n^{-(2v+m-1)}$$

$$+ (d_{m-3})\, z_n^{-(2v+m)} + (d_{m-2})\, z_n^{-(2v+m-1)} \qquad (212)$$

Expression (212) is the finite series representation of the step error needed to calculate the sum of the squared-error samples of the error pulse sequence. Substituting from (212) into (206) to evaluate the step error sum yields

$$\sum_{k=0}^{\infty} [e(kT)]^2 = (2v+1) + \left(d_0 - v - \frac{1}{2}\right)^2 + (d_1 - v - 1)^2 + (d_2 - d_0)^2$$

$$+ (d_3 - d_1)^2 + (d_4 - d_2)^2 + \cdots +$$

$$+ (d_{m-4} - d_{m-2})^2 + (d_{m-3})^2 + (d_{m-2})^2 \qquad (213)$$

The d_k's can be chosen to minimize the above sum by applying the conventional calculus techniques

$$\frac{\partial}{\partial d_b}\left\{\sum_{k=0}^{\infty} [e(kT)]^2\right\} = 0 \qquad b = 0, 1, 2, \cdots, m$$

to yield the following set of simultaneous equations

$$
\begin{aligned}
2d_0 - d_2 &= v + \frac{1}{2} \\
2d_1 - d_3 &= v + 1 \\
-d_0 + 2d_2 - d_4 &= 0 \\
-d_1 + 2d_3 - d_5 &= 0 \\
&\vdots \\
-d_{m-6} + 2d_{m-4} - d_{m-2} &= 0 \\
-d_{m-5} + 2d_{m-3} &= 0 \\
-d_{m-4} - 2d_{m-2} &= 0
\end{aligned}
\tag{214}
$$

A general solution for the above set of equations is possible. (Actually, two solutions exist – one being valid for m odd and the other for m even.) However, such expressions are cumbersome, and straightforward solution of the equations is trivial for reasonable values of m, and it is unlikely that very many terms will be added to the digital controller.

Now consider the specific system of Example 3. The important parameters of this system are $v = 14$, $T_1/T = 0.25$, and $T = 280$ sec. The minimum settling time controller was derived in which

$$
\begin{aligned}
m &= 2 \\
a_0 &= 15.5 \\
a_1 &= -15
\end{aligned}
\tag{215}
$$

and the step and ramp responses of the minimum settling time system are given by

$$
C(z_n)_{step} = 15.5\, z_n^{-29} + 16\, z_n^{-30} + z_n^{-31} + z_n^{-32} + z_n^{-33} + \cdots \tag{216}
$$

$$C(z_n)_{ramp} = T\left(15.5\, z_n^{-31} + 16\, z_n^{-32} + 16.5\, z_n^{-33} + 17\, z_n^{-34} + \cdots + \frac{k}{2}\, z_n^{-k} + \cdots\right) \qquad (217)$$

If one term is added to the digital controller,

$$m = 2$$

and minimization of the sum of the squared-error samples of the step response error pulse sequence yields

$$d_0 = 7.25$$

$$a_0 = 8.25$$

$$a_1 = -0.5$$

$$a_2 = -7.25$$

and the step and ramp responses are given by

$$C(z_n)_{step} = 8.25\, z_n^{-29} + 16\, z_n^{-30} + 8.25\, z_n^{-31} + z_n^{-32} + z_n^{-33} + z_n^{-34} + \cdots$$

$$C(z_n)_{ramp} = T\left(8.25\, z_n^{-31} + 16\, z_n^{-32} + 16.5\, z_n^{-33} + \cdots + \frac{k}{2}\, z_n^{-k} + \cdots\right)$$

The numerical value of the sum of the squared-error samples after the digital controller initiates control for step and ramp inputs is given by

$$\sum_{k=29}^{\infty} \left[e\left(k\, \frac{T}{2}\right)\right]^2_{step} = 330.125$$

$$\sum_{k-31}^{\infty} \left[e\left(k\frac{T}{2}\right)\right]^2_{ramp} = 52.5625$$

Similarly for other additional terms we have

$$m = 3$$

$$d_0 = 7.25$$

$$d_1 = 7.5$$

$$a_0 = 8.25$$

$$a_1 = -8.00$$

$$a_2 = 7.75$$

$$a_3 = -7.5$$

$$C(z_n)_{step} = 8.25\, z_n^{-29} + 8.5\, z_n^{-30} + 8.25\, z_n^{-31} + 8.5\, z_n^{-32} + z_n^{-33} + z_n^{-34} + \cdots$$

$$C(z_n)_{ramp} = T\left(8.25\, z_n^{-31} + 8.5\, z_n^{-32} + 16.5\, z_n^{-33} + 17\, z_n^{-34} + \cdots + \frac{k}{2}\, z_n^{-k} + \cdots\right)$$

$$\sum_{k=29}^{\infty} [e(kT)]^2_{step} = 217.625$$

$$\sum_{k=31}^{\infty} [e(kT)]^2_{ramp} = 108.8125$$

As in the case of the single-rate controller, a compromise between step and ramp error must be decided upon. A plot of the sum of the squared-error samples for step and ramp inputs vs the number of extra terms in the digital controller will aid this choice. Figure 26 shows such a plot for the system under consideration. The step and ramp responses for the multirate controlled system above are plotted in Fig. 27 to 30 for several different "number of extra terms in the digital controller."

VIII. DESIGN FOR STATISTICAL INPUTS

It is possible to design an "optimum digital controller" on statistical principles when the statistical properties of the inputs, including noise, are known. The conventional design for continuous systems using the Wiener–Kolmogoroff theory for optimum filter synthesis has been extended to sampled-data systems (Ref. 22). Minimization of the mean-square sampled error yields the following equations for the digital controller transfer function $D(z)$ as derived in Appendix B:

$$D(z) = \frac{W(z)}{1 - W(z)\,G(z)} \tag{218}$$

where

$$W(z) = \frac{\left\{ \dfrac{G(z^{-1})\,G_d(z)\,[\phi_{ss}(z) + \phi_{ns}(z)]}{[G(z)\,G(z^{-1})]^{-}\,[\phi_t(z)]^{-}} \right\}_{+}}{[G(z)\,G(z^{-1})]^{+}\,[\phi_t(z)]^{+}} \tag{219}$$

and

$D(z)$ is the digital controller transfer function

$G(z)$ is the plant transfer function

$G(z^{-1}) = G(z)\big|_{z=z^{-1}}$

$G_d(z)$ is the desired closed-loop transfer function

$\phi_{ss}(z)$ is the pulse-auto-spectral density of the input signal

$\phi_{sn}(z)$ is the pulse-cross-spectral density of the signal to noise

$\phi_{ns}(z)$ is the pulse-cross-spectral density of the noise to signal

$\phi_{nn}(z)$ is the pulse-auto-spectral density of the noise

$\phi_t(z) = \phi_{ss}(z) + \phi_{sn}(z) + \phi_{ns}(z) + \phi_{nn}(z)$

$[F(z)]^{+}$, $\{F(z)\}_{+}$ denote the factors of $F(z)$ with poles and zeros inside the unit circle of the z-plane

$[F(z)]^{-}$ denote the factors of $F(z)$ with poles and zeros on or outside the unit circle of the z-plane

When the plant transfer function $G(z)$ contains an z^{-n} factor (a pure time lag in most cases when $n > 1$), the physical realizability of the digital controller can be preserved by including the z^{-n} factor in the desired closed-loop transfer function $G_d(z)$.

If the plant transfer function is stable, Eq. (219) reduces to

$$W(z) = \frac{\left\{\dfrac{G_d(z)\,[\phi_{ss}(z) + \phi_{ns}(z)]}{[\phi_t(z)]^{-}}\right\}_{+}}{G(z)\,[\phi_t(z)]^{+}} \tag{220}$$

where the quantities are defined as for Eq. (219).

It should be noted that since the desired closed-loop transfer function must also contain the pure time delay and the actual closed-loop transfer function will usually fall short of the desired, the resulting controller will not be a predictor controller, and the output will usually lag the input by at least the pure time delay. This may or may not be acceptable. When the system is subjected to noisy inputs, the above design procedure may be necessary to achieve any degree of control in the presence of the noise.

IX. SUBRATE CONTROLLER

A variation of the multirate controller will be called the subrate controller. The subrate controller output sampler operates at a slower rate than the system error sampler. Thus the controller has more information about the system's operating conditions before it sends a control command to the plant to be controlled. In some cases, this method should yield a better "predictor" controller than the multirate or single-rate controller. However, the overall system response may be slightly slower than a multirate controlled system response, since the control command rate may be slower with the subrate controller.

An error-sampled subrate controlled digital system is shown in Fig. 31. The sampling period T of the controller sampler S_2 shall be considered the basic sampling period for the system. The error sampler S_1 has a sampling period of T/n, where n is assumed to be an integer to simplify the analysis and design procedures. The conventional approach to the analysis of such a system is to consider the equivalent single-rate system shown in Fig. 32.

The error sampler is replaced by n fictitious samplers with a sampling period T arranged in parallel. The inputs to the samplers are advanced in time, and the outputs of the fictitious samplers are delayed an equal amount of time. Immediately it is seen that the analysis is complicated by the fact that the system is no longer an error-sampled system. In such a case, it is usually possible to obtain only the z-transform of the system output rather than an overall system (closed-loop) transfer function.

It can be shown that the z-transform of the output of the subrate controller is given by

$$C(z) = \frac{G(z) \sum_{k=0}^{n-1} \mathcal{Z}\{\epsilon^{kTs/n} R(s)\} \mathcal{Z}\{\epsilon^{-kTs/n} D^*(s)\}}{1 + \sum_{k=0}^{n-1} \mathcal{Z}\{\epsilon^{kTs/n} G(s)\} \mathcal{Z}\{\epsilon^{-kTs/n} D^*(s)\}} \tag{221}$$

where

$\mathcal{Z}\{\epsilon^{-kTs/n} F(s)\}$ denotes the z-transform of the delayed function $\epsilon^{-kTs/n} F(s)$

Expression (221) is reasonably convenient for analysis of subrate systems for given system parameters and a specific input. However, it is not a convenient tool for the design of a digital controller since the terms $D(z)$ or $D^*(s)$, describing the digital controller, do not appear explicitly. This problem is "solved" in the design of the multirate controller by the introduction of the z_n-transform denoting the z-transformation with respect to the high rate-sampling period T/n. This approach will be applied to the subrate controller after a brief review of the z_n- and z-transform properties.

The z-transform $\mathfrak{z}[X(s)]$, and the z_n-transform $\mathfrak{z}_n\ [X(s)]$ are defined by the following expressions

$$\mathfrak{z}[X(s)] = \sum_{r=0}^{\infty} X(kT)\ z^{-k}$$

$$\mathfrak{z}_n[X(s)] = \sum_{r=0}^{\infty} \frac{X(kT)}{n}\ z_n^{-k} \tag{222}$$

There are three combinations of the above transform to be considered for the development of the subrate controller.

First consider the case shown in Fig. 33a, where the continuous input is first sampled by a high-rate sampler with sampling period T/n, passed through a pulse network with a transfer function $D(z_n)$ $(D(z_n) = \mathfrak{z}_n[D^*(s)])$, and then sampled by the basic rate sampler with sampling period T. The output of a pulse network is a sequence of "sampled" pulses appearing at the same rate as the input samples. If $X(z_n)$ denotes the output of the pulse network $D(z_n)$, then the output $x(z)$ of the basic rate sampler is given by

$$X(z) = \mathfrak{z}[X(z_n)] \tag{223}$$

where $\mathfrak{z}[X(z_n)]$ denotes the operation of z-transforming with respect to the basic sampling period T, a pulse function $X(z_n)$ for which the z_n-transform with respect to the high-rate sampler is given. This amounts to taking every nth pulse of the original pulse function $X(z_n)$. It has been shown that

$$\mathcal{Z}[F(z_n)] = \frac{1}{n} \sum_{k=1}^{n} F(z_n \epsilon^{-j2\pi k/n}) \tag{224}$$

Now consider the case shown in Fig. 33b, where the input is sampled at the basic rate T and passed through a pulse network $D(z)$. The pulse output $X(z)$ of the network is sampled by a high-rate sampler with sampling period T/n to produce the output $X(z_n)$. This output is given by

$$X(z_n) = \mathcal{Z}_n[X(z)] \tag{225}$$

where $\mathcal{Z}_n[X(z)]$ denotes the operation of z_n-transforming with respect to the high rate-sampling period T/n, a pulse function $X(z_n)$ for which the z-transform with respect to the basic sampler is given. However, since $X(z)$ is a pulse function (sequence) with nonzero values only at $t = T, 2T, 3T, \cdots$, only every nth pulse can appear in $X(z_n)$. Thus

$$X(z_n) = X(z)\big|_{z=z_n^n} \tag{226}$$

However, if the pulse network $D(z) = \mathcal{Z}[D^*(s)]$ is replaced by a continuous network $D(z) = \mathcal{Z}[D(s)]$ as shown in Fig. 33c, then the expression no longer holds. In this case the output shall be denoted by

$$X(z_n) = \mathcal{Z}_n[X(z,m)] \tag{227}$$

since the modified z-transform $X(z,m)$ is necessary to describe the output of the continuous network at the high rate-sampling instants. By definition, the modified z-transform is given by

$$X(z,m) = \sum_{k=0}^{\infty} X(kT + mT)\, z^{-(k+1)} \tag{228}$$

where

$$0 < m \leq 1$$

It can be shown that the output of the high rate sampler is given by

$$X(z_n) = \mathfrak{Z}_n\,[X(z, m)]$$

$$= X(z_n^n) + \sum_{k=1}^{n-1} z_n^k\, X\left(z_n^n, \frac{k}{n}\right) \tag{229}$$

where

$$X\left(z_n^n, \frac{k}{n}\right) = X(z, m)\bigg|_{z = z_n^n,\; m = k/n}$$

when the input to the high rate sampler is continuous.

