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by James C. Howard

Ames Research Center

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SUMMARY

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Absolute or intrinsic differentiation has been shown to be an effective tool for solving a certain class of vector problems. In many cases, the solution of problems involving the rates of change of vectors may be obtained more directly by this method than by conventional methods. A single formula with sufficient generality to handle a wide range of problems renders application a purely mechanical process, requiring little or no ingenuity on the part of the analyst. In order to demonstrate its utility, the method has been used to process certain fundamental vectors associated with the orbits of planetary bodies or space vehicles. The processing of these vectors leads to Lagrange's planetary equations. It is shown that the method has distinct advantages when used for obtaining the rate of change of the argument of perifocus of an orbiting body.

AUTHOR

INTRODUCTION

It is well known to students of tensor analysis that problems involving the rates of change of vectors of any variance are conveniently solved by the method of covariant and intrinsic differentiation. This method of dealing with vectors, and the more general entities called tensors, has found little or no application to the problems of ordinary vector analysis probably because all such problems are amenable to solution by conventional methods. However, when covariant and intrinsic differentiation are divorced from the complexities inherent in tensor analysis, it is found that certain problems can be solved more directly and with much less ingenuity on the part of the analyst than is required by conventional methods. This is a consequence of the geometrical simplification inherent in the method when curvilinear coordinates are used. For example, intrinsic differentiation may be used to determine the influence of perturbing forces on the time rates of change of certain fundamental vectors associated with the orbits of planetary bodies or space vehicles. When this is done, a more direct determination of the rates of change of some of the orbital elements is possible. In particular, the intrinsic derivative of a vector which lies in the orbital plane, and points in the direction of the perifocus, leads more directly to a rate of change of the argument of perifocus which includes the influence of changes in the longitude of the ascending node. Furthermore, since the orientation of a plane is uniquely determined by the normal to its surface, the orientation of an orbital plane in space is determined by the angular momentum vector. This fact may be used to advantage in finding the time rates of change of the elements defining the orientation of an orbit plane in the presence of perturbing forces. The intrinsic derivative may be used to obtain the rate of change of the angular momentum vector and hence the rates of change of orbital plane inclination and nodal longitude. These two examples have been chosen to illustrate the utility of the method. Moreover, the method may be applied with equal facility to any situation in which vector changes are involved.

NOMENCLATURE

$\bar{A}$	vector
$A^i(x)$	contravariant vector components in the x coordinate system
$A_i(x)$	covariant vector components in the x coordinate system
a	the semimajor axis
$\hat{a}$	unit vector which lies in the orbital plane and points in the direction of the perifocus
$\bar{a}^i(x)$	system of base vectors reciprocal to $\bar{a}_i(x)$
$\bar{a}_i(x)$	system of base vectors in the x coordinate system
$B^j(y)$	contravariant vector components in the y coordinate system
$B_j(y)$	covariant vector components in the y coordinate system
$\bar{b}^j(y)$	system of base vectors reciprocal to $\bar{b}_j(y)$
$\bar{b}_j(y)$	system of base vectors in the y coordinate system
e	eccentricity
$\bar{e}$	vector lying in the orbital plane and pointing in the direction of the perifocus
g_{ij}	$\bar{a}_i \cdot \bar{a}_j$
g^{ij}	$\bar{a}^i \cdot \bar{a}^j$
$\bar{h}$	angular momentum vector
i	orbital plane inclination
M	mass of central body
m	mass of space vehicle or planet
$\bar{P}$	perturbing force vector

$\bar{r}$	position vector
T	coordinate transformation
t	time
$\bar{V}$	velocity vector
x^i	components of the position vector in the x coordinate system
y^j	components of the position vector in the y coordinate system
$[ij,k]$	Christoffel symbol of the first kind
$\left\{ {i \atop jk} \right\}$	Christoffel symbol of the second kind
α^i_j	constant coefficients
γ	true anomaly
δ^i_k	Kronecker delta
μ	dynamical constant, $G(M + m)$
Ω	longitude of ascending node line
ω	argument of perifocus as measured from ascending node line
$\tilde{\omega}$	$\omega + \Omega$

Superscripts

i,j,k,l	indices of contravariance

Subscripts

i,j,k,l	indices of covariance

ANALYTICAL CONSIDERATIONS

Transformation Laws for Scalar Components and Base Vectors

Scalar components.- When referred to a general curvilinear coordinate system, a vector $\bar{A}$ may be expressed in the following form

$$\bar{A} = A^i \bar{a}_i \tag{1}$$

If in some expression a certain index occurs twice, this means that the expression is to be summed with respect to that index for all admissible values of the index, that is,

$$A^i\bar{a}_i = \sum_{i=1}^{n} A^i\bar{a}_i$$

where A^i are the tensor components of the vector $\bar{A}$, and $\bar{a}_i$, a system of base vectors. In accordance with established notation, tensor components will be denoted by superscripts and the corresponding base vectors by subscripts. In the literature, these vectors are referred to as contravariant vectors, to distinguish them from other vectors which are denoted by subscripts. For the problems considered in the present report, the distinction between these vectors disappears. However, it is necessary to keep the distinction in mind, because if general coordinate transformations are contemplated the transformation law for the components of a contravariant vector denoted by superscripts, differs from that for a vector denoted by subscripts. The latter vectors are referred to as covariant vectors. For a coordinate transformation T from a coordinate system x to a coordinate system y given by

$$y^i = y^i(x^1 x^2, \ldots, x^n) \tag{2}$$

the law of transformation for the components of a contravariant vector A^i is given by (see appendix A and ref. 1):

$$B^j(y) = \frac{\partial y^j}{\partial x^i} A^i(x) \tag{3}$$

where $A^i(x)$ are the contravariant components in the x coordinate system and $B^j(y)$ are the components when referred to the y coordinate system. For the same transformation of coordinates, other vectors, such as the gradient of a scalar point function, obey a different transformation law. These are the covariant vectors denoted by subscripts. The appropriate transformation law for these vector components is (see appendix A)

$$B_j(y) = \frac{\partial x^i}{\partial y^j} A_i(x) \tag{4}$$

where $A_i(x)$ are the covariant components in the x coordinate frame and $B_j(y)$ are the covariant components when referred to the y coordinate frame. As the following argument shows, the distinction between these two transformation laws vanishes when the transformation T is orthogonal Cartesian. Let x^i be the components of a position vector $\bar{r}$ when referred to the x coordinate system which is orthogonal Cartesian. Likewise, let y^j be components of the same vector when referred to another orthogonal Cartesian system. The transformation of coordinates T is given by

$$y^i = \alpha^i_j x^j \tag{5}$$

where the α^i_j are constants. The position vector $\bar{r}$ is invariant with respect to coordinate transformations. Hence the square of the vector is also invariant. Therefore

$$x^j x^j = y^i y^i = \alpha^i_j \alpha^i_k x^j x^k = \delta^j_k x^j x^k$$

therefore

$$\alpha^i_j \alpha^i_k = \delta^j_k \tag{6}$$

where δ^j_k is the Kronecker delta, that is, (see ref. 2)

$$\delta^j_k = \begin{matrix} 1 & \text{for } j = k \\ 0 & \text{for } j \neq k \end{matrix}$$

Equation (6) is the orthogonality condition which may be used to solve equation (5) for x^j. If both sides of equation (5) are multiplied by α^i_k,

$$\alpha^i_j \alpha^i_k x^j = \alpha^i_k y^i$$

therefore

$$\delta^j_k x^j = x^k = \alpha^i_k y^i$$

Therefore,

$$x^j = \alpha^i_j y^i \tag{7}$$

From equation (5), it is seen that

$$\frac{\partial y^i}{\partial x^j} = \alpha^i_j \tag{8}$$

and from equation (7)

$$\frac{\partial x^j}{\partial y^i} = \alpha^i_j \tag{9}$$

It follows from equations (8) and (9) that

$$\frac{\partial y^i}{\partial x^j} = \frac{\partial x^j}{\partial y^i} \tag{10}$$

As a consequence of equation (10), the distinction between contravariant and covariant vectors disappears, when coordinate transformations are confined to orthogonal Cartesians systems. This also explains why there is no preoccupation with these vectors in the study of ordinary vector analysis.

