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Progress Report
GIMBALLESS INERTIAL NAVIGATION SYSTEMS

Semi-Annual Progress Report
for the period

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Moore School Report No. 65-26

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1. INTRODUCTION

This is the first semi-annual report on a study of Gimballess Inertial Navigation Systems. All of the tasks which were originally proposed for inclusion in this study are being actively pursued. They are: (1) a comparative study of several methods of measuring angular velocity without the use of gyroscopes, (2) a study of methods of error damping for flights that are not earth-bound, (3) the construction of a breadboard model of a rotating-accelerometer device for measuring angular velocity, (4) the determination of the computer requirements for performing the computations necessary for gimballess inertial navigation systems, (5) the simulation of one or more mechanizations of gimballess systems and (6) an extensive error study of gimballess systems.

A summary of the work of the first six months is given in the next section of this report and details are given in the appendices.

The technical staff has consisted of two Research Fellows, each at 75% of full time, and one Associate Professor at 20% of full time. In addition, there has been part time programming assistance. It is anticipated that by the end of the first year of this study, both of the Research Fellows will have completed all of their work for the Ph.D. degree and that further work on gimballess inertial navigation systems would be carried out by other advanced graduate students.

2. SUMMARY OF WORK

2.1 Measuring Angular Velocity without Using Gyroscopes

Most inertial navigation systems use accelerometers to sense linear acceleration and gyroscopes to sense angular velocity. The use of gyroscopes may be avoided, however, by using six or more accelerometers that are fixed to the vehicle or by mounting an accelerometer on each of two perpendicular rotating rings.

Many configurations of fixed accelerometers will allow the determination of both linear acceleration and angular acceleration. Several of these configurations are discussed in some detail in Appendixes A and B. The minimum number of fixed accelerometers for an all-accelerometer inertial navigation system is six. With six accelerometers there is an ambiguity in the sign of the angular velocity. This ambiguity may be avoided with eight or nine accelerometers. Also, the use of eight or more accelerometers provides greater freedom in the placement of the accelerometers. It is shown in Appendix B that the linear and angular acceleration can be readily determined even though the accelerometers are not placed symmetrically about the vehicle's center of mass and even though the center of mass may move due to fuel consumption.

Section 2.3 discusses the use of accelerometers mounted on rotating rings.

2.2 Error Damping

Bodner and Seleznev[1] and Krishnan[2] have studied the mechanization of the navigation equations for gimballess navigation systems and have found that mechanization in an inertial frame of reference is more suitable than mechanization in a vehicular frame of reference.

However, a simple mechanization of the equations in an inertial frame gives a characteristic equation of the form

$$(s^2 + \omega_s^2)\ (s^2 - 2\omega_s^2) = 0$$

corresponding to a transient error that contains a bounded sinusoidal term and an unbounded hyperbolic cosine term. Appendix E treats three methods of achieving a stable mechanization of the navigation equations. They are (1) damping by means of reference trajectory information, (2) damping by means of external velocity information and (3) elimination of diverging errors by means of altimeter information.

For many flights the reference (desired) trajectory of the vehicle is accurately specified and the actual trajectory does not deviate far from this reference trajectory. With the reference trajectory information available on board the vehicle, damping can be obtained without the use of any auxiliary sensors; the accelerometers required for an undamped system suffice. The accuracy of a system with reference trajectory damping is dependent upon the closeness of the actual and reference trajectories.

Doppler radar has been used to obtain velocity information for use in airborne terrestrial navigation systems. For space navigation, Doppler radar is less satisfactory. Franklin and Birx[3] have reported encouraging results from feasibility and accuracy studies of optical Doppler. Doppler-damped navigation system equations for space flight are presented in Appendix E.

The distance from the vehicle to a celestial body may be found by a variety of distance measuring schemes. The measured value of this distance may be used to eliminate the diverging error that occurs in the undamped navigation system. However, it will not eliminate the sinusoidal oscillation.

2.3 Rotating Accelerometer Experimental Studies

Krishnan[2] has shown that two or three linear accelerometers mounted on mutually perpendicular rotating rings can be used to determine both linear acceleration and angular velocity. A rotating disk, mounted on a dividing head and carrying an accelerometer, is under construction. The experimental study to determine the feasibility of a rotating accelerometer sensing system has not yet been initiated due to delays in obtaining transformers for the accelerometer's loop-closing amplifier and the difficulty in obtaining satisfactory operation of the loop-closing amplifier. It is expected that the use of very carefully matched diodes will remove the remaining difficulty in the loop-closing amplifier and that experimental studies will commence during May.

2.4 The Computer Simulation

The navigation system is to be simulated on a digital computer. This simulation will permit a more adequate analysis of the system than is possible by purely analytic means. It will also permit a study of the navigation system errors and will provide the necessary information for determining the requirements for an on-board computer.

In the simulation of the navigation system, the inputs are an angular velocity pattern and a linear acceleration pattern. With these inputs, the simulated system - and the actual system - will provide the vehicle's position coordinates, which due to errors in the sensors, errors in the calculation of the gravity compensation, errors in the direction cosine calculations and errors in the numerical integrations, will deviate from the true value.

Figure 1 shows the block diagram for the simulation of the navigation system on a digital computer. The linear acceleration pattern $\bar{a}_I(t)$ enters in the upper left hand corner of the figure. It is integrated twice, combined with initial conditions and the gravity terms and fed to Box 1 where the exact direction cosine matrix $[D(T_i)]_E^{-1}$ is formed. This direction cosine matrix relates the inertial coordinate system and the vehicular coordinate system. The inputs to Box 1 are referred to the inertial coordinate system and the outputs are referred to the vehicular coordinate system. Box 2 combines the linear acceleration $\bar{A}_V(t)$, the angular velocity $\bar{\Omega}_V(t)$ and the acceleration $\ddot{\bar{r}}_V(t)$ of the origin of the vehicular coordinate system [V] with respect to the origin of the coordinate system [m] about which the accelerometers are centered. The outputs of Box 2 are the accelerations that are read by the accelerometers. Box 3 simulates the accelerometers. Its inputs are the true accelerations and noise and its outputs are the accelerometer readings. The calculated values of the angular velocity $\bar{\Omega}_{V_c}$ and of linear acceleration $\bar{A}_{V_c}$ are formed in Box 4. Since the accelerometer

readings are noisy a linear filter and estimator is included (Box 5). Box 6 forms the calculated direction cosine matrix $[D(T_i)]_c$. The inputs to Box 6 are referred to the vehicular coordinate system and the output $\overline{A}_{I_c}(T_i)$ is referred to the inertial coordinate system. This linear acceleration, $\overline{A}_{I_c}(T_i)$, is combined with the calculated gravity term $\overline{g}_{I_c}$ and the resultant is integrated twice to give the vehicle's calculated position $\overline{R}_{I_c}(T_i)$. A linear predictor (Box 12) gives the estimated position at the next sampling instant, T_{i+1}, and the gravity computer (Box 8) gives the corresponding acceleration of gravity. Boxes 13 and 14 give the calculated and exact Euler Angles respectively.

Portions of the simulation have been successfully tested. The exact form of some boxes, such as the linear filter and estimator, have not been determined, nor has a closed loop test been attempted.

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2. Krishnan, V.: "Design and Mathematical Analysis of Gimballess Inertial Navigation Systems," Ph.D. Dissertation, University of Pennsylvania, 1963.

3. Franklin, G. R. and D. L. Birx: "Optical Doppler for Space Navigation in Guidance and Control," Edited by R. E. Robertson and J. S. Farrior, Acad. Press, 1962.

APPENDIX A

ALL ACCELEROMETER TECHNIQUES FOR MEASURING LINEAR ACCELERATION AND ANGULAR ACCELERATION

A. R. Schuler

Currently, most inertial navigation systems use linear accelerometers to sense linear accelerations and gyroscopes to sense angular velocity or angular position. It is possible, however, to determine both linear acceleration and angular acceleration (or angular velocity) without the use of gyroscopes. This report presents four configurations of linear accelerometers which permit the determination of both linear acceleration and angular acceleration (or angular velocity). Although there are many possible configurations, only those that seem to be of major importance are treated here. It is to be noted that the possible configurations differ in the number of accelerometers required and the mathematical form of the outputs. In this section, it is assumed that the accelerometers are placed on a coordinate system that has as its origin the center of gravity of the vehicle.

Two coordinate systems are used:

a) the inertial system [I], and

b) the vehicular coordinate system [V] which has its origin at the center of mass of the vehicle and its axes V_1, V_2 and V_3 along the principal axes of the vehicle.

Figure A-1 shows the coordinate system [V] which, in general, is rotating and accelerating with respect to the inertial coordinate system [I]. The angular velocity of [V] is designated as:

$$\bar{\Omega} = \Omega_{V_1} \bar{i}_V + \Omega_{V_2} \bar{j}_V + \Omega_{V_3} \bar{k}_V \tag{A-1}$$

It has been shown[1] that the inertial acceleration of an arbitrary point P accelerating with respect to a moving reference frame is given by (a dot represents differentiation with respect to time):

$$\bar{A}_I = \ddot{\bar{R}}_I + \dot{\bar{\Omega}} \times \bar{r}_V + \ddot{\bar{r}}_V + 2\,\bar{\Omega} \times \dot{\bar{r}}_V + \bar{\Omega} \times (\Omega \times \bar{r}_V) \tag{A-2}$$

where:

$\bar{R}_I$ is the vector from the origin of the inertial frame to the origin of the vehicular frame.

$\bar{r}_V$ is the vector from the origin of the vehicular frame to the point P.

If the point P is fixed in the vehicular system then $\dot{\bar{r}}_V = \ddot{\bar{r}}_V = 0$ and the acceleration of the point is given by:

$$\bar{A}_I = \ddot{\bar{R}}_I + \dot{\bar{\Omega}} \times \bar{r}_V + \bar{\Omega} \times (\bar{\Omega} \times \bar{r}_V) \tag{A-3}$$

$\bar{R}_I$ may be written as a sum of components along the vehicular axes, i.e.:

$$\bar{R}_I = R_{V_1} \bar{i}_V + R_{V_2} \bar{j}_V + R_{V_3} \bar{k}_V \tag{A-4}$$

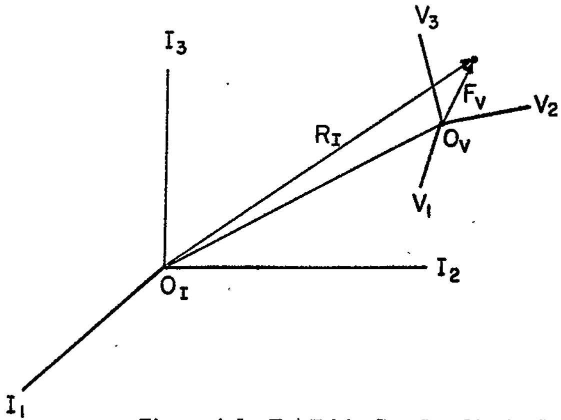

Figure A-1 The Vehicular Coordinate System [V]

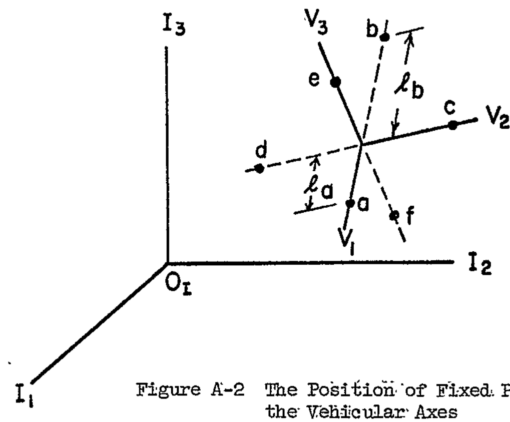

Figure A-2 The Position of Fixed Points on the Vehicular Axes

Then

$$\ddot{\bar{R}}_I = \ddot{R}_{V_1}\,\bar{i}_V + \ddot{R}_{V_2}\,\bar{j}_V + \ddot{R}_{V_3}\,\bar{k}_V$$

$$= A_{V_1}\,\bar{i}_V + A_{V_2}\,\bar{j}_V + \bar{A}_{V_3}\,\bar{k}_V = \bar{A}_V \tag{A-5}$$

Note that $\ddot{\bar{R}}_I$ is an inertial acceleration whose components have been resolved along the vehicular system axes. We introduced the notation $\ddot{\bar{R}}_{V_i} = A_{V_i}$ $(i = 1, 2, 3)$.

The acceleration in inertial space $(\bar{A}_I)$ will now be found for the six points, a, b, c, d, e, and f, fixed on the vehicular axis as shown in Fig. A-2.

To find the acceleration of point e with respect to inertial space, let

$$\bar{r}_V = \ell_e\,\bar{k}_V \tag{A-6}$$

Using Eqs. (A-1), (A-5), and (A-6) in Eq. (A-3),

$$(\bar{A}_I)_{pt\ e} = A_{V_1}\,\bar{i}_V + A_{V_2}\,\bar{j}_V + A_{V_3}\,\bar{k}_V + (\dot{\Omega}_{V_1}\,\bar{i}_V + \dot{\Omega}_{V_2}\,\bar{j}_V + \dot{\Omega}_{V_3}\,\bar{k}_V) X\, \ell_e\,\bar{k}_V$$

$$+ (\Omega_{V_1}\,\bar{i}_V + \Omega_{V_2}\,\bar{j}_V + \Omega_{V_3}\,\bar{k}_V)\ X\ [(\Omega_{V_1}\,\bar{i}_V + \Omega_{V_2}\,\bar{j}_V + \Omega_{V_3}\bar{k}_V) X\, \ell_e\bar{k}_V]$$

$$= \bar{i}_V\ [A_{V_1} + \ell_e\,\dot{\Omega}_{V_2} + \ell_e\,\Omega_{V_1}\,\Omega_{V_3}]$$

$$+ \bar{j}_V\ [A_{V_2} - \ell_e\,\dot{\Omega}_{V_1} + \ell_e\,\Omega_{V_2}\,\Omega_{V_3} \tag{A-7}$$

$$+ \bar{k}_V\ [A_{V_3} - \ell_e\,\Omega^2_{V_2} - \ell_e\,\Omega^2_{V_1}]$$

To obtain the inertial acceleration at the other five points, let $\bar{r}_V$ take on the desired coordinate value.