The error-sampled subrate-controlled system is shown in z_n- and z-transform notation in Fig. 34. The digital controller transfer function $D(z_n)$ is in z_n-transform notation since its input is sampled at the higher sampling rate (n/T samples per sec). The plant transfer function $G(z)$ is in z-transform notation, indicating that its input is sampled at the basic rate ($1/T$ samples per sec).

The error equation is given by

$$E(z_n) = R(z_n) - C(z_n) \tag{230}$$

Since $C(z_n)$ is the output of a high rate sampler preceded by a continuous network preceded by a basic rate sampler,

$$C(z_n) = C(z_n^n) + \sum_{k=1}^{n-1} z_n^k\, C\left(z_n^n, \frac{k}{n}\right) \tag{231}$$

by direct application of Eq. (229).

Now

$$C(z) = \quad [D(z_n)\, E(z_n)] \cdot G(z)$$

$$C(z, m) = \mathcal{Z}[D(z_n)\, E(z_n)] \cdot G(z, m) \tag{232}$$

Thus

$$C(z_n) = \mathcal{Z}[D(z_n)\, E(z_n)]\, G(z_n^n) + \mathcal{Z}[D(z_n)\, E(z_n^n)] \sum_{k=0}^{n-1} z^k\, G\left(z_n^n, \frac{k}{n}\right) \tag{233}$$

or

$$C(z_n) = \mathcal{Z}[D(z_n)\, E(z_n)] \cdot \left[G(z_n^n) + \sum_{k=0}^{n-1} z^k\, G\left(z_n^n, \frac{k}{n}\right)\right] \tag{234}$$

Now let

$$\overline{G(z_n)} = G(z_n^n) + \sum_{k=0}^{n-1} z^k\, G\left(z_n^n, \frac{k}{n}\right) \tag{235}$$

so

$$C(z_n) = \mathcal{Z}[D(z_n)\, E(z_n)]\, \overline{G(z_n)} \tag{236}$$

Thus

$$E(z_n) = R(z_n) - \mathcal{Z}[D(z_n)\, E(z_n)]\, \overline{G(z_n)} \tag{237}$$

In order to extend the previous design methods for deadbeat-response with a subrate controller, it is necessary to develop an expression for $E(z_n)$. Equation (237) is of a form similar to Eq. (160) in the development of the multirate controller. The multirate controller equation was solved with the aid of Eq. (224) and matrix manipulations. This approach was applied to the subrate controller Eq. (237). Unfortunately, the result could not be reduced to a convenient form for design.

This phase of the research on digital control of systems with long time delays was concluded at this point. It is hoped that later research will yield a satisfactory design procedure for the development of the subrate controller. It is felt that this type of controller will yield the best type of control when the effect of the time delay must be offset by a predictor controller.

X. INSTRUMENTATION OF THE DIGITAL CONTROLLER

The single-rate digital controller transfer function $D(z)$ can easily be instrumented with a simple digital computer program. The computer must store past values of the controller (computer) input and output. Of course, the input and output are discrete values (samples).

Considering the single-rate controller, let $E(z)$ denote the z-transform of the input pulse sequence and $M(z)$ denote the z-transform of the output pulse sequence. Thus

$$D(z) = \frac{M(z)}{E(z)} \tag{238}$$

It has been shown that the digital controller transfer function is of the form

$$D(z) = \frac{\sum_{j=0}^{a} d_j z^{-j}}{\sum_{j=0}^{b} D_j z^{-j}} \tag{239}$$

Substituting from (239) into (238) and cross-multiplying yields

$$M(z) + \frac{1}{d_0} \sum_{j=1}^{b} D_j M(z) z^{-j} = \frac{1}{d_0} \sum_{j=0}^{a} d_j E(z) z^{-j} \tag{240}$$

Taking the inverse z-transform of both sides of (240) yields (at the sampling instants)

$$m(t) = \frac{1}{d_0} \sum_{j=0}^{a} d_j e(t - jT) - \frac{1}{d_0} \sum_{i=1}^{b} D_i m(t - iT) \tag{241}$$

or

$$m(kT) = \frac{1}{d_0} \sum_{j=0}^{a} d_j e(kT - jT) - \frac{1}{d_0} \sum_{i=1}^{b} D_i m(kT - iT) \tag{242}$$

Thus the output of the computer at the kth sampling instant is just a linear combination of the present and past values of the input and past values of the output. This is indeed an elementary digital computer program that can be performed by the simplest of digital computers with sufficient memory, addition, and multiplication capacity.

Example 7. Digital Computer Program for a Single-Rate Deadbeat Controller

Consider the digital controller required to yield a minimum settling time deadbeat response for a ramp input as derived in Example 2. In this case

$$D(z) = \frac{\left(1 - \epsilon^{-T/T_1} z^{-1}\right)\left(16 - 15 z^{-1}\right)}{K\left(1 - \epsilon^{-T/T_1}\right)\left(1 - 16 z^{-15} + 15 z^{-16}\right)} \tag{243}$$

Thus

$$M(z) = \frac{16\left[1 - \left(\dfrac{15}{16} + \epsilon^{-T/T_1}\right) z^{-1} + \dfrac{15}{16}\,\epsilon^{-T/T_1} z^{-2}\right]}{K\left(1 - \epsilon^{-T/T_1}\right)\left[1 - 16 z^{-15} + 15 z^{-16}\right]}$$

and

$$m(kT) = \frac{16}{K\left(1 - \epsilon^{-T/T_1}\right)}\left[e(kT) - \left(\frac{15}{16} + \epsilon^{-T/T_1}\right) e(kT - T) + \frac{15}{16}\,\epsilon^{-T/T_1} e(kT - 2T)\right]$$

$$+\ 16\, m(kT - 15T) - 15\, m(kT - 16T) \tag{244}$$

Equation (244) describes the digital computer program necessary to provide minimum settling time deadbeat response for the single-rate system of Example 2.

Now consider the multirate controller problem. It will be shown that the multirate controller usually cannot be instrumented by a simple digital computer program in the same manner as the single-rate controller. The multirate controller has the general transfer function $D(z_n)$ given by

$$D(z_n) = \frac{\sum_{i=0}^{a} D_i \, z_n^{-i}}{\sum_{i=0}^{b} d_j \, z_n^{-j}}$$

$$= \frac{M(z_n)}{E(z_n)} \tag{245}$$

Thus

$$d_0 M(z_n) + \sum_{j=1}^{a} d_j M(z_n) z_n^{-j} = \sum_{i=0}^{b} D_i \, E(z_n) \, z_n^{-i}$$

$$m \quad k\frac{T}{n} = \frac{1}{d_0} \sum_{i=0}^{b} D_i \, e\left(k\frac{T}{n} - i\frac{T}{n}\right) - \frac{1}{d_0} \sum_{j=1}^{a} d_j \, m\left(k\frac{T}{n} - j\frac{T}{n}\right) \tag{246}$$

However, $e(k^{T/n} - i^{T/n})$ is available to the computer only when $i = n + d$, where d is an integer depending on relationship between k and the sampling instant for the low-rate error sampler. This condition exists since the error sampler passes only every nth sample with respect to the high-rate sampling period T/n.

This indicates that a new form for $K(z_n)$ is needed if a simple digital computer is to be used to instrument the multirate controller. Possibly, the controller can be instrumented in its present form by a continuous network and the sampler by converting $D(z_n)$ to $D(s)$ in the same manner that $K(s)$ was obtained from $K(z_n)$ to obtain $K(z)$ in the multirate controller development. Table 1 may be useful in converting $D(z_n)$.

Table 1. $Z[F(z_n)]$

$F(z_n)$	$Z[F(z_n)]$
$G(z_n)$	$Z[G(z_n)]$
$kG(z_n)$	$KZ[G(z_n)]$
$z_n^{-p} \quad p \neq kn$	0
$z_n^{-p} \quad p = kn$	z^{-k}
$\dfrac{1}{z_n - a}$	$\dfrac{a^{n-1}}{z - a^n}$
$\dfrac{1}{z_n - a^2}$	$\dfrac{a^{2n-2} + (n-1)\,a^{n-2}\,z}{z_n - a^2}$
$\dfrac{1}{z_n - a^3}$	$\dfrac{2\,a^{3n-3} + (n+4)\,(n-1)\,a^{2n-3}\,z + (n-1)\,(n-2)\,a^{n-3}\,z}{2!\,(z - a^n)^3}$
$\dfrac{z_n}{z_n^2 + 2az_n + a^2 + b^2}$	$\dfrac{(a^2+b^2)^{n/2}\sin(n\theta)\,z}{b\,[z^2 - 2(a^2+b^2)^{n/2}\cos(n\theta)\,z + (a^2+b^2)^n]}$ where $\theta = \tan^{-1}\dfrac{b}{-a}$
$\dfrac{1}{z_n^2 + 2az_n + a^2 + b^2}$	$\dfrac{b\,(a^2+b^2)^{n-1} + (a^2+b^2)^{(n-1)/2}\sin[(n-1)\,\phi]\,z}{b\,[z^2 - 2(a^2+b^2)^{n/2}\cos(n\,\phi)z + (a^2+b^2)^n]}$ where $\phi = \tan^{-1}\dfrac{b}{a}$

Note: This table has been reproduced from Ref. 21.

NOMENCLATURE

ω	frequency
ω_e	error frequency
ω_o	output frequency
ω_i	input frequency
ω_c	center frequency
ω_1, ω_2	phase detector frequencies
ω_v	mixer frequency
$\Delta\omega_b$	filter half bandwidth
θ	phase
θ_e	error phase
θ_o	output phase
θ_i	input phase
θ_1, θ_2	phase detector input phases
ϕ, ϕ_v, ϕ_q, ϕ_c, ϕ_g, ψ_1	phase shifts
$\phi_n, \phi_{lj}, \psi_{lj}$	phase shifts in Fourier series representations of nonlinear element outputs
ϕ_1, ϕ_2	phase detector input phases
t	time
$R(t)$	system input time function
$c(t)$	system output time function

NOMENCLATURE (Cont'd)

$v(t)$	time variable	
$Q(t)$	time variable	
$I(t)$	total input to nonlinear element	
$m(t)$	nonlinear element output	
$h(t)$	amplifier output	
$K(t)$	high-gain limiter output	
$g(t)$	narrow-band filter output	
$e(t)$	system error time function	
$R_1(t)$, $R_2(t)$	multiplier inputs	
$E_m(t)$	multiplier output	
$e_o(t)$	filter output	
$e_1(t)$, $e_2(t)$	phase detector inputs	
$u(t)$	unit step	
$r(t)$	system input	
R	signal amplitudes	system input
C	signal amplitudes	system output
$V(C)$	signal amplitudes	system internal signal
Q	signal amplitudes	system internal signal
M^*_{lj}	signal amplitudes	nonlinear element output
M_{lj}	signal amplitudes	nonlinear element output
H	signal amplitudes	amplifier output
K_n	signal amplitudes	high-gain limiter output
r_1, r_2, E	signal amplitudes	multiplier input
E_1, E_2	signal amplitudes	phase detector input
E_D	signal amplitudes	discriminator output

NOMENCLATURE (Cont'd)

R_1, R_2	resistances
C_1, C_2, C_3	capacitances
L_1, L_2, L_3	inductances
K, K_d, K_p, K_g	overall system gain
v	time delay ratio
q	time constant ratio
kT	kth sampling instant
s	Laplace variable
z	z-transform variable
z_n	z_n-transform variable
$D(s)$	digital controller Laplace transfer function
$D^*(s)$	digital controller starred Laplace transfer function
$D(z)$	digital controller z-transfer function
$D(z_n)$	digital controller z_n-transfer function
$G(s)$	plant to be controlled Laplace transfer function
$G(z)$	plant z-transfer function
$G(z, m)$	plant modified z-transfer function
$G(z_n)$	plant n-transfer function
$W_e(z)$	error transfer function
$G_o(z)$	overall transfer function
$W(z)$	statistical development function
$R(z)$	z-transform of system input
$F(z^{-1}), A(z^{-1})$	polynomials in z^{-1}
$R(z)_{step}$	z-transform of unit step

NOMENCLATURE (Cont'd)

$R(z)_{ramp}$	z-transform of unit ramp
$C(z, m)_{step}$	modified z-transform of step response
$C(z, m)_{ramp}$	modified z-transform of ramp response
$\mathcal{Z}\{X\}$	z-transform of X
$\mathcal{Z}_m\{X\}$	modified z-transform of X
$\mathcal{Z}_n\{X\}$	z_n-transform of X
$C(kT)$	output at kth sampling instant
T_d	time delay
T	basic sampling period
T/n	high-rate sampling period
T_1	time constant
$\epsilon^{-T_d s}$	$\exp(-T_d s)$

$$\left.\begin{array}{r} a_0, a_1, a_2, a_3, \cdots \\ b_0, b_1, b_2, b_3, \cdots \\ f_0, f_1, f_2, f_3, \cdots \\ \alpha_0, \alpha_1, \alpha_2, \alpha_3, \cdots \\ d_0, d_1, d_2, d_3, \cdots \\ D_0, D_1, D_2, D_3, \cdots \end{array}\right\} \text{polynomial coefficients}$$

Appendix A. Evaluation of $K(z)$ from $K(z_n)$

The problem is to find $K(z)$ from $K(z_n)$ for a multirate controller yielding minimum settling time for a ramp input to a system with a pure time delay and a first-order lag. We have defined

$$K(z) = \mathfrak{z}\{K(z_n)\} \tag{A-1}$$

where $\mathfrak{z}\{f(z_n)\}$ denotes the operation of z-transforming a function with respect to the sampling period T, a function for which the z-transform with respect to T/n is known. A straightforward method of finding $K(z)$ is to obtain the function $K(s)$ from which $K(z_n)$ is obtained. Then $K(z)$ is found by z-transform $K(s)$.