Base vectors.- Subscripts assigned to a system of base vectors $\bar{a}_i$ indicate that they are covariant in character, and obey the convariant transformation law. See equation (4) and appendix A. Therefore, if $\bar{a}_i(x)$ are a system of base vectors in the x coordinate system, and $\bar{b}_j(y)$ are the corresponding base vectors in the y coordinate system, then

$$\bar{b}_j(y) = \frac{\partial x^i}{\partial y^j} \bar{a}_i(x) \tag{11}$$

In this connection it should be noted that to every system of base vector $\bar{a}_i$, there exists a reciprocal system of base vectors $\bar{a}^i$ with the following property

$$\bar{a}^j \cdot \bar{a}_i = \delta_i^j = \bar{a}_i \cdot \bar{a}^j \tag{12}$$

where

$$\delta_i^j = \begin{matrix} 1 & \text{for} & j = i \\ 0 & \text{for} & j \neq i \end{matrix}$$

A superscript is assigned to the reciprocal base vectors to indicate their contravariant character, and to emphasize the fact that they obey the contravariant transformation law. (See eq. (3) and appendix A.) Hence, if $\bar{a}^i(x)$ are the reciprocal base vectors in the x coordinate system and $\bar{b}^j(y)$ are the corresponding vectors in the y coordinate system, then

$$\bar{b}^j(y) = \frac{\partial y^j}{\partial x^i} \bar{a}^i(x) \tag{13}$$

In a curvilinear coordinate system the base vectors are, in general, not unit vectors, but are functions of the coordinates; that is,

$$\bar{a}_i = \bar{a}_i(x^1 x^2, \ldots, x^n) \tag{14}$$

$$\bar{a}^j = \bar{a}^j(x^1 x^2, \ldots, x^n) \tag{15}$$

The base vectors may be obtained as follows: let $d\bar{r}$ be the differential of a position vector $\bar{r}$ and let dx^i be the corresponding differentials of the position components. Then by substituting $d\bar{r}$ for $\bar{A}$, and dx^i for A^i in equation (1), we have

$$d\bar{r} = dx^i \bar{a}_i \tag{16}$$

From equation (16) the base vectors $\bar{a}_i$ are given by

$$\bar{a}_i = \frac{\partial \bar{r}}{\partial x^i} \tag{17}$$

In an orthogonal Cartesian frame of reference, the base vectors $\bar{a}_i$ constitute a triad of mutually orthogonal unit vectors, that is, vectors of unit length. However, in problem formulation, it is usually convenient to use a more general curvilinear coordinate system. When this is done, the magnitudes of the base vectors generally differ from unity.

Vector Derivatives and the Christoffel Symbols

The scalar product of any two base vectors $\bar{a}_i$ and $\bar{a}_j$ may be defined as follows:

$$\bar{a}_i \cdot \bar{a}_j = g_{ij} = \bar{a}_j \cdot \bar{a}_i \tag{18}$$

Likewise, the scalar product of the reciprocal base vectors $\bar{a}^i$ and $\bar{a}^j$ may be defined as

$$\bar{a}^i \cdot \bar{a}^j = g^{ij} = \bar{a}^j \cdot \bar{a}^i \tag{19}$$

The symmetry of g_{ij} and g^{ij} follows from the nature of the scalar product. Certain combinations of the partial derivatives of the scalar products with respect to the system coordinates are useful in obtaining the derivative of a vector, or in formulating the equations of motion in a general curvilinear coordinate system. The definitions that follow are ascribed to Christoffel and are called Christoffel symbols (see ref. 3). There are two of these symbols, the first of which is defined as

$$[ij, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) \tag{20}$$

The Christoffel symbol of the second kind is

$$\left\{ {k \atop ij} \right\} = g^{kl}[ij, l] \tag{21}$$

The utility of the Christoffel symbols is immediately apparent when an attempt is made to find the partial derivative of a base vector, or its reciprocal, with respect to any system coordinate. Any vector $\bar{A}$ may be expressed in the form of equation (1). Furthermore, since the base vectors are in general functions of the coordinates, it follows that the derivative of $\bar{A}$ with respect to any coordinate must involve the Christoffel symbols. From equation (1), the partial derivative of the vector $\bar{A}$ with respect to the coordinate x^k is given by

$$\frac{\partial \bar{A}}{\partial x^k} = \frac{\partial A^i}{\partial x^k} \bar{a}_i + A^i \frac{\partial \bar{a}^i}{\partial x^k} \tag{22}$$

However, since $\bar{a}_i \cdot \bar{a}_j = g_{ij}$

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial \bar{a}_i}{\partial x^k} \cdot \bar{a}_j + \bar{a}_i \cdot \frac{\partial \bar{a}_j}{\partial x^k} \tag{23}$$

Likewise,

$$\frac{\partial g_{jk}}{\partial x^i} = \frac{\partial a^j}{\partial x^i} \cdot \bar{a}_k + \bar{a}_j \cdot \frac{\partial \bar{a}_k}{\partial x^i} \tag{24}$$

and

$$\frac{\partial g_{ik}}{\partial x^j} = \frac{\partial \bar{a}_i}{\partial x^j} \cdot \bar{a}_k + \bar{a}_i \cdot \frac{\partial \bar{a}_k}{\partial x^j} \tag{25}$$

Since

$$\bar{a}_i = \frac{\partial \bar{r}}{\partial x^i}$$

it follows that

$$\frac{\partial \bar{a}_i}{\partial x^j} = \frac{\partial}{\partial x^j}\left(\frac{\partial \bar{r}}{\partial x^i}\right) = \frac{\partial}{\partial x^i}\left(\frac{\partial \bar{r}}{\partial x^j}\right) = \frac{\partial \bar{a}_j}{\partial x^i} \tag{26}$$

From equations (23) through (26)

$$\frac{\partial \bar{a}_i}{\partial x^j} \cdot \bar{a}_k = [ij, k] \tag{27}$$

From equation (12), the rate of change of the base vector $\bar{a}_i$ with respect to x_j assumes the form

$$\frac{\partial \bar{a}_i}{\partial x^j} = [ij, k]\bar{a}^k \tag{28}$$

Equation (28) gives the required rate of change of the base vector, with respect to a system coordinate, in terms of the Christoffel symbol of the first kind and the reciprocal base vectors. A more convenient form is obtained if both sides of equation (28) are multiplied scalarly by the reciprocal base vector $\bar{a}^l$ to yield

$$\frac{\partial \bar{a}_i}{\partial x^j} \cdot \bar{a}^l = [ij, k]\bar{a}^k \cdot \bar{a}^l \tag{29}$$

