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FREE AND FORCED VIBRATIONS OF CANTILEVER BEAMS WITH VISCOUS DAMPING

by Floyd J. Stanek

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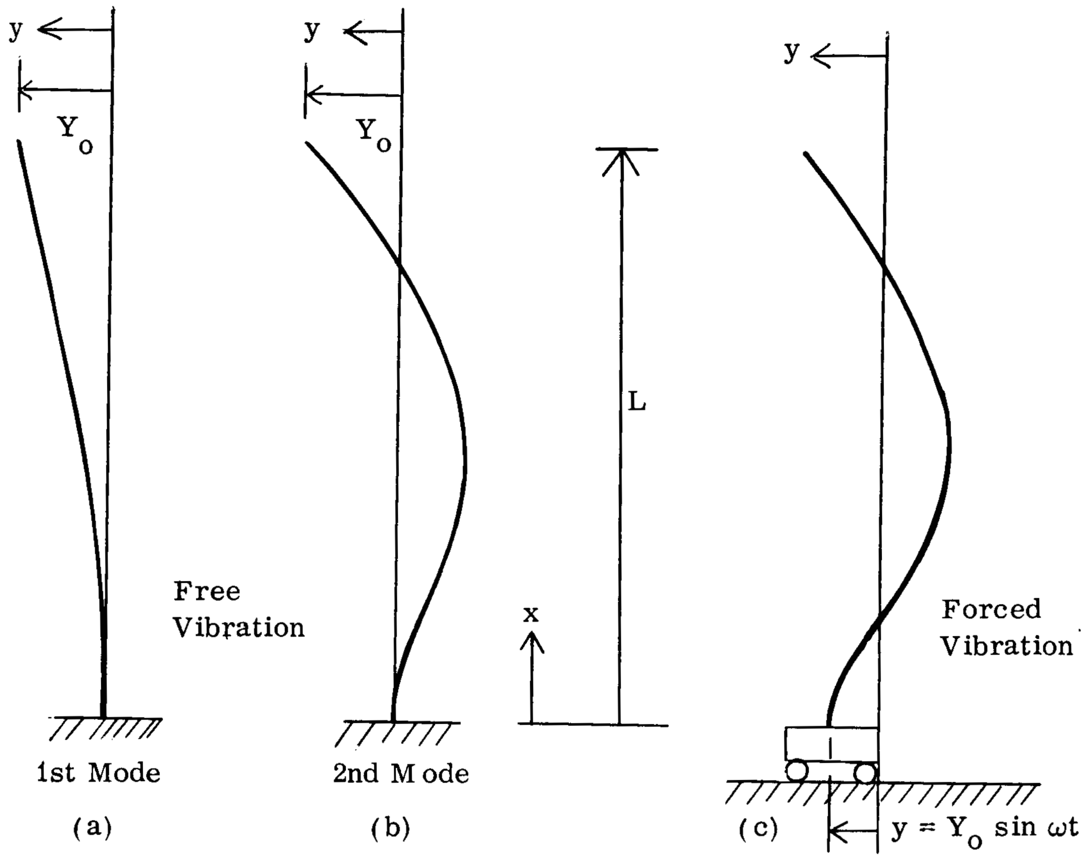

FIGURE 1. CONFIGURATION OF A VIBRATING CANTILEVER BEAM

Symbol	Definition
x	axial coordinate of beam (in.)*
L	beam length (in.)
z	$= x/L$ dimensionless axial coordinate of beam
y	lateral deflection of beam (in.)
Y_o	initial displacement of free end of beam in free vibration (in.) (See top of next page for definition in forced vibration).

* These are the most common set of units used in the U.S.A.; however, any consistent set may be used as the formulas are presented in dimensionless form.

DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
Y_o	amplitude of lateral displacement of "fixed" end of beam in forced vibration (in.) (See previous page for definition in free vibration).
W	weight of beam (lb)
I	moment of inertia of beam cross-section ($in.^4$)
E	modulus of elasticity of the material of the beam ($lb/in.^2$)
g	acceleration of gravity. ($in./sec^2$) (taken equal to 386.4 $in./sec^2$ in sample examples)
K	$= \sqrt{\frac{EIg}{WL^3}}$, beam parameter in free vibration (sec^{-1})
ω	circular frequency of excitomotor (rad/sec)
β	$= \left(\frac{WL^3\,\omega^2}{EIg}\right)^{1/4} = \sqrt{\frac{\omega}{K}}$, beam parameter in forced vibration (dimensionless)
C	coefficient of viscous damping (lb-sec/in.)
t	time (sec)
θ	dimensionless time
	$=$ Kt, in free vibration
	$=$ ωt, in forced vibration
λ	characteristic root; first five values tabulated in Section II (dimensionless)
α	damping factor (dimensionless)
	$= \frac{Cg}{2WK}$, in free vibration
	$= \frac{Cg}{W\omega}$, in forced vibration

FREE AND FORCED VIBRATIONS OF CANTILEVER BEAMS WITH VISCOUS DAMPING

SUMMARY

The fundamental theory for analyzing an undamped vibrating cantilever beam is presented in elementary texts on mechanical vibrations. The necessary formulas for a complete study of the state of motion and the behavior of a vibrating cantilever beam with viscous damping, however, have not been developed and presented for design.

This report provides the equations of motion for the free and the forced vibrations of a cantilever beam with viscous damping. The equations of motion include formulas for the bending moment, lateral shearing force, deflection, velocity, and the acceleration at any desired point of the beam for any chosen time. The coefficient of viscous damping is assumed to be constant throughout the length of the beam.

It is assumed during free vibration that the free end of the beam is initially displaced some arbitrary distance Y_0 and then released. The mode of vibration and natural frequency is defined by the value of a characteristic root. The first five values of this root are tabulated in this report.

It is assumed during forced vibration that the normally fixed end of the cantilever beam is subjected to a lateral displacement of the form $y = Y_0 \sin \omega t$ and that the zero slope is maintained. The equations of motion for forced vibration are for the steady-state condition only.

The equations of motion are presented in dimensionless form in a convenient and usable manner. This report also contains the derivations, design curves to evaluate some of the parameters, and a set of dimensionless results for each of several sample examples. The intent of this report is to bridge the gap between theory and design.

SECTION I. INTRODUCTION

Necessary formulas for the state of motion for the free and the forced vibration of a cantilever beam are presented in this report. These formulas include necessary equations for evaluating bending moment, lateral shearing force, deflection, velocity, and acceleration at any desired point along the beam for any desired instant.

During free vibration it is assumed that the free end of the cantilever beam is initially displaced some arbitrary distance Y_0 and then released. Natural frequency of the vibrating cantilever beam is dependent upon the mode of vibration, characterized by the value of a characteristic root. The first five values of this root are included in this report.

During forced vibration it is assumed that the normally fixed end of the beam is subjected to a lateral displacement of the form $y = Y_0 \sin \omega t$, where Y_0 is an arbitrary constant and ω is the circular frequency of the excitomotor, and the zero slope is maintained. Formulas for forced vibration are for the steady-state conditions only; that is, equations which define the state of motion after sufficient time has elapsed for the transient terms to decay out of range of significant value.

Equations of motion are expressed in terms of dimensionless coordinates and parameters. These coordinates are axial z and time θ. These parameters are defined in the tabulation of the formulas for each type of vibration. One of these parameters is the damping factor α which is dependent upon the coefficient of viscous damping. This coefficient is assumed constant throughout the length of the beam. Practically any beam of uniform cross-section may be analyzed by the procedure presented.

This procedure and the necessary formulas for each type of vibration are presented in a convenient and usable manner in Section II. Design curves for the evaluation of the beam parameters and the damping factor are presented in Section III. Also, in Section III is a set of dimensionless results for the deflection and bending moment for each of several sample cases. The derivation of the formulas is presented in the Appendix. It is not necessary to understand this derivation to understand its application.

SECTION II. APPLICATION

A. FREE VIBRATION

1. Procedure. The necessary formulas for the state of motion, bending moment, and lateral shearing force in a cantilever beam during free vibration are tabulated in this section.

The evaluation of these formulas is straightforward when done in a step-wise manner. A brief account of the physical aspects of some of these steps is given in the remainder of this section.

The free vibration of a cantilever beam is characterized by the mode of vibration. This mode of vibration is defined by the value of the characteristic root λ. The values of the first five roots are given in Formula 3. The first two modes of vibration are shown in Figure 1 parts (a) and (b) in the definition of symbols.

Values of the characteristic roots are solutions of a characteristic equation representing four boundary conditions; namely the lateral deflection and slope are zero at the fixed end, and the bending moment and lateral shearing force are zero at the free end of the beam.

Only two parameters are required for the initial evaluation of the formulas; the damping factor α (Formula 2) and the characteristic root λ (Formula 3). The value of the beam parameter K (Formula 1) is used to convert the dimensionless results to physical quantities. This parameter is also included in the definition of the damping factor α. The values of the parameters K and α may be obtained from the design curves given in Section III.

The values of λ and α determine the value of the vibration parameter γ (Formula 4). The significance of this parameter is that the product γK is the natural circular frequency of the cantilever beam for the mode of vibration characterized by the value of λ. The value of γ reduces to λ^2 when the beam is undamped ($\alpha = 0$).

The integration constant B (Formula 5) is one of two elements of a characteristic vector corresponding to the particular value of the characteristic root λ. The other element (integration constant) does not appear in the formulas as it was taken equal to one.

The Z-function (Formula 6) is a function of the dimensionless axial coordinate z only. The value of the Z-function at $z = 1$ is the value of the constant Z_1 (Formula 7). This constant appears in the T-function (Formula 8); a function of dimensionless time θ only. The integration constants in the T-function were evaluated in terms of Z_1 by the initial conditions that the displacement is equal to Y_0 and the velocity (dy/dt) is zero at the free end of the beam when $t = 0$.

The product of the Z- and T-functions is the dimensionless deflection y/Y_0 (Formula 9). The remaining results, also dimensionless, are obtained by evaluating Formulas 10 through 13. A set of dimensionless results for the deflection and for the bending moment (Formulas 9 and 12, respectively) are given in Section III.

The set of formulas for the free vibration of a cantilever beam are now tabulated in the next subsection. The procedure and the formulas for forced vibrations are given after this tabulation.

2. Formulas.

Beam parameter

$$K = \sqrt{\frac{EIg}{WL^3}} \tag{1}$$

Damping factor

$$\alpha = \frac{Cg}{2WK} \tag{2}$$

Characteristic root, λ*

Mode	λ
1	1.87510
2	4.69409
3	7.85476
4	10.99554
5	14.13717

(3)

Vibration parameter

$$\gamma = \sqrt{\lambda^4 - \alpha^2} \tag{4}$$

Integration constant in Z-function

$$B = -\frac{\cosh \lambda + \cos \lambda}{\sinh \lambda + \sin \lambda} \tag{5}$$

The Z-function

$$Z = \cosh \lambda z - \cos \lambda z + B(\sinh \lambda z - \sin \lambda z) \tag{6}$$

The Z_1 constant

$$Z_1 = \text{value of Z-function at } z = 1$$

$$= \cosh \lambda - \cos \lambda + B(\sinh \lambda - \sin \lambda) \tag{7}$$

The T-function

$$T = \frac{1}{\gamma Z_1} e^{-\alpha\theta} (\gamma \cos \gamma\theta + \alpha \sin \gamma\theta) \tag{8}$$

* λ is the root of $1 + \cosh \lambda \cos \lambda = 0$.

Deflection, y

$$\frac{y}{Y_o} = ZT \tag{9}$$

Velocity, V

$$\frac{V}{KY_o} = \frac{-\lambda^4}{\gamma Z_1} Ze^{-\alpha\theta} \sin\gamma\theta \tag{10}$$

Acceleration, $\underline{A}$

$$\frac{\underline{A}}{K^2 Y_o} = \frac{-\lambda^4}{\gamma Z_1} Ze^{-\alpha\theta} (\gamma \cos\gamma\theta - \alpha \sin\gamma\theta) \tag{11}$$

Bending moment, $\underline{M}$

$$\frac{\underline{M}\, L^2}{EIY_o} = \lambda^2 [\cosh\lambda z + \cos\lambda z + B(\sinh\lambda z + \sin\lambda z)]\, T \tag{12}$$

Shearing force, Q

$$\frac{QL^3}{EIY_o} = \lambda^3 [\sinh\lambda z - \sin\lambda z + B(\cosh\lambda z + \cos\lambda z)]\, T \tag{13}$$

B. FORCED VIBRATION

1. Procedure. The necessary formulas for the state of motion, bending moment, and lateral shearing force during the steady-state condition of the forced vibration of a cantilever beam are given in this section. The normally fixed end of the cantilever beam is subjected to a lateral displacement $y = Y_o \sin\omega t$, but the zero slope is maintained. A steady-state condition is established when sufficient time has elapsed for the transient terms to decay out of range of significant value.

In the case of forced vibration, the beam parameter β (Formula 14) is dimensionless and equivalent to $\sqrt{\omega/K}$ where K is the beam parameter for free vibration (Formula 1 in the previous subsection). The damping factor α for forced vibration (Formula 15) is not the same as for free vibration (Formula 2) however, it is still dimensionless.