It can be shown that

$$\mathfrak{z}\left\{\frac{z_n^{-k}}{(1 - z_n^{-1})^2}\right\} = \frac{n}{T}\ \mathfrak{z}_n\left\{\frac{\epsilon^{-\frac{(k-1)}{n}Ts}}{s^2}\right\} \tag{A-2}$$

$$= \frac{nz^{-a}}{(1 - z^{-1})^2}\left[m + (1 - m)\,z^{-1}\right]\Bigg|_{m = a-(k-1/n)} \tag{A-3}$$

where

$$a - 1 \leq \frac{k-1}{n} \leq a$$

In the development of the multirate minimum settling time deadbeat controller, it was shown that (for $n = 2$)

$$K(z_n) = \frac{(1 - z_n^{-2})^2}{(1 - z_n^{-1})^2}\left[\left(v + \frac{3}{2}\right) - (v + 1)z_n^{-1}\right] z_n^{-2v-1} \tag{A-4}$$

$$K(z_n) = \frac{\left(v + \frac{3}{2}\right) z_n^{-2v-1}}{(1 - z_n^{-1})^2} - \frac{2\left(v + \frac{3}{2}\right) z_n^{-2v-3}}{(1 - z_n^{-1})^2} + \frac{\left(v + \frac{3}{2}\right) z_n^{-2v-5}}{(1 - z_n^{-1})^2}$$

$$- \frac{(v + 1) z_n^{-2v-2}}{(1 - z_n^{-1})^2} + \frac{2(v + 1) z_n^{-2v-4}}{(1 - z_n^{-1})^2} - \frac{(v + 1) z_n^{-2v-6}}{(1 - z_n^{-1})^2} \qquad \text{(A-5)}$$

Applying Eq. (A-3) to (A-5) and collecting terms yields

$$K(z) = (2v + 3) z^{-v} - (v + 1) z^{-(v+1)} - (v + 1) z^{-(v+2)} \qquad \text{(A-6)}$$

Equation (A-6) is the desired expression for $K(z)$.

Appendix B. Minimization of the Mean-Square Sampled Error

The conventional techniques for minimization of the rms error for continuous systems has been extended to sampled-data systems (Ref. 22). This criterion will be applied to the error-sampled unity feedback system shown in Fig. B-1 , where $G(z)$ is the transfer function of the plant to be controlled and $D(z)$ is the digital controller. The system has two inputs – an input signal r_s and noise r_n. The closed-loop transfer function (output/input) after digital compensation will be denoted by $G_0(z)$ and the desired closed-loop transfer function will be denoted by $G_d(z)$.

An important extention of the spectral densities for continuous signals is the pulse-auto-spectral density and the pulse-cross-spectral density given by (Ref. 7, p. 543):

$$\phi_{xx}(z) = \sum_{R=-\infty}^{\infty} \psi_{xx}(RT)\, z^{-k}$$

$$\phi_{xy}(z) = \sum_{R=-\infty}^{\infty} \psi_{xy}(RT)\, z^{-k} \qquad \text{(B-1)}$$

where

$$\psi_{xx}(kT) = \operatorname*{limit}_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(nT)\, x(nT+kT)$$

$$\psi_{xy}(kT) = \operatorname*{limit}_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(nT)\, y(nT+kT) \qquad \text{(B-2)}$$

It has been shown that the mean-square sampled error is given by

$$\overline{e^2(nT)} = \frac{1}{2\pi j} \oint_{\Gamma} \phi_{ee}(z)\, z^{-1}\, dz \qquad \text{(B-3)}$$

where

$$\phi_{ee}(z) = [G_0(z) - G_d(z)]\,[G_0(z^{-1}) - G_d(z^{-1})]\,\phi_{ss}(z) + G_0(z)\,[G_0(z^{-1}) - G_d(z^{-1})]\,\phi_{sn}(z)$$

$$+ G_0(z^{-1})\,[G_0(z) - G_d(z)]\,\phi_{ns}(z) + G_0(z)\,G_0(z^{-1})\,\phi_{nn}(z) \qquad \text{(B-4)}$$

Let

$$W(z) = \frac{D(z)}{1 + D(z)\,G(z)} \qquad \text{(B-5)}$$

so

$$G_0(z) = D(z)\,W(z) \qquad \text{(B-6)}$$

and

$$D(z) = \frac{W(z)}{1 + W(z)\,G(z)} \qquad \text{(B-7)}$$

Now apply variation of parameter techniques so that

$$W(z) = \lambda\, n(z)$$

$$W(z^{-1}) = \lambda\, n(z^{-1}) \qquad \text{(B-8)}$$

$$\left[e^2(nT)\right]\Big|_{\delta} = \overline{e^2(nT)} + \overline{\delta\, e^2(nT)}$$

$$\frac{\partial}{\partial \lambda}\left[\overline{e^2(nT)} + \overline{\delta\, e^2(nT)}\right]\Bigg|_{\lambda=0} = 0 \qquad \text{(B-9)}$$

Substituting from (B-3), (B-4), (B-6), and (B-8) into (B-9) yields

$$\frac{1}{2\pi j} \oint_{\Gamma} \eta(z^{-1})\, G(z^{-1}) \{W(z)\, G(z)\, \phi_t(z) - G_d(z)\, [\phi_{ss}(z^{-1}) + \phi_{ns}(z^{-1})]\} z^{-1}\, dz$$

$$+ \frac{1}{2\pi j} \oint_{\Gamma} \eta(z)\, G(z) \{W(z^{-1})\, G(z^{-1})\, \phi_t(z) - G_d(z^{-1})\, [\phi_{ss}(z) + \phi_{ns}(z)]\} z^{-1}\, dz = 0$$

(B-10)

where

$$\phi_t(z) = \phi_{ss}(z) + \phi_{nn}(z) + \phi_{sn}(z) + \phi_{ns}(z)$$

$$F(z^{-1}) = F(z)\big|_{z=z^{-1}}$$

and

$\phi_{ss}(z)$ is the pulse-auto-spectral density of the input signal r_s

$\phi_{nn}(z)$ is the pulse-auto-spectral density of the noise r_n

$\phi_{sn}(z)$ is the pulse-cross-spectral density of the input signal to the noise

$\phi_{ns}(z)$ is the pulse-cross-spectral density of the noise to the input signal

Now let

$$\phi_t(z) = [\phi_t(z)]^+ [\phi_t(z)]^-$$

$$G(z)\, G(z^{-1}) = [G(z)\, G(z^{-1})]^+ [G(z)\, G(z^{-1})]^- \qquad \text{(B-11)}$$

where

$[\quad]^+$ denotes factors with poles and zeros inside the unit circle of the z-plane

$[\quad]^-$ denotes factors with poles and zeros on or outside the unit circle of the z-plane

Substitution of (B-11) into (B-10) yields

$$\frac{1}{2\pi j}\oint_{\Gamma} \eta(z^{-1})[\phi_t(z)]^{-}[G(z)\,G(z^{-1})]^{-}\left\{[G(z)\,G(z^{-1})]^{+}\cdot W(z)[\phi_t(z)]^{+} - \left\{\frac{G(z^{-1})\,G_d(z)[\phi_{ss}(z) + \phi_{ns}(z)]}{[G(z)\,G(z^{-1})]^{-}[\phi_t(z)]^{-}}\right\}\right\} z^{-1}\,dz$$

$$+\frac{1}{2\pi j}\oint_{\Gamma} \eta(z)[\phi_t(z)]^{+}[G(z)\,G(z^{-1})]^{+}\left\{[G(z)\,G(z^{-1})]^{-}W(z^{-1})\cdot [\phi_t(z)]^{-} - \left\{\frac{G(z)\,G_d(z^{-1})[\phi_{ss}(z^{-1}) + \phi_{ns}(z^{-1})]}{[G(z)\,G(z^{-1})]^{+}[\phi_t(z)]^{+}}\right\}\right\} z^{-1}\,dz = 0 \qquad \text{(B-12)}$$

Expand the terms within $\{\;\}$ into a partial fraction expansion such that

$$\{Q\} = \{Q\}^{+} + \{Q\}^{-}$$

where

$\{Q\}^{+}$ denotes portion with poles inside the unit circle of the z-plane

$\{Q\}^{-}$ denotes the portion with poles on or outside the unit circle in the z-plane

Thus

$$\frac{1}{2\pi j}\oint_{\Gamma} \eta(z^{-1})[\phi_t(z)]^{-}[G(z)\,G(z^{-1})]^{-}\{[G(z)\,G(z^{-1})]^{+}W(z)\cdot[\phi_t(z)]^{+} - \{Q_1\}^{+} - \{Q_1\}^{-}\}\,z^{-1}\,dz$$

$$+\frac{1}{2\pi j}\oint_{\Gamma} \eta(z)[\phi_t(z)]^{+}[G(z)\,G(z^{-1})]^{+}\{[G(z)\,G(z^{-1})]^{-}W(z^{-1})\cdot[\phi_t(z)]^{-} - \{Q_2\}^{+} - \{Q_2\}^{-}\}\,z^{-1}\,dz$$

(B-13)

Reference 7 states (page 551): "It can be readily shown that the counter (contour) integral vanishes if the integrand has its poles either all inside the unit circle or all outside the unit circle (of the z-plane)." Thus it can be shown that

$$\frac{1}{2\pi j} \oint_{\Gamma} \eta(z^{-1})[\phi_t(z)]^{-}[G(z)\,G(z^{-1})]^{-}\{[G(z)\,G(z^{-1})]^{+}W(z)[\phi_t(z)]^{+} - \left\{Q_1\right\}^{+}\}z^{-1}\,dz$$

$$+\frac{1}{2\pi j} \oint_{\Gamma} \eta(z)[\phi_t(z)]^{+}[G(z)\,G(z^{-1})]^{+}\{[G(z)\,G(z^{-1})]^{-}W(z)[\phi_t(z)]^{-} - \left\{Q_2\right\}^{-}\}z^{-1}\,dz = 0$$

(B-14)

If (B-14) is to be true independent of $\eta(z)$ (from variation of parameter calculus), then

$$[G(z)\,G(z^{-1})]^{+}W(z)[\phi_t(z)]^{+} - \left\{\frac{G(z^{-1})\,G_d(z)[\phi_{ss}(z) + \phi_{ns}(z)]}{[G(z)\,G(z^{-1})]^{-}[\phi_t(z)]^{-}}\right\}^{+} = 0 \qquad \text{(B-15)}$$

and

$$[G(z)\,G(z^{-1})]^{-}W(z)[\phi_t(z)]^{-} - \left\{\frac{G(z)\,G_d(z^{-1})[\phi_{ss}(z^{-1}) + \phi_{ns}(z^{-1})]}{[G(z)\,G(z^{-1})]^{+}[\phi_t(z)]^{+}}\right\}^{-} = 0 \qquad \text{(B-16)}$$

It can be shown that (B-15) and (B-16) are identical so that

$$W(z) = \frac{\left\{\dfrac{G(z^{-1})\,G_d(z)[\phi_{ss}(z) + \phi_{ns}(z)]}{[G(z)\,G(z^{-1})]^{-}[\phi_t(z)]^{-}}\right\}^{+}}{[G(z)\,G(z^{-1})]^{+}[\phi_t(z)]^{+}}$$

If $G(z)$ has no poles or zeros outside the unit circle, then

$$[G(z)\,G(z^{-1})]^{+} = G(z)$$

$$[G(z)\,G(z^{-1})]^{-} = G(z^{-1}) \qquad \text{(B-17)}$$

and

$$W(z) = \frac{1}{G(z)\,[\phi_t(z)]^{+}} \left\{ \frac{G_d(z)\,[\phi_{ss}(z) + \phi_{ns}(z)]}{[\phi_t(z)]^{-}} \right\}^{+} \qquad \text{(B-18)}$$

$$G_0(z) = \frac{1}{[\phi_t(z)]^{+}} \left\{ \frac{G_d(z)\,[\phi_{ss}(z) + \phi_{ns}(z)]}{[\phi_t(z)]^{-}} \right\}^{+} \qquad \text{(B-19)}$$

which agrees with the established design equation of Ref. 7, page 551.