From equation (19), it is seen that

$$\bar{a}^k \cdot \bar{a}^l = g^{kl}$$

therefore

$$\frac{\partial \bar{a}_i}{\partial x^j} \cdot \bar{a}^l = [ij, k]g^{kl} \tag{30}$$

In terms of the defining formula (21), equation (30) may be rewritten as follows:

$$\frac{\partial \bar{a}_i}{\partial x^j} \cdot \bar{a}^l = \begin{Bmatrix} l \\ ij \end{Bmatrix} \tag{31}$$

Therefore,

$$\frac{\partial \bar{a}_i}{\partial x^j} = \begin{Bmatrix} l \\ ij \end{Bmatrix} \bar{a}_l \tag{32}$$

By substitution of equation (32) in equation (22) the partial derivative of a vector $\bar{A}$ with respect to the system coordinate x^k is

$$\frac{\partial \bar{A}}{\partial x^k} = \frac{\partial A^i}{\partial x^k} \bar{a}_i + A^i \begin{Bmatrix} l \\ ik \end{Bmatrix} \bar{a}_l \tag{33}$$

The indices i and l in the second term on the right side of equation (33) are dummy[1] indices, and may therefore be replaced by any other convenient indices. In order to have a common base vector $\bar{a}_i$, equation (33) may be rewritten as follows

$$\frac{\partial \bar{A}}{\partial x^k} = \left(\frac{\partial A^i}{\partial x^k} + A^j \begin{Bmatrix} i \\ jk \end{Bmatrix}\right) \bar{a}_i \tag{34}$$

Furthermore, since

$$\frac{\partial \bar{A}}{\partial x^k} \frac{dx^k}{dt} = \frac{d\bar{A}}{dt}$$

and

$$\frac{\partial A^i}{\partial x^k} \frac{dx^k}{dt} = \frac{dA^i}{dt}$$

the intrinsic derivative, or the derivative with respect to time, may be obtained from equation (34) in the following form:

$$\frac{d\bar{A}}{dt} = \left(\frac{dA^i}{dt} + \begin{Bmatrix} i \\ jk \end{Bmatrix} A^j \frac{dx^k}{dt}\right) \bar{a}_i \tag{35}$$

[1]As already indicated, a repeated index implies summation with respect to that index. Since the summation index can be changed at will, it is usually referred to as a dummy index. Of course, the range of admissible values of the index must be preserved.

In an orthogonal Cartesian reference frame

$$g_{ij} = \bar{a}_i \cdot \bar{a}_j = \delta^i_j = \bar{a}^i \cdot \bar{a}^j = g^{ij}$$

Therefore, since all these scalar products are constants, it follows that the Christoffel symbols vanish. In this case, the covariant derivative, (34) reduces to the sum of the partial derivatives of the components along a set of fixed axes

$$\frac{\partial \bar{A}}{\partial x^k} = \frac{\partial A^i}{\partial x^k}\bar{a}_i \qquad i = 1, 2, 3$$

Likewise, the intrinsic derivative of a vector reduces to the ordinary time rates of change of the components along a set of fixed axes.

For a general space of three dimensions, equation (35) assumes the form

$$\frac{d\bar{A}}{dt} = \left[\left(\frac{dA^1}{dt} + f_1\right)\bar{a}_1 + \left(\frac{dA^2}{dt} + f_2\right)\bar{a}_2 + \left(\frac{dA^3}{dt} + f_3\right)\bar{a}_3\right] \tag{36}$$

$$f_1 = \left[A^1\left(\begin{Bmatrix}1\\11\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}1\\12\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}1\\13\end{Bmatrix}\frac{dx^3}{dt}\right) + A^2\left(\begin{Bmatrix}1\\21\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}1\\22\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}1\\23\end{Bmatrix}\frac{dx^3}{dt}\right)\right.$$

$$\left. + A^3\left(\begin{Bmatrix}1\\31\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}1\\32\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}1\\33\end{Bmatrix}\frac{dx^3}{dt}\right)\right] \tag{37}$$

$$f_2 = \left[A^1\left(\begin{Bmatrix}2\\11\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}2\\12\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}2\\13\end{Bmatrix}\frac{dx^3}{dt}\right) + A^2\left(\begin{Bmatrix}2\\21\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}2\\22\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}2\\23\end{Bmatrix}\frac{dx^3}{dt}\right)\right.$$

$$\left. + A^3\left(\begin{Bmatrix}2\\31\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}2\\32\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}2\\33\end{Bmatrix}\frac{dx^3}{dt}\right)\right] \tag{38}$$

$$f_3 = \left[A^1\left(\begin{Bmatrix}3\\11\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}3\\12\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}3\\13\end{Bmatrix}\frac{dx^3}{dt}\right) + A^2\left(\begin{Bmatrix}3\\21\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}3\\22\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}3\\23\end{Bmatrix}\frac{dx^3}{dt}\right)\right.$$

$$\left. + A^3\left(\begin{Bmatrix}3\\31\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}3\\32\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}3\\33\end{Bmatrix}\frac{dx^3}{dt}\right)\right] \tag{39}$$

The formidable looking equations (36) through (39) for the intrinsic derivative of a vector in a general space of three dimensions contain 27 Christoffel symbols. Because of the symmetry of the Christoffel symbols,

$$\begin{Bmatrix}k\\ij\end{Bmatrix} = \begin{Bmatrix}k\\ji\end{Bmatrix} \tag{40}$$

and the number of independent Christoffel symbols reduces to 18. Furthermore, for the three-dimensional spaces most commonly used, equation (36) reduces to a manageable form. If a base vector of unit length be denoted by $\hat{a}_i$, then in a cylindrical coordinate system,

$$\left.\begin{aligned} \bar{a}_1 &= \hat{a}_1 , & g_{11} &= 1 \\ \bar{a}_2 &= x^1 \hat{a}_2 , & g_{22} &= (x^1)^2 \\ \bar{a}_3 &= \hat{a}_3 , & g_{33} &= 1 \end{aligned}\right\} \tag{41}$$

As a consequence of equations (41), there are only two nonzero Christoffel symbols in a cylindrical coordinate system, embedded in a space of three dimensions. These are

$$\left.\begin{aligned} \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} &= -x^1 \\ \text{and} \qquad \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} &= \frac{1}{x^1} \end{aligned}\right\} \tag{42}$$

Hence, a vector referred to this coordinate system has a time rate of change as follows:

$$\frac{d\bar{A}}{dt} = \left\{ \left(\frac{dA^1}{dt} + A^2 \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} \frac{dx^2}{dt} \right) \bar{a}_1 + \left[\frac{dA^2}{dt} + \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} \left(A^1 \frac{dx^2}{dt} + A^2 \frac{dx^1}{dt} \right) \right] \bar{a}_2 + \frac{dA^3}{dt} \bar{a}_3 \right\} \tag{43}$$