The parameters β and α are the only two parameters required for evaluation of a complete set of dimensionless results. The value of the parameter β may be obtained from the design curves given in Section III. The results are obtained in a step-wise manner. The physical aspects of some of these steps and the nomenclature used in this presentation are explained in the following discussion.

The parameters β and α define the four vibration parameters ϕ, μ, a, and b (Formula 16). The vibration parameters a and b define the symbols s_i and t_i ($i = 1, 2, 3, 4$) which, in turn, define the symbols S_{ij} and T_{ij} (Formula 17). The symbols S_{ij} and T_{ij} are used to evaluate the elements of a four by four matrix (Formula 18).

The solution of this matrix equation evaluates the integration constants A, B, C, and D and in turn, evaluates the four integration constants E, F, G, and H (Formula 19). Various linear combinations of these eight integration constants will be required later. The presentation of these combinations is explained below (in the following paragraphs).

The general form of the expression for the dimensionless deflection y/Y_o is as follows

$$y/Y_o = Z^s \sin\theta + Z^c \cos\theta,$$

where θ is the dimensionless time ωt, Z^s and Z^c are functions of the dimensionless axial coordinate z only. The significance of the superscripts s and c becomes apparent when each is correlated with the sine and the cosine term, respectively.

The Z^s function is defined as follows:

$$\begin{aligned} Z^s = {} & A \cosh az \cos bz + B \sinh az \sin bz + \\ & C \sinh az \cos bz + D \cosh az \sin bz + \\ & E \cosh bz \cos az + F \sinh bz \sin az + \\ & G \sinh bz \cos az + H \sinh bz \sin az, \end{aligned}$$

where the symbols A through H are the eight integration constants evaluated by Formulas 17 and 18.

The expression for Z^s is transformed to the following matrix equation

$$Z^s = [\cosh az \quad \sinh az] M^s \begin{bmatrix} \cos bz \\ \sin bz \end{bmatrix} + [\cosh bz \quad \sinh bz] N^s \begin{bmatrix} \cos az \\ \sin az \end{bmatrix},$$

where the eight integration constants are grouped into the matrices M^s and N^s as follows

$$M^s = \begin{bmatrix} A & D \\ C & B \end{bmatrix} \qquad N^s = \begin{bmatrix} E & H \\ G & F \end{bmatrix}$$

The form of the Z^c function is the same as the form of the Z^s function, except for a different set of eight integration constants. This set is designated by the superscript c. The expression for the derivative d^nZ/dz^n, of any order n, is also the same form as the Z-function, except with a different set of eight constants. The second and the third derivative of Z with respect to z is required for the evaluation of Formulas 25 and 26. An integer subscript is added to provide the scripted symbols Z_i^j, M_i^j, and N_i^j, where j is s or c and the integer i designates the order of the derivative with respect to z. This scripted notation on the matrices M and N also applies to each of the elements within the matrix.

The non-scripted symbols A through H defined by Formulas 18 and 19 are equivalent to the corresponding scripted symbols with j = s and i = 0. The value of each of the other scripted symbols (that is, the elements of the other matrices M and N) is obtained by taking certain linear combinations of the original eight integration constants A through H. This linear combination is presented as the linear combination of two, 2 by 2 matrices (Formula 20). This is illustrated below for the elements A_2^s and G_3^c (see matrix equation for M_2^s and N_3^c in Formula 20).

$$A_2^s = A \cos 2\phi + B \sin 2\phi \equiv A_o^s \cos 2\phi + B_o^s \sin 2\phi ,$$

and, similarly

$$G_3^c = F \sin 3\phi - E \cos 3\phi \equiv F_o^s \sin 3\phi - E_o^s \cos 3\phi$$

After the evaluation of all the scripted elements (constants), the desired result is obtained by evaluating the appropriate of Formulas 22 through 26; noting that the particular Z_i^j function is established by substituting the corresponding matrices M_i^j and N_i^j into Formula 21.

The initial results are dimensionless as defined by the expression on the left of the equal sign in each of Formulas 22 through 26. The dimensionless time is converted to physical time with the relationship $\theta = \omega t$. A set of dimensionless values for the deflection and bending moment for several examples are presented in Section III.

The formulas for forced vibration of a cantilever beam are tabulated in the remainder of this section. The equations of motion presented in this section are derived in the Appendix.

2. Formulas.

Beam parameter, β

$$\beta = \left(\frac{WL^3 \omega^2}{EIg}\right)^{1/4} \tag{14}$$

Damping factor, α

$$\alpha = \frac{Cg}{W\omega} \tag{15}$$

Vibration parameters

$$\begin{aligned} \phi &= \frac{1}{4}\tan^{-1} \alpha \\ \mu &= \beta \, (1 + \alpha^2)^{1/8} \\ a &= \mu \cos \phi \\ b &= \mu \sin \phi \end{aligned} \tag{16}$$

Definition of symbols for the elements of the matrix to evaluate the first four integration constants.

$$\begin{aligned} s_1 &= \cosh a & t_1 &= \cosh b \\ s_2 &= \sinh a & t_2 &= \sinh b \\ s_3 &= \cos b & t_3 &= \cos a \\ s_4 &= \sin b & t_4 &= \sin a \\ S_{13} &= s_1 s_3 & T_{13} &= t_1 t_3 \\ S_{14} &= s_1 s_4 & T_{14} &= t_1 t_4 \\ S_{23} &= s_2 s_3 & T_{23} &= t_2 t_3 \\ S_{24} &= s_2 s_4 & T_{24} &= t_2 t_4 \end{aligned} \tag{17}$$

The matrix equation

$$\begin{bmatrix} S_{13}+T_{13} & S_{24}-T_{24} & S_{23}+T_{14} & S_{14}+T_{23} \\ -S_{24}+T_{24} & S_{13}+T_{13} & -S_{14}-T_{23} & S_{23}+T_{14} \\ S_{23}-T_{14} & S_{14}-T_{23} & S_{13}+T_{13} & S_{24}-T_{24} \\ -S_{14}+T_{23} & S_{23}-T_{14} & -S_{24}+T_{24} & S_{13}+T_{13} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} T_{13} \\ T_{24} \\ -T_{14} \\ T_{23} \end{bmatrix} \tag{18}$$

Integration constants E, F, G, and H

$$E = 1 - A, \quad F = B, \quad G = -D, \quad H = -C. \tag{19}$$

Matrices M_i^s, N_i^s, M_i^c, and N_i^c. ($i = 0$, 2, and 3)

$$M_o^s = \begin{bmatrix} A & D \\ C & B \end{bmatrix} \qquad N_o^s = \begin{bmatrix} E & H \\ G & F \end{bmatrix}$$

$$M_o^c = \begin{bmatrix} B & -C \\ D & -A \end{bmatrix} \qquad N_o^c = \begin{bmatrix} -F & G \\ -H & E \end{bmatrix}$$

$$\begin{aligned} M_2^s &= \cos 2\phi \begin{bmatrix} A & D \\ C & B \end{bmatrix} + \sin 2\phi \begin{bmatrix} B & -C \\ D & -A \end{bmatrix} \\ N_2^s &= -\cos 2\phi \begin{bmatrix} E & H \\ G & F \end{bmatrix} + \sin 2\phi \begin{bmatrix} F & -G \\ H & -E \end{bmatrix} \\ M_2^c &= \cos 2\phi \begin{bmatrix} B & -C \\ D & -A \end{bmatrix} - \sin 2\phi \begin{bmatrix} A & D \\ C & B \end{bmatrix} \\ N_2^c &= -\cos 2\phi \begin{bmatrix} -F & G \\ -H & E \end{bmatrix} + \sin 2\phi \begin{bmatrix} E & H \\ G & F \end{bmatrix} \end{aligned} \tag{20}$$

$$M_3^s = \cos 3\phi \begin{bmatrix} C & B \\ A & D \end{bmatrix} + \sin 3\phi \begin{bmatrix} D & -A \\ B & -C \end{bmatrix}$$

$$N_3^s = -\sin 3\phi \begin{bmatrix} G & F \\ E & H \end{bmatrix} - \cos 3\phi \begin{bmatrix} H & -E \\ F & -G \end{bmatrix}$$

(20 Concluded)

$$M_3^c = \cos 3\phi \begin{bmatrix} D & -A \\ B & -C \end{bmatrix} - \sin 3\phi \begin{bmatrix} C & B \\ A & D \end{bmatrix}$$

$$N_3^c = -\sin 3\phi \begin{bmatrix} -H & E \\ -F & G \end{bmatrix} - \cos 3\phi \begin{bmatrix} G & F \\ E & H \end{bmatrix}$$

The Z_i^j - function ($i = 0, 2, 3;\ j = s, c$)*

$$Z_i^j = [\cosh az \quad \sinh az]\, M_i^j \begin{bmatrix} \cos & bz \\ \sin & bz \end{bmatrix} + [\cosh bz \quad \sinh bz]\, N_i^j \begin{bmatrix} \cos & az \\ \sin & az \end{bmatrix} \qquad (21)$$

Deflection, y

$$\frac{y}{Y_o} = Z_o^s \sin\theta + Z_o^c \cos\theta \qquad (22)$$

Velocity, V

$$\frac{V}{\omega Y_o} = Z_o^s \cos\theta - Z_o^c \sin\theta \qquad (23)$$

Acceleration, A

$$\frac{A}{\omega^2 Y_o} = -\left(Z_o^s \sin\theta + Z_o^c \cos\theta\right) \qquad (24)$$

---

* The non-scripted symbols in Formulas 18, 19, and 20 are equivalent to the scripted symbols when $j = s$ and $i = 0$.

Bending moment, M

$$\frac{\underline{M}\ L^2}{EI\ Y_o} = \mu^2 \left(Z_2^S \sin\theta + Z_2^C \cos\theta \right) \tag{25}$$

Shearing force, Q

$$\frac{QL^3}{EI\ Y_o} = \mu^3 \left(Z_3^S \sin\theta + Z_3^C \cos\theta \right) \tag{26}$$

SECTION III. SAMPLE RESULTS

A set of dimensionless values for the deflection and bending moment of a vibrating cantilever beam are presented in this section. Design curves for the evaluation of the beam parameters K and β and for the damping factor α for free vibration are also given. The use of these curves is illustrated in each Figure.

The set of dimensionless results for the cases of free vibration immediately follow Figure 4. These cases are for the first two modes of vibration (first two values of λ) for each of the damping factor α equal to 0 and to 0.2. These results were obtained by evaluating Formulas 9 and 12 of Section II and are presented in Tables 1A through 8A.

Similar data for the cases of forced vibration are presented after Table 8A. These cases are for the beam parameter β equal to 5 and to 10 for each value of the damping factor α for forced vibration equal to 0 and 0.2. These results were obtained by evaluating Formulas 22 and 25 of Section II and are presented in Tables 1B through 8B.

The format of each table is the same; with the dimensionless time coordinate θ (in angular degrees) in the first column and with the dimensionless axial coordinate z as columnar headings of the remaining columns. In the case of forced vibration, it is recalled that the formulas are for the steady state condition only. That is, the point of zero time ($\theta = 0$) is merely a beginning reference point of a repetitious cycle. This explains why the results are not zero everywhere along the beam at $\theta = 0$ when the beam is subjected to viscous damping ($\alpha \neq 0$). The derivation of the formulas in Section II is given in the Appendix which follows Table 8B.