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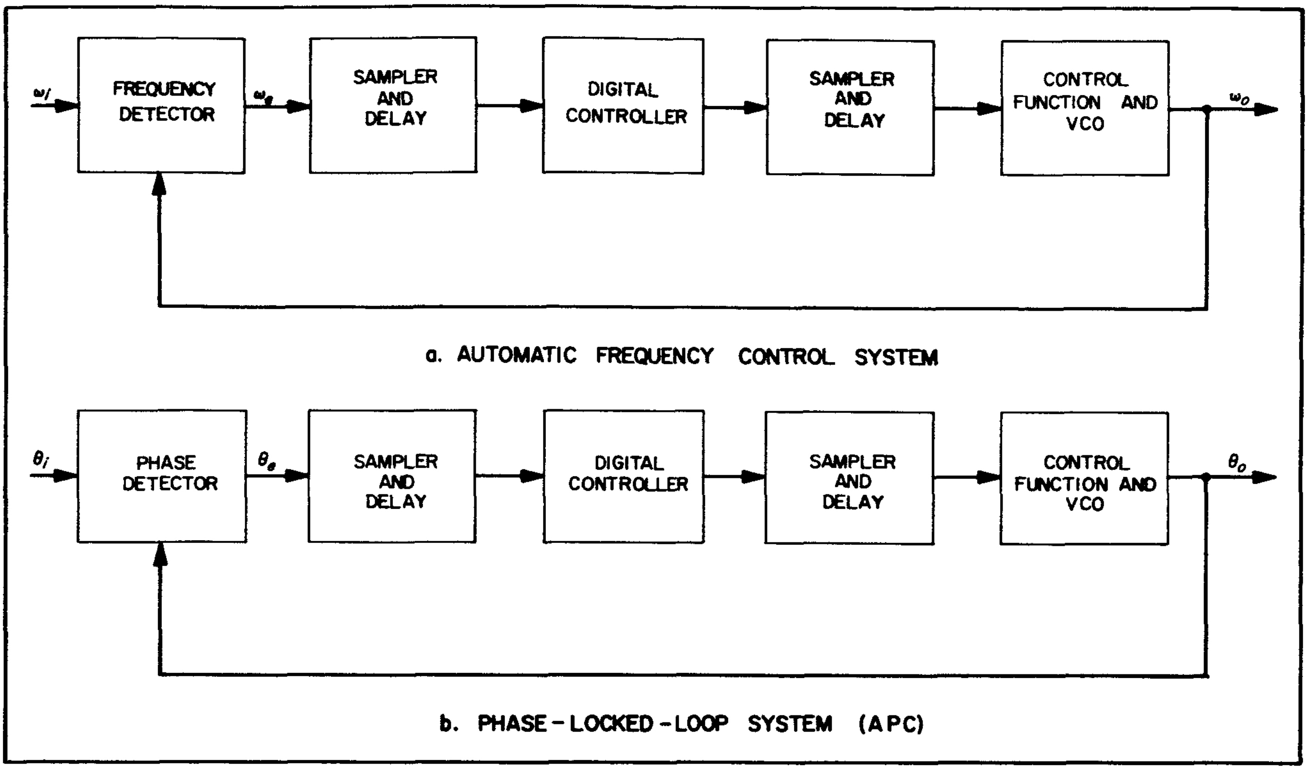