In three-dimensional spherical coordinates,

$$\left.\begin{aligned} \bar{a}_1 &= \hat{a}_1 , & g_{11} &= 1 \\ \bar{a}_2 &= x^1 \hat{a}_2 , & g_{22} &= (x^1)^2 \\ \bar{a}_3 &= x^1 \sin x^2 \hat{a}_3 , & g_{33} &= (x^1 \sin x^2)^2 \end{aligned}\right\} \tag{44}$$

In this case there are six nonzero Christoffel symbols. These are

$$\left.\begin{aligned} \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} &= -x^1 & \begin{Bmatrix} 2 \\ 33 \end{Bmatrix} &= -\sin x^2 \cos x^2 \\ \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} &= \frac{1}{x^1} & \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 31 \end{Bmatrix} &= \frac{1}{x^1} \\ \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} &= -x^1 \sin^2 x^2 & \begin{Bmatrix} 3 \\ 23 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 32 \end{Bmatrix} &= \cot x^2 \end{aligned}\right\} \tag{45}$$

When these values of the Christoffel symbols are substituted in equation (35), the time rate of change of a vector, referred to a three-dimensional spherical coordinate system, assumes the following form:

$$\frac{d\bar{A}}{dt} = \left(\frac{dA^1}{dt} + A^2 \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} \frac{dx^2}{dt} + A^3 \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} \frac{dx^3}{dt}\right) \bar{a}_1$$

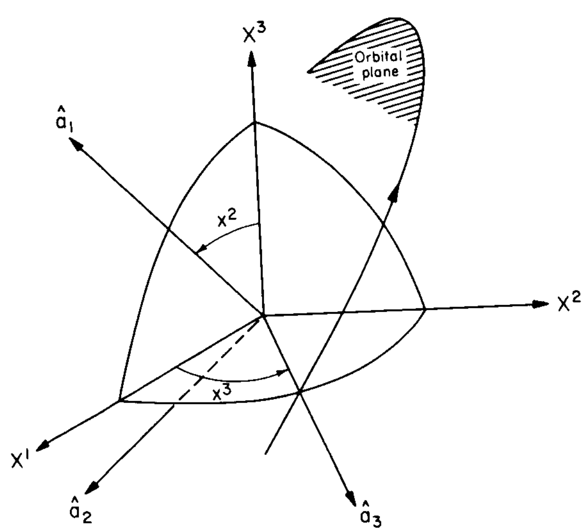
$$+ \left[\frac{dA^2}{dt} + \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} \left(A^1 \frac{dx^2}{dt} + A^2 \frac{dx^1}{dt}\right) + A^3 \begin{Bmatrix} 2 \\ 33 \end{Bmatrix} \frac{dx^3}{dt}\right] \bar{a}_2$$

$$+ \left[\frac{dA^3}{dt} + \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} \left(A^1 \frac{dx^3}{dt} + A^3 \frac{dx^1}{dt}\right) + \begin{Bmatrix} 3 \\ 23 \end{Bmatrix} \left(A^2 \frac{dx^3}{dt} + A^3 \frac{dx^2}{dt}\right)\right] \bar{a}_3 \quad (46)$$

APPLICATIONS

Intrinsic Derivatives of Certain Fundamental Vectors Associated With the Orbits of Planetary Bodies and Space Vehicles

Certain fundamental vectors associated with an orbit in space, and which are subject to change in the presence of perturbing forces, provide a useful area of application for intrinsic differentiation. One example of such a vector is the angular momentum vector $\bar{h}$ which lies in the direction of the normal to the orbital plane. This vector may be used to determine the orientation of an orbital plane in space. Furthermore, the rate of change of this vector in the presence of perturbing torques may be used to determine the rates of change of orbital plane inclination and nodal longitude. The intrinsic derivative of a vector $\bar{e}$, which lies in the orbital plane and is directed to the perifocus, may be used to determine the rates of change of the argument of perifocus and the eccentricity. A third vector may be used to complete the orthogonal triad.



Sketch (a)

The angular momentum vector.- Equation (35) may be used to obtain the rates of change of the angular momentum vector, and hence the rates of change of the orbital elements defining the orientation of an orbital plane in space. In spherical coordinates, equation (35) assumes the form shown in equation (46). The coordinate uses are chosen as follows (see sketch (a)):

1. The $\hat{a}_1$ axis is taken to be coincident with the angular momentum vector $\bar{h}$.

2. The $\hat{a}_2$ axis is in the direction of increasing polar angle, that is, in the direction of increasing orbital plane inclination.

3. The third axis $\hat{a}_3$ is in the direction of the ascending node line and completes the mutually orthogonal triad of axes.

In this coordinate system the base vectors are given by equations (44). Let the vector $\bar{A}$ appearing in equation (46) have components as follows

$$A^1 = h \ , \qquad A_2 = A_3 = 0 \tag{47}$$

When substitutions are made from equations (44), (45), and (47) in equation (46), it is found that

$$\frac{d\bar{A}}{dt} = \frac{d\bar{h}}{dt} = \left[\frac{dh}{dt}\,\hat{a}_1 + \left(\frac{1}{x^1}\,h\,\frac{dx^2}{dt}\right)x^1\hat{a}_2 + \frac{1}{x^1}\left(h\,\frac{dx^3}{dt}\right)x^1\sin x^2\hat{a}_3\right]$$

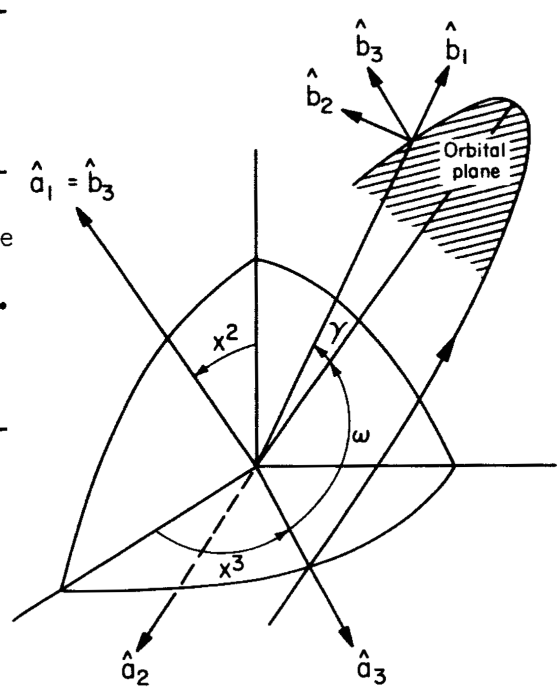
therefore

$$\frac{d\bar{h}}{dt} = \left(\frac{dh}{dt}\,\hat{a}_1 + h\,\frac{dx^2}{dt}\,\hat{a}_2 + h\sin x^2\,\frac{dx^3}{dt}\,\hat{a}_3\right) \tag{48}$$

These are the component rates of change of the vector $\bar{h}$, referred to the x coordinate system with base vectors $\bar{a}_i$. However, it may be more convenient to use some other reference frame. Assuming that the perturbing forces are referred to the radial and transverse directions in the plane of the orbit, and in the direction of the normal to the orbital plane, the vector $d\bar{h}/dt$ should be transformed to a reference frame having these directions. If the coordinates in this reference frame be denoted by y^i, where $i = 1, 2, 3$, a coordinate transformation T_1 must be determined before the components of $d\bar{h}/dt$ can be referred to the y coordinate frame. Let $\bar{b}_j(y)$ be a system of base vectors in the y coordinate system (see sketch (b)).