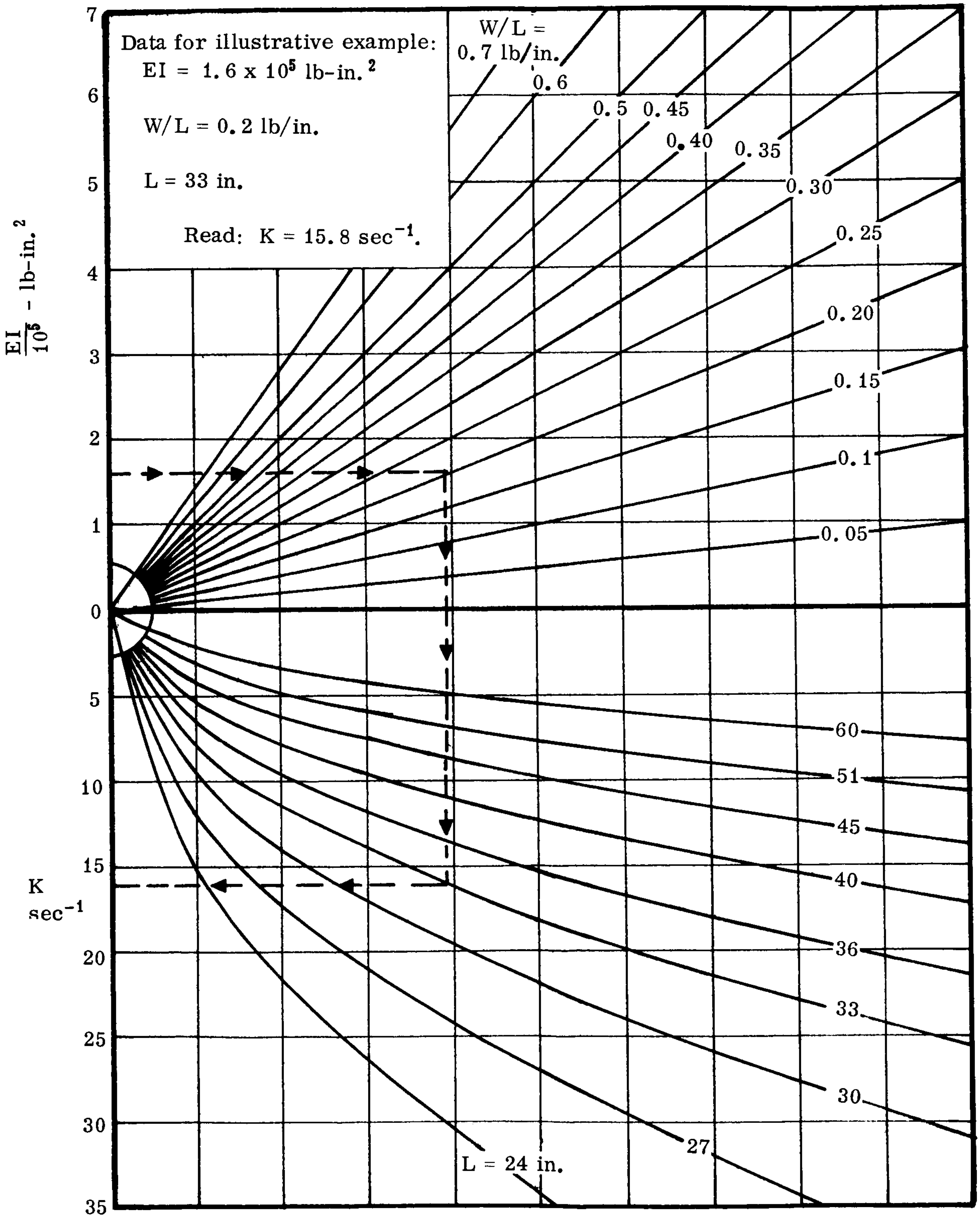

FIGURE 2. THE BEAM PARAMETER K

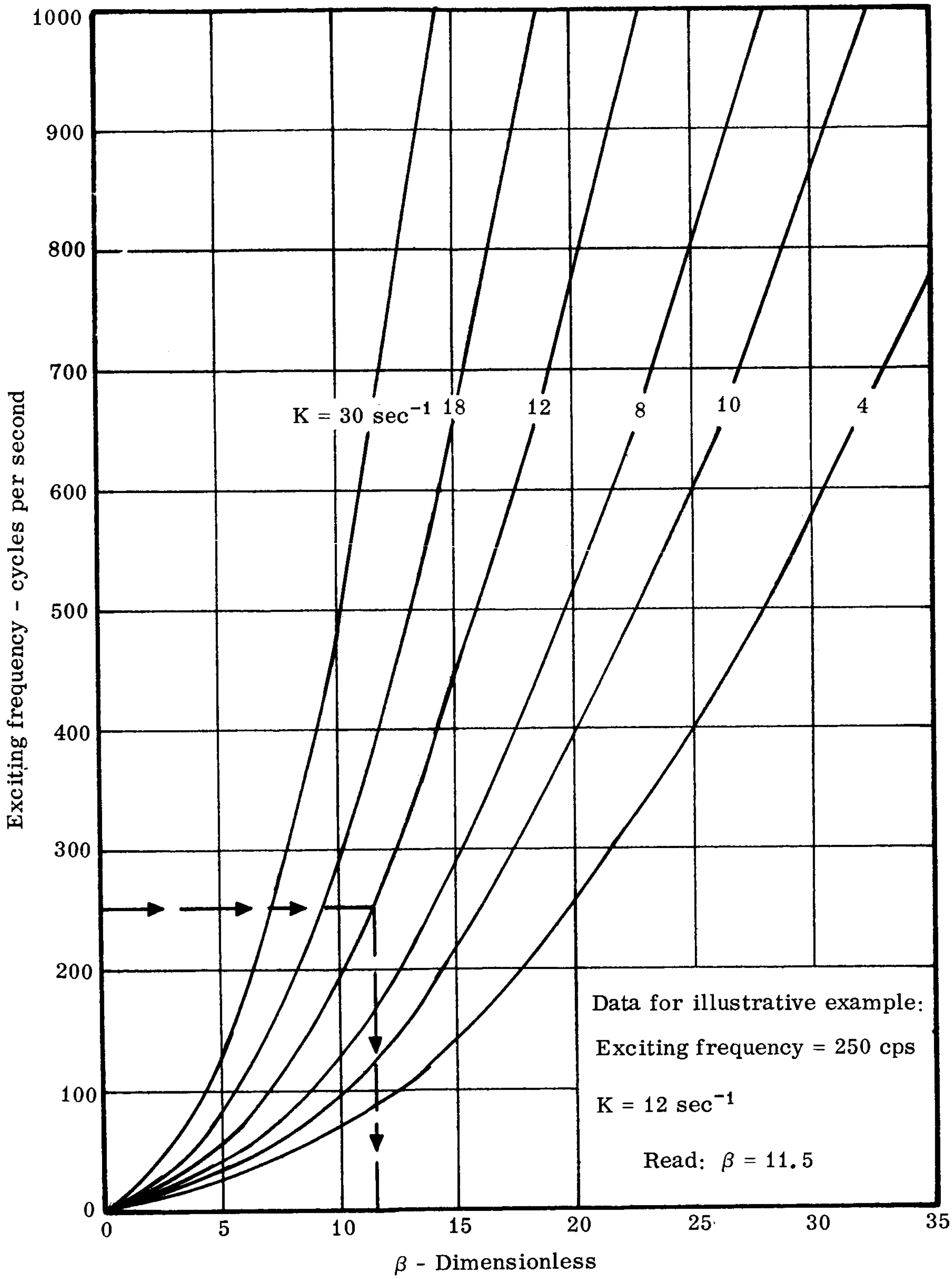

FIGURE 3. THE BEAM PARAMETER β

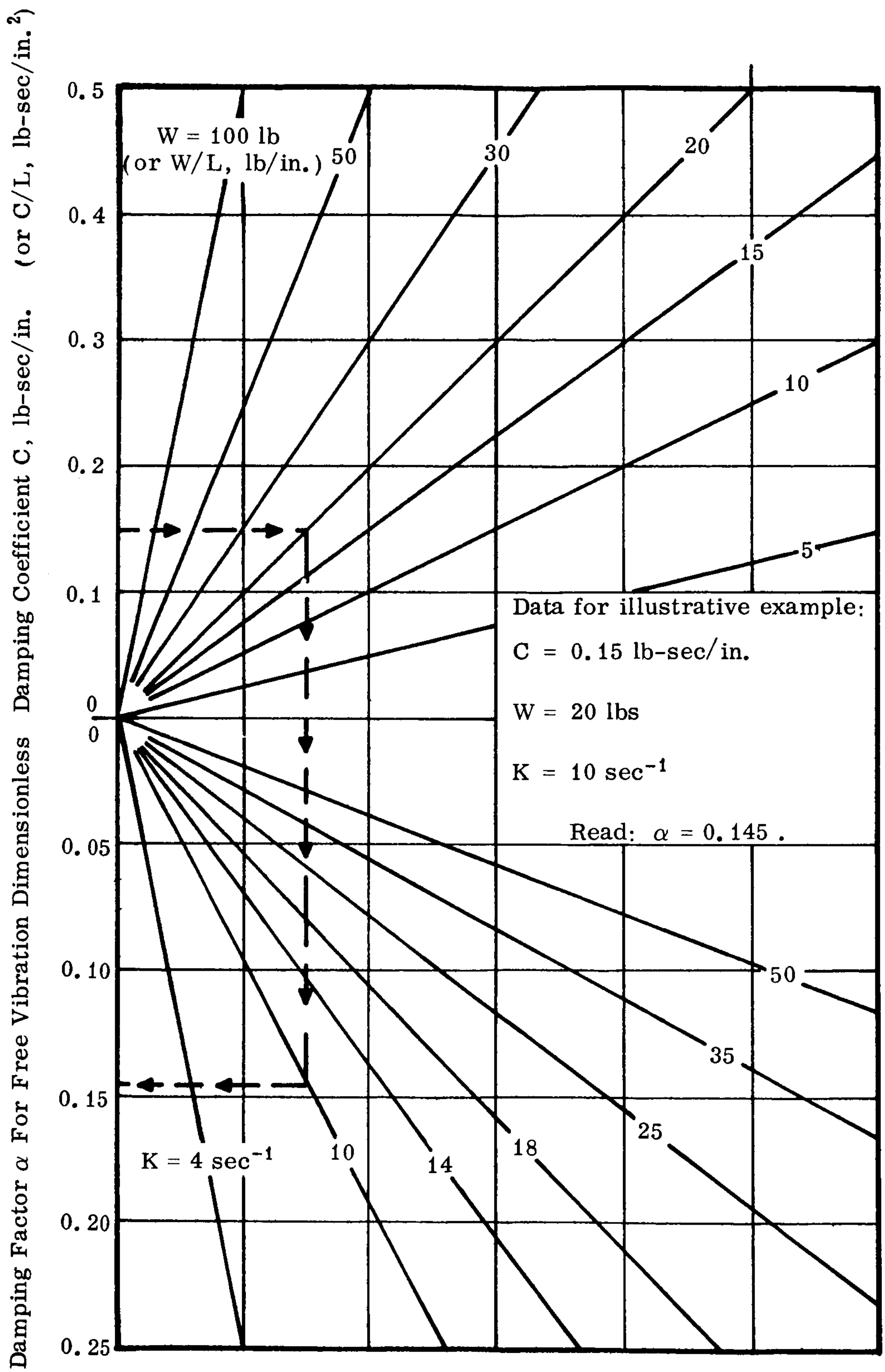

FIGURE 4. THE DAMPING FACTOR α FOR FREE VIBRATION

TABLES

TABLE 1A. DEFLECTION OF CANTILEVER BEAM - FREE VIBRATION

y/Y_0

$\lambda = 1.8751$ $\alpha = 0.0$

θ \ z	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	0.000	.016	.063	.136	.229	.339	.461	.590	.725	.862	.999
15	0.000	.016	.061	.131	.222	.327	.445	.570	.700	.833	.965
30	0.000	.014	.055	.118	.199	.294	.399	.511	.628	.746	.866
45	0.000	.011	.045	.096	.162	.240	.326	.417	.512	.609	.707
60	0.000	.008	.031	.068	.114	.169	.230	.295	.362	.431	.500
75	0.000	.004	.016	.035	.059	.087	.119	.152	.187	.223	.258
90	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
105	0.000	-.004	-.016	-.035	-.059	-.087	-.119	-.152	-.187	-.223	-.258
120	0.000	-.008	-.031	-.068	-.114	-.169	-.230	-.295	-.362	-.431	-.499
135	0.000	-.011	-.045	-.096	-.162	-.240	-.326	-.417	-.512	-.609	-.707
150	0.000	-.014	-.055	-.118	-.199	-.294	-.399	-.511	-.628	-.746	-.866
165	0.000	-.016	-.061	-.131	-.222	-.327	-.445	-.570	-.700	-.833	-.965
180	0.000	-.016	-.063	-.136	-.229	-.339	-.461	-.590	-.725	-.862	-.999
195	0.000	-.016	-.061	-.131	-.222	-.327	-.445	-.570	-.700	-.833	-.965
210	0.000	-.014	-.055	-.118	-.199	-.294	-.399	-.511	-.628	-.746	-.866
225	0.000	-.011	-.045	-.096	-.162	-.240	-.326	-.417	-.512	-.609	-.707
240	0.000	-.008	-.031	-.068	-.114	-.169	-.230	-.295	-.362	-.431	-.500
255	0.000	-.004	-.016	-.035	-.059	-.087	-.119	-.152	-.187	-.223	-.258
270	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
285	0.000	.004	.016	.035	.059	.087	.119	.152	.187	.223	.258
300	0.000	.008	.031	.068	.114	.169	.230	.295	.362	.431	.499
315	0.000	.011	.045	.096	.162	.240	.326	.417	.512	.609	.707
330	0.000	.014	.055	.118	.199	.294	.399	.511	.628	.746	.866
345	0.000	.016	.061	.131	.222	.327	.445	.570	.700	.833	.965

