7 ap

COMMENTS ON SYNTHESIS TECHNIQUES EMPLOYING
THE DIRECT METHOD

By: Richard V. Monopoli
Richard V. Monopoli
Research Assistant

Approved by: DPLindorff
David P. Lindorff
Principal Investigator

Date: May 3, 1965

UNIVERSITY OF CONNECTICUT
ELECTRICAL ENGINEERING
DEPARTMENT

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION PROJECT

This report was prepared under NASA
Research Grant NSG - 309 - S3

COMMENTS ON SYNTHESIS TECHNIQUES EMPLOYING THE DIRECT METHOD

by R. V. Monopoli

Synthesis techniques based on Liapunov's direct method often require control signals which are sign functions.[1-5] The argument of the sign function in each case is a linear combination of system states. Because Flügge-Lotz[6] has shown that the desired solution may not exist for equations which have sign functions as forcing functions, authors customarily replace the discontinuous sign function by a continuous function, usually the saturation function defined as

$$\text{sat } k\gamma = \begin{cases} 1 \text{ for } \gamma > 1/k \\ k\gamma \text{ for } -1/k \leq \gamma \leq 1/k \\ -1 \text{ for } \gamma < -1/k \end{cases} \tag{1}$$

where $k > o$ is a constant.

This avoids the theoretical difficulty discussed by Flügge-Lotz, but, in some cases, leads to another concerning the validity of conclusions reached regarding stability in the region where the saturation function has a smaller magnitude than the theoretically specified sign function. This problem is investigated here in relation to the synthesis technique employing a model reference[2] for control of single input, single output, linear time-varying plants with parameters varying in an unknown manner within known, finite bounds.

The class of plants considered can be described by the equation

$$\dot{\underline{x}} = (A + \Delta A)\,\underline{x} + \underline{b}u \tag{2}$$

in which x is the plant output, u its input (the control signal), $\underline{x}^T = (x_1, x_2, \ldots x_n)$, $x_{i+1} = \dot{x}_i$ for $i = 1, 2, \ldots n-1$, A is an nxn constant stable matrix (all its eigenvalues have negative real parts) of the form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \cdots & 0 \\ 0 & & & \cdots & 1 \\ -a_1 & -a_2 & & \cdots & -a_n \end{bmatrix}$$

and $\underline{b}^T = [0, 0, 0, \ldots b_n]$.

It is assumed that $\infty > b_{n(max)} \geq b_n \geq b_{n(min)} > 0$, and that $b_{n(min)}$ is known. The elements of the matrix $\Delta A(t)$ are the variations of plant parameters from their desired values which are the elements of the A matrix. Because of the forms of A and $\underline{x}$, $\Delta A(t)$ is of the form

$$\Delta A(t) = \begin{bmatrix} & & \bigcirc & & \\ \Delta a_1(t) & & \Delta a_2(t) & \dots & \Delta a_n(t) \end{bmatrix}$$

where the Δa_i's may very in an unknown manner within known finite bounds.

The desired behavior for the plant is given by the equation of the model reference which is

$$\dot{\underline{x}}_d = A\,\underline{x}_d + \underline{c}\,r \tag{3}$$

where $\underline{x}_d^T = \left[\,x_{d1}, x_{d2}, \dots x_{dn}\right]$, $x_{d1} = x_d$ which is the model output, $x_{d(i+1)} = \dot{x}_{di}$ for $i = 1, 2, \dots n-1$, $\underline{c}^T = \left[0, 0, 0\dots c_n\right]$, and r is the reference input to the system.

An error for the system is defined by

$$\underline{e} = \underline{x}_d - \underline{x} \tag{4}$$

Subtracting (2) from (3) gives

$$\dot{\underline{e}} = A\underline{e} + \underline{c}\,r - \underline{b}\,u - \Delta A(t)\,\underline{x} \tag{5}$$

A function of quadratic form

$$V(\underline{e}) = \underline{e}^T P\underline{e} \tag{6}$$

has time derivative

$$\dot{V}(\underline{e}, r, u, \underline{x}) = \underline{e}^T (A^T P + PA)\,\underline{e} + 2\underline{e}^T P\left[\,-\underline{b}\,u + \underline{c}\,r - \Delta A(t)\underline{x}\,\right] \tag{7}$$

If

$$A^T P + PA = -Q \tag{8}$$

and Q is a symmetric positive definite matrix, then P is a symmetric positive definite matrix because of the assumptions on A.[7] For convenience, let Q = I, the identity matrix.