In terms of the notation already established, the transformation of base vectors assumes the following form

$$\left.\begin{aligned} \hat{b}_1 &= -\sin(\gamma + \omega)\hat{a}_2 + \cos(\gamma + \omega)\hat{a}_3 \\ \hat{b}_2 &= -\cos(\gamma + \omega)\hat{a}_2 - \sin(\gamma + \omega)\hat{a}_3 \\ \hat{b}_3 &= \hat{a}_1 \end{aligned}\right\} \tag{49}$$



Sketch (b)

If $A^i(x)$ are the components of $d\bar{h}/dt$ in the x coordinate system and $B^j(y)$ are the corresponding components in the y coordinate system, the transformation law, equation (3) gives

$$B^j(y) = \frac{\partial y^j}{\partial x^i} A^i(x)$$

However,

$$\hat{b}^j \cdot \hat{a}_i = \frac{\partial y^j}{\partial x^i} = \hat{b}_j \cdot \hat{a}_i$$

See appendix A. Therefore, equation (3) may be written in the following alternative form

$$B^j(y) = (\hat{b}_j \cdot \hat{a}_i)A^i(x) \tag{50}$$

From equation (48)

$$A^1 = \frac{dh}{dt}, \quad A^2 = h\frac{dx^2}{dt}, \quad A^3 = h \sin x^2 \frac{dx^3}{dt} \tag{51}$$

Substituting from equations (51) in equation (50), and using equations (49) we obtain the transformed components of the vector dh/dt as follows

$$\left.\begin{aligned} B^1(y) &= -h \sin(\gamma + \omega)\frac{dx^2}{dt} + h \cos(\gamma + \omega) \sin x^2 \frac{dx^3}{dt} \\ B^2(y) &= -h \cos(\gamma + \omega)\frac{dx^2}{dt} - h \sin(\gamma + \omega) \sin x^2 \frac{dx^3}{dt} \\ B^3(y) &= \frac{dh}{dt} \end{aligned}\right\} \tag{52}$$

However,

$$\frac{d\bar{h}}{dt} = \bar{r} \times \bar{P}$$

where $\bar{r}$ is the position vector of the planet or space vehicle in the y coordinate system with base vectors b_j, and $\bar{P}$ is the perturbing force vector. Therefore,

$$\left.\begin{aligned} \bar{r} \times \bar{P} &= r\hat{b}_1 \times (\hat{b}_j\hat{b}_j) \cdot \bar{P} \\ \bar{r} \times \bar{P} &= r\hat{b}_1 \times (\hat{b}_1\hat{b}_1 + \hat{b}_2\hat{b}_2 + \hat{b}_3\hat{b}_3) \cdot \bar{P} \\ \bar{r} \times \bar{P} &= r(\hat{b}_3\hat{b}_2 - \hat{b}_2\hat{b}_3) \cdot \bar{P} \end{aligned}\right\} \tag{53}$$

From equations (52) and (53), it follows that

$$h \cos(\gamma + \omega) \sin x^2 \frac{dx^3}{dt} - h \sin(\gamma + \omega) \frac{dx^2}{dt} = 0$$

$$h \sin(\gamma + \omega) \sin x^2 \frac{dx^3}{dt} + h \cos(\gamma + \omega) \frac{dx^2}{dt} = r(\hat{b}_3 \cdot \bar{P})$$

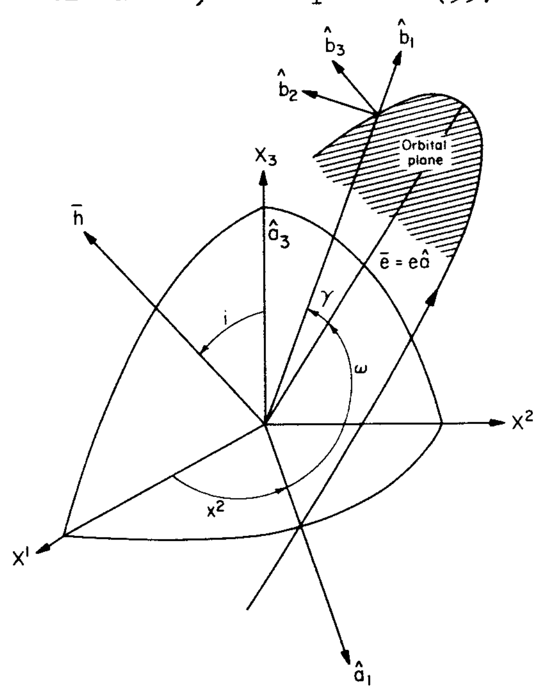
$$\frac{dh}{dt} = r(\hat{b}_2 \cdot \bar{P})$$

These equations give

$$\frac{dx^2}{dt} = \frac{r \cos(\gamma + \omega)}{h} (\hat{b}_3 \cdot \bar{P}) \tag{54}$$

$$\frac{dx^3}{dt} = \frac{r \sin(\gamma + \omega)}{h \sin x^2} (\hat{b}_3 \cdot \bar{P}) \tag{55}$$

Equation (54) gives the rate of change of inclination, and equation (55) the rate of change of nodal longitude.



Sketch (c)

<u>The orbital vector $\bar{e}$</u>.- Consider the vector $\bar{e}$, which lies in the orbital plane and is directed to the perifocus. (See sketch (c) and ref. 4.) In a cylindrical reference frame with coordinates denoted by x^i, the base vectors $\bar{a}_i$ have the following values:

$$\left.\begin{aligned} \bar{a}_1 &= \hat{a}_1 \\ \bar{a}_2 &= x^1\hat{a}_2 \\ \bar{a}_3 &= \hat{a}_3 \end{aligned}\right\} \tag{56}$$

In this case, it should be noted that x^2 is the nodal longitude. In the case previously considered, x^2 was the orbital plane inclination. In this reference frame, which is chosen because the influence of nodal longitude and orbital plane inclination appear in the formulation, the vector $\bar{e}$ has components as follows (see appendix B and sketch (c)):

$$\bar{e} = e\left(\cos \omega\bar{a}_1 + \frac{\sin \omega \cos i}{x^1} \bar{a}_2 + \sin \omega \sin i\bar{a}_3\right)$$

that is,

$$\bar{e} = A^i\bar{a}_i = A^1\bar{a}_1 + A^2\bar{a}_2 + A^3\bar{a}_3$$

where

$$\left.\begin{aligned} A^1 &= e \cos \omega \\ A^2 &= (e \sin \omega \cos i)/x^1 \\ A^3 &= e \sin \omega \sin i \end{aligned}\right\} \tag{57}$$