Fig. 1. Block diagrams of digital AFC and APC systems with delays

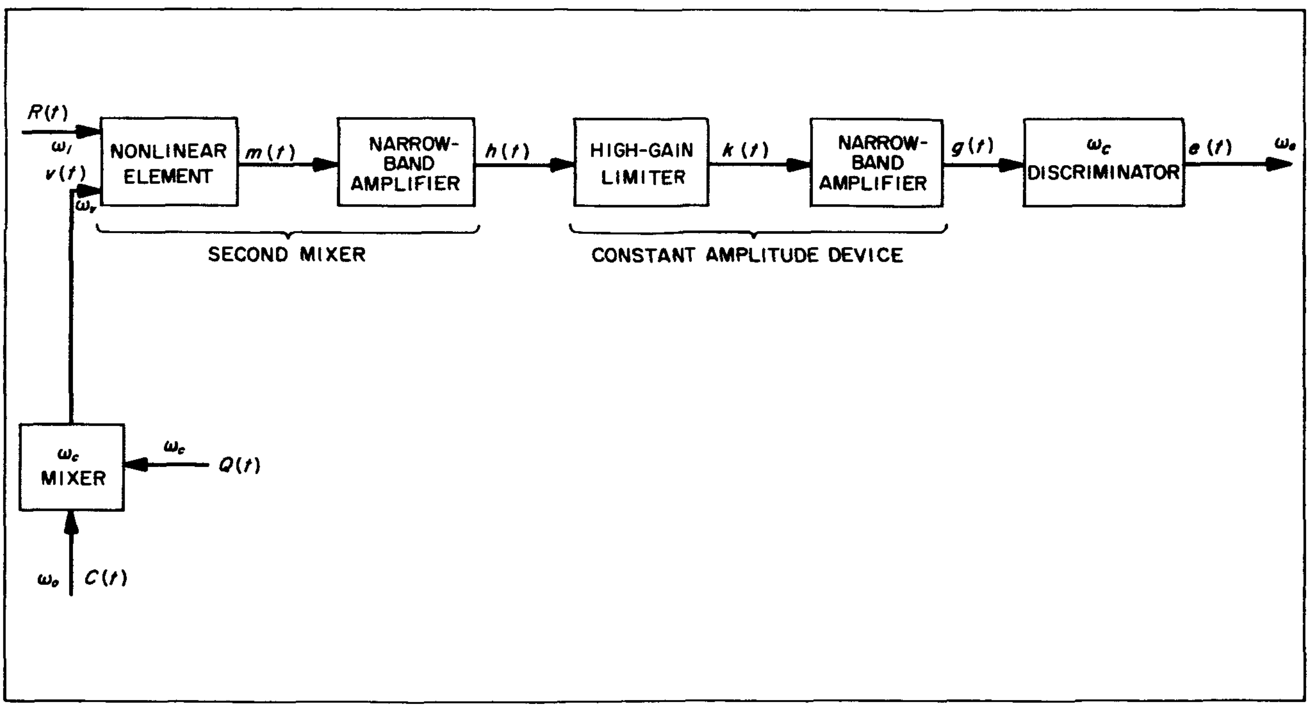

Fig. 2. Frequency detector

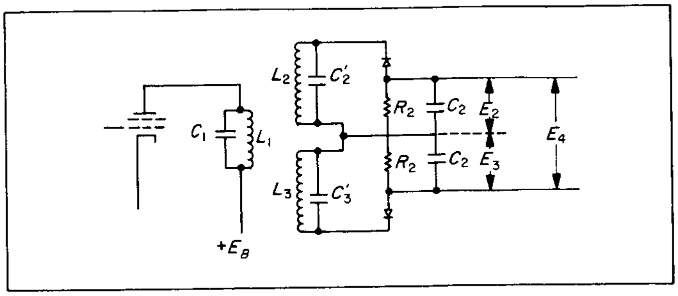

Fig. 3. Discriminator circuit

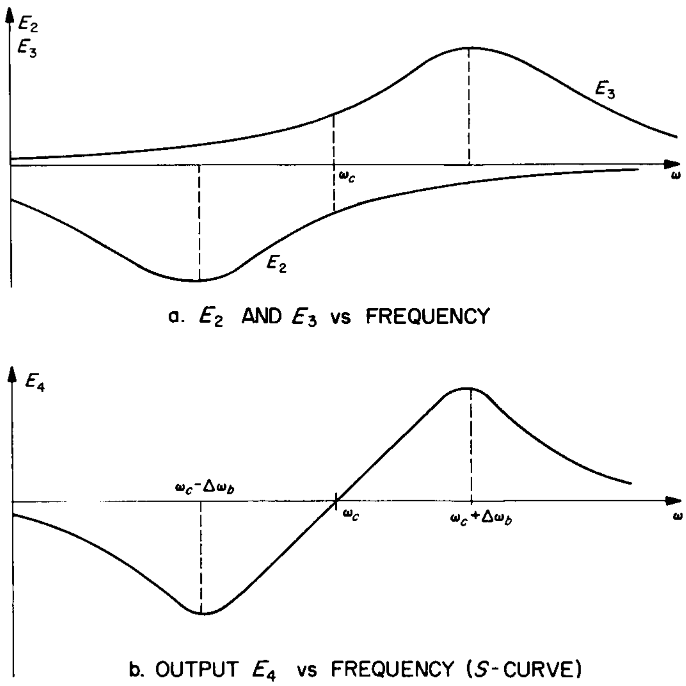

Fig. 4. Discriminator input-output characteristic

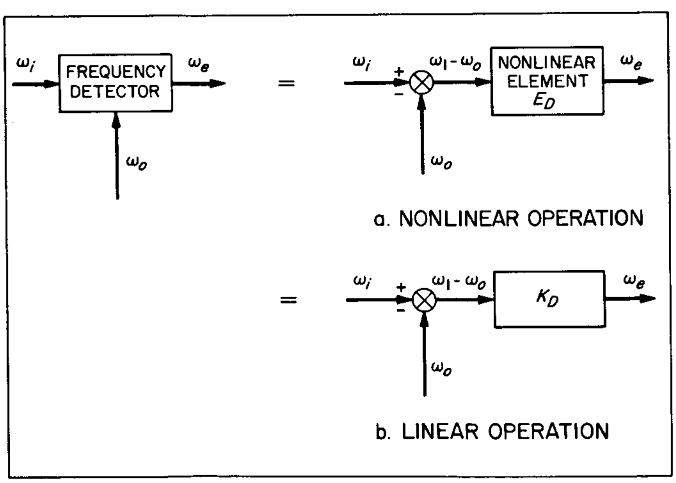

Fig. 5. Frequency detector representation

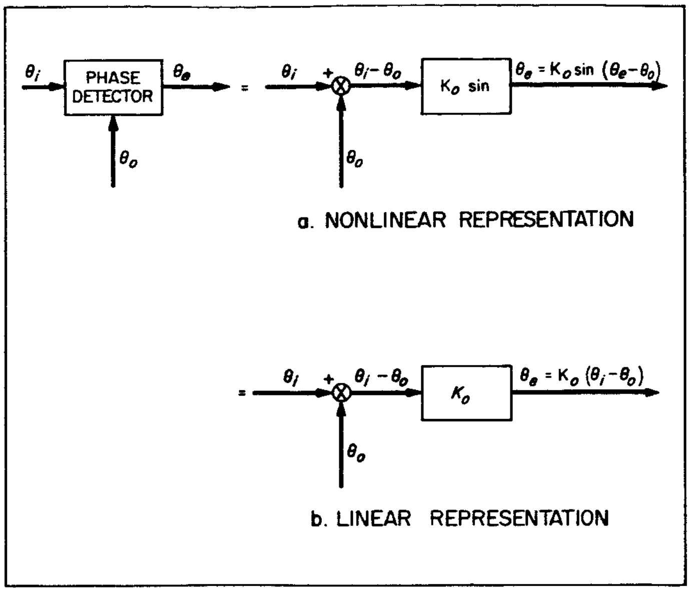

Fig. 6. Phase detection representation

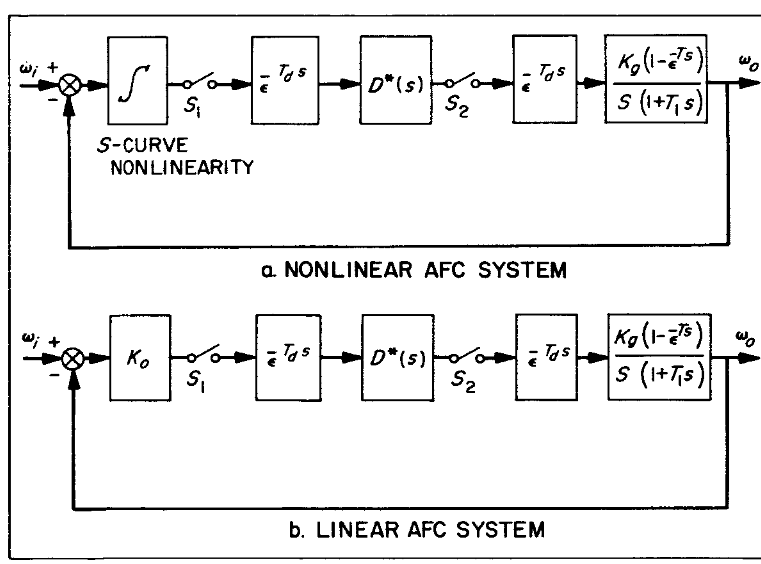

Fig. 7. Block diagram representation of a digital AFC system

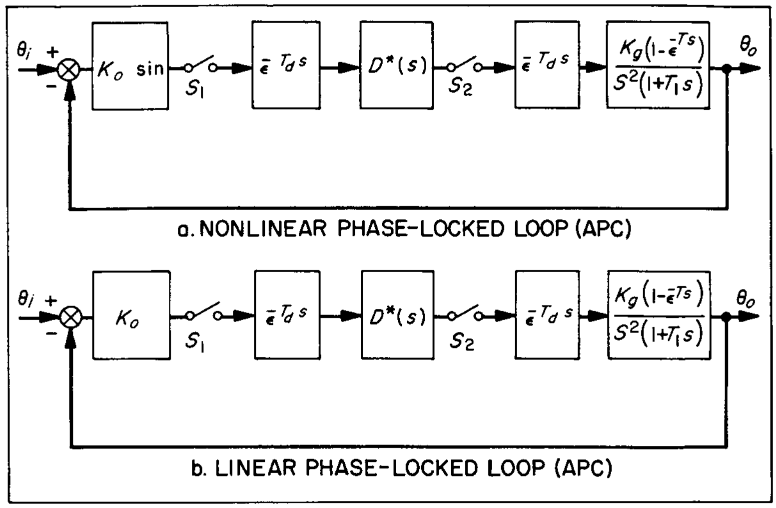

Fig. 8. Block diagram representation of a digital phase-locked loop

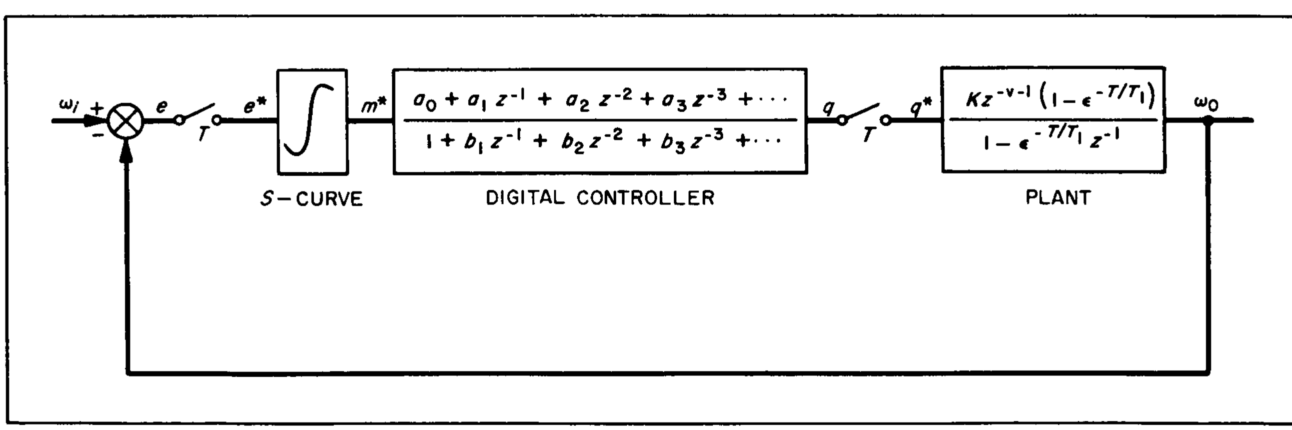

Fig. 9. Nonlinear digital AFC system

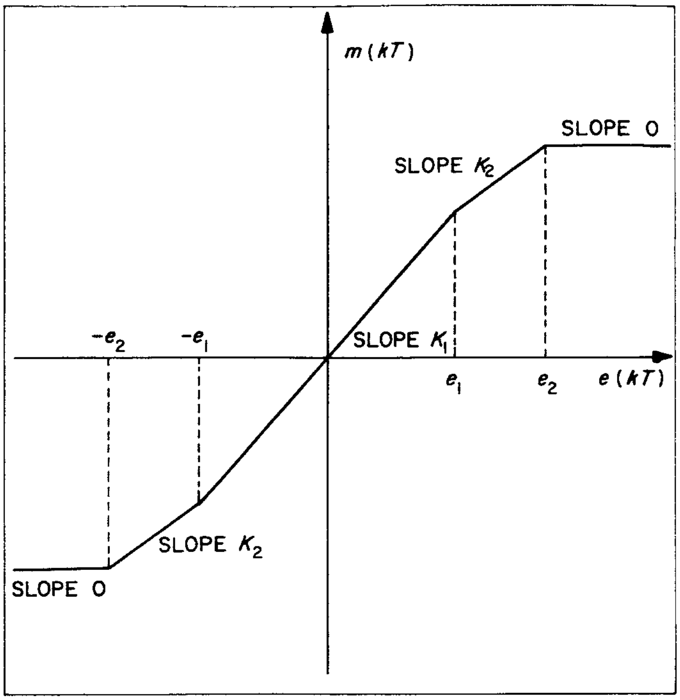

Fig. 10. Piecewise linear approximation to S-curve

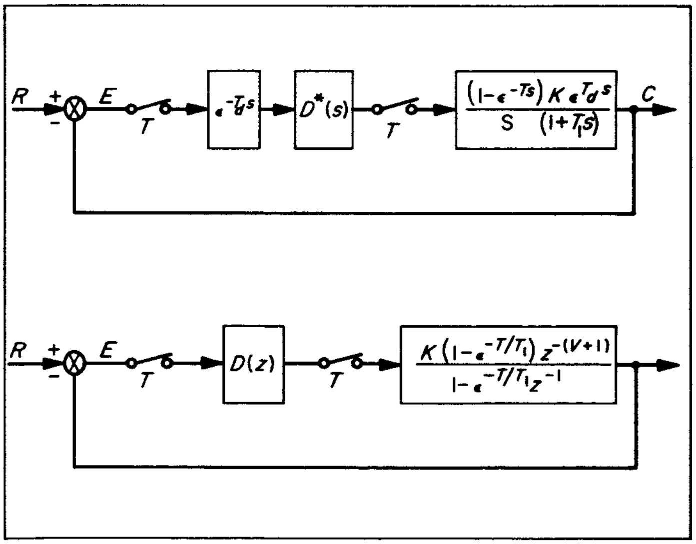

Fig. 11. Digital AFC system

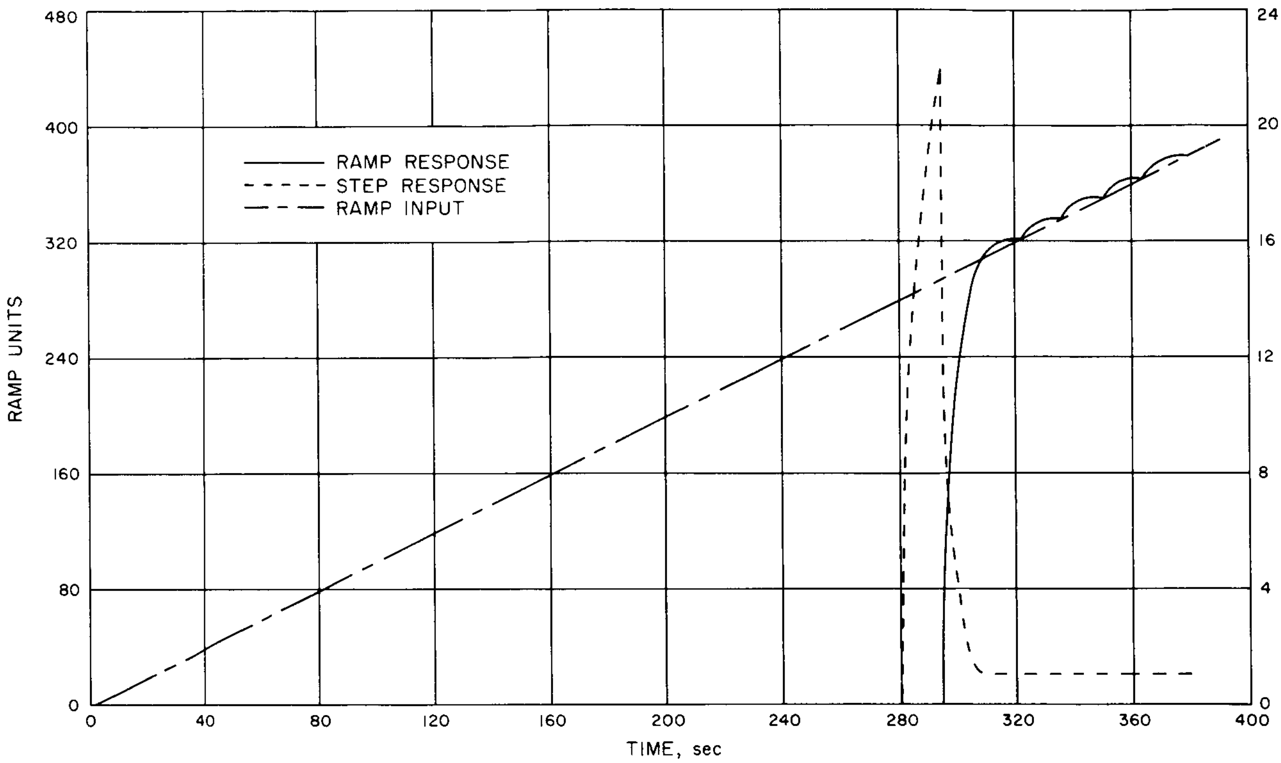

Fig. 12. Step and ramp response (v = 20, q = 0.25)

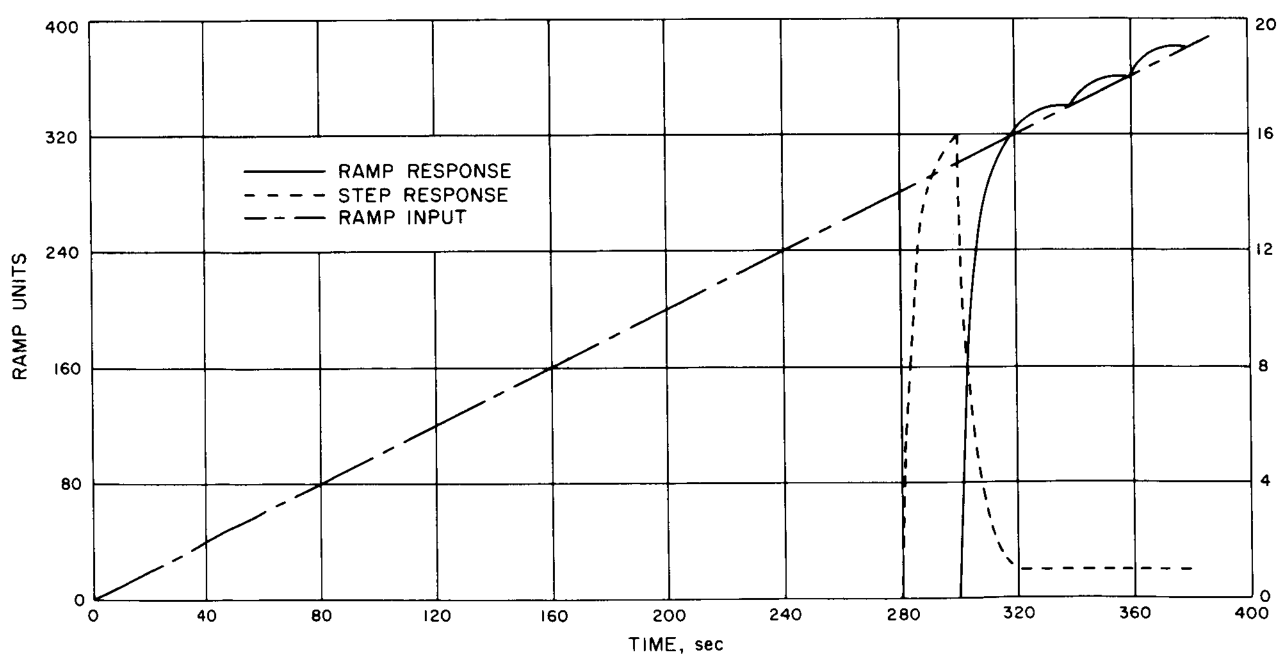

Fig. 13. Step and ramp response (v = 14, q = 0.25)

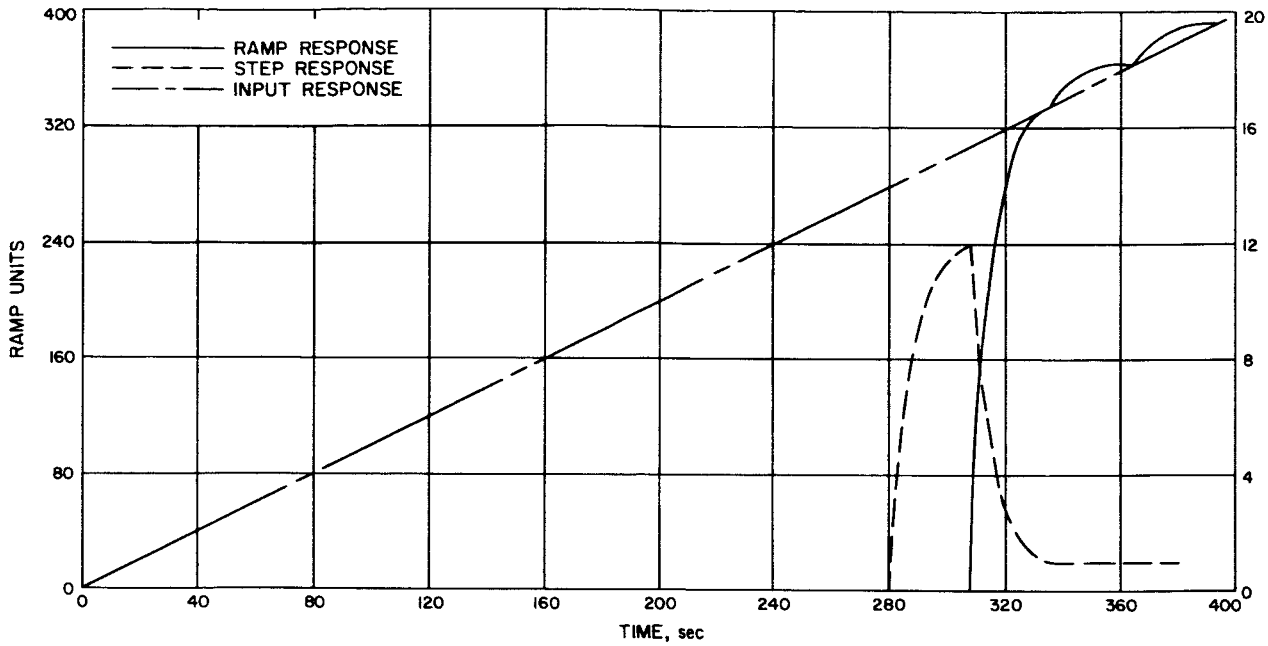

Fig. 14. Step and ramp response ($v = 10$, $q = 0.25$)

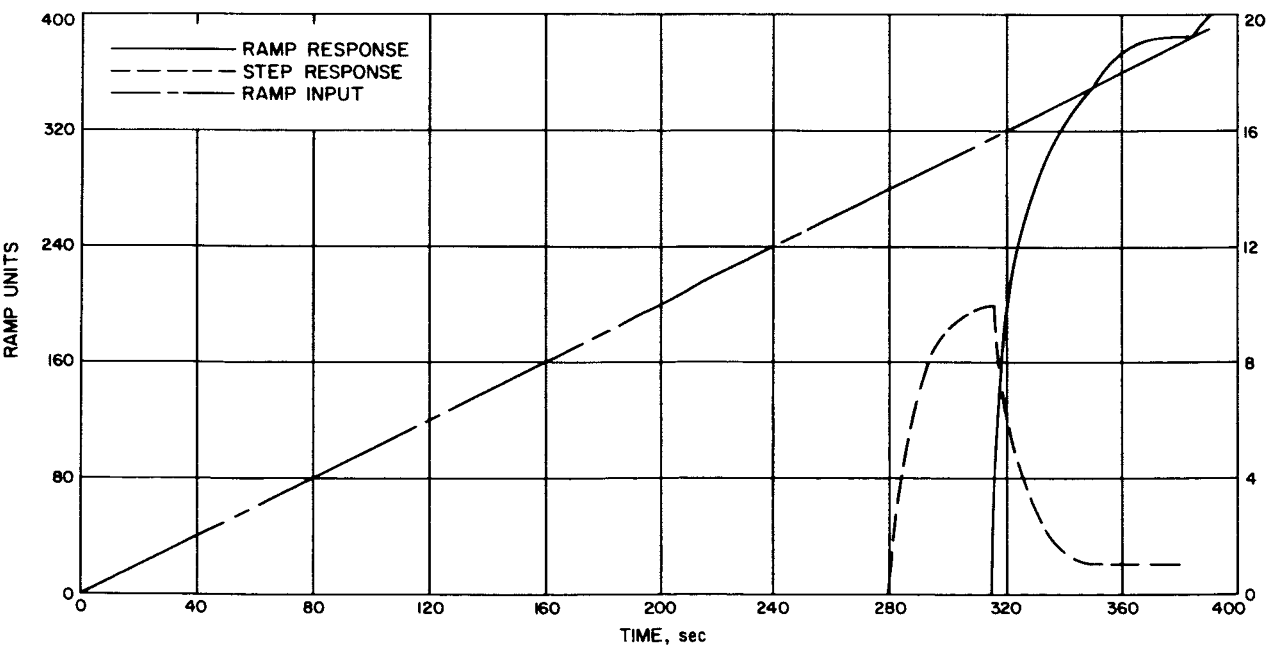

Fig. 15. Step and ramp response ($v = 8$, $q = 0.25$)