According to two theorems given in Reference 8, the following statement can be made concerning (5). If

$$u = \left| -\frac{c_n}{b_n} r + \sum_{i=1}^{n} \frac{\Delta a_i}{b_n} x_i \right|_{max} \operatorname{sgn} (\mathrm{grad}_e\ V)_n \tag{9}$$

then (5) is asymptotically stable, i.e., $e \to 0$ as $t \to \infty$. But there is the difficulty posed by Flügge-Lotz to contend with, so (9) must be replaced by

$$u = \left| -\frac{c_n}{b_n} r + \sum_{i=1}^{n} \frac{\Delta a_i}{b_n} x_i \right|_{max} \operatorname{sat} k \gamma \tag{10}$$

where $\gamma = (\mathrm{grad}_e\ V)_n$

Now it no longer can be argued that (5) is asymptotically stable in the region where

$$|\gamma| < 1/k \tag{11}$$

since u as given by (10) may not have sufficient magnitude to guarantee that $\dot{V} < 0$ in this region. What, then, can be said regarding stability? The following discussion considers this question. It is to be understood throughout that u as given by (10) is being considered, not that of (9).

The term $-2\ \underline{e}^T P\ \underline{b}\ u$ in (7) is always negative because of (10). Therefore, neglecting this term, $\dot{V}$ satisfies the inequality

$$\dot{V} \leq -\sum_{i=1}^{n} e_i^{\,2} + \gamma \left[c_n r - \sum_{i=1}^{n} \Delta a_i (x_{di} - e_i)\right] \tag{12}$$

The variables x_{di} and e_i are introduced into the square bracket of (12) by adding and subtracting $\Delta A(t)\ \underline{x}_d$ within the square brackets of (7). This removes the components of $\underline{x}$ from the problem. The reason for doing this is explained below.

If r is bounded, then so are the components of $\underline{x}_d$ since a stable model is used. Thus

$$\left| c_n r - \sum_{i=1}^{n} \Delta a_i x_{di} \right| \leq M < \infty \tag{13}$$

An upper bound cannot be established for

$$\left| c_n r - \sum_{i=1}^{n} \Delta a_i x_i \right| \tag{14}$$

without making the unwarranted assumption that the system is stable. This explains the need for eliminating components of $\underline{x}$ from the problem. Another upper bound which can be established and which is useful is

$$\left| \sum_{i=1}^{n} \Delta a_i e_i \right| \leq \left| \Delta a \right|_m \sum_{i=1}^{n} \left| e_i \right| \leq \left| \Delta a \right|_m \left(n \sum_{i=1}^{n} e_i^2 \right)^{\frac{1}{2}} \tag{15}$$

where $\left| \Delta a \right|_m = \underset{i}{\mathrm{Max}} \left\{ \left| \Delta a_i \right| \right\}$; $i=1, 2, \ldots n$

Use of (13) and (15) leads to a more conservative inequality for $\dot{V}$ which is

$$\dot{V} \leq -R^2 + \gamma \left(M + k_1 R \right) \tag{16}$$

where $R = \left(\sum_{i=1}^{n} e_i^2 \right)^{\frac{1}{2}}$

and $k_1 = \sqrt{n} \left| \Delta a \right|_{max}$

In the region where $\left| \gamma \right| < 1/k$

$$\dot{V} \leq -R^2 + \frac{k_1}{k} R + \frac{M}{k} \tag{17}$$

From (17) it can be seen that

$$\dot{V} < 0 \tag{18}$$

everywhere outside the spherical region

$$R\,(k) < \frac{k_1}{2k} + \frac{1}{2} \sqrt{ \left(\frac{k_1}{k} \right)^2 + \frac{4M}{k} } \tag{19}$$

By choosing k large enough $R\,(k)$ can be made arbitrarily small.

It can be concluded that as $t \to \infty$, u will cause $\underline{e}$ to be within the region common to $R\,(k)$ and $\left| \gamma \right| < 1/k$. For $n = 2$, the situation is as shown in Figure 1. Asymptotic stability of (5) cannot be concluded. Limit cycles may exist, but can be made arbitrarily small be choosing k large enough.

ACKNOWLEDGEMENT

The author would like to thank Professor D. P. Lindorff and Mr. T. M. Taylor for their helpful discussions relating to this work