In cylindrical coordinates, equation (35) assumes the form shown in equation (43). Substituting from equations (42) and (57) in equation (43) gives

$$\begin{aligned} \frac{d\bar{e}}{dt} = \Bigg\{ & \left[\frac{d}{dt}(e \cos \omega) - x^1 \frac{dx^2}{dt} \cdot \frac{e \sin \omega \cos i}{x^1}\right] \hat{a}_1 \\ & + \left[\frac{d}{dt}\left(\frac{e \sin \omega \cos i}{x^1}\right) + \frac{1}{x^1}\left(e \frac{dx^2}{dt} \cos \omega + e \frac{dx^1}{dt} \frac{\sin \omega \cos i}{x^1}\right)\right] x^1 \hat{a}_2 \\ & + \frac{d}{dt}(e \sin \omega \sin i)\hat{a}_3 \Bigg\} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d\bar{e}}{dt} = \Bigg[& \left(\dot{e} \cos \omega - e \sin \omega\dot{\omega} - e \frac{dx^2}{dt} \sin \omega \cos i\right) \hat{a}_1 \\ & + \left(\dot{e} \sin \omega \cos i + e\dot{\omega} \cos \omega \cos i - e \frac{di}{dt} \sin \omega \sin i + e \frac{dx^2}{dt} \cos \omega\right) \hat{a}_2 \\ & + \left(\dot{e} \sin \omega \sin i + e\dot{\omega} \cos \omega \sin i + e \frac{di}{dt} \sin \omega \cos i\right) \hat{a}_3 \Bigg] \end{aligned} \tag{58}$$

These are the component rates of change of the vector $\bar{e}$ referred to the coordinate system with base vectors $\bar{a}_i$. As in the case of the angular momentum vector, it may be more convenient to use some other reference frame. Assuming again that the perturbing forces are referred to the radial and transverse directions in the plane of the orbit, and in the direction of the normal to the orbital plane, the vector $d\bar{e}/dt$ must be transformed to a reference frame having these directions. A coordinate transformation T_2 must be determined before the components of $d\bar{e}/dt$ can be referred to this coordinate frame. Let the base vectors in this reference frame be again denoted by $\bar{b}_j$, then the base vectors $\bar{b}_j$ and $\bar{a}_i$ are related as follows:

$$\left.\begin{aligned} \hat{b}_1 &= \cos(\gamma + \omega)\hat{a}_1 + \sin(\gamma + \omega) \cos i\hat{a}_2 + \sin(\gamma + \omega) \sin i\hat{a}_3 \\ \hat{b}_2 &= -\sin(\gamma + \omega)\hat{a}_1 + \cos(\gamma + \omega) \cos i\hat{a}_2 + \cos(\gamma + \omega) \sin i\hat{a}_3 \\ \hat{b}_3 &= -\sin i\hat{a}_2 + \cos i\hat{a}_3 \end{aligned}\right\} \tag{59}$$

Corresponding to the transformation of coordinates T_2, the transformation law for the components of the vector $d\bar{e}/dt$ is given by equation (50). In this case, the components $A^i(x)$ appearing in the right side of equation (50) may be obtained from equation (58). They are

$$\left.\begin{aligned}
A^1 &= \left(\dot{e}\cos\omega - e\dot{\omega}\sin\omega - e\frac{dx^2}{dt}\sin\omega\cos i\right) \\
A^2 &= \left(\dot{e}\sin\omega\cos i + e\dot{\omega}\cos\omega\cos i - e\frac{di}{dt}\sin\omega\sin i + e\frac{dx^2}{dt}\cos\omega\right) \\
A^3 &= \left(\dot{e}\sin\omega\sin i + e\dot{\omega}\cos\omega\sin i + e\frac{di}{dt}\sin\omega\cos i\right)
\end{aligned}\right\} \quad (60)$$

When substitutions are made from equations (59) and (60) in equation (50), it is found that the vector $d\overline{e}/dt$ has the following components in the directions of the base vector $\overline{b}_j$

$$\left.\begin{aligned}
B^1(y) &= \left(\dot{e}\cos\gamma + e\dot{\omega}\sin\gamma + e\frac{dx^2}{dt}\sin\gamma\cos i\right) \\
B^2(y) &= \left(-\dot{e}\sin\gamma + e\dot{\omega}\cos\gamma + e\frac{dx^2}{dt}\cos\gamma\cos i\right) \\
B^3(y) &= \left(e\frac{di}{dt}\sin\omega - e\frac{dx^2}{dt}\sin i\cos\omega\right)
\end{aligned}\right\} \quad (61)$$

In order to express the rate of change of the vector $\overline{e}$ as a function of the perturbing force vector $\overline{P}$, it is necessary to examine the equation of motion of a particle in a noncentral, gravitational force field. It is shown in equation (B19) that

$$\frac{d\overline{e}}{dt} = \left[\frac{2h}{\mu}\hat{b}_1\hat{b}_2 - \left(\frac{re}{h}\sin\gamma\hat{b}_2\hat{b}_2 + \frac{h}{\mu}\hat{b}_2\hat{b}_1\right) - \frac{re}{h}\sin\gamma\hat{b}_3\hat{b}_3\right]\cdot\overline{P} \quad (62)$$

By equating components from equations (61) and (62), and remembering that in this case $dx^2/dt = d\Omega/dt$, we obtain the following equations:

$$\dot{e}\cos\gamma + e\dot{\omega}\sin\gamma = \frac{2h}{\mu}(\hat{b}_2\cdot\overline{P}) - e\frac{d\Omega}{dt}\sin\gamma\cos i \quad (63)$$

$$e\dot{\omega}\cos\gamma - \dot{e}\sin\gamma = -\left(\frac{re}{h}\sin\gamma\hat{b}_2 + \frac{h}{\mu}\hat{b}_1\right)\cdot\overline{P} - e\frac{d\Omega}{dt}\cos\gamma\cos i \quad (64)$$

$$e\frac{di}{dt}\sin\omega - e\frac{d\Omega}{dt}\sin i\cos\omega = -\frac{re}{h}\sin\gamma(\hat{b}_3\cdot\overline{P}) \quad (65)$$

The rate of change of the eccentricity $\dot{e}$ and the rate of the change of the argument of perifocus may be obtained by solving equations (63) and (64). When this is done, it is found that

$$\dot{\omega} = \left[-\frac{h}{\mu e}\cos\gamma(\hat{b}_1\cdot\overline{P}) + \frac{r}{eh}\sin\gamma(2 + e\cos\gamma)(\hat{b}_2\cdot\overline{P}) - \frac{d\Omega}{dt}\cos i\right] \quad (66)$$

It is seen that the influence of changes in the longitude of the ascending node is given by the third term on the right side of equation (66), that is,

$$\frac{\partial\dot{\omega}}{\partial\dot{\Omega}} = -\cos i \tag{67}$$

However, if the argument of perifocus as measured from the inertially fixed X^1 axis is denoted by $\tilde{\omega}$, then

$$\dot{\tilde{\omega}} = \dot{\omega} + \dot{\Omega} \tag{68}$$

In this case, the total contribution from the rate of change of the nodal longitude is

$$\frac{\partial\dot{\tilde{\omega}}}{\partial\dot{\Omega}}\dot{\Omega} = \dot{\Omega}(1 - \cos i) = 2\dot{\Omega}\sin^2\left(\frac{i}{2}\right) \tag{69}$$

From equation (55)

$$\dot{\Omega} = \frac{r\sin(\gamma + \omega)}{h\sin i}(\hat{b}_3 \cdot \bar{P})$$

therefore

$$\frac{\partial\dot{\tilde{\omega}}}{\partial\dot{\Omega}}\dot{\Omega} = \frac{r}{h}\sin(\gamma + \omega)\tan\left(\frac{1}{2}i\right)(\hat{b}_3 \cdot \bar{P}) \tag{70}$$

Substituting from equation (70) in equation (68) gives

$$\dot{\tilde{\omega}} = \left\{-\left(\frac{h}{\mu e}\cos\gamma\right)\hat{b}_1 + \frac{r}{eh}\sin\gamma(2 + e\cos\gamma)\hat{b}_2 + \left[\frac{r}{h}\sin(\gamma + \omega)\tan\left(\frac{1}{2}i\right)\right]b_3\right\}\cdot\bar{P} \tag{71}$$

The derivation of this result should be compared with the approach used in reference 4, where spherical geometry had to be used to determine the influence of inclination and nodal longitude. In that case, the problem had to be solved in two parts, whereas in the present case the problem is solved in one step without appeal to system geometry, subsequent to the choice of curvilinear coordinates.