TABLE 2A. DEFLECTION OF CANTILEVER BEAM - FREE VIBRATION

$\lambda = 1.8751$ $\alpha = 0.2$ y/Y_0

θ \ z	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	0.000	.016	.063	.136	.229	.339	.461	.590	.725	.862	.999
15	0.000	.016	.061	.131	.222	.328	.445	.570	.700	.833	.966
30	0.000	.014	.055	.118	.199	.294	.400	.513	.629	.748	.868
45	0.000	.011	.045	.097	.164	.242	.329	.422	.518	.616	.714
60	0.000	.008	.033	.070	.118	.175	.238	.305	.375	.446	.517
75	0.000	.004	.018	.039	.066	.098	.134	.172	.211	.251	.291
90	0.000	0.000	.003	.007	.011	.017	.024	.030	.037	.044	.052
105	0.000	-.003	-.011	-.025	-.042	-.062	-.084	-.108	-.133	-.158	-.183
120	0.000	-.006	-.025	-.054	-.091	-.135	-.184	-.236	-.290	-.344	-.399
135	0.000	-.009	-.037	-.079	-.134	-.197	-.268	-.344	-.422	-.502	-.583
150	0.000	-.012	-.046	-.098	-.165	-.244	-.332	-.426	-.523	-.622	-.721
165	0.000	-.013	-.051	-.110	-.185	-.274	-.372	-.476	-.585	-.696	-.807
180	0.000	-.014	-.053	-.114	-.192	-.283	-.385	-.494	-.606	-.721	-.836
195	0.000	-.013	-.051	-.110	-.185	-.274	-.372	-.477	-.586	-.696	-.807
210	0.000	-.012	-.046	-.099	-.166	-.246	-.334	-.428	-.526	-.626	-.725
225	0.000	-.010	-.038	-.081	-.137	-.202	-.275	-.353	-.433	-.515	-.597
240	0.000	-.007	-.027	-.059	-.099	-.146	-.199	-.255	-.313	-.373	-.432
255	0.000	-.004	-.015	-.033	-.055	-.082	-.112	-.143	-.176	-.210	-.243
270	0.000	0.000	-.002	-.005	-.010	-.014	-.020	-.025	-.031	-.037	-.043
285	0.000	.002	.009	.020	.035	.052	.070	.090	.111	.132	.153
300	0.000	.005	.021	.045	.076	.113	.154	.197	.242	.288	.334
315	0.000	.008	.031	.066	.112	.165	.224	.288	.353	.420	.487
330	0.000	.010	.038	.082	.138	.204	.278	.356	.437	.520	.603
345	0.000	.011	.043	.092	.155	.229	.311	.398	.489	.582	.674

TABLE 3A. DEFLECTION OF CANTILEVER BEAM - FREE VIBRATION

$\lambda = 4.6941$ $\alpha = 0.0$ y/Y_0

θ \ z	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	0.000	-.092	-.301	-.526	-.683	-.713	-.589	-.317	.070	.523	.999
15	0.000	-.089	-.290	-.508	-.660	-.689	-.569	-.306	.067	.505	.965
30	0.000	-.080	-.260	-.455	-.591	-.618	-.510	-.274	.060	.453	.866
45	0.000	-.065	-.212	-.372	-.483	-.504	-.416	-.224	.049	.370	.707
60	0.000	-.046	-.150	-.263	-.341	-.356	-.294	-.158	.035	.261	.500
75	0.000	-.023	-.077	-.136	-.176	-.184	-.152	-.082	.018	.135	.258
90	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
105	0.000	.023	.077	.136	.176	.184	.152	.082	-.018	-.135	-.258
120	0.000	.046	.150	.263	.341	.356	.294	.158	-.035	-.261	-.499
135	0.000	.065	.212	.372	.483	.504	.416	.224	-.049	-.370	-.707
150	0.000	.080	.260	.455	.591	.618	.510	.274	-.060	-.453	-.866
165	0.000	.089	.290	.508	.660	.689	.569	.306	-.067	-.505	-.965
180	0.000	.092	.301	.526	.683	.713	.589	.317	-.070	-.523	-.999
195	0.000	.089	.290	.508	.660	.689	.569	.306	-.067	-.505	-.965
210	0.000	.080	.260	.455	.591	.618	.510	.274	-.060	-.453	-.866
225	0.000	.065	.212	.372	.483	.504	.416	.224	-.049	-.370	-.707
240	0.000	.046	.150	.263	.341	.356	.294	.158	-.035	-.261	-.500
255	0.000	.023	.077	.136	.176	.184	.152	.082	-.018	-.135	-.258
270	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
285	0.000	-.023	-.077	-.136	-.176	-.184	-.152	-.082	.018	.135	.258
300	0.000	-.046	-.150	-.263	-.341	-.356	-.294	-.158	.035	.261	.499
315	0.000	-.065	-.212	-.372	-.483	-.504	-.416	-.224	.049	.370	.707
330	0.000	-.080	-.260	-.455	-.591	-.618	-.510	-.274	.060	.453	.866
345	0.000	-.089	-.290	-.508	-.660	-.689	-.569	-.306	.067	.505	.965

TABLE 4A. DEFLECTION OF CANTILEVER BEAM - FREE VIBRATION

$\lambda = 4.6941$ $\alpha = 0.2$ y/Y_0

θ \ z	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	0.000	-.092	-.301	-.526	-.683	-.713	-.589	-.317	.070	.523	.999
15	0.000	-.089	-.290	-.508	-.660	-.689	-.569	-.306	.067	.505	.965
30	0.000	-.080	-.260	-.455	-.592	-.618	-.510	-.274	.060	.453	.866
45	0.000	-.065	-.213	-.372	-.484	-.505	-.417	-.224	.049	.371	.708
60	0.000	-.046	-.151	-.264	-.343	-.359	-.296	-.159	.035	.263	.503
75	0.000	-.024	-.079	-.139	-.180	-.188	-.155	-.083	.018	.138	.264
90	0.000	0.000	-.002	-.004	-.006	-.006	-.005	-.002	0.000	.004	.008
105	0.000	.022	.074	.129	.168	.175	.144	.077	-.017	-.128	-.245
120	0.000	.044	.145	.254	.330	.344	.284	.153	-.033	-.252	-.482
135	0.000	.063	.206	.360	.468	.489	.404	.217	-.048	-.359	-.685
150	0.000	.077	.253	.442	.574	.600	.495	.266	-.058	-.440	-.841
165	0.000	.086	.282	.493	.641	.669	.553	.297	-.065	-.491	-.938
180	0.000	.090	.292	.511	.664	.693	.572	.308	-.068	-.509	-.971
195	0.000	.086	.282	.493	.641	.670	.553	.297	-.065	-.491	-.938
210	0.000	.078	.253	.443	.575	.600	.496	.266	-.058	-.441	-.842
225	0.000	.063	.207	.362	.470	.491	.405	.218	-.048	-.360	-.638
240	0.000	.045	.147	.257	.334	.348	.288	.155	-.034	-.256	-.488
255	0.000	.023	.077	.135	.175	.183	.151	.081	-.017	-.134	-.256
270	0.000	0.000	.002	.004	.005	.006	.005	.002	0.000	-.004	-.008
285	0.000	-.022	-.071	-.125	-.163	-.170	-.140	-.075	.016	.125	.239
300	0.000	-.043	-.141	-.246	-.320	-.334	-.276	-.148	.032	.245	.469
315	0.000	-.061	-.200	-.350	-.455	-.475	-.392	-.211	.046	.349	.666
330	0.000	-.075	-.246	-.430	-.558	-.583	-.481	-.259	.057	.428	.817
345	0.000	-.084	-.274	-.480	-.623	-.651	-.537	-.289	.063	.477	.912

TABLE 5A. BENDING MOMENT IN CANTILEVER BEAM - FREE VIBRATION

$\frac{ML^2}{EIY_0}$

$\lambda = 1.8751$ $\alpha = 0.0$

θ \ z	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	3.516	3.032	2.550	2.077	1.621	1.193	.808	.479	.224	.058	0.000
15	3.396	2.928	2.463	2.006	1.566	1.153	.780	.463	.216	.056	0.000
30	3.044	2.625	2.209	1.799	1.404	1.033	.699	.415	.194	.051	0.000
45	2.486	2.144	1.803	1.469	1.146	.844	.571	.339	.158	.041	0.000
60	1.758	1.516	1.275	1.038	.810	.596	.404	.239	.112	.029	0.000
75	.910	.784	.660	.537	.419	.308	.209	.124	.058	.015	0.000
90	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
105	-.910	-.784	-.660	-.537	-.419	-.308	-.209	-.124	-.058	-.015	0.000
120	-1.758	-1.516	-1.275	-1.038	-.810	-.596	-.404	-.239	-.112	-.029	0.000
135	-2.486	-2.144	-1.803	-1.469	-1.146	-.844	-.571	-.339	-.158	-.041	0.000
150	-3.044	-2.625	-2.209	-1.799	-1.404	-1.033	-.699	-.415	-.194	-.051	0.000
165	-3.396	-2.928	-2.463	-2.006	-1.566	-1.153	-.780	-.463	-.216	-.056	0.000
180	-3.516	-3.032	-2.550	-2.077	-1.621	-1.193	-.808	-.479	-.224	-.058	0.000
195	-3.396	-2.928	-2.463	-2.006	-1.566	-1.153	-.780	-.463	-.216	-.056	0.000
210	-3.044	-2.625	-2.209	-1.799	-1.404	-1.033	-.699	-.415	-.194	-.051	0.000
225	-2.486	-2.144	-1.803	-1.469	-1.146	-.844	-.571	-.339	-.158	-.041	0.000
240	-1.758	-1.516	-1.275	-1.038	-.810	-.596	-.404	-.239	-.112	-.029	0.000
255	-.910	-.784	-.660	-.537	-.419	-.308	-.209	-.124	-.058	-.015	0.000
270	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
285	.910	.784	.660	.537	.419	.308	.209	.124	.058	.015	0.000
300	1.758	1.516	1.275	1.038	.810	.596	.404	.239	.112	.029	0.000
315	2.486	2.144	1.803	1.469	1.146	.844	.571	.339	.158	.041	0.000
330	3.044	2.625	2.209	1.799	1.404	1.033	.699	.415	.194	.051	0.000
345	3.396	2.928	2.463	2.006	1.566	1.153	.780	.463	.216	.056	0.000

TABLE 6A. BENDING MOMENT IN CANTILEVER BEAM - FREE VIBRATION

$$\frac{ML^2}{EIY_0}$$

$\lambda = 1.8751$ $\alpha = 0.2$

θ \ Z	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	3.516	3.032	2.550	2.077	1.621	1.193	.808	.479	.224	.058	0.000
15	3.397	2.929	2.464	2.007	1.566	1.153	.780	.463	.216	.056	0.000
30	3.052	2.632	2.214	1.803	1.407	1.036	.701	.416	.194	.051	0.000
45	2.512	2.167	1.823	1.484	1.158	.853	.577	.342	.160	.042	0.000
60	1.819	1.569	1.320	1.075	.839	.617	.418	.248	.116	.030	0.000
75	1.024	.883	.743	.605	.472	.347	.235	.139	.065	.017	0.000
90	.183	.157	.132	.108	.084	.062	.042	.025	.011	.003	0.000
105	-.645	-.556	-.468	-.381	-.297	-.219	-.148	-.088	-.041	-.010	0.000
120	-1.406	-1.212	-1.020	-.830	-.648	-.477	-.323	-.191	-.089	-.023	0.000
135	-2.050	-1.767	-1.487	-1.211	-.945	-.696	-.471	-.279	-.130	-.034	0.000
150	-2.536	-2.187	-1.840	-1.498	-1.169	-.861	-.583	-.346	-.162	-.042	0.000
165	-2.838	-2.447	-2.059	-1.677	-1.308	-.963	-.652	-.387	-.181	-.047	0.000
180	-2.939	-2.535	-2.132	-1.737	-1.355	-.998	-.675	-.401	-.187	-.049	0.000
195	-2.840	-2.449	-2.060	-1.678	-1.309	-.964	-.652	-.387	-.181	-.047	0.000
210	-2.552	-2.201	-1.851	-1.508	-1.176	-.866	-.586	-.348	-.163	-.042	0.000
225	-2.101	-1.811	-1.524	-1.241	-.968	-.713	-.482	-.286	-.134	-.035	0.000
240	-1.521	-1.312	-1.103	-.898	-.701	-.516	-.349	-.207	-.097	-.025	0.000
255	-.856	-.738	-.621	-.505	-.394	-.290	-.196	-.116	-.054	-.014	0.000
270	-.153	-.132	-.111	-.090	-.070	-.052	-.035	-.020	-.009	-.002	0.000
285	.539	.465	.391	.318	.248	.183	.124	.073	.034	.009	0.000
300	1.175	1.014	.853	.694	.542	.399	.270	.160	.075	.019	0.000
315	1.714	1.478	1.243	1.012	.790	.581	.394	.233	.109	.028	0.000
330	2.120	1.829	1.538	1.253	.978	.720	.487	.289	.135	.035	0.000
345	2.373	2.046	1.721	1.402	1.094	.805	.545	.323	.151	.039	0.000