<u>For point f let $\bar{r}_V = -\ell_f \bar{k}_V$</u>

$$
\begin{aligned}
(\bar{A}_I)_{\text{pt f}} = \; & \bar{i}_V \left[A_{V_1} - \dot{\Omega}_{V_2} \ell_f - \Omega_{V_1} \Omega_{V_3} \ell_f\right] \\
& + \bar{j}_V \left[A_{V_2} + \dot{\Omega}_{V_1} \ell_f - \Omega_{V_2} \Omega_{V_3} \ell_f\right] \qquad \text{(A-8)} \\
& + \bar{k}_V \left[A_{V_3} + \Omega_{V_1}^2 \ell_f + \Omega_{V_2}^2 \ell_f\right]
\end{aligned}
$$

<u>For point a let $\bar{r}_V = \ell_a \bar{i}_V$</u>

$$
\begin{aligned}
(A_I)_{\text{pt a}} = \; & \bar{i}_V \left[A_{V_1} - \Omega_{V_2}^2 \ell_a - \Omega_{V_3}^2 \ell_a\right] \\
& + \bar{j}_V \left[A_{V_2} + \dot{\Omega}_{V_3} \ell_a + \Omega_{V_1} \Omega_{V_2} \ell_a\right] \qquad \text{(A-9)} \\
& + \bar{k}_V \left[A_{V_3} - \dot{\Omega}_{V_2} \ell_a + \Omega_{V_1} \Omega_{V_3} \ell_a\right]
\end{aligned}
$$

<u>For point b, let $\bar{r}_V = -\ell_b \bar{i}_V$</u>

$$
\begin{aligned}
(A_I)_{\text{pt b}} = \; & \bar{i}_V \left[A_{V_1} + \Omega_{V_2}^2 \ell_b + \Omega_{V_3}^2 \ell_b\right] \\
& + \bar{j}_V \left[A_{V_2} - \dot{\Omega}_{V_3} \ell_b - \Omega_{V_1} \Omega_{V_2} \ell_b\right] \qquad \text{(A-10)} \\
& + \bar{k}_V \left[A_{V_3} + \dot{\Omega}_{V_2} \ell_b - \Omega_{V_1} \Omega_{V_3} \ell_b\right]
\end{aligned}
$$

For point c let $\bar{r}_V = \ell_c \bar{j}_V$

$$(A_I)_{pt\ c} = \bar{i}_V [A_{V_1} - \dot{\Omega}_{V_3} \ell_c + \Omega_{V_1} \Omega_{V_2} \ell_c]$$

$$+ \bar{j}_V [A_{V_2} - \Omega^2_{V_1} \ell_c - \Omega^2_{V_3} \ell_c] \qquad \text{(A-11)}$$

$$+ \bar{k}_V [A_{V_3} + \dot{\Omega}_{V_1} \ell_c + \Omega_{V_2} \Omega_{V_3} \ell_c]$$

For point d let $\bar{r}_V = -\ell_d \bar{j}_V$

$$(A_I)_{pt\ d} = \bar{i}_V [A_{V_1} + \dot{\Omega}_{V_3} \ell_d - \Omega_{V_1} \Omega_{V_2} \ell_d]$$

$$+ \bar{j}_V [A_{V_2} + \Omega^2_{V_1} \ell_d + \Omega^2_{V_3} \ell_d] \qquad \text{(A-12)}$$

$$+ \bar{k}_V [A_{V_3} - \dot{\Omega}_{V_1} \ell_d - \Omega_{V_2} \Omega_{V_3} \ell_d]$$

Also, if $r_V = 0$

$$(A_I)_{origin} = \bar{i}_V A_{V_1} + \bar{j}_V A_{V_2} \quad \bar{k}_V A_{V_3} \qquad \text{(A-13)}$$

Consider first the configuration of linear accelerometers shown in Fig. A-3 (configuration A). The six accelerometers are oriented about the vehicle's center of gravity and are mounted on the principal axes. Two accelerometers are placed on each of the three

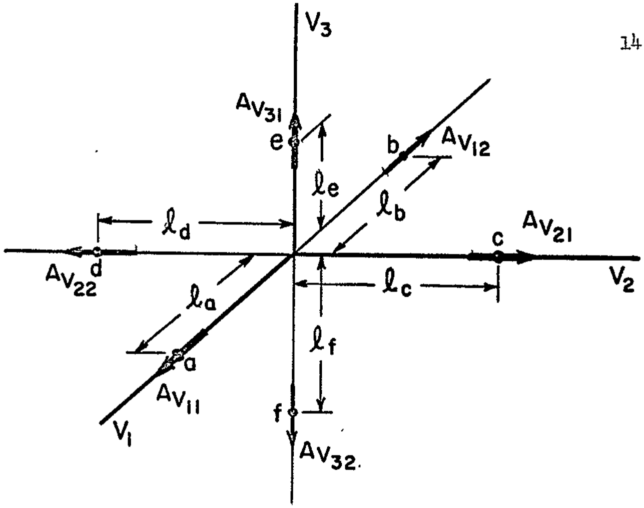

Figure A-3 Configuration A

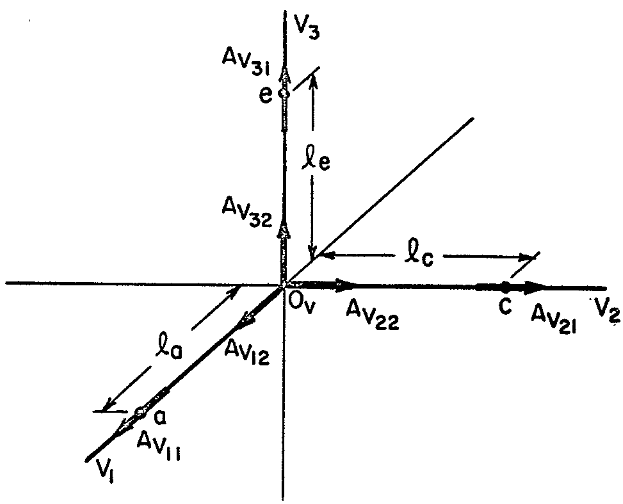

Figure A-4 Configuration B

vehicular axes with the sensitive axis of one accelerometer along the axis in the positive sense and the sensitive axis of the other along the axis in the negative sense. The arrows in the figure indicate the sensitive axes of the accelerometers.

Since $A_{V_{11}}$ is at point a and oriented in the $\bar{i}_V$ direction, it measures, from Eq. (A-9):

$$A_{V_{11}} = A_{V_1} - (\Omega_{V_2}^2 + \Omega_{V_3}^2)\, \ell_a \tag{A-14}$$

By referring to Fig. A-3 and Eqs. (A-7) through (A-12), the remaining 5 accelerometer readings can be determined.

$$A_{V_{12}} = A_{V_1} + (\Omega_{V_2}^2 + \Omega_{V_3}^2)\, \ell_b \tag{A-15}$$

$$A_{V_{21}} = A_{V_2} - (\Omega_{V_1}^2 + \Omega_{V_3}^2)\, \ell_c \tag{A-16}$$

$$A_{V_{22}} = A_{V_2} + (\Omega_{V_1}^2 + \Omega_{V_3}^2)\, \ell_d \tag{A-17}$$

$$A_{V_{31}} = A_{V_3} - (\Omega_{V_2}^2 + \Omega_{V_1}^2)\, \ell_e \tag{A-18}$$

$$A_{V_{32}} = A_{V_3} + (\Omega_{V_2}^2 + \Omega_{V_1}^2)\, \ell_f \tag{A-19}$$

These equations are now combined algebraically

$$x_1 = \frac{A_{V_{12}} - A_{V_{11}}}{\ell_a + \ell_b} = (\Omega_{V_2}^2 + \Omega_{V_3}^2) \tag{A-20}$$

$$x_2 = \frac{A_{V_{22}} - A_{V_{21}}}{\ell_c + \ell_d} = (\Omega_{V_1}^2 + \Omega_{V_3}^2) \tag{A-21}$$

$$x_3 = \frac{A_{V_{32}} - A_{V_{31}}}{\ell_e + \ell_f} = (\Omega_{V_1}^2 + \Omega_{V_2}^2) \tag{A-22}$$

Then

$$\Omega_{V_1}^2 = \left(\frac{-x_1 + x_2 + x_3}{2} \right) \tag{A-23}$$

$$\Omega_{V_2}^2 = \left(\frac{x_1 - x_2 + x_3}{2} \right) \tag{A-24}$$

$$\Omega_{V_3}^2 = \left(\frac{x_1 + x_2 - x_3}{2} \right) \tag{A-25}$$

Thus

$$\Omega_{V_1} = \left(\frac{-x_1 + x_2 + x_3}{2} \right)^{1/2} \tag{A-26}$$

$$\Omega_{V_2} = \left(\frac{x_1 - x_2 + x_3}{2} \right)^{1/2} \tag{A-27}$$

$$\Omega_{V_3} = \left(\frac{x_1 + x_2 - x_3}{2} \right)^{1/2} \tag{A-28}$$

These equations, being wholly algebraic, are easy to evaluate on a digital computer. A difficulty arises, however, in evaluating $\overline{\Omega}$ since the square root of Eqs. (A-23), (A-24) and (A-25) may have either a plus sign or a minus sign. This sign difficulty can be resolved through the use of auxiliary devices that may be less accurate and less costly than accelerometers.

Then, from Eqs. (A-14), (A-16), (A-18), (A-20), (A-21), and (A-22)

$$A_{V_1} = A_{V_{11}} + x_1 \ell_a \tag{A-29}$$

$$A_{V_2} = A_{V_{21}} + x_2 \ell_c \tag{A-30}$$

$$A_{V_3} = A_{V_{31}} + x_3 \ell_e \tag{A-31}$$

If either the condition $\ell_a = \ell_b$, $\ell_c = \ell_d$, and $\ell_e = \ell_f$, or $\ell_a = \ell_b = \ell_c = \ell_d = \ell_e = \ell_f$ were satisfied, the equations presented above would simplify.

In configuration B, the accelerometers mounted parallel to the negative vehicular axes are brought to the origin of the vehicular axis as shown in Fig. A-4. In practice this is impossible unless it were feasible to mount a three degree of freedom accelerometer at the origin. Nevertheless, the configuration is analyzed here. The six accelerometers measure the following quantities:

$$A_{V_{11}} = A_{V_1} - (\Omega_{V_2}^2 + \Omega_{V_3}^2)\, \ell_a \tag{A-32}$$

$$A_{V_{12}} = A_{V_1} \tag{A-33}$$

$$A_{V_{21}} = A_{V_2} - (\Omega_{V_1}^2 + \Omega_{V_3}^2)\, \ell_c \tag{A-34}$$

$$A_{V_{22}} = A_{V_2} \tag{A-35}$$

$$A_{V_{31}} = A_{V_3} - (\Omega_{V_2}^2 + \Omega_{V_1}^2)\, \ell_e \tag{A-36}$$

$$A_{V_{32}} = A_{V_3} \tag{A-37}$$

Since A_{V_1}, A_{V_2} and A_{V_3} are known [Eqs. (A-33), (A-35) and (A-37)] we can write

$$\Omega_{V_2}^2 + \Omega_{V_3}^2 = \frac{A_{V_1} - A_{V_{11}}}{\ell_a} = x_1' \tag{A-38}$$

$$\Omega_{V_1}^2 + \Omega_{V_3}^2 = \frac{A_{V_2} - A_{V_{21}}}{\ell_c} = x_2' \tag{A-39}$$

$$\Omega_{V_2}^2 + \Omega_{V_1}^2 = \frac{A_{V_3} - A_{V_{31}}}{\ell_e} = x_3' \tag{A-40}$$

Then, as before,

$$\Omega_{V_1} = \left(\frac{-x_1' + x_2' + x_3'}{2} \right)^{1/2} \tag{A-41}$$

$$\Omega_{V_2} = \left(\frac{x_1' - x_2' + x_3'}{2} \right)^{1/2} \tag{A-42}$$

$$\Omega_{V_3} = \left(\frac{x_1' + x_2' - x_3'}{2} \right)^{1/2} \tag{A-43}$$

Configuration C utilizes 9 stationary accelerometers. It eliminates the ambiguity in the sign of the angular velocity Ω. Also, no accelerometers are needed at the center of gravity. This method

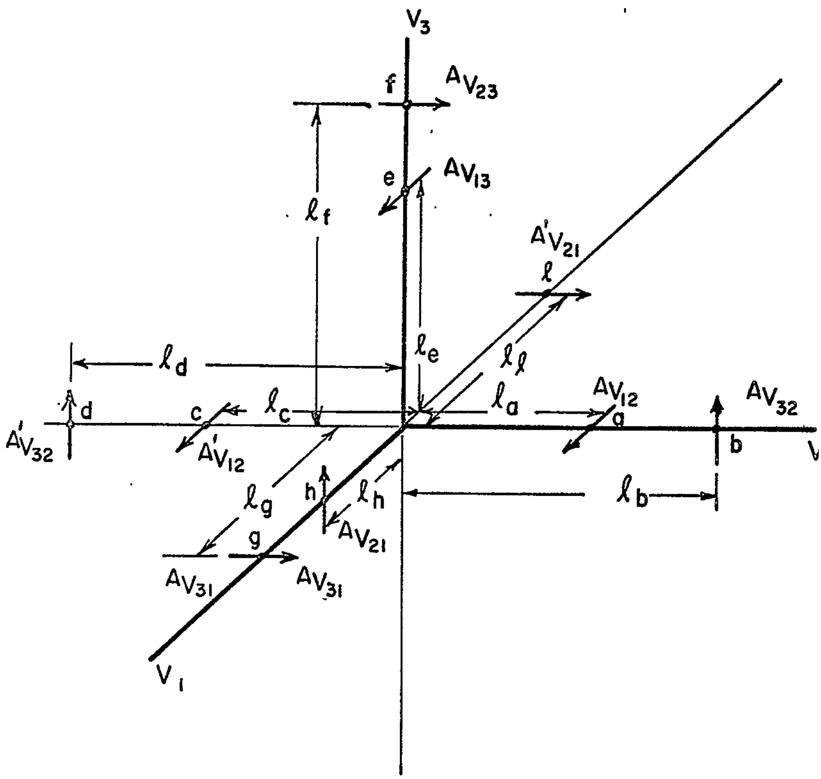

Figure A-5 Configuration C with Arbitrary Accelerometer Spacing gives the output as A_{V_1}, A_{V_2}, A_{V_3}, Ω_{V_1}, Ω_{V_2} and $\dot{\Omega}_{V_3}$. Note that the mounting of the accelerometers must still satisfy the requirement that they be placed on the principal vehicular axis about the center of gravity. The mounting scheme is shown in Fig. A-5.

Using Eqs. (A-7) through (A-12), we can write the outputs of these 9 accelerometers as (note that the alphabetic points no longer coincide)

$$A_{V_{12}} = A_{V_1} - (\dot{\Omega}_{V_3} - \Omega_{V_1} \Omega_{V_2}) \ell_a \tag{A-44-1}$$

$$A_{V_{32}} = A_{V_3} + (\dot{\Omega}_{V_1} + \Omega_{V_2} \Omega_{V_3}) \ell_b \tag{A-44-2}$$

$$A'_{V_{12}} = A_{V_1} + (\dot{\Omega}_{V_3} - \Omega_{V_1} \Omega_{V_2}) \ell_c \tag{A-44-3}$$

$$A'_{V_{32}} = A_{V_3} - (\dot{\Omega}_{V_1} + \Omega_{V_2} \Omega_{V_3}) \ell_d \tag{A-44-4}$$

$$A_{V_{13}} = A_{V_1} + (\dot{\Omega}_{V_2} + \Omega_{V_1} \Omega_{V_3}) \ell_e \tag{A-44-5}$$

$$A_{V_{23}} = A_{V_2} - (\dot{\Omega}_{V_1} - \Omega_{V_2} \Omega_{V_3}) \ell_f \tag{A-44-6}$$

$$A_{V_{31}} = A_{V_3} - (\dot{\Omega}_{V_2} - \Omega_{V_1} \Omega_{V_3}) \ell_g \tag{A-44-7}$$

$$A_{V_{21}} = A_{V_2} + (\dot{\Omega}_{V_3} + \Omega_{V_1} \Omega_{V_2}) \ell_h \tag{A-44-8}$$

$$A'_{V_{21}} = A_{V_2} - (\dot{\Omega}_{V_3} + \Omega_{V_1} \Omega_{V_2}) \ell_\ell \tag{A-44-9}$$

Letting $\underline{\ell_b = \ell_d}$ and adding (A-44-2) and (A-44-4),

$$\frac{A_{V_{32}} + A'_{V_{32}}}{2} = A_{V_3} \tag{A-45}$$

Letting $\underline{\ell_a = \ell_c}$ and adding (A-44-1) and (A-44-3),

$$\frac{A_{V_{12}} + A'_{V_{12}}}{2} \qquad A_{V_1} \tag{A-46}$$

Letting $\underline{\ell_h = \ell_\ell}$ and adding (A-44-8) and (A-44-9),

$$\frac{A_{V_{21}} \quad A'_{V_{21}}}{2} \qquad A_{V_2} \tag{A-47}$$

Three summations have yielded linear acceleration.