Likewise, on solving equations (63) and (64) for $\dot{e}$, it is found that

$$\dot{e} = \left\{\left(\frac{h}{\mu}\sin\gamma\right)\hat{b}_1 + \frac{r}{h}\left[\cos\gamma(2 + e\cos\gamma) + e\right]\hat{b}_2\right\}\cdot\bar{P} \tag{72}$$

If required, the rate of change of the semimajor axis may be obtained with the aid of equation (72) and the following relationship

$$h^2 = \mu a(1 - e^2)$$

therefore

$$2h \frac{dh}{dt} = \mu(1 - e^2)\dot{a} - 2\mu ae\dot{e}$$

therefore

$$\dot{a} = \frac{2\mu ae}{\mu(1 - e^2)} \dot{e} + \frac{2h}{\mu(1 - e^2)} \frac{dh}{dt}$$

therefore

$$\dot{a} = \frac{2ae}{(1 - e^2)} \dot{e} + \frac{2hr}{\mu(1 - e^2)} (\hat{b}_2 \cdot \overline{P})$$

Therefore,

$$\dot{a} = \frac{2a^2}{h} \left[e \sin \gamma \hat{b}_1 + (1 + e \cos \gamma)\hat{b}_2 \right] \cdot \overline{P} \tag{73}$$

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Feb. 4, 1965

APPENDIX A

TRANSFORMATION FORMULAS FOR THE BASE VECTORS AND THEIR RECIPROCALS

The transformation formulas and, hence, the covariant or contravariant character of the base vectors and their reciprocals may be obtained as follows: Let the differential of a position vector be denoted by $d\bar{r}$. Then if $\bar{a}_i(x)$ are the base vectors in the x coordinate system, and $\bar{b}_j(y)$ the base vectors in the y coordinate system, the differential $d\bar{r}$ may be expressed in the following alternative forms

$$d\bar{r} = \bar{a}_i(x)dx^i = \bar{b}_j(y)dy^j = \bar{b}_j(y)\frac{\partial y^j}{\partial x^i}dx^i \tag{A1}$$

therefore

$$\bar{a}_i(x) = \frac{\partial y^j}{\partial x^i}\bar{b}_j(y) \tag{A2}$$

Likewise,

$$\bar{a}_i(x)\frac{\partial x^i}{\partial y^j}dy^j = \bar{b}_j(y)dy^j$$

therefore

$$\bar{b}_j(y) = \frac{\partial x^i}{\partial y^j}\bar{a}_i(x) \tag{A3}$$

It is seen from equations (A2) and (A3) that the base vectors $\bar{a}_i$ and $\bar{b}_j$ obey the covariant transformation law; consequently, the use of subscripts is justified.

Reciprocal Base Vectors

To each system of base vectors $\bar{a}_i$ there exists a reciprocal system of vectors $\bar{a}^j$ with the following property

$$\bar{a}_i \cdot \bar{a}^j = \delta_i^j = \bar{a}^j \cdot \bar{a}_i \tag{A4}$$

where δ_i^j is the Kronecker delta, that is,

$$\delta_i^j = \begin{matrix} 1 & \text{for } j = i \\ 0 & \text{for } j \neq i \end{matrix}$$

Scalar multiplication of each side of equation (A2) by $\bar{b}^j(y)$ gives on using (A4)(see ref. 5)

$$\bar{b}^j(y) \cdot \bar{a}_i(x) = \frac{\partial y^j}{\partial x^i} \tag{A5}$$

Similarly, from equations (A3) and (A4) it is seen that

$$\bar{a}^i(x) \cdot \bar{b}_j(y) = \frac{\partial x^i}{\partial y^j} \tag{A6}$$

Equation (A1) referred to the reciprocal system of base vectors assumes the form

$$d\bar{r} = \bar{a}^i(x)dx_i = \bar{b}^j(y)dy_j \tag{A7}$$

therefore

$$dy_j = \bar{b}_j(y) \cdot \bar{a}^i(x)dx_i$$

therefore

$$dy_j = \frac{\partial x^i}{\partial y^j} dx_i \tag{A8}$$

and

$$dx_i = \bar{b}^j(y) \cdot \bar{a}_i(x)dy_j$$

therefore

$$dx_i = \frac{\partial y^j}{\partial x^i} dy_j \tag{A9}$$

From equations (A7) and (A8)

$$\bar{a}^i(x)dx_i = \bar{b}^j(y) \frac{\partial x^i}{\partial y^j} dx_i$$

therefore

$$\bar{a}^i(x) = \frac{\partial x^i}{\partial y^j} \bar{b}^j(y) \tag{A10}$$

Likewise, from equations (A7) and (A9)

$$\bar{b}^j(y) = \frac{\partial y^j}{\partial x^i} \bar{a}^i(x) \tag{A11}$$

From equations (A10) and (A11), it is seen that the reciprocal base vectors $\bar{a}^i$, obey the contravariant law of transformation; therefore, the superscript notation is justified.

Vector Transformations

Equations (A10) and (A11) may be used to obtain the transformation law for a vector $\bar{A}$, where

$$\bar{A} = A^i\bar{a}_i = A_j\bar{a}^j \tag{A12}$$

If $\bar{A} = A^i(x)\bar{a}_i(x)$ when the vector $\bar{A}$ is referred to the x coordinate system, and if $\bar{A} = B^j(y)\bar{b}_j(y)$ when referred to the y coordinate system, invariance of $\bar{A}$ requires that

$$B^j(y)\bar{b}_j(y) = A^i(x)\bar{a}_i(x) \tag{A13}$$

From equations (A2) and (A13), the appropriate transformation law is obtained as follows:

$$B^j(y) = \frac{\partial y^j}{\partial x^i} A^i(x) \tag{A14}$$

Equation (A14) is the contravariant transformation law for the components of the vector $\bar{A}$. When $\bar{A}$ is referred to the x coordinate system with base vectors $\bar{a}_i(x)$, which obey the covariant transformation law, the components $A^i(x)$ obey the contravariant transformation law, and, hence, the use of superscripts is justified. If $\bar{A}$ is referred to the reciprocal base system $\bar{a}^i$, then from equation (A12)

$$\bar{A} = A_i\bar{a}^i$$

On a transformation of coordinates from the x coordinate system to the y coordinate system, the invariance of $\bar{A}$ requires that

$$A_i(x)\bar{a}^i(x) = B_j(y)\bar{b}^j(y) \tag{A15}$$

From equations (A10) and (A15), the appropriate transformation law is obtained as follows

$$B_j(y) = \frac{\partial x^i}{\partial y^j} A_i(x) \tag{A16}$$

It is seen that when a vector $\bar{A}$ is referred to a coordinate system by use of reciprocal base vectors, which obey the contravariant law, the corresponding components of $\bar{A}$ obey the covariant law, and the use of subscripts is therefore justified.