TABLE 7A. BENDING MOMENT IN CANTILEVER BEAM - FREE VIBRATION

$$\frac{ML^2}{EIY_0}$$

$\lambda = 4.6941$ $\alpha = 0.0$

θ \ z	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	-22.034	-11.540	-1.543	6.986	12.988	15.725	15.059	11.593	6.633	2.040	0.000
15	-21.283	-11.147	-1.490	6.748	12.546	15.189	14.546	11.198	6.407	1.971	0.000
30	-19.082	-9.994	-1.336	6.050	11.248	13.618	13.042	10.039	5.744	1.767	0.000
45	-15.580	-8.160	-1.091	4.939	9.184	11.119	10.648	8.197	4.690	1.443	0.000
60	-11.017	-5.770	-.771	3.493	6.494	7.862	7.529	5.796	3.316	1.020	0.000
75	-5.702	-2.986	-.399	1.808	3.361	4.070	3.897	3.000	1.716	.528	0.000
90	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
105	5.702	2.986	.399	-1.808	-3.361	-4.070	-3.897	-3.000	-1.716	-.528	0.000
120	11.017	5.770	.771	-3.493	-6.494	-7.862	-7.529	-5.796	-3.316	-1.020	0.000
135	15.580	8.160	1.091	-4.939	-9.184	-11.119	-10.648	-8.197	-4.690	-1.443	0.000
150	19.082	9.994	1.336	-6.050	-11.248	-13.618	-13.042	-10.039	-5.744	-1.767	0.000
165	21.283	11.147	1.490	-6.748	-12.546	-15.189	-14.546	-11.198	-6.407	-1.971	0.000
180	22.034	11.540	1.543	-6.986	-12.988	-15.725	-15.059	-11.593	-6.633	-2.040	0.000
195	21.283	11.147	1.490	-6.748	-12.546	-15.189	-14.546	-11.198	-6.407	-1.971	0.000
210	19.082	9.994	1.336	-6.050	-11.248	-13.618	-13.042	-10.039	-5.744	-1.767	0.000
225	15.580	8.160	1.091	-4.939	-9.184	-11.119	-10.648	-8.197	-4.690	-1.443	0.000
240	11.017	5.770	.771	-3.493	-6.494	-7.862	-7.529	-5.796	-3.316	-1.020	0.000
255	5.702	2.986	.399	-1.808	-3.361	-4.070	-3.897	-3.000	-1.716	-.528	0.000
270	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
285	-5.702	-2.986	-.399	1.808	3.361	4.069	3.897	3.000	1.716	.528	0.000
300	-11.017	-5.770	-.771	3.493	6.494	7.862	7.529	5.796	3.316	1.020	0.000
315	-15.580	-8.160	-1.091	4.939	9.184	11.119	10.648	8.197	4.690	1.443	0.000
330	-19.082	-9.994	-1.336	6.050	11.248	13.618	13.042	10.039	5.744	1.767	0.000
345	-21.283	-11.147	-1.490	6.748	12.546	15.189	14.546	11.198	6.407	1.971	0.000

TABLE 8A. BENDING MOMENT IN CANTILEVER BEAM - FREE VIBRATION

$\lambda = 4.6941 \qquad \alpha = 0.2 \qquad \dfrac{ML^2}{EIY_0}$

θ \ z	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	-22.034	-11.540	-1.543	6.986	12.988	15.725	15.059	11.593	6.633	2.040	0.000
15	-21.284	-11.147	-1.490	6.748	12.546	15.190	14.547	11.198	6.407	1.971	0.000
30	-19.091	-9.999	-1.337	6.052	11.253	13.624	13.048	10.044	5.747	1.768	0.000
45	-15.610	-8.176	-1.093	4.949	9.202	11.140	10.669	8.213	4.699	1.445	0.000
60	-11.084	-5.805	-.776	3.514	6.534	7.910	7.575	5.831	3.337	1.026	0.000
75	-5.826	-3.051	-.408	1.847	3.434	4.158	3.982	3.065	1.754	.539	0.000
90	-.197	-.103	-.013	.062	.116	.140	.134	.103	.059	.018	0.000
105	5.418	2.838	.379	-1.718	-3.194	-3.867	-3.703	-2.851	-1.631	-.501	0.000
120	10.639	5.572	.745	-3.373	-6.271	-7.593	-7.272	-5.597	-3.203	-.985	0.000
135	15.112	7.915	1.058	-4.791	-8.908	-10.785	-10.329	-7.951	-4.549	-1.399	0.000
150	18.536	9.708	1.298	-5.877	-10.926	-13.228	-12.669	-9.752	-5.580	-1.716	0.000
165	20.684	10.833	1.448	-6.557	-12.192	-14.761	-14.136	-10.882	-6.227	-1.915	0.000
180	21.415	11.216	1.499	-6.789	-12.623	-15.283	-14.636	-11.267	-6.447	-1.983	0.000
195	20.686	10.834	1.448	-6.558	-12.194	-14.763	-14.138	-10.883	-6.227	-1.916	0.000
210	18.554	9.718	1.299	-5.882	-10.937	-13.241	-12.681	-9.762	-5.585	-1.718	0.000
225	15.171	7.946	1.062	-4.810	-8.943	-10.827	-10.369	-7.982	-4.567	-1.405	0.000
240	10.772	5.642	.754	-3.415	-6.350	-7.688	-7.363	-5.668	-3.243	-.997	0.000
255	5.662	2.965	.396	-1.795	-3.338	-4.041	-3.870	-2.979	-1.704	-.524	0.000
270	.191	.100	.013	-.060	-.112	-.136	-.130	-.100	-.057	-.017	0.000
285	-5.266	-2.758	-.368	1.669	3.104	3.758	3.599	2.770	1.585	.487	0.000
300	-10.340	-5.415	-.724	3.278	6.095	7.379	7.067	5.440	3.113	.957	0.000
315	-14.687	-7.692	-1.028	4.656	8.658	10.482	10.038	7.727	4.421	1.360	0.000
330	-18.015	-9.435	-1.261	5.711	10.619	12.857	12.313	9.478	5.423	1.668	0.000
345	-20.102	-10.528	-1.407	6.373	11.850	14.346	13.739	10.576	6.051	1.862	0.000

TABLE 1B. DEFLECTION OF CANTILEVER BEAM - FORCED VIBRATION

$\beta = 5.0$ $\alpha = 0.0$ y/Y_0

θ \ z	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	.258	.167	-.050	-.306	-.512	-.602	-.538	-.319	.021	.435	.874
30	.500	.323	-.098	-.591	-.990	-1.163	-1.039	-.617	.041	.840	1.689
45	.707	.457	-.139	-.837	-1.400	-1.644	-1.470	-.873	.058	1.189	2.388
60	.866	.559	-.170	-1.025	-1.715	-2.014	-1.800	-1.070	.071	1.456	2.925
75	.965	.624	-.190	-1.143	-1.913	-2.247	-2.008	-1.193	.079	1.624	3.263
90	1.000	.646	-.197	-1.183	-1.980	-2.326	-2.079	-1.235	.082	1.681	3.378
105	.965	.624	-.190	-1.143	-1.913	-2.247	-2.008	-1.193	.079	1.624	3.263
120	.866	.559	-.170	-1.025	-1.715	-2.014	-1.800	-1.070	.071	1.456	2.925
135	.707	.457	-.139	-.837	-1.400	-1.644	-1.470	-.873	.058	1.189	2.388
150	.500	.323	-.098	-.591	-.990	-1.163	-1.039	-.617	.041	.840	1.689
165	.258	.167	-.050	-.306	-.512	-.602	-.538	-.319	.021	.435	.874
180	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
195	-.258	-.167	.050	.306	.512	.602	.538	.319	-.021	-.435	-.874
210	-.499	-.323	.098	.591	.990	1.163	1.039	.617	-.041	-.840	-1.689
225	-.707	-.457	.139	.837	1.400	1.644	1.470	.873	-.058	-1.189	-2.388
240	-.866	-.559	.170	1.025	1.715	2.014	1.800	1.070	-.071	-1.456	-2.925
255	-.965	-.624	.190	1.143	1.913	2.247	2.008	1.193	-.079	-1.624	-3.263
270	-1.000	-.646	.197	1.183	1.980	2.326	2.079	1.235	-.082	-1.681	-3.378
285	-.965	-.624	.190	1.143	1.913	2.247	2.008	1.193	-.079	-1.624	-3.263
300	-.866	-.559	.170	1.025	1.715	2.014	1.800	1.070	-.071	-1.456	-2.925
315	-.707	-.457	.139	.837	1.400	1.644	1.470	.873	-.058	-1.189	-2.388
330	-.500	-.323	.098	.591	.990	1.163	1.039	.617	-.041	-.840	-1.689
345	-.258	-.167	.050	.306	.512	.602	.538	.319	-.021	-.435	-.874

TABLE 2B. DEFLECTION OF CANTILEVER BEAM - FORCED VIBRATION

$\beta = 5.0$ $\alpha = 0.2$ y/Y_0

θ \ z	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	0.000	-.146	-.470	-.809	-1.038	-1.071	-.876	-.465	.108	.775	1.474
15	.258	.057	-.400	-.905	-1.277	-1.388	-1.178	-.658	.101	1.002	1.950
30	.500	.258	-.303	-.939	-1.429	-1.611	-1.401	-.807	.088	1.160	2.293
45	.707	.441	-.185	-.909	-1.484	-1.723	-1.527	-.900	.068	1.240	2.479
60	.866	.594	-.055	-.817	-1.438	-1.718	-1.550	-.932	.044	1.234	2.497
75	.965	.706	.078	-.669	-1.293	-1.596	-1.467	-.900	.017	1.145	2.345
90	.999	.770	.207	-.476	-1.061	-1.365	-1.284	-.807	-.010	.978	2.033
105	.965	.782	.322	-.250	-.756	-1.041	-1.014	-.659	-.038	.743	1.582
120	.866	.741	.414	-.007	-.399	-.646	-.674	-.467	-.063	.459	1.023
135	.707	.648	.479	.235	-.015	-.207	-.289	-.242	-.084	.143	.395
150	.500	.512	.510	.463	.368	.245	.116	0.000	-.099	-.182	-.260
165	.258	.341	.507	.659	.728	.681	.513	.240	-.107	-.496	-.897
180	0.000	.146	.470	.809	1.038	1.071	.876	.465	-.108	-.775	-1.474
195	-.258	-.057	.400	.905	1.277	1.388	1.178	.658	-.101	-1.002	-1.950
210	-.499	-.258	.303	.939	1.429	1.611	1.401	.807	-.088	-1.160	-2.293
225	-.707	-.441	.185	.909	1.484	1.723	1.527	.900	-.068	-1.240	-2.479
240	-.866	-.594	.055	.817	1.438	1.718	1.550	.932	-.044	-1.234	-2.497
255	-.965	-.706	-.078	.669	1.293	1.596	1.467	.900	-.017	-1.145	-2.345
270	-1.000	-.770	-.207	.476	1.061	1.365	1.284	.807	.010	-.978	-2.033
285	-.965	-.782	-.322	.250	.756	1.041	1.014	.659	.038	-.743	-1.582
300	-.866	-.741	-.414	.007	.399	.646	.674	.467	.063	-.459	-1.023
315	-.707	-.648	-.479	-.235	.015	.207	.289	.242	.084	-.143	-.395
330	-.500	-.512	-.510	-.463	-.368	-.245	-.116	0.000	.099	.182	.260
345	-.258	-.341	-.507	-.659	-.728	-.681	-.513	-.240	.107	.496	.897

TABLE 3B. DEFLECTION OF CANTILEVER BEAM - FORCED VIBRATION

$\beta = 10.0$ $\alpha = 0.0$ y/Y_0

θ \ z	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	.258	.282	.204	-.004	-.187	-.191	-.018	.166	.183	-.010	-.308
30	.500	.545	.394	-.008	-.362	-.369	-.035	.322	.354	-.019	-.596
45	.707	.772	.557	-.011	-.512	-.522	-.050	.456	.501	-.027	-.843
60	.866	.945	.682	-.014	-.627	-.640	-.061	.558	.614	-.034	-1.032
75	.965	1.054	.761	-.016	-.700	-.714	-.068	.623	.685	-.038	-1.151
90	1.000	1.091	.788	-.016	-.725	-.739	-.070	.645	.709	-.039	-1.192
105	.965	1.054	.761	-.016	-.700	-.714	-.068	.623	.685	-.038	-1.151
120	.866	.945	.682	-.014	-.627	-.640	-.061	.558	.614	-.034	-1.032
135	.707	.772	.557	-.011	-.512	-.522	-.050	.456	.501	-.027	-.843
150	.500	.545	.394	-.008	-.362	-.369	-.035	.322	.354	-.019	-.596
165	.258	.282	.204	-.004	-.187	-.191	-.018	.166	.183	-.010	-.308
180	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
195	-.258	-.282	-.204	.004	.187	.191	.018	-.166	-.183	.010	.308
210	-.499	-.545	-.394	.008	.362	.369	.035	-.322	-.354	.019	.596
225	-.707	-.772	-.557	.011	.512	.522	.050	-.456	-.501	.027	.843
240	-.866	-.945	-.682	.014	.627	.640	.061	-.558	-.614	.034	1.032
255	-.965	-1.054	-.761	.016	.700	.714	.068	-.623	-.685	.038	1.151
270	-1.000	-1.091	-.788	.016	.725	.739	.070	-.645	-.709	.039	1.192
285	-.965	-1.054	-.761	.016	.700	.714	.068	-.623	-.685	.038	1.151
300	-.866	-.945	-.682	.014	.627	.640	.061	-.558	-.614	.034	1.032
315	-.707	-.772	-.557	.011	.512	.522	.050	-.456	-.501	.027	.843
330	-.500	-.545	-.394	.008	.362	.369	.035	-.322	-.354	.019	.596
345	-.258	-.282	-.204	.004	.187	.191	.018	-.166	-.183	.010	.308