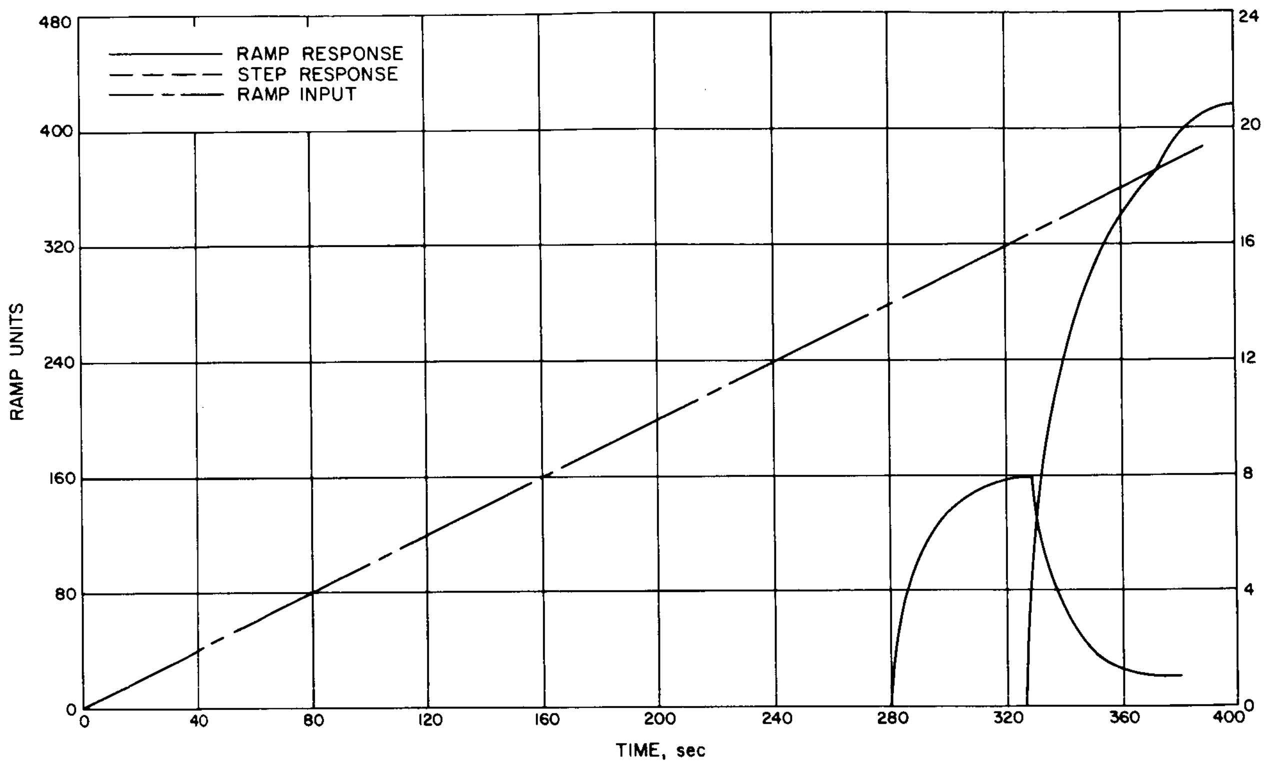

Fig. 16. Step and ramp response ($v = 6$, $q = 0.25$)