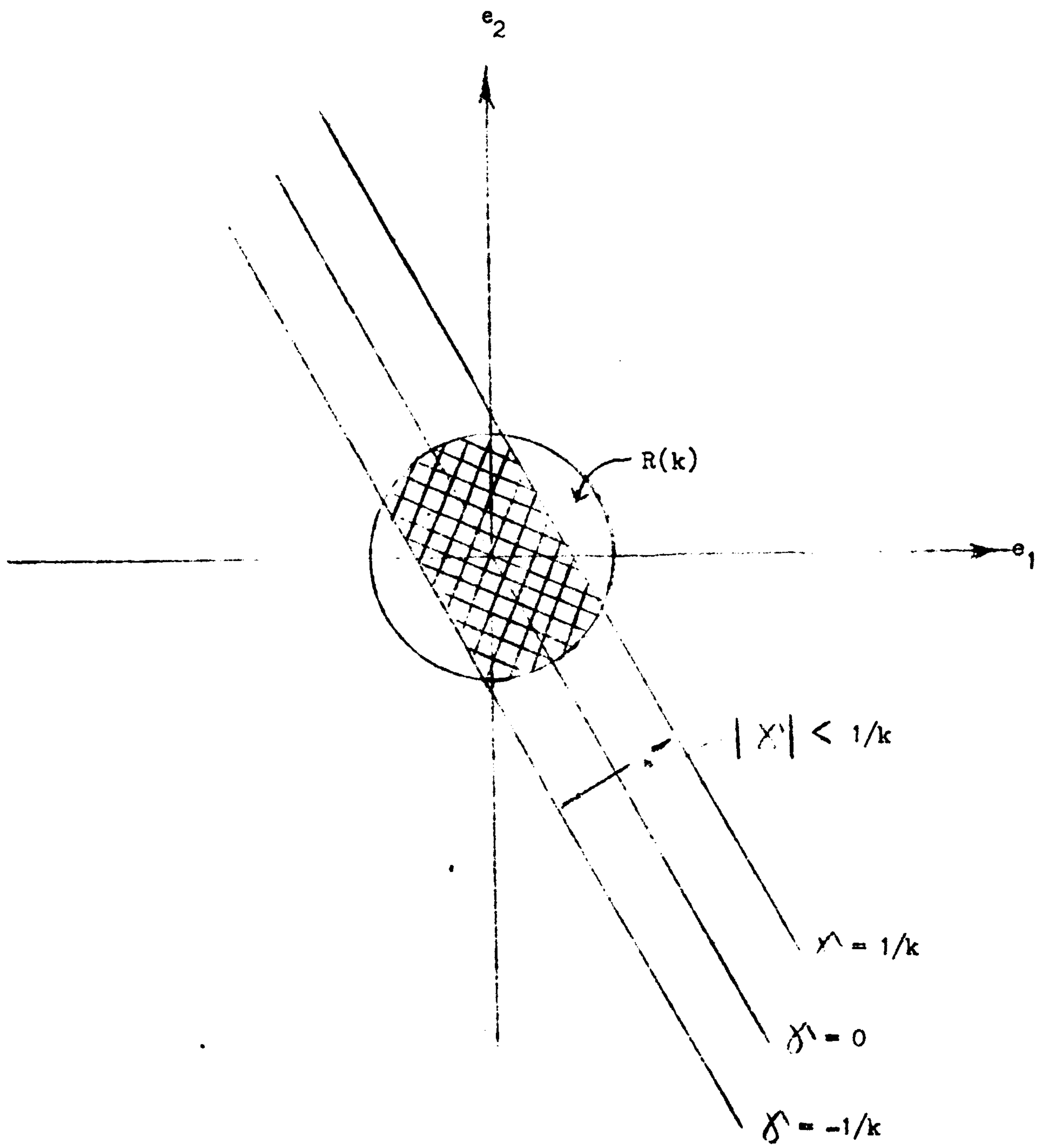

Figure 1. Cross Hatched Area Indicating Where $\dot{V}$ May Be Positive

REFERENCES

1. R. W. Bass, discussion of a paper by A. M. Letov, Proc. Heidelberg Conf. on Automatic Control ("Regelungstechnik: Moderne Theorien und ihre Verwendbarkeit" by R. Oldenbourg, Munich, 1957) pp. 209-210.

2. L. P. Grayson, "Design via Lyapunov's Second Method," Proc. Fourth Joint Conference on Automatic Control, pp. 589-595; 1963.

3. G. W. Johnson, "Synthesis of Control Systems with Stability Constraints; via the Direct Method of Lyapunov," IEEE Trans. on Automatic Control, Vol. AC-9, pp. 270-273; July, 1964.

4. R. V. Monopoli, "The Direct Method of Liapunov Applied to the Design of Controllers for a Class of Nonlinear and Time-Varying Processes," research reported under NASA Research Grant No. NSG-308-63; August, 1963.

5. R. V. Monopoli, "Controller Design for Nonlinear and Time Varying Plants," NASA Contractor Report NASA CR-152; January, 1965.

6. I. Flügge-Lotz, "Discontinuous Automatic Control" (book) Princeton University Press, Princeton, New Jersey; 1953.

7. W. Hahn, "Theory and Application of Liapunov's Direct Method," (book) Prentice-Hall, Inc., Englewood Cliffs, New Jersey; 1963.

8. L. P. Grayson, "Two Theorems on the Second Method," (correspondence) IEEE Trans. on Automatic Control, Vol. AC-9, p.587; October, 1964.