TABLE 4B. DEFLECTION OF CANTILEVER BEAM - FORCED VIBRATION

$\beta = 10.0$ $\alpha = 0.2$

y/Y_0

θ \ z	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	0.000	-.204	-.391	-.245	.094	.284	.159	-.112	-.220	-.033	.317
15	.258	.068	-.205	-.255	-.076	.117	.147	.034	-.062	-.042	.051
30	.499	.336	-.005	-.248	-.242	-.057	.125	.179	.099	-.049	-.217
45	.707	.581	.195	-.224	-.392	-.229	.094	.312	.255	-.053	-.471
60	.866	.787	.382	-.185	-.515	-.384	.057	.423	.393	-.053	-.693
75	.965	.939	.543	-.133	-.603	-.514	.016	.505	.505	-.049	-.868
90	1.000	1.027	.668	-.072	-.649	-.608	-.025	.553	.581	-.042	-.983
105	.965	1.045	.746	-.006	-.652	-.661	-.065	.563	.619	-.032	-1.032
120	.866	.992	.774	.060	-.610	-.669	-.101	.535	.614	-.020	-1.010
135	.707	.871	.749	.122	-.526	-.631	-.130	.471	.567	-.006	-.919
150	.500	.691	.673	.176	-.406	-.550	-.150	.374	.482	.007	-.766
165	.258	.463	.551	.218	-.259	-.432	-.160	.251	.363	.020	-.560
180	0.000	.204	.391	.245	-.094	-.284	-.159-	.112	.220	.032	-.317
195	-.258	-.068	.205	.255	.076	-.117	-.147	-.034	.062	.042	-.051
210	-.499	-.336	.005	.248	.242	.057	-.125	-.179	-.099	.049	.217
225	-.707	-.581	-.195	.224	.392	.229	-.094	-.312	-.255	.053	.471
240	-.866	-.787	-.382	.185	.515	.384	-.057	-.423	-.393	.053	.693
255	-.965	-.939	-.543	.133	.603	.514	-.016	-.505	-.505	.049	.868
270	-1.000	-1.027	-.668	.072	.649	.608	.025	-.553	-.581	.042	.983
285	-.965	-1.045	-.746	.006	.652	.661	.065	-.563	-.619	.032	1.032
300	-.866	-.992	-.774	-.060	.610	.669	.101	-.535	-.614	.020	1.010
315	-.707	-.871	-.749	-.122	.526	.631	.130	-.471	-.567	.006	.919
330	-.500	-.691	-.673	-.176	.406	.550	.150	-.374	-.481	-.007	.766
345	-.258	-.463	-.551	-.218	.259	.432	.160	-.251	-.363	-.020	.560

TABLE 5B. BENDING MOMENT IN CANTILEVER BEAM - FORCED VIBRATION

$\beta = 5.0$ $\alpha = 0.0$ $\mu^2 = 25.0$

$$\frac{ML^2}{EIY_0 \mu^2}$$

θ \ z	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	-.835	-.510	-.147	.202	.477	.626	.627	.498	.291	.090	0.000
30	-1.613	-.986	-.284	.392	.922	1.210	1.213	.962	.562	.175	0.000
45	-2.281	-1.395	-.401	.554	1.304	1.711	1.715	1.361	.795	.248	0.000
60	-2.794	-1.709	-.492	.679	1.597	2.095	2.101	1.667	.974	.304	0.000
75	-3.117	-1.906	-.549	.757	1.782	2.337	2.343	1.859	1.086	.339	0.000
90	-3.226	-1.973	-.568	.784	1.845	2.420	2.426	1.925	1.125	.351	0.000
105	-3.117	-1.906	-.549	.757	1.782	2.337	2.343	1.859	1.086	.339	0.000
120	-2.794	-1.709	-.492	.679	1.597	2.095	2.101	1.667	.974	.304	0.000
135	-2.281	-1.395	-.401	.554	1.304	1.711	1.715	1.361	.795	.248	0.000
150	-1.613	-.986	-.284	.392	.922	1.210	1.213	.962	.562	.175	0.000
165	-.835	-.510	-.147	.202	.477	.626	.627	.498	.291	.090	0.000
180	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
195	.835	.510	.147	-.202	-.477	-.626	-.627	-.498	-.291	-.090	0.000
210	1.613	.986	.284	-.392	-.922	-1.210	-1.213	-.962	-.562	-.175	0.000
225	2.281	1.395	.401	-.554	-1.304	-1.711	-1.715	-1.361	-.795	-.248	0.000
240	2.794	1.709	.492	-.679	-1.597	-2.095	-2.101	-1.667	-.974	-.304	0.000
255	3.117	1.906	.549	-.757	-1.782	-2.337	-2.343	-1.859	-1.086	-.339	0.000
270	3.226	1.973	.568	-.784	-1.845	-2.420	-2.426	-1.925	-1.125	-.351	0.000
285	3.117	1.906	.549	-.757	-1.782	-2.337	-2.343	-1.859	-1.086	-.339	0.000
300	2.794	1.709	.492	-.679	-1.597	-2.095	-2.101	-1.667	-.974	-.304	0.000
315	2.281	1.395	.401	-.554	-1.304	-1.711	-1.715	-1.361	-.795	-.248	0.000
330	1.613	.986	.284	-.392	-.922	-1.210	-1.213	-.962	-.562	-.175	0.000
345	.835	.510	.147	-.202	-.477	-.626	-.627	-.498	-.291	-.090	0.000

TABLE 6B. BENDING MOMENT IN CANTILEVER BEAM - FORCED VIBRATION

$\beta = 5.0 \qquad \alpha = 0.2 \qquad \mu^2 = 25.2463 \qquad \dfrac{ML^2}{EIY_0\,\mu^2}$

θ \ z	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	-1.405	-.692	-.055	.455	.790	.924	.863	.652	.367	.111	0.000
15	-1.881	-1.015	-.177	.544	1.057	1.296	1.247	.963	.551	.169	0.000
30	-2.229	-1.269	-.287	.597	1.252	1.579	1.547	1.208	.698	.216	0.000
45	-2.424	-1.436	-.378	.608	1.362	1.755	1.741	1.371	.796	.247	0.000
60	-2.455	-1.506	-.442	.579	1.378	1.811	1.816	1.440	.841	.262	0.000
75	-2.318	-1.472	-.477	.509	1.301	1.744	1.767	1.411	.828	.259	0.000
90	-2.023	-1.339	-.479	.405	1.135	1.558	1.598	1.286	.759	.238	0.000
105	-1.590	-1.114	-.448	.274	.892	1.266	1.320	1.074	.638	.201	0.000
120	-1.049	-.813	-.387	.123	.588	.887	.952	.788	.473	.150	0.000
135	-.436	-.457	-.299	-.034	.244	.448	.519	.448	.276	.089	0.000
150	.205	-.070	-.191	-.191	-.116	-.021	.051	.078	.061	.022	0.000
165	.834	.322	-.070	-.334	-.469	-.489	-.420	-.296	-.158	-.046	0.000
180	1.405	.692	.055	-.455	-.790	-.924	-.863	-.652	-.367	-.111	0.000
195	1.881	1.015	.177	-.544	-1.057	-1.296	-1.247	-.963	-.551	-.169	0.000
210	2.229	1.269	.287	-.597	-1.252	-1.579	-1.547	-1.208	-.698	-.216	0.000
225	2.424	1.436	.378	-.608	-1.362	-1.755	-1.741	-1.371	-.796	-.247	0.000
240	2.455	1.506	.442	-.579	-1.378	-1.811	-1.816	-1.440	-.841	-.262	0.000
255	2.318	1.472	.477	-.509	-1.301	-1.744	-1.767	-1.411	-.828	-.259	0.000
270	2.023	1.339	.479	-.405	-1.135	-1.558	-1.598	-1.286	-.759	-.238	0.000
285	1.590	1.114	.448	-.274	-.892	-1.266	-1.320	-1.074	-.638	-.201	0.000
300	1.049	.813	.387	-.123	-.588	-.887	-.952	-.788	-.473	-.150	0.000
315	.436	.457	.299	.034	-.244	-.448	-.519	-.448	-.276	-.089	0.000
330	-.205	.070	.191	.191	.116	.021	-.051	-.078	-.061	-.022	0.000
345	-.834	-.322	.070	.334	.469	.489	.420	.296	.158	.046	0.000

TABLE 7B. BENDING MOMENT IN CANTILEVER BEAM - FORCED VIBRATION

$$\frac{ML^2}{EIY_0\,\mu^2}$$

$\beta = 10.0$ $\alpha = 0.0$ $\mu^2 = 100.0$

θ \ z	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	.168	-.125	-.146	.025	.194	.192	.013	-.181	-.225	-.103	0.000
30	.324	-.242	-.282	.048	.376	.371	.026	-.351	-.435	-.199	0.000
45	.458	-.343	-.399	.069	.532	.525	.037	-.497	-.615	-.282	0.000
60	.562	-.420	-.489	.084	.651	.643	.045	-.608	-.754	-.345	0.000
75	.626	-.468	-.546	.094	.726	.717	.051	-.679	-.841	-.385	0.000
90	.649	-.485	-.565	.097	.752	.742	.053	-.702	-.870	-.399	0.000
105	.626	-.468	-.546	.094	.726	.717	.051	-.679	-.841	-.385	0.000
120	.562	-.420	-.489	.084	.651	.643	.045	-.608	-.754	-.345	0.000
135	.458	-.343	-.399	.069	.532	.525	.037	-.497	-.615	-.282	0.000
150	.324	-.242	-.282	.048	.376	.371	.026	-.351	-.435	-.199	0.000
165	.168	-.125	-.146	.025	.194	.192	.013	-.181	-.225	-.103	0.000
180	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
195	-.168	.125	.146	-.025	-.194	-.192	-.013	.181	.225	.103	0.000
210	-.324	.242	.282	-.048	-.376	-.371	-.026	.351	.435	.199	0.000
225	-.458	.343	.399	-.069	-.532	-.525	-.037	.497	.615	.282	0.000
240	-.562	.420	.489	-.084	-.651	-.643	-.045	.608	.754	.345	0.000
255	-.626	.468	.546	-.094	-.726	-.717	-.051	.679	.841	.385	0.000
270	-.649	.485	.565	-.097	-.752	-.742	-.053	.702	.870	.399	0.000
285	-.626	.468	.546	-.094	-.726	-.717	-.051	.679	.841	.385	0.000
300	-.562	.420	.489	-.084	-.651	-.643	-.045	.608	.754	.345	0.000
315	-.458	.343	.399	-.069	-.532	-.525	-.037	.497	.615	.282	0.000
330	-.324	.242	.282	-.048	-.376	-.371	-.026	.351	.435	.199	0.000
345	-.168	.125	.146	-.025	-.194	-.192	-.013	.181	.225	.103	0.000

TABLE 8B. BENDING MOMENT IN CANTILEVER BEAM - FORCED VIBRATION

$\beta = 10.0$ $\alpha = 0.2$ $\mu^2 = 100.9853$ $\dfrac{ML^2}{EIY_0\,\mu^2}$

θ \ Z	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	-.667	.050	.372	.210	-.166	-.344	-.157	.179	.319	.162	0.000
15	-.530	-.070	.249	.247	.010	-.182	-.153	.021	.130	.077	0.000
30	-.358	-.187	.110	.267	.187	-.007	-.139	-.138	-.067	-.013	0.000
45	-.161	-.291	-.036	.269	.350	.167	-.114	-.288	-.261	-.103	0.000
60	.046	-.376	-.181	.252	.490	.331	-.083	-.419	-.437	-.185	0.000
75	.251	-.434	-.313	.219	.597	.472	-.045	-.521	-.583	-.255	0.000
90	.439	-.463	-.424	.170	.662	.581	-.004	-.587	-.689	-.308	0.000
105	.596	-.461	-.506	.110	.683	.650	.036	-.614	-.748	-.340	0.000
120	.713	-.426	-.554	.042	.657	.675	.074	-.598	-.756	-.348	0.000
135	.782	-.363	-.563	-.028	.586	.654	.107	-.542	-.713	-.333	0.000
150	.797	-.275	-.535	-.096	.475	.589	.134	-.449	-.621	-.295	0.000
165	.757	-.169	-.469	-.159	.332	.483	.151	-.325	-.486	-.236	0.000
180	.667	-.050	-.372	-.210	.166	.344	.157	-.179	-.319	-.162	0.000
195	.530	.070	-.249	-.247	-.010	.182	.153	-.021	-.130	-.077	0.000
210	.358	.187	-.110	-.267	-.187	.007	.139	.138	.067	.013	0.000
225	.161	.291	.036	-.269	-.350	-.167	.114	.288	.261	.103	0.000
240	-.046	.376	.181	-.252	-.490	-.331	.083	.419	.437	.185	0.000
255	-.251	.434	.313	-.219	-.597	-.472	.045	.521	.583	.255	0.000
270	-.439	.463	.424	-.170	-.662	-.581	.004	.587	.689	.308	0.000
285	-.596	.461	.506	-.110	-.683	-.650	-.036	.614	.748	.340	0.000
300	-.713	.426	.554	-.042	-.657	-.675	-.074	.598	.756	.348	0.000
315	-.782	.363	.563	.028	-.586	-.654	-.107	.542	.713	.333	0.000
330	-.797	.275	.535	.096	-.475	-.589	-.134	.449	.621	.295	0.000
345	-.757	.169	.469	.159	-.332	-.483	-.151	.325	.486	.236	0.000

APPENDIX. DERIVATIONS

A. INTRODUCTION

The derivations of the formulas presented in Section II are given in this section. These derivations stem from the solution of a differential equation whose development for undamped vibration is presented in texts on elementary strength of materials and mechanical vibrations. The presentation in this report treats primarily the development of the formulas from the solution of this differential equation.