APPENDIX B

EQUATION OF MOTION OF A PARTICLE IN A NONCENTRAL GRAVITATIONAL FORCE FIELD

The equation of motion of a particle of unit mass moving in an inverse-square law, central force field is

$$\frac{d^2\bar{r}}{dt^2} = -\frac{\mu}{r^2}\hat{b}_1 \tag{B1}$$

Vector multiplication of each side of equation (B1) by the angular momentum vector $\bar{h}$ gives

$$\frac{d^2\bar{r}}{dt^2} \times \bar{h} = -\frac{\mu}{r^2}(\hat{b}_1 \times \bar{h}) \tag{B2}$$

$$\bar{h} = \bar{r} \times \bar{V} = (r^2\dot{\gamma})\hat{b}_3 \tag{B3}$$

On substitution from equation (B3) in equation (B2) it is seen that

$$\frac{d^2\bar{r}}{dt^2} \times \bar{h} = (\mu\dot{\gamma})\hat{b}_2$$

therefore

$$\frac{d}{dt}\left(\frac{\bar{V} \times \bar{h}}{\mu}\right) = \frac{d}{dt}\hat{b}_1 \tag{B4}$$

The integral of equation (B4) is given by

$$\frac{\bar{V} \times \bar{h}}{\mu} = \hat{b}_1 + \bar{e} \tag{B5}$$

where $\bar{e}$ is a constant vector of integration. The vector $\bar{e}$ may be expressed in terms of its scalar magnitude and a vector of unit length as follows:

$$\bar{e} = e\hat{a} \tag{B6}$$

where $\hat{a}$ is a vector of unit length. See sketch (c). Substituting equation (B6) in equation (B5) gives

$$\frac{\bar{V} \times \bar{h}}{\mu} = \hat{b}_1 + e\hat{a} \tag{B7}$$

Equation (B7) may be solved to obtain the position vector $\bar{r}$. Scalar multiplication of each side of equation (B7) by $\bar{r}$ gives the following equation for $\bar{r}$:

$$\bar{r} \cdot \left(\frac{\bar{V} \times \bar{h}}{\mu}\right) = r + e(\bar{r} \cdot \hat{a})$$

Therefore

$$\frac{(\bar{r} \times \bar{V}) \cdot \bar{h}}{\mu} = r(1 + e \cos \gamma)$$

and

$$r = \frac{h^2/\mu}{1 + e \cos \gamma} \tag{B8}$$

It is seen that e, the magnitude of the constant vector of integration appearing in equation (B6), is the orbital eccentricity. Vector multiplication of each side of equation (B7) by h gives

$$\bar{h} \times (\bar{V} \times \bar{h}) = \mu\bar{h} \times (\hat{b}_1 + e\hat{a})$$

Therefore,

$$\bar{V} = \frac{\mu}{h}\left[\hat{b}_2 + e(\hat{b}_3 \times \hat{a})\right] \tag{B9}$$

With the notation of sketch (c), the unit vector $\hat{a}$ may be expressed in the following form:

$$\hat{a} = (\cos \gamma)\hat{b}_1 - (\sin \gamma)\hat{b}_2 \tag{B10}$$

If the assumption of an inverse-square law central force field is not satisfied, the equation of motion must be modified accordingly. In the presence of a perturbing force $\bar{P}$, the equation of motion becomes

$$\frac{d\bar{V}}{dt} = \bar{P} - \frac{\mu\bar{r}}{r^3} \tag{B11}$$

Furthermore, in the presence of the perturbing force vector $\bar{P}$, the assumption of constancy no longer applies to the vector $\bar{e}$. Hence,

$$\mu \frac{d\bar{e}}{dt} = \left\{\frac{d\bar{V}}{dt} \times \bar{h} + \bar{V} \times (\bar{r} \times \bar{P}) - \frac{\mu}{r^3}\left[(\bar{r} \times \bar{V}) \times \bar{r}\right]\right\} \tag{B12}$$

Therefore,

$$\mu \frac{d\bar{e}}{dt} = \left(\frac{d\bar{V}}{dt} + \frac{\mu\bar{r}}{r^3}\right) \times \bar{h} + \bar{V} \times (\bar{r} \times \bar{P})$$

and

$$\mu \frac{d\bar{e}}{dt} = \bar{P} \times \bar{h} + \bar{V} \times (\bar{r} \times \bar{P}) \tag{B13}$$

The first term on the right side of equation (B13) may be written in the following alternative form:

$$\bar{P} \times \bar{h} = h(\hat{b}_1\hat{b}_2 - \hat{b}_2\hat{b}_1) \cdot \bar{P} \tag{B14}$$

Likewise,

$$\bar{r} \times \bar{P} = r(\hat{b}_3\hat{b}_2 - \hat{b}_2\hat{b}_3) \cdot \bar{P} \tag{B15}$$

When substitutions are made from equations (B9) and (B15), the second term on the right side of equation (B13) assumes the form

$$\bar{V} \times (\bar{r} \times \bar{P}) = \frac{\mu r}{h} \left\{\hat{b}_1\hat{b}_2 + e\left[\hat{a}\hat{b}_2 + (\hat{b}_2 \cdot \hat{a})\hat{b}_3\hat{b}_3\right]\right\} \cdot \bar{P} \tag{B16}$$

Substituting for $\hat{a}$ from equation (B10) in (B16) gives

$$\bar{V} \times (\bar{r} \times \bar{P}) = \frac{\mu r}{h} [(1 + e \cos \gamma)\hat{b}_1\hat{b}_2 - e \sin \gamma(\hat{b}_2\hat{b}_2 + \hat{b}_3\hat{b}_3)] \cdot \bar{P} \tag{B17}$$

From equations (B14) and (B17) it follows that

$$\bar{P} \times \bar{h} + \bar{V} \times (\bar{r} \times \bar{P}) = \left[(2h)\hat{b}_1\hat{b}_2 - \left(\frac{\mu re}{h} \sin \gamma\hat{b}_2\hat{b}_2 + h\hat{b}_2\hat{b}_1\right) - \frac{\mu re}{h} \sin \gamma\hat{b}_3\hat{b}_3\right] \cdot \bar{P} \tag{B18}$$

Therefore,

$$\frac{d\bar{e}}{dt} = \left\{\frac{2h}{\mu} \hat{b}_1\hat{b}_2 - \left[\frac{re}{h} \sin \gamma(\hat{b}_2\hat{b}_2 + \hat{b}_3\hat{b}_3) + \frac{h}{\mu} \hat{b}_2\hat{b}_1\right]\right\} \cdot \bar{P} \tag{B19}$$

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