Letting $\underline{\ell_d \quad \ell_f}$ and adding (A-44-4) and (A-44-6) yields:

$$A'_{V_{32}} + A_{V_{23}} = A_{V_3} + A_{V_2} - \dot{\Omega}_{V_1}\,\ell_d - \Omega_{V_2}\,\Omega_{V_3}\,\ell_d - \dot{\Omega}_{V_1}\,\ell_d + \Omega_{V_2}\,\Omega_{V_3}\,\ell_d$$

$$= A_{V_3} + A_{V_2} - 2\,\dot{\Omega}_{V_1}\,\ell_d$$

$$\dot{\Omega}_{V_1} = \frac{-A_{V_{32}} - A_{V_{23}} + A_{V_3} + A_{V_2}}{2\,\ell_d} \tag{A-48}$$

Letting $\underline{\ell_e = \ell_g}$ and subtracting (A-44-7) from (A-44-5) yields:

$$A_{V_{13}} - A_{V_{31}} = A_{V_1} - A_{V_3} + \dot{\Omega}_{V_2} \ell_e + \Omega_{V_1} \Omega_{V_3} \ell_e + \dot{\Omega}_{V_2} \ell_e - \Omega_{V_1} \Omega_{V_3} \ell_e$$

$$= A_{V_1} - A_{V_3} + 2 \dot{\Omega}_{V_2} \ell_e$$

$$\dot{\Omega}_{V_2} = \frac{A_{V_{13}} - A_{V_{31}} - A_{V_1} + A_{V_3}}{2 \ell_e} \qquad \text{(A-49)}$$

Letting $\underline{\ell_a = \ell_h}$ and subtracting (A-44-8) from (A-44-1) yields:

$$A_{V_{12}} - A_{V_{21}} = A_{V_1} - \dot{\Omega}_{V_3} \ell_a + \Omega_{V_1} \Omega_{V_2} \ell_a - A_{V_2} - \dot{\Omega}_{V_3} \ell_a - \Omega_{V_1} \Omega_{V_2} \ell_a$$

$$= A_{V_1} - A_{V_2} - 2 \dot{\Omega}_{V_3} \ell_a$$

$$\dot{\Omega}_{V_3} = \frac{A_{V_1} - A_{V_2} - A_{V_{12}} + A_{V_{21}}}{2 \ell_a} \qquad \text{(A-50)}$$

For Eqs. (A-45) through (A-50), the following equalities apply

$$\ell_a = \ell_c = \ell_h = \ell_{K_1}$$

$$\ell_b = \ell_d = \ell_f = \ell_{K_2}$$

$$\ell_g = \ell_e = \ell_{K_3} \qquad \text{(A-51)}$$

Then Fig. A-5 evolves as shown in Fig. A-6. Note that the arrangement is such that no two accelerometers must overlap at a point (which is of course a physical impossibility).

Configuration D is of interest in that it utilizes eight accelerometers as shown in Fig. A-7. The equations governing these outputs are:

$$A_{V_{31}} = A_{V_3} - (\dot{\Omega}_{V_2} - \Omega_{V_1}\Omega_{V_3})\,\ell_a \tag{A-52-1}$$

$$A'_{V_{31}} = A_{V_3} + (\dot{\Omega}_{V_2} - \Omega_{V_1}\Omega_{V_3})\,\ell_b \tag{A-52-2}$$

$$A_{V_{32}} = A_{V_3} + (\dot{\Omega}_{V_1} + \Omega_{V_2}\Omega_{V_3})\,\ell_c \tag{A-52-3}$$

$$A'_{V_{32}} = A_{V_3} - (\dot{\Omega}_{V_1} + \Omega_{V_2}\Omega_{V_3})\,\ell_d \tag{A-52-4}$$

$$A_{V_{13}} = A_{V_1} + (\dot{\Omega}_{V_2} + \Omega_{V_1}\Omega_{V_3})\,\ell_e \tag{A-52-5}$$

$$A'_{V_{13}} = A_{V_1} - (\dot{\Omega}_{V_2} + \Omega_{V_1}\Omega_{V_3})\,\ell_f \tag{A-52-6}$$

$$A_{V_{23}} = A_{V_2} - (\dot{\Omega}_{V_1} - \Omega_{V_2}\Omega_{V_3})\,\ell_g \tag{A-52-7}$$

$$A'_{V_{23}} = A_{V_2} + (\dot{\Omega}_{V_1} - \Omega_{V_2}\Omega_{V_3})\,\ell_h \tag{A-52-8}$$

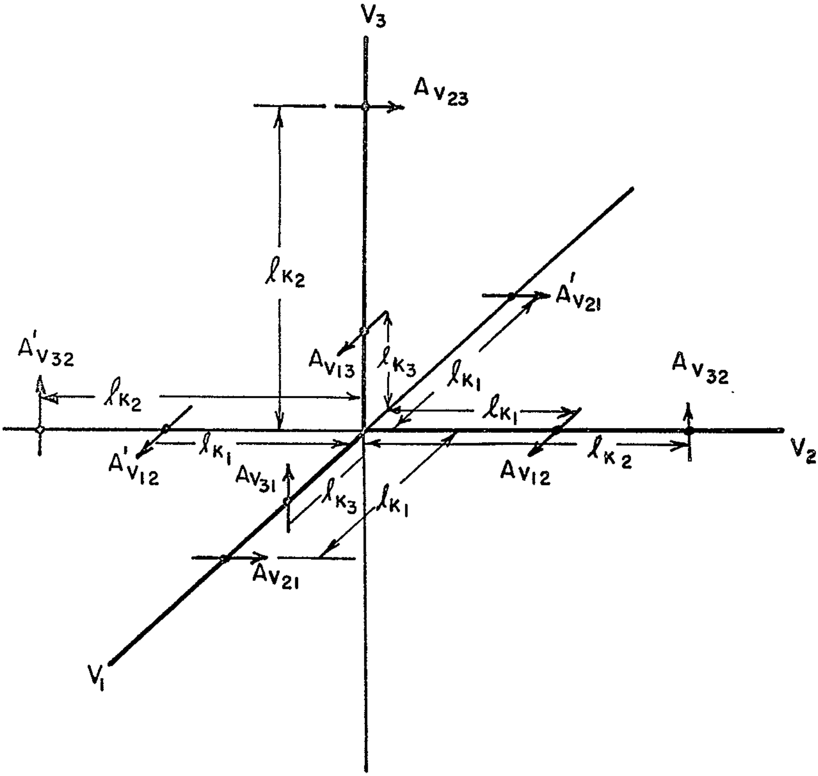

Figure A-6 Configuration C with Symmetrical Spacing of the Accelerometers

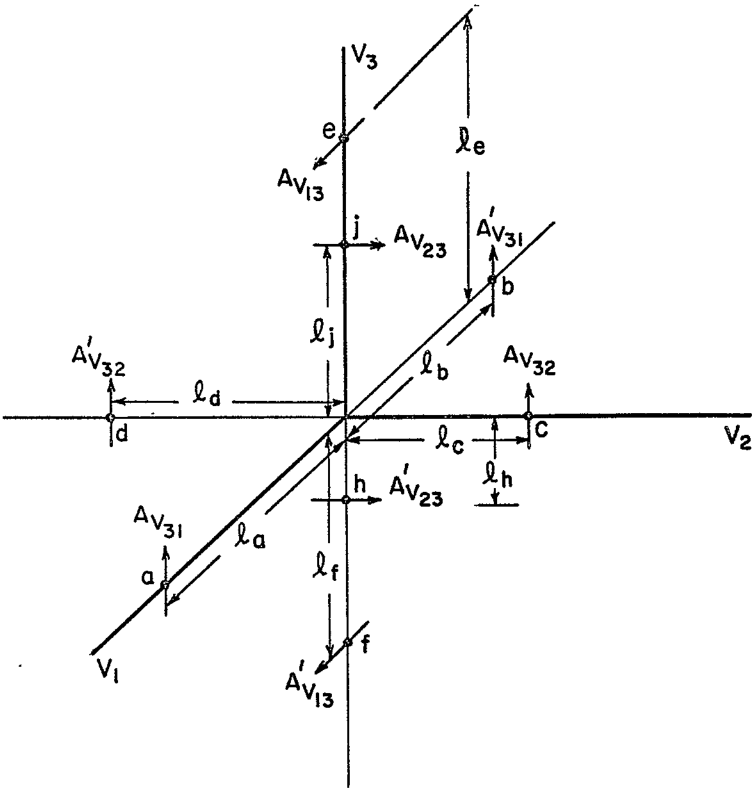

Figure A-7 Configuration D

Combining algebraically

if $\ell_e = \ell_f = \alpha$:

$$\frac{A_{V_{13}} + A'_{V_{13}}}{2} = A_{V_1} \tag{A-53}$$

if $\ell_g = \ell_h = \beta$:

$$\frac{A_{V_{23}} + A'_{V_{23}}}{2} = A_{V_2} \tag{A-54}$$

if $\ell_c = \ell_d = \gamma$:

$$\frac{A_{V_{32}} + A'_{V_{32}}}{2} = A_{V_3} \tag{A-55}$$

also, let $\ell_a = \ell_b = \Delta$

Then we can write the following equations using (A-52-1) through (A-52-8)

$$\frac{A'_{V_{31}} - A_{V_{31}}}{2\Delta} = \dot{\Omega}_{V_2} - \Omega_{V_1}\,\Omega_{V_3} = x''_1 \tag{A-56}$$

$$\frac{A_{V_{32}} - A'_{V_{32}}}{2\gamma} = \dot{\Omega}_{V_1} + \Omega_{V_2}\,\Omega_{V_3} = x''_2 \tag{A-57}$$

$$\frac{A_{V_{13}} - A'_{V_{13}}}{2\alpha} = \dot{\Omega}_{V_2} + \Omega_{V_1}\,\Omega_{V_2} = x''_3 \qquad \text{(A-58)}$$

$$\frac{A'_{V_{23}} - A_{V_{23}}}{2\beta} = \dot{\Omega}_{V_1} - \Omega_{V_2}\,\Omega_{V_3} = x''_4 \qquad \text{(A-59)}$$

Then

$$\dot{\Omega}_{V_1} \quad \frac{x''_2 + x''_4}{2} \quad \frac{A_{V_{32}} - A'_{V_{32}}}{4\gamma} + \frac{A'_{V_{23}} - A_{V_{23}}}{4\beta} \qquad \text{(A-60)}$$

$$\dot{\Omega}_{V_2} = \frac{x''_1 + x''_3}{2} = \frac{A'_{V_{31}} - A_{V_{31}}}{4\Delta} + \frac{A_{V_{13}} - A'_{V_{13}}}{4\alpha} \qquad \text{(A-61)}$$

Now from (A-56)

$$\dot{\Omega}_{V_3} = \frac{\dot{\Omega}_{V_2} - x_1}{\Omega_{V_1}} = \frac{\dot{\Omega}_{V_2} - \dfrac{A'_{V_{31}} - A_{V_{31}}}{2\Delta}}{\Omega_{V_1}} \qquad \text{(A-62)}$$

Note that the angular information is $\dot{\Omega}_{V_1}$, $\dot{\Omega}_{V_2}$ and $\dot{\Omega}_{V_3}$. Equation (A-62) requires the integration of $\dot{\Omega}_{V_1}$ [integration of Equation (A-60)] before it can be solved.

REFERENCES

(1) Page, Leigh: "Introduction to Theoretical Physics," D. Van Nostrand Company, 1947.

APPENDIX B

THE EFFECT OF LOCATING THE SENSORS AT A POINT OTHER THAN THE VEHICLE'S CENTER OF MASS.

Alfred R. Schuler

It is not usually convenient to locate the sensors of an inertial navigation system at the vehicle's center of mass. This appendix considers the effect, in an all-accelerometer system, of placing the accelerometers symmetrically about the origin of a coordinate system that is remote from and moving with respect to th center of mass of the vehicle.

The equation governing the inertial acceleration at an arbitrary point P accelerating with respect to a moving reference frame has already been given by Eq. (A-2). Restated for convenience, it is:

$$\bar{A}_I = \ddot{\bar{R}}_I + \dot{\bar{\Omega}} \times \bar{r}_V + \ddot{\bar{r}}_V + 2\bar{\Omega} \times \dot{\bar{r}}_V + \bar{\Omega} \times (\bar{\Omega} \times \bar{r}_V) \qquad \text{(B-1)}$$

where

$\bar{A}_I$ = inertial acceleration of arbitrary point P

$\ddot{\bar{R}}_I$ = linear acceleration of origin of vehicular coordinate system with respect to inertial space

$\bar{\Omega}$ = angular velocity of vehicular system

$\bar{r}_V$ = vector from vehicular origin to point P.

Figure B-1 shows the physical picture. The following two assumptions are made:

1. The [m] axes, about which the accelerometers are placed remains in an orientation on the vehicle that is paralle to the vehicle's major axes. The vehicular axes [V],

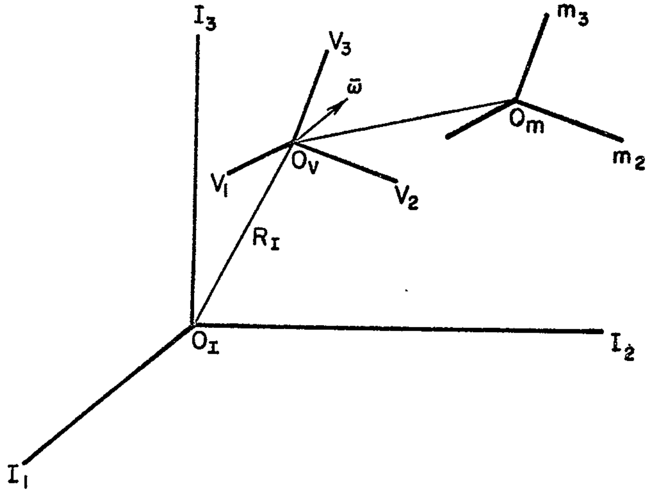

Figure B-1 The [m] Coordinate System

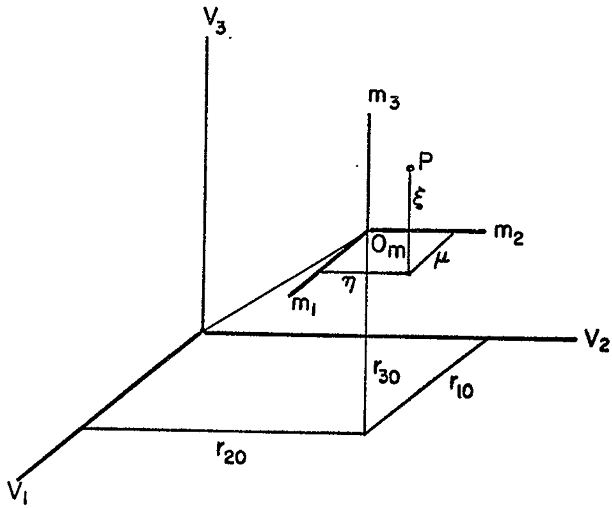

Figure B-2 Coordinate System Notation

however, translates as a known function of time. Note that [V] and [m] always remain parallel to their respective coordinate axes.