B. INITIAL DIFFERENTIAL EQUATION

A brief account of the development of the initial differential equation for a vibrating cantilever beam is given in this section. This development begins with the equation of equilibrium and the moment-curvature relationship written below. *

1. Equilibrium Equations

$$\frac{\partial Q}{\partial x} = - m \frac{\partial^2 y}{\partial t^2} - c \frac{\partial y}{\partial t}$$

$$\frac{\partial \underline{M}}{\partial x} = Q \tag{1}$$

2. Moment-Curvature Relationship

$$\underline{M} = E I \frac{\partial^2 y}{\partial x^2} \tag{2}$$

In the above relationships, Q is the lateral shearing force and $\underline{M}$ is the bending moment in the beam. The symbols m and c are the mass and the viscous damping coefficient per unit length of the beam, respectively.

The initial differential equation is developed by eliminating $\underline{M}$ and Q from Equations 1 and 2. This yields

$$E I \frac{\partial^4 y}{\partial x^4} = - m \frac{\partial^2 y}{\partial t^2} - c \frac{\partial y}{\partial t} \quad . \tag{3}$$

The formulas for each type of vibration are developed from a solution of this differential equation (Eq. 3). The first step in this development is the transformation of Equation 3 to dimensionless form. This transformation is slightly different for each type of vibration. This transformation and the subsequent derivations for each type of vibration is given in separate subsections. The formulas for free vibration are developed in the next subsection.

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* These relationships for the undamped vibrations of a cantilever beam are developed and presented by W. T. Thomson, "Mechanical Vibrations," Prentice-Hall, Inc. 1948, Chapter 6. In this report, the positive direction of the coordinate y is changed.

C. FREE VIBRATION

The derivation of the formulas for free vibration is given in this section. This derivation begins by introducing the following coordinate transformations and symbol definitions.

$$x = zL, \qquad m = \frac{W}{gL}, \qquad c = \frac{C}{L}$$

$$K^2 = \frac{EIg}{WL^3}, \quad \alpha = \frac{Cg}{2WK} \tag{4}$$

$$\theta = Kt.$$

The above definition of K and α is the same as Formulas 1 and 2 of Section II.

The definitions and coordinate transformations in Equation 4 transform Equation 3 to the following form.

$$\frac{\partial^4 y}{\partial z^4} + \frac{\partial^2 y}{\partial \theta^2} + 2\,\alpha \frac{\partial y}{\partial \theta} = 0. \tag{5}$$

The solution of Equation 5 is obtained by the method of separation of variables, that is, the deflection y is taken to be of the form

$$y/Y_o = ZT, \tag{6}$$

where Z is a function of the dimensionless axial coordinate z only, T is a function of the dimensionless time θ only and Y_o is an arbitrary constant.

Substituting Equation 6 into Equation 5 and separating the variables yields the following two ordinary differential equations.

$$\frac{d^4Z}{dz^4} - \lambda^4 Z = 0, \qquad \text{(a)}$$

$$\frac{d^2T}{d\theta^2} + 2\,\alpha \frac{dT}{d\theta} + \lambda^4 T = 0, \qquad \text{(b)} \tag{7}$$

where λ is a constant, which will be evaluated later from the specific conditions imposed upon the vibrating beam.

Equation 7a is written below in a form which satisfies the boundary conditions that the deflection and slope are zero at the fixed end of the cantilever beam; in other words, $z = dZ/dz = 0$ at $z = 0$.

$$Z = A(\cosh \lambda z - \cos \lambda z) + B(\sinh \lambda z - \sin \lambda z). \tag{8}$$

The remaining two boundary conditions are that the bending moment and lateral shearing force are zero at the free end of the beam. This corresponds to setting the second and the third derivatives of Z with respect to z equal to zero at z = 1. These conditions yield the following matrix equation.

$$\begin{bmatrix} \cosh\lambda + \cos\lambda & \sinh\lambda + \sin\lambda \\ \sinh\lambda - \sin\lambda & \cosh\ + \cos\lambda \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0. \tag{9}$$

A non-trival solution of Equation 9 is obtained only when λ is the characteristic root of the coefficient matrix. Equating the deteriment of this coefficient matrix to zero, yields the following expression for the evaluation of the characteristic root.

$$1 + \cosh\lambda \cos\lambda = 0. \tag{10}$$

The first five roots of Equation 10 are tabulated in Section II as Formula 3. Accuracy greater than eight significant digits is required to obtain the higher roots because of the characteristic behavior of Equation 10 at these higher values. The mode of vibration is characterized by the value of the characteristic root, λ. The first two modes are shown in Figure 1a and 1b in the definition of symbols.

For each characteristic root there corresponds a characteristic vector whose elements are the integration constants A and B. The value of the integration constant A is arbitrarily taken equal to one. The integration constant B is evaluated from the first of Equation 9; this yields:

$$B = -\frac{\cosh\lambda + \cos\lambda}{\sinh\lambda + \sin\lambda}. \tag{11}$$

Equation 11 is Formula 5 of Section II. The evaluation of A and B completes the definition of the Z-function in Equation 8 and is Formula 6 of Section II. The constant Z_1 (Formula 7 of Section II) is the value of the Z-function at z = 1 and will be used later. This discussion continues with the solution of Equation 7b.

The general solution of Equation 7b is

$$T = e^{-\alpha\theta}\ (E \cos\gamma\,\theta + F \sin\gamma\,\theta), \tag{12}$$

where E and F are integration constants, and $\gamma = \sqrt{\lambda^4 - \alpha^2}$. During this solution it is assumed that λ^4 is greater than α^2 which must be true in order to have oscillatory motion.

It is assumed that the initial conditions for free vibration are that the deflection is equal to an arbitrary displacement Y_0 and that the velocity is equal to zero at the free end of the beam when t = 0; mathematically,

$$\left.\begin{aligned} y &= Y_o, \\ \text{and} \quad \frac{dy}{d\theta} &= 0, \end{aligned}\right\} \quad \text{when } \theta = 0 \text{ and } z = 1. \tag{13}$$

Imposing these conditions upon Equation 6 yields

$$E = 1/Z_1, \qquad \text{and } F = \alpha/\gamma\, Z_1. \tag{14}$$

Substituting Equations 14 into Equation 12 completes the definition of the T-function; that is,

$$T = \frac{e^{-\alpha\theta}}{\gamma Z_1} (\gamma \cos\gamma\, \theta + \alpha \sin \gamma\theta), \tag{15}$$

which is Formula 8 of Section II.

Introducing the above definitions of the Z-and the T-functions into Equation 6 completes the derivation of Formula 9 of Section II. The derivation of the remaining formulas for the free vibration of a cantilever beam (Formulas 10 through 13) are developed in a straightforward manner by taking the appropriate derivative of Formula 9. The definitions and the corresponding coordinate transformations for this are tabulated below.

Velocity

$$V = \frac{\partial y}{\partial t} = K \frac{\partial y}{\partial \theta} = K\, Y_o\, Z \frac{dT}{d\theta}$$

Acceleration

$$\underline{A} = \frac{\partial^2 y}{\partial t^2} = K^2 \frac{\partial^2 y}{\partial \theta^2} = K^2\, Y_o\, Z \frac{d^2T}{d\theta^2}$$

(16)

Bending Moment

$$\underline{M} = E\, I \frac{\partial^2 y}{\partial x^2} = \frac{EI}{L^2} \frac{\partial^2 y}{\partial z^2} = \frac{EI\, Y_o}{L^2} \frac{d^2Z}{dz^2}\, T$$

Shearing Force

$$Q = EI \frac{\partial^3 y}{\partial x^3} = \frac{E\, I}{L^3} \frac{\partial^3 y}{\partial z^3} = \frac{EI\, Y_o}{L^3} \frac{d^3Z}{dz^3}\, T$$

Formulas 10 through 13 are verified by substituting and performing the operations indicated in Equations 16. These operations are elementary and straightforward.

This concludes the discussion on the derivation of the formulas for the free vibration of a cantilever beam presented in Section II. The formulas for the forced vibration of a cantilever beam is given in the next subsection.

D. FORCED VIBRATION

The formulas for the forced vibration of a cantilever beam are derived in this section. This derivation begins with the initial differential Equation 3.

In the case of forced vibration, it is assumed that the normally fixed end of the cantilever beam is subjected to an exciting displacement of the form $y = Y_o \sin \omega t$, where Y_o is a constant and ω is the circular frequency of the excitomotor. The formulas presented in Section II for this type of vibration pertain only for the steady state condition. This condition is characterized mathematically by a particular rather than by the general solution of Equation 3.

The initial step for the derivation of this particular solution is the transformation of Equation 3 to dimensionless form, by introducing the following symbols and coordinate transformations.

$$x = zL, \quad m = \frac{W}{gL}, \quad \theta = \omega t, \quad c = \frac{C}{L},$$
$$\beta^4 = \frac{WL^3 \omega^2}{EIg} = \frac{\omega^2}{K^2}, \quad \alpha = \frac{Cg}{W\omega}. \tag{17}$$

The above definitions for β and α are Formulas 14 and 15 of Section II, respectively.

Imposing these definitions and transformations upon Equation 3 yields

$$\frac{\partial^4 y}{\partial z} + \beta^4 \left(\frac{\partial^2 y}{\partial \theta^2} + \alpha \frac{\partial y}{\partial \theta} \right) = 0. \tag{18}$$

A particular solution of Equation 18 is assumed to be of the form

$$y/Y_o = Z^s \sin \theta + Z^c \cos \theta, \tag{19}$$

where Z^s and Z^c are functions of the axial coordinate z only and Y_o is a constant.

Equation 19 is a solution of Equation 18 provided that the Z-functions satisfy the following two fourth-order ordinary differential equations simultaneously.

$$\frac{d^4 Z^s}{dz^4} - \beta^4 (Z^s + \alpha Z^c) = 0,$$

$$\frac{d^4 Z^c}{dz^4} - \beta^4 (Z^c - \alpha Z^s) = 0. \tag{20}$$

The following eighth-order differential equation is developed by combining Equations 20 to eliminate Z^c.

$$\frac{d^8 Z^s}{dz^8} - 2\beta^4 \frac{d^4 Z^s}{dz^4} + \beta^8 (1 + \alpha^2) Z^s = 0. \tag{21}$$

The following symbols are introduced for writing the general solution of Equation 21.

$$\mu = \beta\,(1 + \alpha^2)^{1/8} \qquad \phi = \frac{1}{4}\tan^{-1}\alpha$$

$$a = \mu \cos\phi \qquad b = \mu \sin\phi. \tag{22}$$

The definitions in Equations 22 are Formulas 16 and are determined by the roots of the auxiliary equation of the eighth-order differential Equation 21. The general solution of Equation 21, in matrix form is

$$Z^s = [\cosh az \quad \sinh az]\, M^s \begin{bmatrix} \cos bz \\ \sin bz \end{bmatrix} + [\cosh bz \quad \sinh bz]\, N^s \begin{bmatrix} \cos az \\ \sin az \end{bmatrix} \tag{23}$$

where the eight integration constants are presented as the elements of the following two matrices M^s and N^s.

$$M^s = \begin{bmatrix} A & D \\ C & B \end{bmatrix}^s, \qquad N^s = \begin{bmatrix} E & H \\ G & F \end{bmatrix}^s, \tag{24}$$

where the single superscript on the right is applicable to each of the elements in the matrix.

The form of the Z^c function is identical to the form in Equation 23, except for a different set of eight integration constants. This set is designated with the superscript c. In order to satisfy Equations 20 simultaneously, one set of the eight integration constants must be dependent upon the other set. To show this and then to evaluate the remaining eight integration constants it will become necessary to develop the first four derivatives of the Z-functions.

The form of the derivative of the Z-function, of any order, is the same form as Equation 23, except with a different set of eight constants. These other sets of constants and the order of the derivative are designated with an integer subscript on the matrices M and N. This creates the scripted symbols Z_i^j, M_i^j, and N_i^j, where j is s or c and $i = 0, 1, 2, 3$, or 4. The non-scripted symbols developed above are equivalent to the scripted symbols when j = s and i = 0. The superscripts s and c are omitted when the relationship is for both as is the case in the following discussion.

The elements of the sequential matrices (corresponding to the sequential derivatives) are obtained from a recurrence formula. Two transformation matrices, designated with the symbols J and K, are introduced for writing this recurrence formula.

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{and } K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} . \tag{25}$$

The recurrence formulas for the sequential matrices M and N are written below.

$$\left.\begin{aligned} M_i &= aJM_{i-1} + bM_{i-1}K \\ N_i &= bJN_{i-1} + aN_{i-1}K \end{aligned}\right\} \quad (i = 1, 2, 3, 4) \quad . \tag{26}$$

The elements of the matrices for $i \geq 2$ are expressed in terms of the matrices M_o and N_o by a sequential substitution process. This process is illustrated for the first of Equation 26 with i = 1 and 2.

$$\begin{aligned} M_1 &= aJM_o + bM_oK , \\ M_2 &= aJM_1 + bM_1K . \end{aligned} \tag{27}$$

Before substituting the first into the second of Equation 27, the following property of matrices J and K is recognized.

$$JJ = -KK = I, \tag{28}$$

where I is the identity matrix of order two.