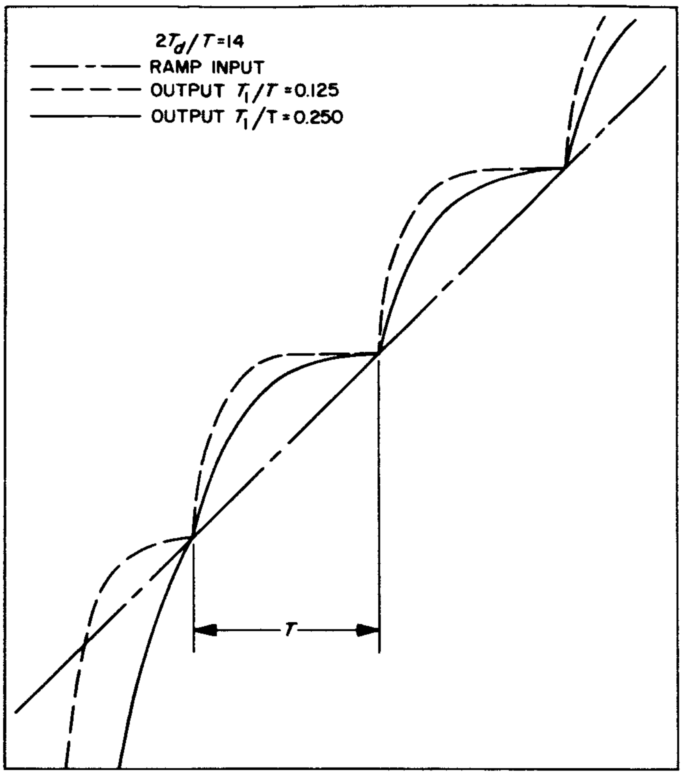

Fig. 17. Exploded view of ramp response

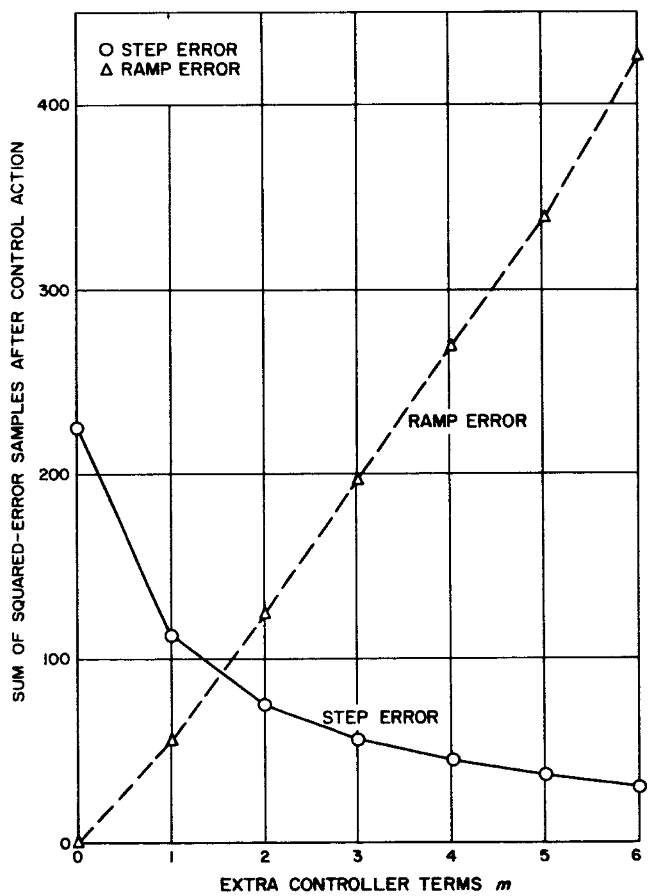

Fig. 18. $\Sigma\ [e(kT)]^2$ after control

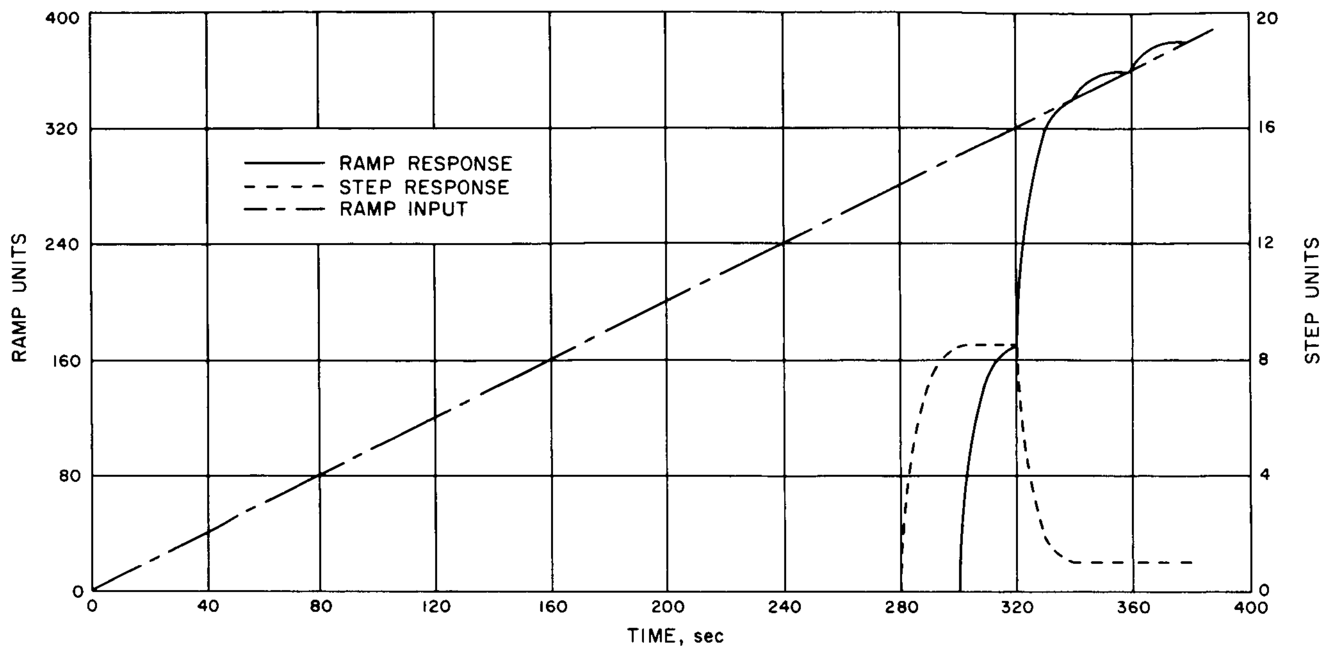

Fig. 19. Single-rate system response – one extra term

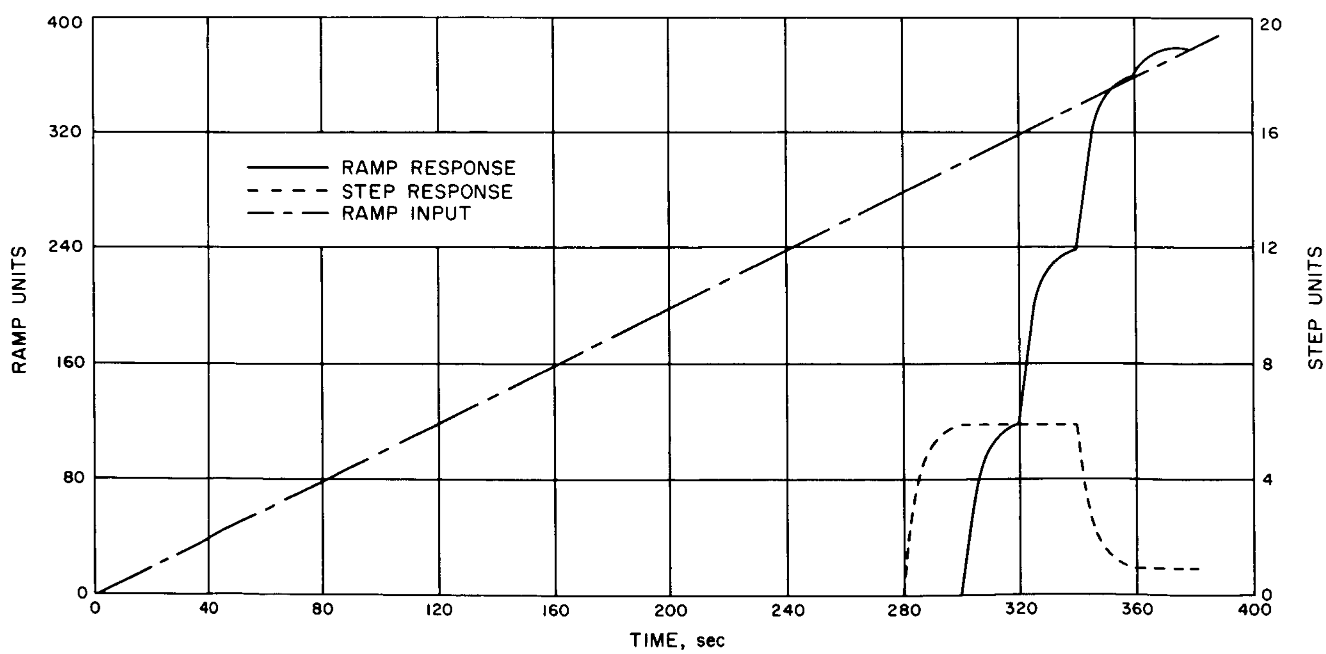

Fig. 20. Single-rate system response – two extra terms

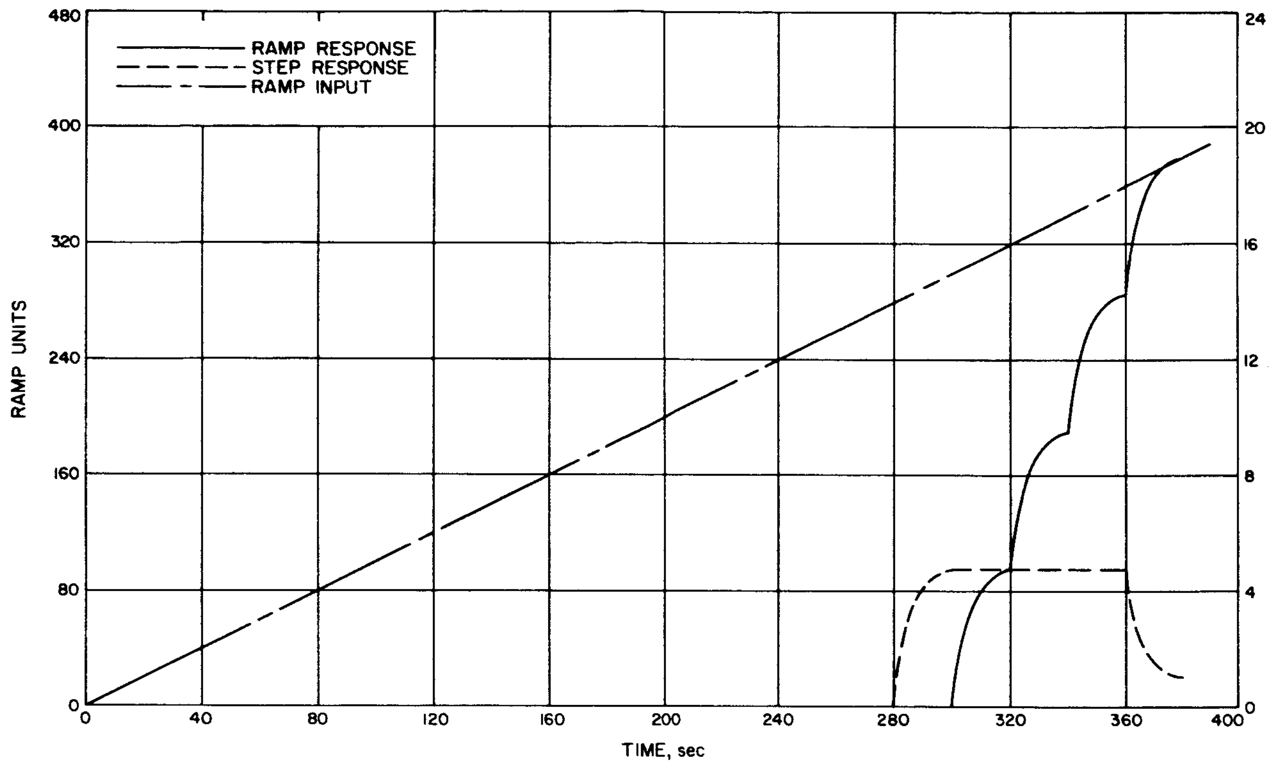

Fig. 21. Single-rate system response – three extra terms

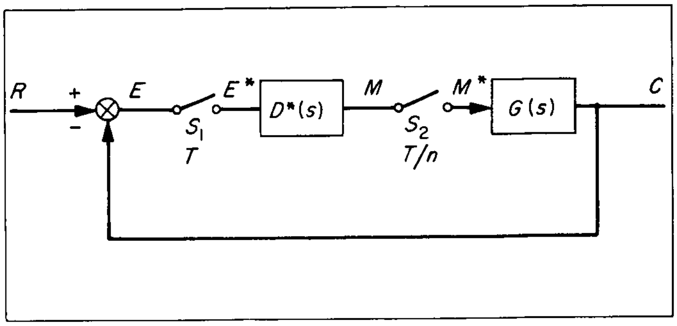

Fig. 22. A multirate controller system

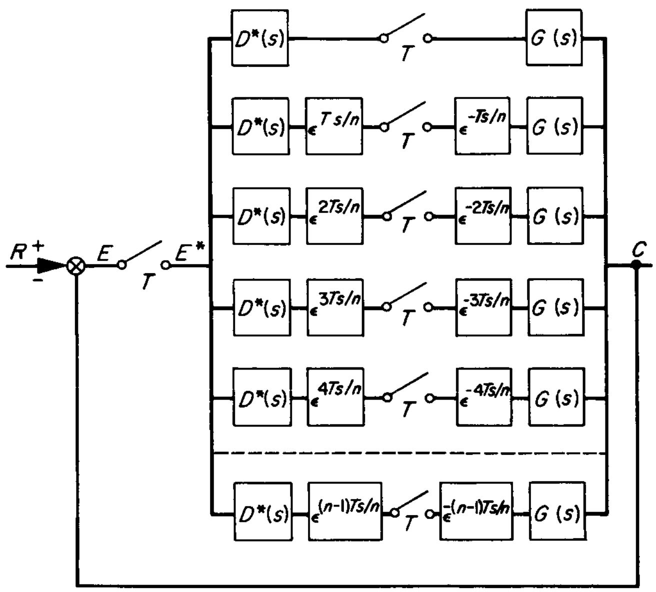

Fig. 23. Equivalent block diagram

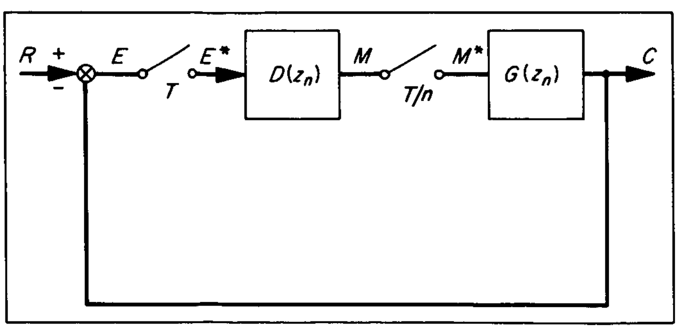

Fig. 24. z_n block diagram

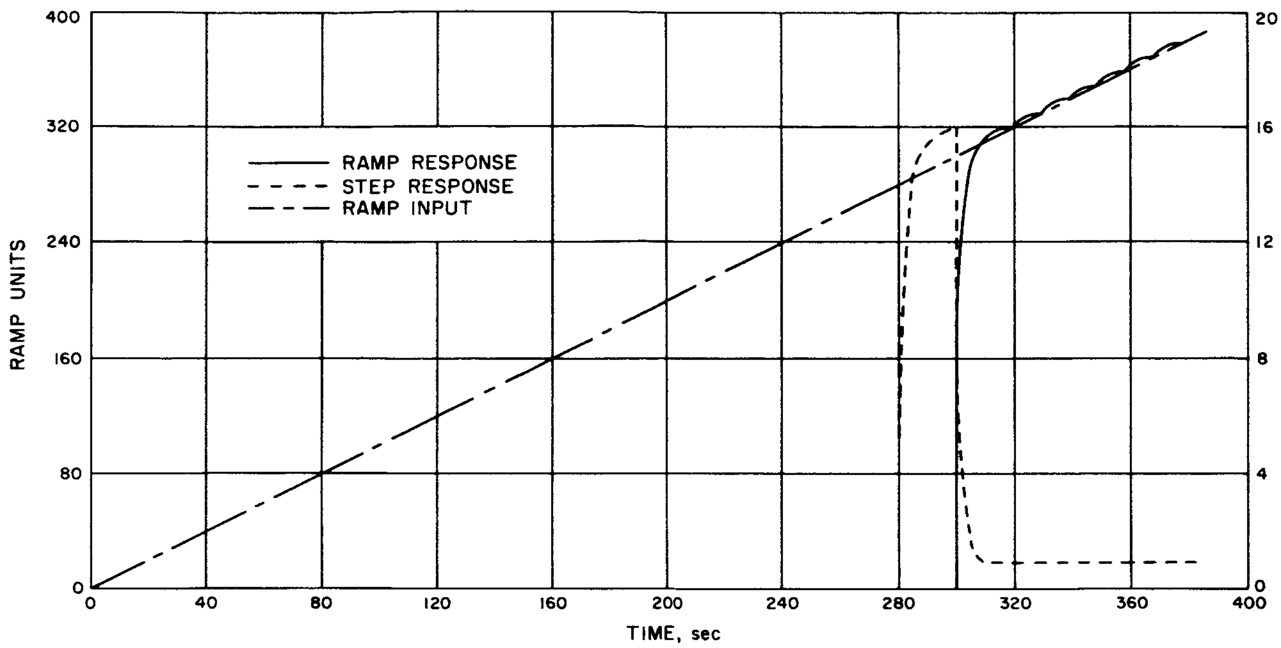

Fig. 25. Multirate system minimum settling time response

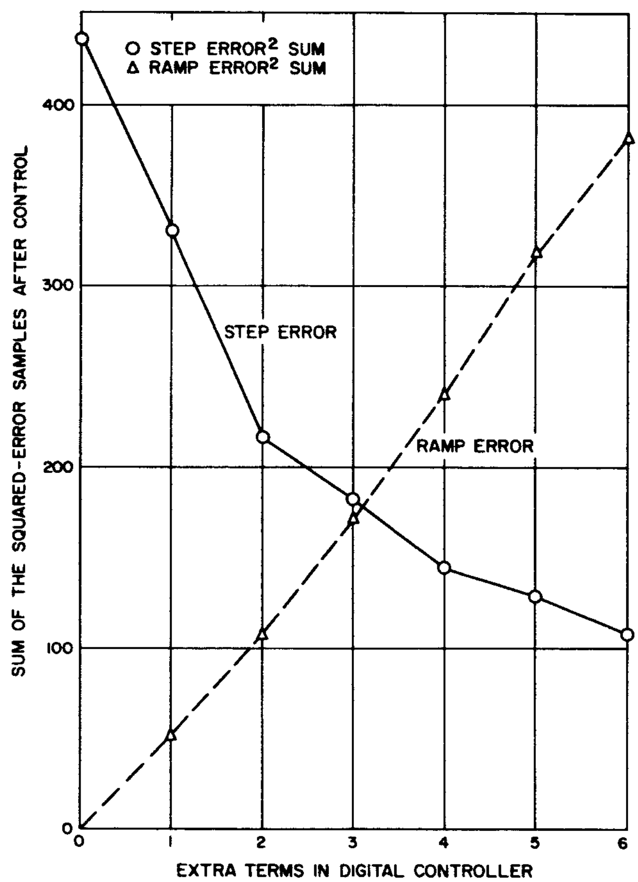