2. $r_V(t) = \bar{i}\, r_{V_1}(t) + \bar{j}\, r_{V_2}(t) + \bar{k}\, r_{V_3}(t)$ is known for all t. $r_V(t)$ is stored in the navigation computer prior to launching or sensing devices are used to permit its calculation while in orbit.

To resolve Eq. (B-1) into components we write:

$$\ddot{\bar{R}}_I = A_{V_1}\bar{i} + A_{V_2}\bar{j} + A_{V_3}\bar{k}$$

$$\bar{\Omega} = \Omega_{V_1}\bar{i} + \Omega_{V_2}\bar{j} + \Omega_{V_3}\bar{k}$$

$$\bar{r}_V = (r_{10} + \mu)\bar{i} + (r_{20} + \eta)\bar{j} + (r_{30} + \zeta)\bar{k}$$

$$\dot{\bar{r}}_V = \dot{r}_{10}\bar{i} + \dot{r}_{20}\bar{j} + \dot{r}_{30}\bar{k}$$

$$\ddot{\bar{r}}_V = \ddot{r}_{10}\bar{i} + \ddot{r}_{20}\bar{j} + \ddot{r}_{30}\bar{k}$$

where r_{10}, r_{20}, r_{30} are the magnitudes of the origin of the [m] coordinate system with respect to the origin (center of mass) of the vehicular coordinate system. μ, η and ζ are the magnitudes of the three coordinates defining a point in space with respect to the [m] coordinate system and are assumed to be constants. See Fig. B-2. Equation (B-1) can now be expressed in the following form:

Then

$$\overline{A}_I = A_{V_1}\overline{i} + A_{V_2}\overline{j} + A_{V_3}\overline{k} + (\Omega_{V_1}\overline{i} + \Omega_{V_2}\overline{j} + \Omega_{V_3}\overline{k}) \times [(r_{10} + \mu)\overline{i} + (r_{20} + \eta)\overline{j} + (r_{30} + \zeta)\overline{k}]$$
$$+ \ddot{r}_{10}\overline{i} + \ddot{r}_{20}\overline{j} + \ddot{r}_{30}\overline{k}$$
$$+ 2(\Omega_{V_1}\overline{i} + \Omega_{V_2}\overline{j} + \Omega_{V_3}\overline{k}) \times (\dot{r}_{10}\overline{i} + \dot{r}_{20}\overline{j} + \dot{r}_{30}\overline{k})$$
$$+ (\Omega_{V_1}\overline{i} + \Omega_{V_2}\overline{j} + \Omega_{V_3}\overline{k}) \times \{(\Omega_{V_1}\overline{i} + \Omega_{V_2}\overline{j} + \Omega_{V_3}\overline{k}) \times [(r_{10} + \mu)\overline{i} + (r_{20} + \eta)\overline{j} + (r_{30} + \zeta)\overline{k}]\} \tag{B-2}$$

Evaluating the three cross products in Eq. (B-2) and collecting terms gives:

$$\overline{A}_I = \overline{i}[A_{V_1} + \ddot{r}_{10} + \dot{\Omega}_{V_2}(r_{30} + \zeta) - \dot{\Omega}_{V_3}(r_{20} + \eta) + \Omega_{V_2}\dot{r}_{30}$$
$$- \Omega_{V_3}\dot{r}_{20} + \Omega_{V_1}\Omega_{V_2}(r_{20} + \eta) + \Omega_{V_1}\Omega_{V_3}(r_{30} + \zeta)$$
$$- (\Omega_{V_2}^2 + \Omega_{V_3}^2)(r_{10} + \mu)]$$
$$+ \overline{j}[A_{V_2} + \ddot{r}_{20} + \dot{\Omega}_{V_3}(r_{10} + \mu) - \dot{\Omega}_{V_1}(r_{30} + \zeta) + \Omega_{V_3}\dot{r}_{10}$$
$$- \Omega_{V_1}\dot{r}_{30} + \Omega_{V_1}\Omega_{V_2}(r_{10} + \mu) + \Omega_{V_2}\Omega_{V_3}(r_{30} + \zeta)$$
$$- (\Omega_{V_1}^2 + \Omega_{V_3}^2)(r_{20} + \eta)]$$
$$+ \overline{k}[A_{V_3} + \ddot{r}_{30} + \dot{\Omega}_{V_1}(r_{20} + \eta) - \dot{\Omega}_{V_2}(r_{10} + \mu)$$
$$+ \Omega_{V_1}\dot{r}_{20} - \Omega_{V_2}\dot{V}_{10} + \Omega_{V_1}\Omega_{V_3}(r_{10} + \mu)$$
$$+ \Omega_{V_2}\Omega_{V_3}(r_{20} + \eta) - (\Omega_{V_1}^2 + \Omega_{V_2}^2)(r_{30} + \zeta)] \tag{B-3}$$

Equation (B-3) may be mechanized by several methods, two of which are discussed here. First, an accelerometer arrangement identical to configuration C described in Appendix A is utilized . Here, nine accelerometers are placed about the [m] axis. Note from Fig. B-3 that no two accelerometers are located at the same physical point. By inspection of Fig. B-3 and Eq. (B-3), the readings of the nine accelerometers are:

(Note that points ℓ_{K_1}, ℓ_{K_2} and ℓ_{K_3} refer to the distances of the various accelerometers from the origin of [m]).

$$
\begin{aligned}
Am_{21} = {} & A_{V_2} + \ddot{r}_{20} + (r_{10} + \ell_{K_1})(\dot{\Omega}_{V_3} + \Omega_{V_1}\Omega_{V_2}) \\
& + (\Omega_{V_2}\Omega_{V_3} - \dot{\Omega}_{V_1})\, r_{30} + \Omega_{V_3}\dot{r}_{10} - \Omega_{V_1}\dot{r}_{30} \\
& - (\Omega_{V_1}^2 + \Omega_{V_3}^2)\, r_{20}
\end{aligned} \tag{B-4}
$$

$$
\begin{aligned}
A'm_{21} = {} & A_{V_2} + \ddot{r}_{20} + (r_{10} - \ell_{K_1})(\dot{\Omega}_{V_3} + \Omega_{V_1}\Omega_{V_2}) + (\Omega_{V_2}\Omega_{V_3} - \Omega_{V_1}) r_{30} \\
& + \Omega_{V_3}\dot{r}_{10} - \Omega_{V_1}\dot{r}_{30} - (\Omega_{V_1}^2 + \Omega_{V_3}^2)\, r_{20}
\end{aligned} \tag{B-5}
$$

$$
\begin{aligned}
Am_{31} = {} & A_{V_3} + \ddot{r}_{30} + (\Omega_{V_1}\Omega_{V_3} - \dot{\Omega}_{V_2})\,(r_{10} + \ell_{K_3}) + (\Omega_{V_2}\Omega_{V_3} + \dot{\Omega}_{V_1}) r_{20} \\
& - (\Omega_{V_1}^2 + \Omega_{V_2}^2)\, r_{30} + \Omega_{V_1}\dot{r}_{20} - \Omega_{V_2}\dot{r}_{10}
\end{aligned} \tag{B-6}
$$

$$
\begin{aligned}
Am_{13} = {} & A_{V_1} + \ddot{r}_{10} + (\Omega_{V_1}\Omega_{V_3} + \dot{\Omega}_{V_2})\,(r_{30} + \ell_{K_3}) \\
& + \Omega_{V_2}\dot{r}_{30} - \Omega_{V_3}\dot{r}_{20} + (\Omega_{V_1}\Omega_{V_2} - \dot{\Omega}_{V_3})\, r_{20} \\
& - (\Omega_{V_2}^2 + \Omega_{V_3}^2)\, r_{10}
\end{aligned} \tag{B-7}
$$

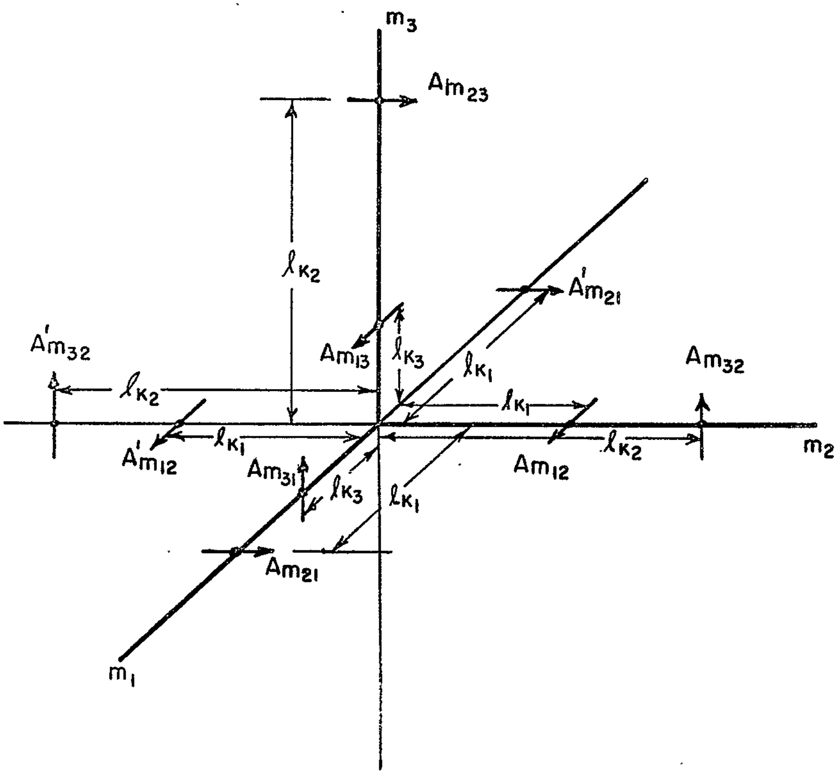

Figure B-3 Configuration C Displaced from the Vehicular Mass Center

$$Am_{23} = A_{V_2} + r_{20} + \Omega_{V_3} r_{10} - \Omega_{V_1}(r_{30} + \ell_{K_2}) + \Omega_{V_3}\dot{r}_{10} - \Omega_{V_1}\dot{r}_{30}$$

$$+ \Omega_{V_1}\Omega_{V_2} r_{10} + \Omega_{V_2}\Omega_{V_3} \quad (r_{30} + \ell_{K_2}) - (\Omega_{V_1}^2 + \Omega_{V_3}^2)$$

$$= A_{V_2} + \ddot{r}_{20} + (\dot{\Omega}_{V_3} + \Omega_{V_1}\Omega_{V_2})r_{10} + (\Omega_{V_2}\Omega_{V_3} - \dot{\Omega}_{V_1})(r_{30} + \ell_{K_2})$$

$$+ \Omega_{V_3}\dot{r}_{10} - \Omega_{V_1}\dot{r}_{30} - (\Omega_{V_1}^2 + \Omega_{V_2}^2)\, r_{20} \tag{B-8}$$

$$Am_{12} = A_{V_1} + \ddot{r}_{10} + \dot{\Omega}_{V_2} r_{30} - \dot{\Omega}_{V_3} \,(r_{20} + \ell_{K_1}) + \Omega_{V_2}\dot{r}_{30} - \Omega_{V_3}\dot{r}_{20}$$

$$+ \Omega_{V_1}\Omega_{V_2} \,(r_{20} + \ell_{K_1}) + \Omega_{V_1}\Omega_{V_3} - (\Omega_{V_2}^2 + \Omega_{V_3}^2)\, r_{10}$$

$$= A_{V_1} + \ddot{r}_{10} + (\dot{\Omega}_{V_2} + \Omega_{V_1}\Omega_{V_3})r_{30} + (\Omega_{V_1}\Omega_{V_2} - \dot{\Omega}_{V_3})(r_{20} + \ell_{K_1})$$

$$+ \Omega_{V_2}\,\dot{r}_{30} - \Omega_{V_3}\,\dot{r}_{20} - (\Omega_{V_2}^2 + \Omega_{V_3}^2)\, r_{10} \tag{B-9}$$

$$A'm_{12} = A_{V_1} + \ddot{r}_{10} + (\dot{\Omega}_{V_2} + \Omega_{V_1}\Omega_{V_3})r_{30} + (\Omega_{V_1}\Omega_{V_2} - \dot{\Omega}_{V_3})(r_{20} - \ell_{K_1})$$

$$+ \Omega_{V_2}\dot{r}_{30} - \Omega_{V_3}\dot{r}_{20} - (\Omega_{V_2}^2 + \Omega_{V_3}^2)\, r_{10} \tag{B-10}$$

$$Am_{32} = A_{V_3} + \ddot{r}_{30} + \dot{\Omega}_{V_1} \,(r_{20} + \ell_{K_2}) - \dot{\Omega}_{V_2} r_{10} + \Omega_{V_1}\dot{r}_{20} - \Omega_{V_2}\dot{r}_{10}$$

$$+ \Omega_{V_1}\Omega_{V_3} r_{10} + \Omega_{V_2}\Omega_{V_3} \,(r_{20} + \ell_{K_2}) - (\Omega_{V_1}^2 + \Omega_{V_2}^2) r_{30}$$

$$= A_{V_3} + \ddot{r}_{30} + (\dot{\Omega}_{V_1} + \Omega_{V_2}\Omega_{V_3})(r_{20} + \ell_{K_2}) + (\dot{\Omega}_{V_1}\Omega_{V_3} - \dot{\Omega}_{V_2})r_{10}$$

$$\Omega_{V_1}\dot{r}_{20} - \Omega_{V_2}\dot{r}_{10} - (\Omega_{V_1}^2 + \Omega_{V_2}^2)\, r_{30} \tag{B-11}$$

$$A'm_{32} = A_{V_3} + \ddot{r}_{30} + (\dot{\Omega}_{V_1} + \Omega_{V_2}\Omega_{V_3})(r_{20} - \ell_{K_2}) + (\Omega_{V_1}\Omega_{V_3} - \dot{\Omega}_{V_3})r_{10}$$

$$+ \Omega_{V_1}\dot{r}_{20} - \Omega_{V_2}\dot{r}_{10} - (\Omega_{V_1}^2 + \Omega_{V_2}^2)\, r_{30} \qquad \text{(B-12)}$$

Then combining Eqs. (B-5) through (B-12), the linear acceleration and angular velocities are obtained.

$$\frac{Am_{21} - A'm_{21}}{2\ell_{K_1}} = \Omega_{V_3} + \Omega_{V_1}\Omega_{V_2} = \Theta_1 \qquad \text{(B-13)}$$

$$\frac{Am_{12} - A'm_{12}}{2\ell_{K_1}} = \Omega_{V_1}\Omega_{V_2} - \dot{\Omega}_{V_3} = \Theta_2 \qquad \text{(B-14)}$$

$$\frac{Am_{32} - A'm_{32}}{2\ell_{K_2}} \quad \dot{\Omega}_{V_1} + \Omega_{V_2}\Omega_{V_3} = \Theta_3 \qquad \text{(B-15)}$$

$$Am_{21} - Am_{23} = \ell_{K_1}(\dot{\Omega}_{V_3} + \Omega_{V_1}\Omega_{V_2}) - \ell_{K_2}(\Omega_{V_2}\Omega_{V_3} - \dot{\Omega}_{V_1}) = \phi_1 \qquad \text{(B-16)}$$

$$Am_{13} - Am_{12} = \ell_{K_3}(\dot{\Omega}_{V_2} - \Omega_{V_1}\Omega_{V_3}) - \ell_{K_1}(\Omega_{V_1}\Omega_{V_2} - \dot{\Omega}_{V_3}) = \phi_2 \qquad \text{(B-17)}$$

$$Am_{31} - Am_{32} = \ell_{K_3}(\Omega_{V_1}\Omega_{V_3} - \dot{\Omega}_{V_2}) - \ell_{K_2}(\dot{\Omega}_{V_1} + \Omega_{V_2}\Omega_{V_3}) = \phi_3 \qquad \text{(B-18)}$$

Now in Eq. (B-16) substitute Eq. (B-13) for Θ_1.