Accordingly,

$$M_2 = (a^2 - b^2)\, M_o + 2abJM_oK,$$

which, upon substituting for a and b from Equations 22 and using known trigonometric identities, transforms to the following form

$$M_2 = \mu^2 (\cos 2\phi\, M_o + \sin 2\phi\, JM_oK) \,. \tag{29}$$

The remaining matrices are developed in a similar manner. The final form of matrices M_i and N_i ($i = 1, 2, 3, 4$) are tabulated below.

$$\begin{aligned}
M_1 &= \mu (\cos \phi\, JM_o + \sin \phi\, M_oK) \\
M_2 &= \mu^2(\cos 2\phi\, M_o + \sin 2\phi\, JM_oK) \\
M_3 &= \mu^3(\cos 3\phi\, JM_o + \sin 3\phi\, M_oK) \\
M_4 &= \mu^4(\cos 4\phi\, M_o + \sin 4\phi\, JM_oK) \\
\\
N_1 &= \mu (\sin \phi\, JN_o + \cos \phi\, N_oK) \\
N_2 &= \mu^2(-\cos 2\phi\, N_o + \sin 2\phi\, JN_oK) \\
N_3 &= -\mu^3(\sin 3\phi\, JN_o + \cos 3\phi\, N_oK) \\
N_4 &= \mu^4(\cos 4\phi\, N_o - \sin 4\phi\, JN_oK) \,.
\end{aligned} \tag{30}$$

Any derivative of the Z-function is obtained by substituting the appropriate matrix M and N of Equations 30 into 23. The order of the derivative is designated by the integer subscript.

The above relationships are used to evaluate the two sets of eight integration constants; that is, the s-set and the c-set. The eight integration constants in each set are designated by the capital letters A through H and are presented as the elements of the matrices M and N; each with its corresponding superscript. The location of these constants within the matrices is as shown in Equations 24.

The integration constants in the c-set are expressed in terms of the s-set by substituting Equation 23, with the appropriate definition in Equation 30, into the first of Equations 20. This yields

$$M_o^c = JM_o^sK \,, \qquad \text{and} \qquad N_o^c = -JN_o^sK, \tag{31}$$

or, in expanded form

$$\begin{bmatrix} A & D \\ C & B \end{bmatrix}^c = \begin{bmatrix} B & -C \\ D & -A \end{bmatrix}^s, \text{ and } \begin{bmatrix} E & H \\ G & F \end{bmatrix}^c = \begin{bmatrix} -F & G \\ -H & E \end{bmatrix}^s .$$

The set of matrix equations in Formulas 20 in Section II is verified by substituting Equation 31 into the appropriate definition in Equations 30 and expanding the matrices and upon noting that the μ^2 and μ^3 terms are omitted in Formulas 20 as they are included in Formulas 25 and 26.

The eight integration constants in the set are evaluated by applying a set of four boundary conditions to each of the two Z-functions. These conditions are tabulated below.

$$\left.\begin{aligned} Z^s &= 1, & Z^c &= 0 \\ \frac{dZ^s}{dz} &= 0, & \frac{dZ^c}{dz} &= 0 \end{aligned}\right\} \text{ at } z = 0 \qquad \text{(a)}$$

$$(32)$$

$$\left.\begin{aligned} \frac{d^2Z^s}{dz^2} &= 0, & \frac{d^2Z^c}{dz^2} &= 0 \\ \frac{d^3Z^s}{dz^3} &= 0, & \frac{d^3Z^c}{dz^3} &= 0 \end{aligned}\right\} \text{ at } z = 1 \qquad \text{(b)}$$

The first two of Equations 32a ensure that the displacement at $z = 0$ is of the form $y = Y_0 \sin\theta$, where Y_0 is the amplitude of the exciting displacement, (see Equation 19). The second two of Equations 32a ensure that the slope dy/dx is zero at $z = 0$. Equations 32b ensure that the bending moment and lateral shearing force are zero at $z = 1$.

The value of the Z-function at $z = 0$ (Eq. 23) reduces to

$$Z = A + E. \qquad (33)$$

Equations 32a transform to the following set of equations.

$$\begin{aligned} A_0^s + E_0^s &= 1 \\ A_0^c + E_0^c &= 0 \\ A_1^s + E_1^s &= 0 \\ A_1^c + E_1^c &= 0\,. \end{aligned} \qquad (34)$$

Equation 34 are expressed in terms of the s-set of integration constants by using Equation 31 and the appropriate definitions in Equations 24, 25, and 30. This result is written below.

$$\left.\begin{aligned} A_o^s + E_o^s &= 1, \\ B_o^s - F_o^s &= 0, \end{aligned}\right\} \qquad \text{(a)}$$

$$\left.\begin{aligned} \cos\phi\ (C_o^s + H_o^s) + \sin\phi\ (D_o^s + G_o^s) &= 0. \\ -\sin\phi\ (C_o^s + H_o^s) + \cos\phi\ (D_o^s + G_o^s) &= 0, \end{aligned}\right\} \qquad \text{(b)} \qquad (35)$$

or, alternatively for Equations 35b.

$$\left.\begin{aligned} C_o^s + H_o^s &= 0 \\ D_o^s + G_o^s &= 0\ . \end{aligned}\right\} \qquad \text{(c)}$$

The following symbols are introduced for writing the value of the Z-function at $z = 1$, and are used in developing the expression for the remaining four boundary conditions in Equations 32b.

$$\begin{array}{ll} s_1 = \cosh a & t_1 = \cosh b \\ s_2 = \sinh a & t_2 = \sinh b \\ s_3 = \cos b & t_3 = \cos a \\ s_4 = \sin b & t_4 = \sin a \\ & \\ S_{13} = s_1 s_3 & T_{13} = t_1 t_3 \\ S_{14} = s_1 s_4 & T_{14} = t_1 t_4 \\ S_{23} = s_2 s_3 & T_{23} = t_2 t_3 \\ S_{24} = s_2 s_4 & T_{24} = t_2 t_4 \end{array} \qquad (36)$$

The definitions in Equations 36 are the same as those of Formulas 17 in Section II.

The value of the Z-function at $z = 1$ is written below in terms of the above symbols; first in matrix form and then in algebraic form.

$$Z = [s_1 \ \ s_2] \ M \begin{bmatrix} s_3 \\ s_4 \end{bmatrix} + [t_1 \ \ t_2] \ N \begin{bmatrix} t_3 \\ t_4 \end{bmatrix},$$

$$Z = AS_{13} + BS_{24} + CS_{23} + DS_{14} + ET_{13} + FT_{24} + GT_{23} + HT_{14} \tag{37}$$

Equations 37 are valid for any set of the scripted symbols introduced previously. In other words, the expression for any Z-function and any of its derivative is obtained by merely applying a consistent set of scripts to each of the symbols Z and A through H in Equations 37. To illustrate, the second derivative of Z^S and Z^C are written below.

$$Z_2^S = [s_1 \ \ s_2] \ M_2^S \begin{bmatrix} s_3 \\ s_4 \end{bmatrix} + [t_1 \ \ t_2] \ N_2^S \begin{bmatrix} t_3 \\ t_4 \end{bmatrix},$$

$$Z_2^C = [s_1 \ \ s_2] \ M_2^C \begin{bmatrix} s_3 \\ s_4 \end{bmatrix} + [t_1 \ \ t_2] \ N_2^C \begin{bmatrix} t_3 \\ t_4 \end{bmatrix}, \tag{38}$$

where, upon substituting Equation 31 into the appropriate definition of Equation 30 and then simplifying with Equation 28,

$$M_2^S = \mu^2 (\cos 2\phi M_O^S + \sin 2\phi J M_O^S K)$$

$$M_2^C = \mu^2 (-\sin 2\phi M_O^S + \cos 2\phi J M_O^S K) \tag{39}$$

$$N_2^S = \mu^2 (-\cos 2\phi N_O^S + \sin 2\phi J N_O^S K)$$

$$N_2^C = \mu^2 (\sin 2\phi \ N_O^S + \cos 2\phi \ J \ N_O^S K) \ .$$

The algebraic representation of the first two conditions in Equations 32b are obtained by substituting the appropriate set in Equations 39 into Equations 38. These initial algebraic equations are simplified by a large degree by taking certain linear combinations. These linear combinations are illustrated below for Equations 38.

$$\cos 2\phi \ Z_2^S - \sin 2\phi \, Z_2^C = [s_1 \ \ s_2] \ M_O^S \begin{bmatrix} s_3 \\ s_4 \end{bmatrix} - [t_1 \ \ t_2] \ N_O^S \begin{bmatrix} t_3 \\ t_4 \end{bmatrix}$$

$$\sin 2\phi \ Z_2^S + \cos 2\phi \ Z_2^C = [s_1 \ \ s_2] J M_O^S K \begin{bmatrix} s_3 \\ s_4 \end{bmatrix} + [t_1 \ \ t_2] \ J \ N_O^S \ K \begin{bmatrix} t_3 \\ t_4 \end{bmatrix} \tag{40}$$

The corresponding relationships for the last two of Equations 32b are developed in a similar manner and are written below.

$$\cos 3\phi\, Z_3^s - \sin 3\phi\, Z_3^c = [s_1 \;\; s_2]\, J\, M_o^s \begin{bmatrix} s_3 \\ s_4 \end{bmatrix} - [t_1 \;\; t_2]\, N_o^s\, K \begin{bmatrix} t_3 \\ t_4 \end{bmatrix}$$

$$\sin 3\phi\, Z_3^s + \cos 3\phi\, Z_3^c = [s_1 \;\; s_2]\, M_o^s\, K \begin{bmatrix} s_3 \\ s_4 \end{bmatrix} - [t_1 \;\; t_2]\, J\, N_o^s \begin{bmatrix} t_3 \\ t_4 \end{bmatrix} \tag{41}$$

Equations 40 and 41 are transformed to the matrix equation written below by substituting Equations 24 for matrices M_o^s and N_o^s and introducing the symbols S_{ij} and T_{ij} as defined in Equations 36.

$$\begin{bmatrix} S_{13} & S_{24} & S_{23} & S_{14} \\ -S_{24} & S_{13} & -S_{14} & S_{23} \\ S_{23} & S_{14} & S_{13} & S_{24} \\ -S_{14} & S_{23} & -S_{24} & S_{13} \end{bmatrix} \begin{bmatrix} A_o^s \\ B_o^s \\ C_o^s \\ D_o^s \end{bmatrix} - \begin{bmatrix} T_{13} & T_{24} & T_{23} & T_{14} \\ T_{24} & -T_{13} & T_{14} & -T_{23} \\ -T_{14} & T_{23} & -T_{24} & T_{13} \\ T_{23} & T_{14} & T_{13} & T_{24} \end{bmatrix} \begin{bmatrix} E_o^s \\ F_o^s \\ G_o^s \\ H_o^s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{42}$$

The elements of the second vector are related to the elements of the first vector in Equation 42 by the following expressions (see Equations 35a and 35c).

$$E = 1-A, \quad F = B, \quad G = -D, \quad \text{and } H = -C. \tag{43}$$

The scripted notation is no longer required. Substituting Equations 43 for the elements of the second vector, transforms Equation 42 to the following four by four matrix.

$$\begin{bmatrix} S_{13}+T_{13} & S_{24}-T_{24} & S_{23}+T_{14} & S_{14}+T_{23} \\ -S_{24}+T_{24} & S_{13}+T_{13} & -S_{14}-T_{23} & S_{23}+T_{14} \\ S_{23}-T_{14} & S_{14}-T_{23} & S_{13}+T_{13} & S_{24}-T_{24} \\ -S_{14}+T_{23} & S_{23}-T_{14} & -S_{24}+T_{24} & S_{13}+T_{13} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} T_{13} \\ T_{24} \\ -T_{14} \\ T_{23} \end{bmatrix} \tag{44}$$

Equations 43 and 44 are Formulas 19 and 18 in Section II, respectively. This verifies each of the Formulas 14 through 22 of Section II. The remaining formulas (23 through 26) are verified by taking appropriate derivatives of the expression for the beam deflection (Eq. 19). This is done in a straightforward manner by substituting and performing the operations indicated in the following definitions.

<u>Velocity</u>

$$V = \frac{\partial y}{\partial t} = \omega \frac{\partial y}{\partial \theta} = \omega Y_o (Z_o^s \cos\theta - Z_o^c \sin\theta) \quad ,$$

<u>Acceleration</u>

$$\underline{A} = \frac{\partial^2 y}{\partial t^2} = \omega^2 \frac{\partial^2 y}{\partial \theta^2} = -\omega^2 Y_o (Z_o^s \sin\theta + Z_o^c \cos\theta) \, ,$$

<u>Bending Moment</u> (45)

$$\underline{M} = E\,I \frac{\partial^2 y}{\partial x^2} = \frac{EI}{L^2} \frac{\partial^2 y}{\partial z^2} = \frac{EI\,Y_o}{L^2} (Z_2^s \sin\theta + Z_2^c \cos\theta) \, ,$$

<u>Shearing Force</u>

$$Q = E\,I \frac{\partial^3 y}{\partial x^3} = \frac{EI}{L^3} \frac{\partial^3 y}{\partial x^3} = \frac{EI\,Y_o}{L^3} (Z_3^s \sin\theta + Z_3^c \cos\theta) \; .$$

The first two of Equations 45 are Formulas 23 and 24, respectively. Formulas 25 and 26 are verified with the last two of Equations 45 by recognizing that the μ^n terms appearing in Equations 30 are omitted in Formulas 20 but are included in Formulas 25 and 26.

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