Fig. 26. Multirate system error

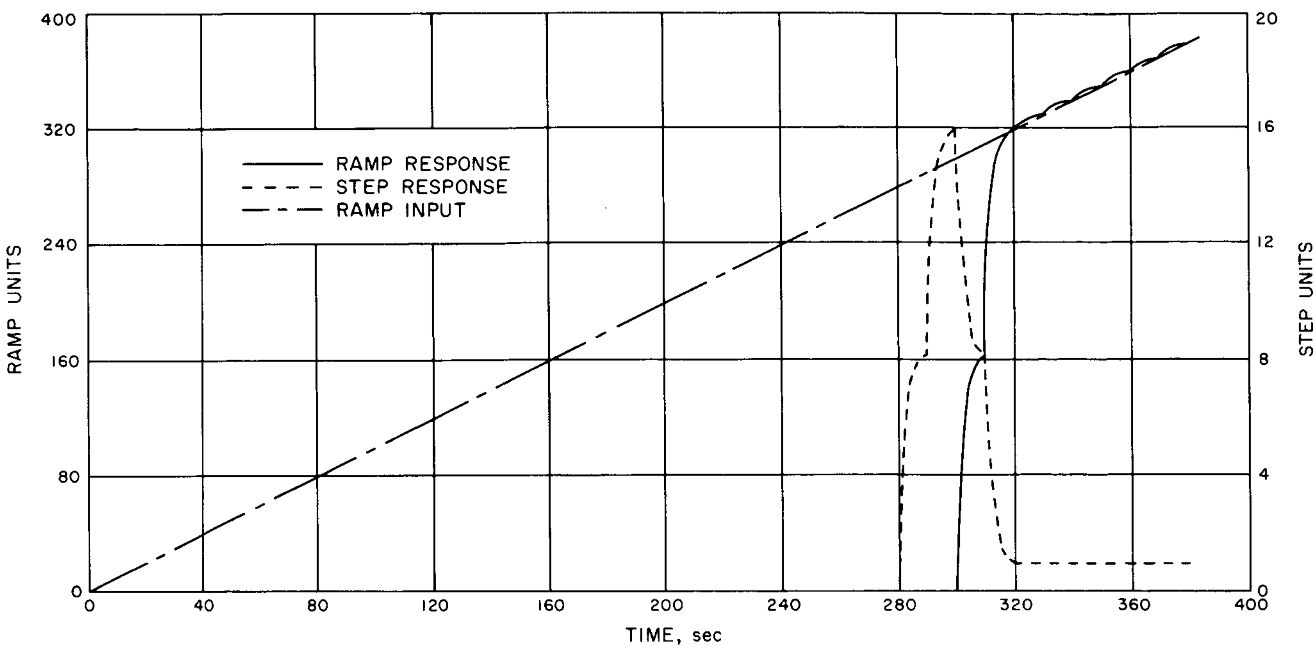

Fig. 27. Multirate system response – one extra term

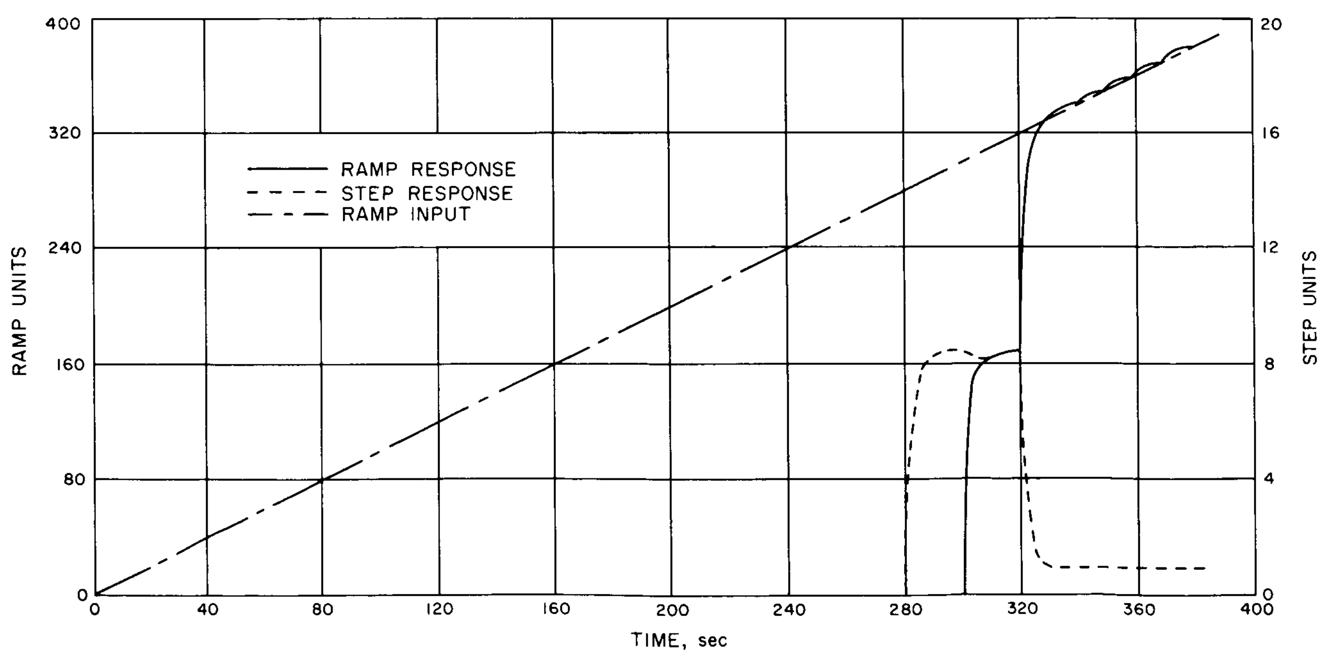

Fig. 28. Multirate system response – two extra terms

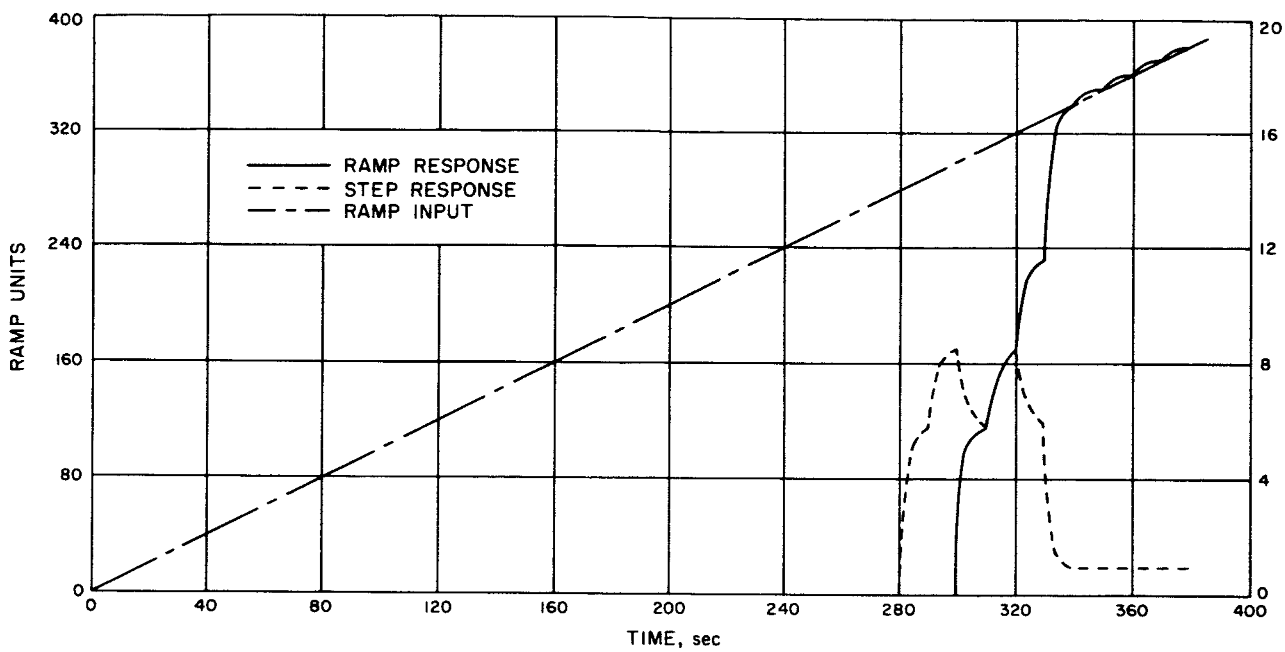

Fig. 29. Multirate system response – three extra terms

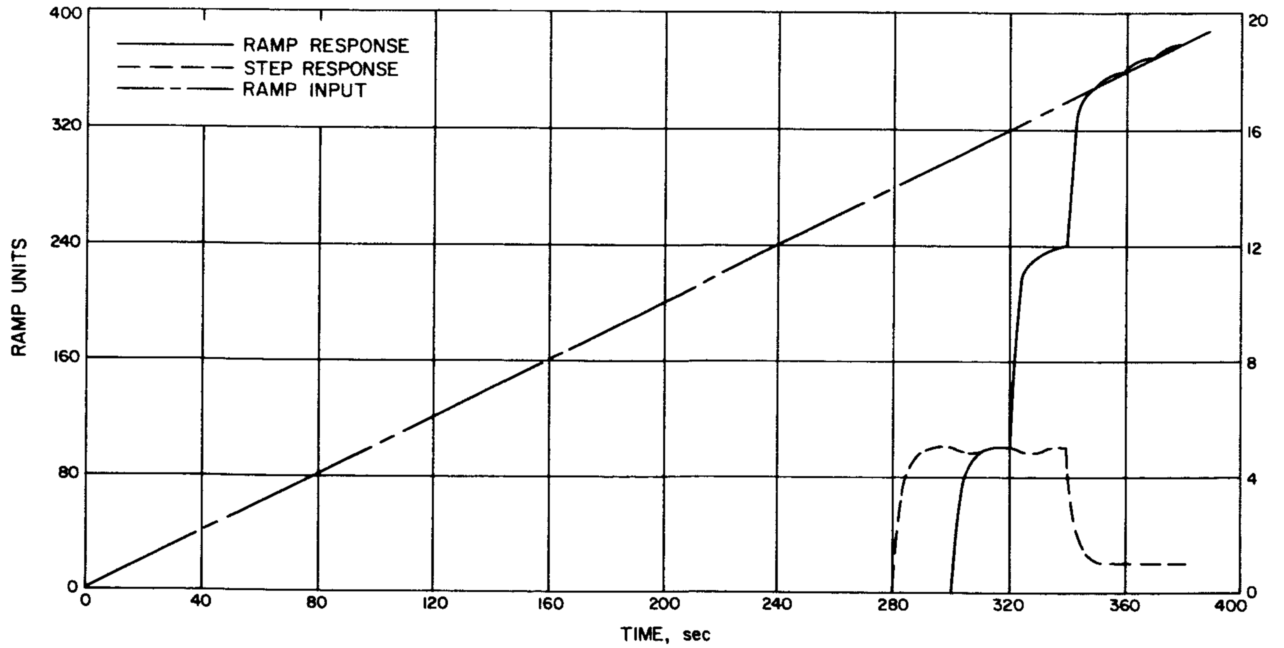

Fig. 30. Multirate system response – four extra terms

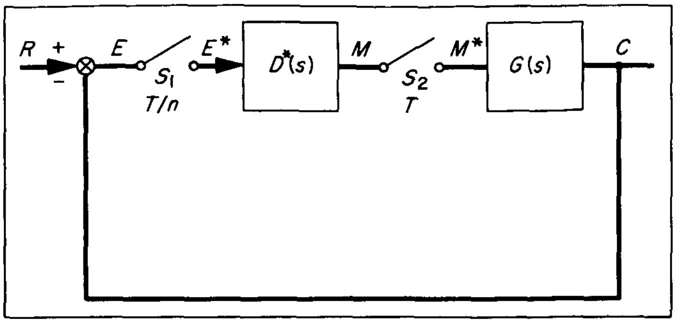

Fig. 31. A subrate controller system

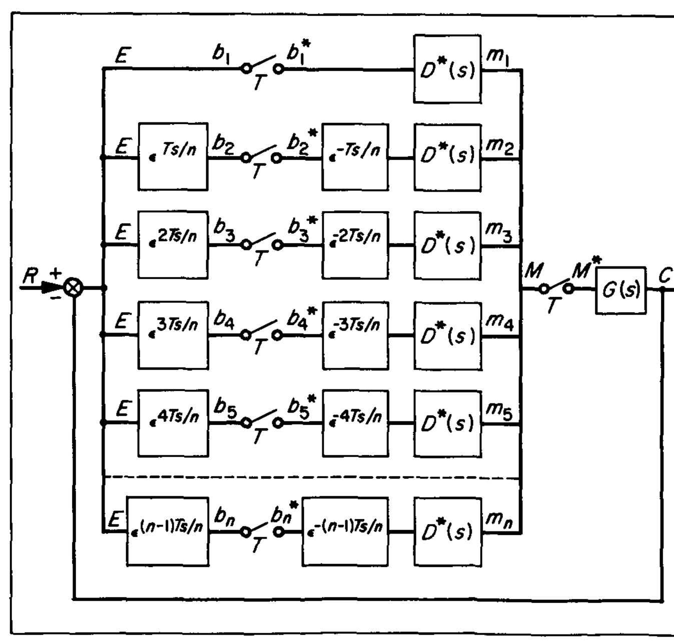

Fig. 32. Equivalent block diagram

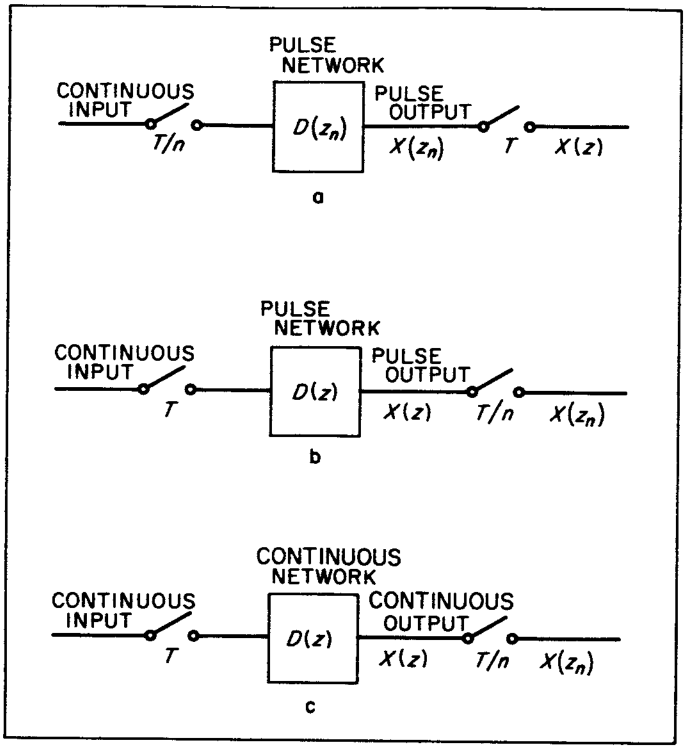

Fig. 33. Multirate sampler combinations

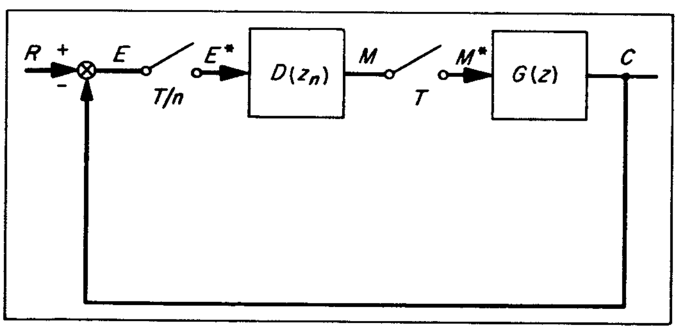

Fig. 34. Equivalent system

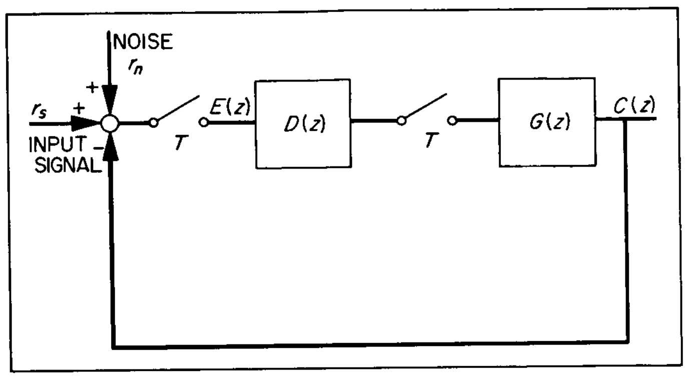

Fig. B-1. Digital system with noise