Then

$$\dot{\phi}_1 = \ell_{K_1}\theta_1 - \ell_{K_2}(\Omega_{V_2}\,\Omega_{V_3} - \dot{\Omega}_{V_1})$$

or

$$\frac{-\dot{\phi}_1 + \ell_{K_1}\,\theta_1}{\ell_{K_2}} = \Omega_{V_2}\,\Omega_{V_3} - \dot{\Omega}_{V_1} = \theta_4 \qquad \text{(B-19)}$$

In Eq. (B-17) substitute Eq. (B-14) for $\dot{\theta}_2$.

Then

$$\dot{\phi}_2 = \ell_{K_3}\;(\Omega_{V_1}\,\Omega_{V_3} + \dot{\Omega}_{V_2}) - \theta_2\;\ell_{K_1}$$

or

$$\frac{\dot{\phi}_2 + \theta_2\,\ell_{K_1}}{\ell_{K_3}} = \Omega_{V_1}\,\Omega_{V_3} + \dot{\Omega}_{V_2} = \theta_5 \qquad \text{(B-20)}$$

In Eq. (B-18) substitute Eq. (B-15) for $\dot{\theta}_3$

Then

$$\phi_3 \quad \ell_{K_3}\,(\Omega_{V_1}\,\Omega_{V_3} - \dot{\Omega}_{V_2}) \quad \ell_{K_2}\,\theta_3$$

or

$$\frac{-\phi_3 + \ell_{K_2}\,\theta_3}{\ell_{K_3}} \quad \Omega_{V_1}\,\Omega_{V_3} - \dot{\Omega}_{V_2} = \theta_6 \qquad \text{(B-21)}$$

The magnitudes of the components of $\dot{\bar{\Omega}}_V$ are given by

$$\frac{\theta_1 - \theta_2}{2} = \dot{\Omega}_{V_3} \qquad \text{(B-22)}$$

$$\frac{\theta_3 - \theta_4}{2} = \dot{\Omega}_{V_1} \tag{B-23}$$

$$\frac{\theta_5 - \theta_6}{2} = \dot{\Omega}_{V_2} \tag{B-24}$$

To obtain the components A_{V_1}, A_{V_2} and A_{V_3}, define the following quantities:

$$\begin{aligned} \Omega_{V_1}^2 + \Omega_{V_2}^2 &= X_3 \\ \Omega_{V_2}^2 + \Omega_{V_3}^2 &= X_1 \\ \Omega_{V_1}^2 + \Omega_{V_3}^2 &= X_2 \end{aligned} \tag{B-25}$$

Note that these Ω's can be evaluated immediately after Eqs. (B-22), (B-23) and (B-24) have been integrated.

Combining Eqs. (B-4) and (B-5) gives:

$$A_{V_2} = \frac{Am_{21} + A'm_{21}}{2} \quad \ddot{r}_{20} - \theta_1 r_{10} - \theta_4 r_{30} + X_3 r_{20}$$

$$\Omega_{V_3} \dot{r}_{10} + \Omega_{V_1} \dot{r}_{30} \tag{B-26}$$

Similarly

$$A_{V_1} = \frac{Am_{12} + A'm_{12}}{2} - \ddot{r}_{10} - \theta_5 r_{30} - \theta_2 r_{20} - \Omega_{V_2} \dot{r}_{30}$$

$$+ \Omega_{V_3} \dot{r}_{20} + X_1 r_{10} \tag{B-27}$$

and

$$A_{V_3} = \frac{Am_{32} + A'm_{32}}{2} - \ddot{r}_{30} - \theta_3 r_{20} - \theta_6 r_{10}$$

$$\Omega_{V_1} \dot{r}_{20} + \Omega_{V_2} \dot{r}_{10} + X_2 r_{30} \qquad \text{(B-28)}$$

If $r_{10}(t) = r_{20}(t) = r_{30}(t) = 0$, then:

$$A_{V_1} = \frac{Am_{12} + A'm_{12}}{2} \qquad \text{(B-29)}$$

$$A_{V_2} \quad \frac{Am_{21} + A'm_{21}}{2} \qquad \text{(B-30)}$$

$$A_{V_3} = \frac{Am_{32} + A'm_{32}}{2} \qquad \text{(B-31)}$$

Thus, this reduces directly to the situation of sensors pla about the origin.

It is noted that although the navigation computer must perform somewhat more arithmetic, there are only 3 integrations to be performed. There are no multiplications that the computer must evaluate (except for 3 squares). Note that if in general:

$$r_{10}(t) \neq 0$$

$$r_{20}(t) \neq 0$$

$$r_{30}(t) \neq 0$$

but

$$v_{10}(t) = v_{20}(t) = v_{30}(t) = 0$$

$$a_{10}(t) = a_{20}(t) = a_{30}(t) = 0$$

Then

$$A_{V_1} = \frac{Am_{12} + A'm_{12}}{2} - \theta_5\, r_{30} - \theta_2\, r_{20} + X_1\, r_{10} \tag{B-32}$$

$$A_{V_2} = \frac{Am_{21} + A'm_{21}}{2} - \theta_1\, r_{10} - \theta_4\, r_{30} + X_3\, r_{20} \tag{B-33}$$

$$A_{V_3} = \frac{Am_{32} + A'm_{32}}{2} - \theta_3\, r_{20} - \theta_6\, r_{10} + X_2\, r_{30} \tag{B-34}$$

This problem can also be approached through the utilization of six accelerometers that are placed with their sensitive axes parallel to the [m] coordinate frame. This arrangement is shown in Fig. 4. Now using Eq. (B-3) and Fig. B-4 we can write down by inspection the quantities measured with the six accelerometers:

$$Am_{11} \quad A_{V_1} + \ddot{r}_{10} + \dot{\Omega}_{V_2} r_{30} - \dot{\Omega}_{V_3} r_{20} + \Omega_{V_2} \dot{r}_{30} - \Omega_{V_3} \dot{r}_{20}$$
$$\Omega_{V_1} \Omega_{V_2} r_{20} + \Omega_{V_1} \Omega_{V_3} r_{30} - (\Omega^2_{V_2} + \Omega^2_{V_3})\,(r_{10} + \ell_a)$$

(B-35)

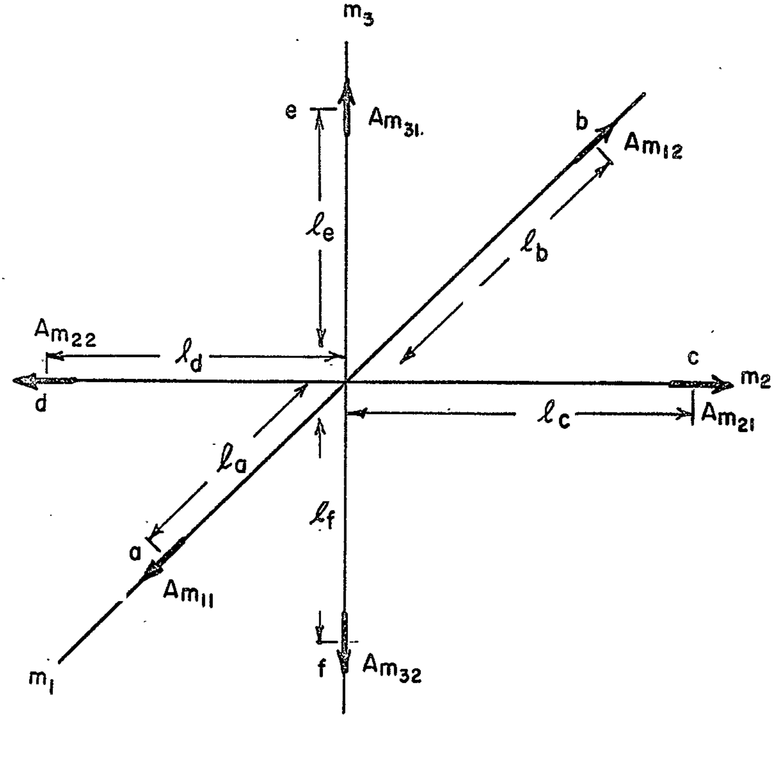

Figure B-4 A Configuration of Six Accelerometers

$$Am_{12} = A_{V_1} + \ddot{r}_{10} + \dot{\Omega}_{V_2} r_{30} - \dot{\Omega}_{V_3} r_{20} + \Omega_{V_2} \dot{r}_{30} - \Omega_{V_3} \dot{r}_{20}$$

$$\Omega_{V_1} \Omega_{V_2} r_{20} + \Omega_{V_1} \Omega_{V_3} r_{30} \quad (\Omega^2_{V_2} + \Omega^2_{V_3}) (r_{10} - \ell_b) \qquad \text{(B-36)}$$

Then subtracting Eq. (B-36) from (B-35) yields:

$$Am_{11} - Am_{12} = -(\Omega^2_{V_2} + \Omega^2_{V_3}) \ell_a - (\Omega^2_{V_2} + \Omega^2_{V_3}) \ell_b$$

or

$$\frac{Am_{12} - Am_{11}}{\ell_a + \ell_b} = \Omega^2_{V_2} + \Omega^2_{V_3} = X_1' \qquad \text{(B-37)}$$

$$Am_{21} = A_{V_2} + \ddot{r}_{20} + \dot{\Omega}_{V_3} r_{10} - \dot{\Omega}_{V_1} r_{30} + \Omega_{V_3} \dot{r}_{10} - \Omega_{V_1} \dot{r}_{30}$$

$$+ \Omega_{V_1} \Omega_{V_2} r_{10} + \Omega_{V_2} \Omega_{V_3} r_{30} - (\Omega^2_{V_1} + \Omega^2_{V_3}) (r_{20} + \ell_c) \qquad \text{(B-38)}$$

$$Am_{22} = A_{V_2} + \ddot{r}_{20} + \dot{\Omega}_{V_3} r_{10} - \dot{\Omega}_{V_1} r_{30} + \Omega_{V_3} \dot{r}_{10} - \Omega_{V_1} \dot{r}_{30}$$

$$+ \Omega_{V_1} \Omega_{V_2} r_{10} + \Omega_{V_2} \Omega_{V_3} r_{30} - (\Omega^2_{V_-} + \Omega^2_{V_3}) (r_{20} - \ell_d) \qquad \text{(B-39)}$$

Then Eq. (B-38) minus (B-39) gives:

$$Am_{22} - Am_{21} = (\Omega^2_{V_1} + \Omega^2_{V_3}) (\ell_d + \ell_c)$$

or

$$\frac{Am_{22} - Am_{21}}{\ell_d + \ell_c} = \Omega^2_{V_1} + \Omega^2_{V_3} = X_2' \qquad \text{(B-40)}$$

$$Am_{31} = A_{V_3} + \ddot{r}_{30} + \dot{\Omega}_{V_1} r_{20} - \dot{\Omega}_{V_2} r_{10} + \Omega_{V_1} \dot{r}_{20} - \Omega_{V_2} \dot{r}_{10} + \Omega_{V_1} \Omega_{V_3} r_{10} + \Omega_{V_2} \Omega_{V_3} r_{20} - (\Omega^2_{V_1} + \Omega^2_{V_2}) (r_{30} + \ell_e) \quad \text{(B-41)}$$

$$Am_{32} = A_{V_3} + \ddot{r}_{30} + \dot{\Omega}_{V_1} r_{20} - \dot{\Omega}_{V_2} r_{10} + \Omega_{V_1} \dot{r}_{20} - \Omega_{V_2} \dot{r}_{10} + \Omega_{V_1} \Omega_{V_3} r_{10} + \Omega_{V_2} \Omega_{V_3} r_{20} - (\Omega^2_{V_1} + \Omega^2_{V_2}) (r_{30} - \ell_f) \quad \text{(B-42)}$$

Therefore Eq. (B-42) minus Eq. (B-41) yields:

$$Am_{32} - Am_{31} = (\Omega^2_{V_1} + \Omega^2_{V_2}) (\ell_f + \ell_e)$$

or

$$\frac{Am_{32} - Am_{31}}{\ell_e + \ell_f} = \Omega^2_{V_1} + \Omega^2_{V_2} = X'_3 \quad \text{(B-43)}$$

Then using Eqs. (B-37), (B-40) and (B-43) we have

$$\Omega_{V_1} = \left(\frac{- X'_1 + X'_2 + X'_3}{2} \right)^{1/2} \quad \text{(B-44)}$$

$$\Omega_{V_2} = \left(\frac{X'_1 - X'_2 + X'_3}{2} \right)^{1/2} \quad \text{(B-45)}$$

$$\Omega_{V_3} = \left(\frac{X'_1 + X'_2 - X'_3}{2} \right)^{1/2} \quad \text{(B-46)}$$

We can now write A_I in terms of known quantities.

From Eq. (B-41)

$$A_{V_3} = Am_{31} - \ddot{r}_{30} - \Omega_{V_1}\dot{r}_{20} + \Omega_{V_2}\dot{r}_{10} + (\dot{\Omega}_{V_2} - \Omega_{V_1}\Omega_{V_3})r_{10} - (\dot{\Omega}_{V_1} + \Omega_{V_2}\Omega_{V_3})r_{20} + X_3' [r_{30} + \ell_e] \quad \text{(B-47)}$$

From Eq. (B-38)

$$A_{V_2} = Am_{21} - \ddot{r}_{20} - \Omega_{V_3}\dot{r}_{10} + \Omega_{V_1}\dot{r}_{30} - (\dot{\Omega}_{V_3} + \Omega_{V_1}\Omega_{V_2})r_{10} + (\dot{\Omega}_{V_1} - \Omega_{V_2}\Omega_{V_3})r_{30} + X_2' [r_{20} + \ell_c] \quad \text{(B-48)}$$

From Eq. (B-35)

$$A_{V_1} = Am_{11} - \ddot{r}_{10} - \Omega_{V_2}\dot{r}_{30} + \Omega_{V_3}\dot{r}_{20} - (\dot{\Omega}_{V_2} + \Omega_{V_1}\Omega_{V_3})r_{30} + (\dot{\Omega}_{V_3} - \Omega_{V_1}\Omega_{V_2})r_{20} + X_1' [r_{10} + \ell_a] \quad \text{(B-49)}$$

Thus, the equations giving the vehicles position are easily mechanized when the accelerometers are placed about axes [m] parallel to the vehicle axes but displaced from the center of gravity. Any change, due to fuel consumption, in the distance between the sensors and the vehicle's center of gravity, is readily programmed into the navigation computer.

APPENDIX C

ROTATING ACCELEROMETERS TO MEASURE LINEAR ACCELERATIONS AND ANGULAR VELOCITIES

Alfred R. Schuler

In this method for determining linear accelerations and angular velocities, each of two rotating rings carries an accelerometer. The two rings are concentric and normal to each other, one rotating about the V_1 axis and the other about the V_2 axis as shown in Fig.(C-1). It is assumed that the rings are placed at the center of gravity and that each accelerometer is mounted with its sensitive axis perpendicular both to the axis of rotation and to a radial line from the axis of rotation to the accelerometer.

Let the velocity of ring 1 be ω_1, that of ring 2 be ω_2. To evaluate the velocities and accelerations, first find the acceleration of a point D on ring 1 located at a radius ℓ_d from the origin of the vehicular system. The radius vector from the origin to point D is given by

$$\bar{\ell}_d = \ell_{d_2} \bar{j}_V + \ell_{d_3} \bar{k}_V \tag{C-1}$$

where ℓ_{d_2} and ℓ_{d_3} are functions of time and can be written

$$\begin{aligned} \ell_{d_2} &= \ell_d \cos \omega_1 t \\ \ell_{d_3} &= \ell_d \sin \omega_1 t \end{aligned} \tag{C-2}$$

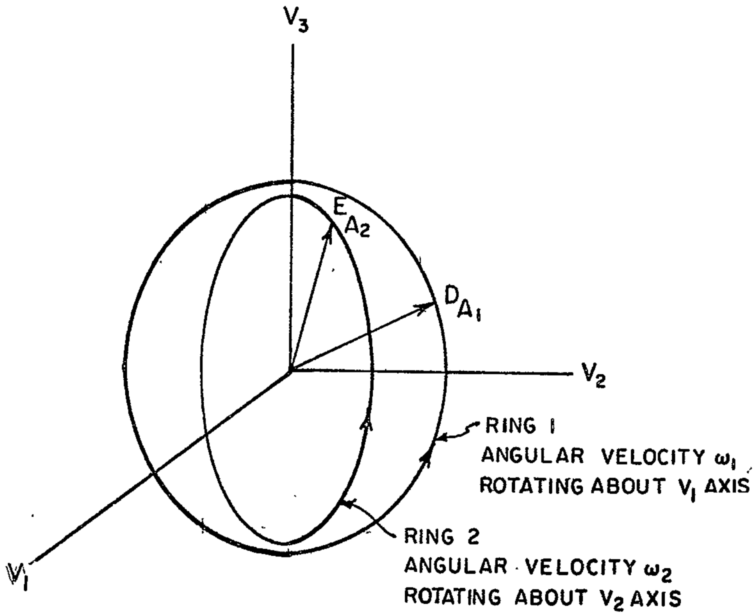

Figure C-1 Diagram Showing the Position of the Two Rotating Rings and the Location of the Accelerometers A_1 and A_2

The velocity of the point D with respect to the vehicular coordinate system is given by

$$\left(\frac{d\,\bar{\ell}_d}{dt} \right)_V = \bar{\omega}_1 \times \bar{\ell}_d \qquad \text{(C-3)}$$

and the acceleration with respect to the vehicular coordinate system is

$$\left(\frac{d^2\,\bar{\ell}_d}{dt^2} \right)_V = \bar{\omega}_1 \times (\bar{\omega}_1 \times \ell_d) \qquad \text{(C-4)}$$

Expressing (C-3) and (C-4) in terms of their projections along the vehicular axis,

$$\left(\frac{d\,\bar{\ell}_d}{dt} \right)_V = \bar{i}_V\,\omega_1 \times (\bar{j}_V\,\ell_{d_2} + \bar{k}_V\,\ell_{d_3})$$

$$= -\bar{j}_V\,\omega_1\,\ell_{d_3} + \bar{k}_V\,\omega_1\,\ell_{d_2}$$

$$= -\bar{j}_V\,\omega_1\,\ell_d \sin \omega_1 t + \bar{k}_V\,\ell_d\,\omega_1 \cos \omega_1 t \qquad \text{(C-5)}$$

$$\left(\frac{d^2\,\bar{\ell}_d}{dt^2} \right)_V = \bar{i}_V\,\omega_1 \times (-\bar{j}_V\,\omega_1\,\ell_{d_3} + \bar{k}_V\,\omega_1\,\ell_{d_2})$$

$$= -\bar{j}_V\,\omega_1^2\,\ell_{d_2} - \bar{k}_V\,\omega_1^2\,\ell_{d_3}$$

$$= -\bar{j}_V\,\omega_1^2\,\ell_d \cos \omega_1 t - \bar{k}_V\,\omega_1^2\,\ell_d \sin \omega_1 t \qquad \text{(C-6)}$$

It is shown by Page[1] that the acceleration measured in a moving system (the vehicular coordinate system [V]) with respect to a fixed inertial frame is

$$(\overline{A})_I = (\ddot{\overline{R}})_I + (\ddot{\overline{r}})_V + \dot{\overline{\Omega}} \times (\overline{r})_V + 2\overline{\Omega} \times (\dot{\overline{r}})_V + \overline{\Omega} \times [\overline{\Omega} \times (\overline{r})_V] \tag{C-7}$$

Here, $\overline{R}$ is the distance from the origin of the inertial frame to the origin of the vehicular system, $\overline{r}$ is the vector distance from the origin of the vehicular system to the point at which the acceleration is being measured, and $\overline{\Omega}$ is the angular velocity of the vehicular system. Using Eq. (C-7) to express the acceleration of point D in inertial coordinates,

$$(\overline{A})_{I \text{ point D}} = (\ddot{\overline{R}})_I + \left(\frac{d^2 \overline{\ell}_d}{dt^2}\right)_V + \dot{\overline{\Omega}} \times (\overline{\ell}_d)_V + 2\overline{\Omega} \times \left(\frac{d \overline{\ell}_d}{dt}\right)_V + \overline{\Omega} \times [\overline{\Omega} \times (\overline{\ell}_d)_V] \tag{C-8}$$

Expressing Eq. (C-8) as projections along the vehicular axis and using Eqs. (C-5) and (C-6), we obtain the following expression for the inertial acceleration of point D:

$$\begin{aligned}(\overline{A})_{I \text{ point D}} = \; & \overline{i}_V \, [A_{V_1} + \dot{\Omega}_{V_2} \ell_d \sin \omega_1 t - \dot{\Omega}_{V_3} \ell_d \cos \omega_1 t) \\ & + 2\omega_1 (\Omega_{V_2} \ell_d \cos \omega_1 t + \Omega_{V_3} \ell_d \sin \omega_1 t) \\ & + \Omega_{V_1} (\Omega_{V_2} \ell_d \cos \omega_1 t + \Omega_{V_3} \ell_d \sin \omega_1 t)] \\ & + \overline{j}_V \, [A_{V_2} - \omega_1^2 \ell_d \cos \omega_1 t - \dot{\Omega}_{V_1} \ell_d \sin \omega_1 t\end{aligned}$$

$$- 2\omega_1 \Omega_{V_1} \ell_d \cos \omega_1 t + \Omega_{V_2} \Omega_{V_3} \ell_d \sin \omega_1 t$$

$$- (\Omega^2_{V_1} + \Omega^2_{V_3}) \ell_d \cos \omega_1 t]$$

$$+ \bar{k}_V [A_{V_3} - \omega_1^2 \ell_d \sin \omega_1 t + \dot{\Omega}_{V_1} \ell_d \cos \omega_1 t$$

$$- 2\omega_1 \Omega_{V_1} \ell_d \sin \omega_1 t + \Omega_{V_2} \Omega_{V_3} \ell_d \cos \omega_1 t$$

$$- (\Omega^2_{V_1} + \Omega^2_{V_2}) \ell_d \sin \omega_1 t] \qquad \text{(C-9)}$$

A_{V_1}, A_{V_2} and A_{V_3} represent the linear acceleration of the origin of the [V] axis with respect to the inertial reference frame but resolved along the vehicular [V] axes.

Now consider the accelerometer A_2 mounted on ring 2 which is rotating with angular velocity ω_2. It is a radial distance ℓ_e from the origin of the vehicular axes. $\bar{\ell}_e$ has projections on the V_1 and V_3 axes and can be expressed as:

$$\bar{\ell}_e = \bar{i}_V \ell_{e_1} + \bar{k}_V \ell_{e_2} \qquad \text{(C-10)}$$

ℓ_{e_1} and ℓ_{e_3} are functions of time and are given by:

$$\begin{aligned} \ell_{e_1} &= \ell_e \cos \omega_2 t \\ \ell_{e_2} &= \ell_e \sin \omega_2 t \end{aligned} \qquad \text{(C-11)}$$

The acceleration of a point E on this ring is:

$$(A)_{I \text{ point E}} = \bar{i}_V [A_{V_1} - \omega_2^2 \ell_e \cos \omega_2 t + \dot{\Omega}_{V_2} \ell_e \sin \omega_2 t$$

$$- 2\omega_2 \Omega_{V_2} \ell_e \cos \omega_2 t + \Omega_{V_1} \Omega_{V_3} \ell_e \sin \omega_2 t$$

$$- (\Omega_{V_2}^2 + \Omega_{V_3}^2)\, \ell_e \cos \omega_2 t]$$

$$+ \bar{j}_V [A_{V_2} + (\dot{\Omega}_{V_3}\, \ell_e \cos \omega_2 t - \Omega_{V_1}\, \ell_e \sin \omega_2 t)$$

$$+ 2\omega_2 (\Omega_{V_3}\, \ell_e \sin \omega_2 t + \Omega_{V_1}\, \ell_e \cos \omega_2 t)$$

$$+ \Omega_{V_2} (\Omega_{V_3}\, \ell_e \sin \omega_2 t + \Omega_{V_1}\, \ell_e \cos \omega_2 t)]$$

$$+ \bar{k}_V [A_{V_3} - \omega_2^2\, \ell_e \sin \omega_2 t - \dot{\Omega}_{V_2}\, \ell_e \cos \omega_2 t$$

$$- 2\omega_2 \Omega_{V_2}\, \ell_e \sin \omega_2 t + \Omega_{V_1} \Omega_{V_3}\, \ell_e \cos \omega_2 t$$

$$- (\Omega_{V_1}^2 + \Omega_{V_2}^2)\, \ell_e \sin \omega_2 t] \qquad \text{(C-12)}$$

Accelerometer A_1 on ring 1 is assumed to have its sensitive axis parallel to V_3 at $t = 0$. Thus, it will be directed along $-V_2$ at $t = \frac{\pi}{2\omega_1}$, along $-V_3$ at $t = \frac{\pi}{\omega_1}$, and along $+V_2$ at $t = \frac{3\pi}{2\omega_1}$, etc. In other words, the accelerometer lines up periodically along the positive and negative V_2 and V_3 axes. Therefore, for $t = 0$, $\frac{2\pi}{\omega_1}$, $\frac{4\pi}{\omega_1}$, $\frac{2n\pi}{\omega_1}$ $n = 1, 2, \ldots\ldots \infty$ only the $\bar{k}_V$ component of Eq. (C-9) is measured with accelerometer A_1

Also

$$\sin\left(\omega_1 \frac{2n\pi}{\omega_1}\right) = \sin 2n\pi = 0$$

and

$$\cos\left(\omega_1 \frac{2n\pi}{\omega_1}\right) = \cos 2n\pi = 1.$$

Therefore, if we designate the quantity measured by A_1 under these circumstances as A_{11}, we have:

$$A_{11} = A_{V_3}' + \Omega_{V_1} \ell_d + \Omega_{V_2}'' \Omega_{V_3} \ell_d' \tag{C-13}$$

For $t = \frac{\pi}{2\omega_1}, \; \frac{5}{2}\frac{\pi}{\omega_1}, \; \frac{9}{2}\frac{\pi}{\omega_1}, \; \cdots \; \frac{(4n+1)\pi}{2\omega_1}$, $n = 1, 2 \ldots \infty$

only the $\bar{j}_V$ component of Eq. (C-9) exists and this is in a negative sense.

Also

$$\sin\left(\omega_1 \frac{(4n+1)\pi}{2\omega_1}\right) \quad 1$$

and

$$\cos\left(\omega_1 \frac{(4n+1)\pi}{2\omega_1}\right) \quad 0$$

Designating the quantity measured by A_1 for the condition that $t = \frac{(4n+1)\pi}{2\omega_1}$ to be A_{12} we have:

$$\begin{aligned} A_{12} &= -[A_{V_2}' - \dot{\Omega}_{V_1} \ell_d + \Omega_{V_2} \Omega_{V_3}'' \ell_d] \\ &= -A_{V_2}' + \dot{\Omega}_{V_1} \ell_d - \Omega_{V_2} \Omega_{V_3} \ell_d \end{aligned} \tag{C-14}$$

For $t = \frac{\pi}{\omega_1}, \; \frac{3\pi}{\omega_1} \; \frac{5\pi}{\omega_1}, \; \ldots . \; \frac{(2n+1)\pi}{\omega_1}$ $\quad n = 1, 2, 3 \ldots$

again only the $\bar{k}_V$ component of Eq. (C-9) exists and it too is in a negative sense. Call this quantity A_{13}.

Also,

$$\sin\left(\omega_1 \quad \frac{(2n+1)\pi}{\omega_1}\right) = 0$$

and

$$\cos\left(\omega_1 \quad \frac{(2n+1)\pi}{\omega_1}\right) = -1$$

Therefore

$$A_{13} = -[A_{V_3} - \dot{\Omega}_{V_1} \ell_d - \Omega_{V_2}\Omega_{V_3} \ell_d] = -A_{V_3} + \dot{\Omega}_{V_1} \ell_d + \Omega_{V_2} \Omega_{V_3} \ell_d \tag{C-15}$$

For $t = \frac{3\pi}{2\omega_1}, \frac{7\pi}{2\omega_1}, \ldots\ldots \frac{(4n+3)\pi}{2\omega_1}$, only the $\bar{J}_V$ component of Eq. (C-9) is detected by accelerometer A_1. Designating this quantity as A_{14} and noting that

$$\sin\left(\omega_1 \frac{(4n+3)\pi}{2\omega_1}\right) = -1$$

and

$$\cos\left(\omega_1 \frac{(4n+3)\pi}{2\omega_1}\right) = 0$$

we then have:

$$A_{11} = A_{V_2} + \dot{\Omega}_{V_1} \ell_d - \Omega_{V_2} \Omega_{V_3} \ell_d \tag{C-16}$$

We now follow the same procedure for accelerometer A_2 at point E on ring 2 as we did for accelerometer A_1 at point D on ring 1.

Thus, for $t = \frac{2n\pi}{\omega_2}$ we note that only the $\bar{K}_V$ component is detected with A_2 which from Eq. (C-12) is:

$$A_{21} = A_{V_3} - \dot{\Omega}_{V_2} \ell_e + \Omega_{V_1} \Omega_{V_3} \ell_e \tag{C-17}$$

Also, for $t = \frac{(4n+1)\pi}{2\omega_2}$, only the minus i_V component detected. Designating it by A_{22}, Eq. (C-12) yields:

$$A_{22} = -A_{V_1} - \Omega_{V_2}\,\ell_e - \Omega_{V_1}\,\Omega_{V_3}\,\ell_e \tag{C-18}$$

For $t = \frac{(2n+1)\pi}{\omega_2}$ only the minus $\bar{k}_V$ component A_{23} is read. From Eq. (C-12):

$$A_{23} = -A_{V_3} - \Omega_{V_2}\,\ell_e + \Omega_{V_1}\,\Omega_{V_3}\,\ell_e \tag{C-19}$$

For $t = \frac{(4n+3)\pi}{2\omega_2}$, the i_V component A_{24} is detected by A_2:

$$A_{24} \quad A_{V_1} - \Omega_{V_2}\,\ell_e - \Omega_{V_1}\,\Omega_{V_3}\,\ell_e \tag{C-20}$$

Assume that $\ell_d = \ell_e$ and also that the ring frequencies ω_1 and ω_2 are equal to ω so that the sampling times are identical. Then repeating the eight equations A_{11} through A_{24} for ease of manipulation:

$$A_{11} = A_{V_3} + \Omega_{V_1}\,\ell_d + \Omega_{V_2}\,\Omega_{V_3}\,\ell_d \tag{C-21}$$

$$A_{12} = -A_{V_2} + \Omega_{V_1}\,\ell_d - \Omega_{V_2}\,\Omega_{V_3}\,\ell_d \tag{C-22}$$

$$A_{13} = -A_{V_3} + \Omega_{V_1}\,\ell_d + \Omega_{V_2}\,\Omega_{V_3}\,\ell_d \tag{C-23}$$

$$A_{14} = A_{V_2} + \Omega_{V_1}\,\ell_d - \Omega_{V_2}\,\Omega_{V_3}\,\ell_d \tag{C-24}$$

$$A_{21} = A_{V_3} - \Omega_{V_2}\,\ell_d + \Omega_{V_1}\,\Omega_{V_3}\,\ell_d \tag{C-25}$$

$$A_{22} = A_{V_1} - \dot{\Omega}_{V_2}\, \ell_d - \Omega_{V_1}\, \Omega_{V_3}\, \ell_d \tag{C-26}$$

$$A_{23} = -A_{V_3} - \dot{\Omega}_{V_2}\, \ell_d + \Omega_{V_1}\, \Omega_{V_3}\, \ell_d \tag{C-27}$$

$$A_{24} = A_{V_1} - \dot{\Omega}_{V_2}\, \ell_d - \Omega_{V_1}\, \Omega_{V_3}\, \ell_d \tag{C-28}$$

Subtracting (C-23) from (C-21) and dividing by two yields:

$$\frac{A_{11} - A_{13}}{2} = A_{V_3} \tag{C-29}$$

Likewise, working with other of the above equations:

$$\frac{A_{24} - A_{22}}{2} = A_{V_1} \tag{C-30}$$

$$\frac{A_{14} - A_{12}}{2} = A_{V_2} \tag{C-31}$$

$$\frac{A_{11} + A_{12} + A_{13} + A_{14}}{4\, \ell_d} = \Omega_{V_1} \tag{C-32}$$

$$\frac{-A_{21} - A_{22} - A_{23} - A_{24}}{4\, \ell_d} = \dot{\Omega}_{V_2} \tag{C-33}$$

To obtain Ω_{V_3}, (C-21) and (C-23) are added

$$\frac{A_{11} + A_{13}}{2\, \ell_d} \qquad \dot{\Omega}_{V_1} + \Omega_{V_2}\, \Omega_{V_3}$$

Then solving for Ω_{V_3}

$$\Omega_{V_3} = \frac{\frac{A_{11} + A_{13}}{2\,\ell_d} - \dot{\Omega}_{V_1}}{\Omega_{V_2}} \tag{C-34}$$

It is seen that A_{V_1}, A_{V_2}, A_{V_3}, $\dot{\Omega}_{V_1}$ and $\dot{\Omega}_{V_2}$ are found directly. A single integration of $\dot{\Omega}_{V_1}$ and $\dot{\Omega}_{V_2}$ yield the angular rates Ω_{V_1} and Ω_{V_2}. Ω_{V_3}, given by Eq. (C-34), must be found by solving a simple algebraic equation.

This method has as its primary advantage over fixed accelerometers the elimination of at least four additional accelerometers and associated electronic gear (amplifiers and torquers). It would be useful if the following conditions are satisfied:

a) Linear acceleration and angular velocity are slowly varying functions of time so that ω is sufficiently low to insure that the time lag in the accelerometer output (due to the restriction on frequency response of the accelerometer) does not introduce errors.

b) Sufficient power is available to drive the rings on which the accelerometers are mounted.

c) Practical sampling circuits are not too difficult to implement.

d) Packaging presents no major difficulties.

REFERENCE

1. Page, Leigh: "Introduction to Theoretical Physics," D. Van Nostrand and Company, 1947.

APPENDIX D

INPUT-OUTPUT ERROR RELATIONS OF THE GRAVITY COMPUTER

Anthony J. Grammaticos

The function of the gravity computer is to generate the gravity acceleration based on the vehicle position information available from the navigation system and the universal law of gravitation which the computer simulates.

The position, however, is not known precisely and therefore position errors enter the computer. As a result, the computed gravity components contain errors. The output gravity errors of the computer are related to the input position errors. This relationship is needed in studying the stability of the navigation system. The present appendix establishes this relationship.

Let m_i, x_i, y_i, z_i ($i = 1, 2, \ldots n$) be the masses and coordinates of the centers of mass of the celestial bodies contributing to the gravity acceleration at the true position x, y, z of the vehicle. Then the true components of the gravity at this point are:

$$g_x = -\sum_{i=1}^{n} \gamma\, m_i \left\{ \frac{x - x_i}{R_i^3} \right\} \tag{D-1a}$$

$$g_y = -\sum_{i=1}^{n} \gamma\, m_i \left\{ \frac{y - y_i}{R_i^3} \right\} \tag{D-1b}$$

$$g_z = -\sum_{i=1}^{n} \gamma\, m_i \left\{ \frac{z - z_i}{R_i^3} \right\} \qquad \text{(D-1c)}$$

where

$$R_i = [\,(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{1/2} \qquad \text{(D-2)}$$

and

γ is the gravitational constant, $6.67 \times 10^{-8}\ \frac{cm^3}{gr.sec^2}$

Since the true position x, y, z of the vehicle is not available, the gravitation acceleration is being calculated on the basis of the approximate position information s_x, s_y, s_z. Then the calculated gravity components are:

$$g_{xc} = -\sum_{i=1}^{n} \gamma\, m_i \left\{ \frac{s_x - x_i}{s_i^3} \right\} \qquad \text{(D-3a)}$$

$$g_{yc} = -\sum_{i=1}^{n} \gamma\, m_i \left\{ \frac{s_y - y_i}{s_i^3} \right\} \qquad \text{(D-3b)}$$

$$g_{zc} = -\sum_{i=1}^{n} \gamma\, m_i \left\{ \frac{s_z - z_i}{s_i^3} \right\} \qquad \text{(D-3c)}$$

where

$$s_i = [\,(s_x-x_i)^2 + (s_y-y_i)^2 + (s_z-z_i)^2]^{1/2} \qquad \text{(D-4)}$$

Let the position and the gravitation errors be:

$$\delta x = s_x - x \qquad \delta y = s_y - y \qquad \delta z = s_z - z \qquad \text{(D-5)}$$

$$\delta g_x = g_{xc} - g_x \qquad \delta g_y = g_{yc} - g_y \qquad \delta g_z = g_{zc} - g_z \tag{D-6}$$

where the g's are given by (D-1) and (D-3).

Now expand Eq. (D-3) in Taylor series about the point (x, y, z). Take for example Eq. (D-3a).

$$g_{xc}(s_x, s_y, s_z) = g_{xc}\Big|_{\substack{s_x = x\\ s_y = y\\ s_z = z}} + \frac{\partial g_{xc}}{\partial s_x}\Big|_{\substack{s_x = x\\ s_y = y\\ s_z = z}}(s_x - x)$$

$$+ \frac{\partial g_{xc}}{\partial s_y}\Big|_{\substack{s_x = x\\ s_y = y\\ s_z = z}}(s_y - y) + \frac{\partial g_{xc}}{\partial s_z}\Big|_{\substack{s_x = x\\ s_y = y\\ s_z = z}}(s_z - z)$$

$$+ \text{higher order terms in } (s_x - x), (s_y - y), (s_z - z) \tag{D-7}$$

where

$$g_{xc}\Big|_{\substack{s_x = x\\ s_y = y\\ s_z = z}} = -\sum_{i=1}^{n} \gamma\, m_i \left\{\frac{s_x - x_i}{s_i^3}\right\}\Bigg|_{\substack{s_x = x\\ s_y = y\\ s_z = z}} = -\sum_{i=1}^{n} \gamma\, m_i \left\{\frac{x - x_i}{R_i^3}\right\} \tag{D-8}$$

$$\frac{\partial g_{xc}}{\partial s_x} = \frac{\partial}{\partial s_x}\left(-\sum_{i=1}^{n} \gamma\, m_i \left\{\frac{s_x - x_i}{s_i^3}\right\}\right)$$

$$= -\sum_{i=1}^{n} \gamma\, m_i \frac{\partial}{\partial s_x}\left\{\frac{s_x - x_i}{[(s_x-x_i)^2 + (s_y-y_i)^2 + (s_z-z_i)^2]^{3/2}}\right\}$$

$$= -\sum_{i=1}^{n} \gamma\, m_i \left\{\frac{1}{[(s_x-x_i)^2 + (s_y-y_i)^2 + (s_z - z_i)^2]^{3/2}}\right.$$

$$\left. - \frac{(s_x-x_i)\,\frac{3}{2}\,[(s_x-x_i)^2 + (s_y-y_i)^2 + (s_z-z_i)^2]^{1/2}\; 2(s_x-x_i)}{[(s_x-x_i)^2 + (s_y-y_i)^2 + (s_z-z_i)^2]^{6/2}}\right\}$$

$$= -\sum_{i=1}^{n} \gamma\, m_i \left[\frac{1}{s_i^3} - \frac{3(s_x-x_i)^2}{s_i^5}\right]$$

hence

$$\left.\frac{\partial g_{xc}}{\partial s_x}\right|_{\substack{s_x = x\\ s_y = y\\ s_z = z}} = -\sum_{i=1}^{n} \gamma\, m_i \left.\left[\frac{1}{s_i^3} - \frac{3(s_x-x_i)^2}{s_i^5}\right]\right|_{\substack{s_x = x\\ s_y = y\\ s_z = z}}$$

$$= -\sum_{i=1}^{n} \frac{\gamma\, m_i}{R_i^3} \qquad \frac{3(x - x_i)^2}{R_i^2} \qquad \text{(D-9)}$$

Similarly

$$\left.\frac{\partial g_{xc}}{\partial s_y}\right|_{\substack{s_x = x\\ s_y = y\\ s_z = z}} \qquad \sum_{i=1}^{n} \frac{\gamma\, m_i}{R_i^3} \left[\frac{-\,3(x-x_i)\,(y-y_i)}{R_i^2} \right] \tag{D-10}$$

$$\left.\frac{\partial g_{xc}}{\partial s_z}\right|_{\substack{s_x = x\\ s_y = y\\ s_z = z}} = -\sum_{i=1} \frac{\gamma\, m_i}{R_i^3} \left[\frac{-\,3(x-x_i)\,(z-z_i)}{R_i^2} \right] \tag{D-11}$$

In view of Eqs. (D-5), (D-9), (D-10) and (D-11) and dropping higher order terms Eq. (D-7) takes the form

$$g_{xc} = \sum_{i=1}^{n} \frac{\gamma\, m_i}{R_i^3} \left\{ (x - x_i) + \left[1 - \frac{3(x - x_i)^2}{R_i^2} \right] \delta x \right.$$
$$\left. \frac{3(x - x_i)\,(y - y_i)}{R_i^2}\ \delta y - \frac{3(x - x_i)\,(z - z_i)}{R_i^2}\ \delta z \right\} \tag{D-12}$$

Then from Eq. (D-1a), (D-12) and (D-6) we obtain

$$\delta g_x = g_{xc} - g_x = -\sum_{i=1}^{n} \frac{\gamma\, m_i}{R_i^3} \left\{ \left[1 - \frac{3(x-x_i)^2}{R_i^2} \right] \delta x \right.$$
$$\left. - \frac{3(x-x_i)\,(y-y_i)}{R_i^2}\ \delta y - \frac{3(x-x_i)\,(z-z_i)}{R_i^2}\ \delta z \right\} \tag{D-13a}$$

and by analogy

$$\delta g_y = g_{yc} - g_y = - \sum_{i=1}^{n} \frac{\gamma\, m_i}{R_i^3} \left\{ - \frac{3(x-x_i)\,(y-y_i)}{R_i^2}\, \delta x + \left[1 - \frac{3(y-y_i)^2}{R_i^2} \right] \delta y - \frac{3(y-y_i)\,(z-z_i)}{R_i^2}\, \delta z \right\} \tag{D-13b}$$

$$\delta g_z = g_{zc} - g_z = - \sum_{i=1}^{n} \frac{\gamma\, m_i}{R_i^3} \left\{ - \frac{3(x-x_i)\,(z-z_i)}{R_i^2}\, \delta x - \frac{3(y-y_i)\,(z-z_i)}{R_i^2}\, \delta y + \left[1 - \frac{3(z-z_i)^2}{R_i^2}\, \delta_z \right] \right\} \tag{D-13c}$$

Let

$$\mu_x = \sum_{i=1}^{n} \frac{\gamma\, m_i}{R_i^3} \left[1 - \frac{3(x-x_i)^2}{R_i^2} \right. \tag{D-14a}$$

$$\mu_y \quad \sum_{i=1}^{n} \frac{\gamma\, m_i}{R_i^3} \left[1 - \frac{3(y-y_i)^2}{R_i^2} \right. \tag{D-14b}$$

$$\mu_z \quad \sum_{i=1}^{n} \frac{\gamma\, m_i}{R_i^3} \left[1 - \frac{3(z-z_i)^2}{R_i^2} \right. \tag{D-14c}$$

$$\nu \quad \sum_{i=1}^{n} \frac{\gamma\, m_i}{R_i^3} \left[\frac{3(x-x_i)\,(y-y_i)}{R_i^2} \right. \tag{D-15}$$

$$\rho \quad \sum_{i=1}^{n} \frac{\gamma\, m_i}{R_i^3} \left[\frac{3(x-x_i)\,(z-z_i)}{R_i^2} \right. \tag{D-16}$$

$$\tau = \sum_{i=1}^{n} \frac{\gamma m_i}{R_i^3} \frac{3(y-y_i)(z-z_i)}{R_i^2} \tag{D-17}$$

Then Eq. (D-13) can be rewritten as follows:

$$\begin{aligned} \delta g_x &= -\mu_x \,\delta x + \nu \,\delta y + \rho \,\delta z \\ \delta g_y &= \nu \,\delta x - \mu_y \,\delta y + \tau \,\delta z \\ \delta g_z &= \rho \,\delta x + \tau \,\delta y - \mu_z \,\delta z \end{aligned} \tag{D-18a}$$

or

$$\begin{bmatrix} \delta g_x \\ \delta g_y \\ \delta g_z \end{bmatrix} = \begin{bmatrix} -\mu_x & \nu & \rho \\ \nu & -\mu_y & \tau \\ \rho & \tau & -\mu_z \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} \tag{D-18b}$$

In most cases of an interplanetary flight the vehicle will be in the sphere of influence of the sun only and equations (D-13) to (D-18) will be simplified considerably. Whenever, several celestial bodies significantly contribute to the total gravitational field in the vicinity of the vehicle, it is convenient to introduce the concept of an equivalent celestial body with an equivalent g/R ratio of

$$\frac{g}{R} = \sum_{i=1}^{n} \frac{\gamma m_i}{R_i^3} \tag{D-19}$$

Note that an error in the measured value of any position coordinate will cause an error in each of the components of gravity errors and that the coefficients of Eq. (D-18) are time-varying.

For a unique center of attraction (the two body problem) the coefficients given by Eqs. (D-14), (D-15), (D-16) and (D-17) have bounded values as follows:

$$-\frac{2g}{R} \leq \mu_x , \mu_y, \mu_z \leq \frac{g}{R} \tag{D-20}$$

$$-\frac{3g}{2R} \leq \nu , \rho , \tau \leq \frac{3g}{2R} \tag{D-21}$$

In the case of a unique center of attraction some of the previous equations can be rewritten as follows:

$$\mu_x = \gamma \left(\frac{m}{R^3}\right) - \mu_x' = \frac{g}{R} - \mu_x' \tag{D-22a}$$

$$\mu_y = \gamma \left(\frac{m}{R^3}\right) \qquad \mu_y' = \frac{g}{R} - \mu_y' \tag{D-22b}$$

$$\mu_z = \gamma \left(\frac{m}{R^3}\right) \qquad \mu_z' = \frac{g}{R} - \mu_z' \tag{D-22c}$$

where

$$\begin{aligned} \mu_x' &= \gamma \frac{m}{R^3} \left(\frac{3x^2}{R^2}\right) \\ \mu_y' &\quad \gamma \frac{m}{R^3} \left(\frac{3y^2}{R^2}\right) \\ \mu_z' &= \gamma \frac{m}{R^3} \left(\frac{3z^2}{R^2}\right) \end{aligned} \tag{D-23}$$

using Eq. (D-23) in Eq. (D-18) yields

$$\begin{bmatrix} \delta g_x \\ \delta g_y \\ \delta g_z \end{bmatrix} \quad \begin{bmatrix} -\frac{g}{R} & 0 & 0 \\ 0 & -\frac{g}{R} & 0 \\ 0 & & -\frac{g}{R} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} \begin{bmatrix} \mu_x' & \nu & \rho \\ \nu & \mu_y' & \tau \\ \rho & \tau & \mu_z' \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} \tag{D-24}$$

A careful examination of the Taylor expansion employed previously reveals that the first term in Eq. (D-24) represents the gravity errors due to the position errors included in the numerator of Eq. (D-3) and that the second part of Eq. (D-24) represents the gravity errors due to the position errors included in the denominator of Eq. (D-3).

Returning to Eq. (D-18) we can write

$$\delta \bar{g} = \bar{G}\ \delta \bar{R} \tag{D-25}$$

where

$$\bar{G} = \begin{bmatrix} -\mu_x & \nu & \rho \\ \nu & -\mu_y & \tau \\ \rho & \tau & -\mu_z \end{bmatrix} \tag{D-26}$$

Equation (D-25) is the basic result of this appendix.

Recapitulating, Eqs. (D-1) and (D-3) can be written as

$$\bar{g} = \bar{G}_1\ (x, y, z, x_i, y_i, z_i, m_i) \tag{D-1}$$

$$\bar{g}_c = \bar{G}_1\ (s_x, s_y, s_z, x_i, y_i, z_i, m_i) \tag{D-3}$$

or in view of Eqs. (D-5) and (D-6)

$$\delta\bar{g} = \bar{g}_c - \bar{g} = \bar{G}_1(x, y, z, \delta x, \delta y, \delta z, x_i, y_i, z_i, m_i)$$

$$- \bar{G}_1(x, y, z, x_i, y_i, z_i, m_i) \qquad \text{(D-27)}$$

Here $\bar{G}_1$ (.) represents the functional form of the universal law of gravitation.

Among others, Eq. (D-27) gives the dependence of $\delta\bar{g}$ upon δx, δy, δz which is needed in the error analysis of the overall navigation system. Such a dependence, however, is nonlinear and cannot be used for an error analysis directly. Then what we do is to find an approximation of Eq. (D-27) by expanding Eq. (D-27) in Taylor series in terms of δx, δy, δz about the point x, y, z, and retaining the linear terms only; this can be done based on the assumption that δx, δy, δz are small.

This procedure leads to Eq. (D-25) which is valid for the errors only. Apparently $\bar{G} \not\equiv \bar{G}_1$. The elements of $\bar{G}$ vary slowly with time and act as gains upon the elements of $\delta\bar{R}$ to generate the elements of $\delta\bar{g}$.

APPENDIX E

A. Grammaticos

E.1 THE MECHANIZATION OF THE NAVIGATION EQUATIONS IN THE INERTIAL FRAME

The problem of the mechanization of the navigation equations in different coordinate frames has been studied by Bodner and Seleznev[1] and Krishnan[2] in their works on gimballess navigation systems. Of all the mechanizations that were studied, the mechanization in the inertial frame was found advantageous and recommended for further study.

Figure E-1 shows such a mechanization whose fundamentals are presented here. Assume that the space vehicle on an interplanetary mission is moving along a path around the sun. The vehicle is subject to the gravitational forces and the thrust of its engines.

The origin of an inertial frame is placed at the center of the sun and its x, y, z axes are fixed with respect to the fixed stars. A vehicular frame is rigidly attached to the vehicle and follows its motion.

The relative orientation of the two frames can be established from on-board acceleration measurements made with body-mounted accelerometers. The inertial acceleration $\bar{a}$ measured along the vehicular axes is resolved along the inertial axes by means of a direction-cosine computer, yielding the components a_x, a_y, a_z of the inertial acceleration.

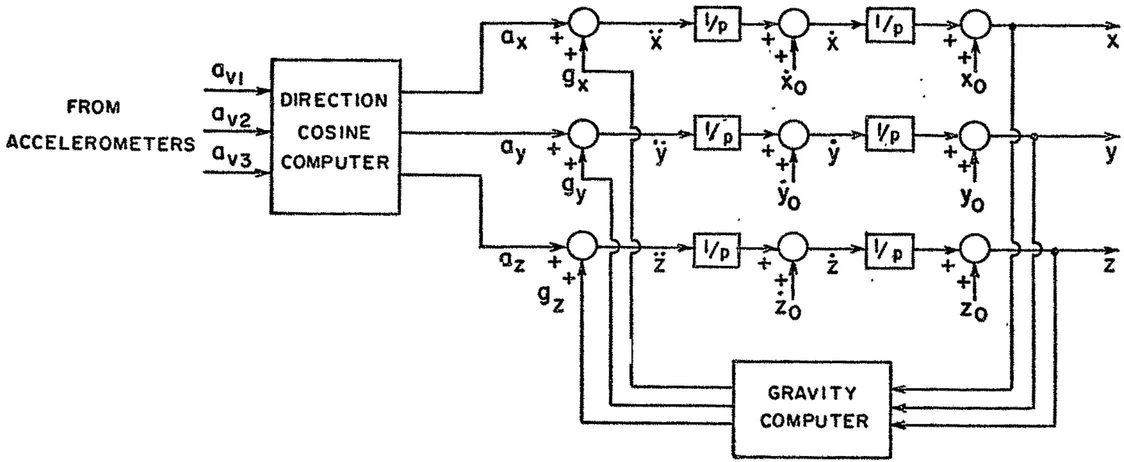

Figure E-1 System Mechanization

Let $\bar{g}(g_x, g_y, g_z)$ be the gravity acceleration at the position occupied by the vehicle. Then the vehicle is subject to a total acceleration

$$\bar{a} + \bar{g} = \ddot{\bar{R}} \tag{E-1}$$

where $\bar{R}(x, y, z)$ is the position vector referred to the inertial frame. Equation (E-1) expresses the Newton law of inertia.

Integrating both sides of Eq. (E-1) once yields the velocity $\dot{\bar{R}}(\dot{x}, \dot{y}, \dot{z})$ and integrating twice yields the position $\bar{R}(x, y, z)$ of the vehicle provided that $\bar{a}$ and $\bar{g}$ are known. $\bar{a}$ can be obtained in the way described above; $\bar{g}$ is obtained from the universal law of gravitation

$$\bar{g} = \frac{\gamma m}{R^3} \bar{R} \tag{E-2}$$

which is simulated by the gravity computer; γ is the gravitation constant and m is the attracting mass. By applying Eq. (E-2) we introduce the feedback loops shown in Fig. E-1.

Finally, assumed initial conditions on velocity and position are shown in Fig. E-1.

Recapitulating the accelerometers measure components along the vehicular axes of the inertial acceleration due to the engine thrust.

These components are transformed into components a_x, a_y, a_z along the inertial frame.

The sum of the thrust acceleration components a_x, a_y, a_z and the corresponding gravity components g_x, g_y, g_z must be equal to the acceleration components $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ of the vehicle.

The vehicle acceleration components $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ are integrated once to give vehicle velocity components $\dot{x}$, $\dot{y}$, $\dot{z}$; proper initial conditions are introduced.

The vehicle velocity components $\dot{x}$, $\dot{y}$, $\dot{z}$ are integrated again to give the vehicle position coordinates x, y, z; proper initial position coordinates are introduced again.

Finally the current vehicle position coordinates are fed into the gravity computer which generates the gravity acceleration needed at the input of the first integrator.

From a system point of view, this mechanization contains two distinct parts cascaded together. The direction-cosine computer is the first part; the other is what one usually calls the navigation loop.

E.2 ERRORS IN THE NAVIGATION SYSTEM

Figure E-1 shows the ideal navigation system. If the accelerometers measure acceleration without errors and the initial alignment of the system is perfect then the system output is identical to the true position of the vehicle.

In practice, however, it is impossible to make error-free acceleration measurements and perfect initial alignment; as a result errors are introduced into the system. Because of the errors $\delta\bar{a}_v(\delta a_{v1}, \delta a_{v2}, \delta a_{v3})$ in the acceleration measurements the output of the direction-cosine computer contains errors $\delta\bar{a}(\delta a_x, \delta a_y, \delta a_z)$ which enter the navigation loop; these acceleration errors together with the initial condition errors give rise to position errors $\delta\bar{R}(\delta x, \delta y, \delta z)$; the position errors enter the gravity computer giving rise to gravity errors $\delta\bar{g}(\delta g_x, \delta g_y, \delta g_z)$ which in turn are fed back into the system. Therefore, a stability analysis of the system is necessary.

Because of the cascaded connection between the direction-cosine computer and the navigation loop the stability analysis of the system can be split into stability analysis of the direction-cosine computer and stability analysis of the navigation loop.

Here we are concerned with the stability analysis of the navigation loop.

Figure E-2 shows the navigation system and the corresponding signals; each signal is represented as the sum of its true value plus an error. By inspection of Fig. E-2 we can write

$$\bar{a} + \delta\bar{a} + \bar{g} + \delta\bar{g} = \ddot{\bar{R}} + \delta\ddot{\bar{R}} \tag{E-3}$$

Using Eq. (E-1) in Eq. (E-3) gives

$$\delta\bar{a} + \delta\bar{g} = \delta\ddot{\bar{R}} \tag{E-4}$$

or

$$\delta\ddot{\bar{R}} - \delta\bar{g} = \delta\bar{a} \tag{E-5}$$

Equation (E-5) is the error equation for the navigation loop. Note that $\delta\bar{a}$ acts as a forcing function whereas $\delta\bar{g}$ depends on $\delta\bar{R}$; the dependence of $\delta\bar{g}$ and $\delta\bar{R}$ was established by Eq.'s (D-18) or (D-25) and (D-26). We recall that:

$$\begin{bmatrix} \delta g_x \\ \delta g_y \\ \delta g_z \end{bmatrix} = \begin{bmatrix} -\mu_x & \nu & \rho \\ \nu & -\mu_y & \tau \\ \rho & \tau & -\mu_z \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} \tag{D-18}$$

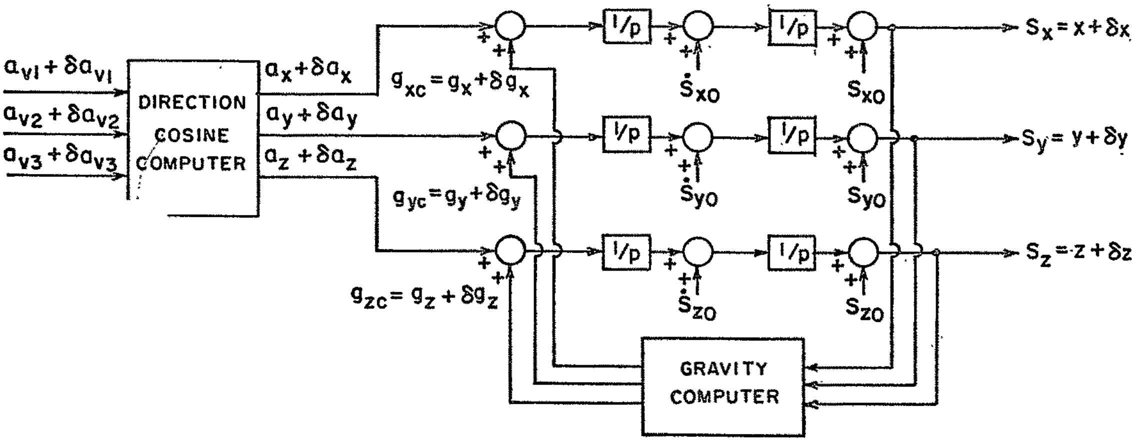

Figure E-2 Errors in the Navigation System or

$$\delta\bar{g} = \bar{G}\,\delta\bar{R} \tag{D-25}$$

where

$$\bar{G} = \begin{bmatrix} -\mu_x & \nu & \rho \\ \nu & -\mu_y & \tau \\ \rho & \tau & -\mu_z \end{bmatrix} . \tag{D-26}$$

Introducing Eq. (D-25) into Eq. (E-5) gives

$$\delta\ddot{\bar{R}} - \bar{G}\,\delta\bar{R} = \delta\bar{a} \tag{E-6}$$

or

$$\begin{bmatrix} \delta\ddot{x} \\ \delta\ddot{y} \\ \delta\ddot{z} \end{bmatrix} \quad \begin{bmatrix} -\mu_x & \nu & \rho \\ \nu & -\mu_y & \tau \\ \rho & \tau & -\mu_z \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} \delta a_x \\ \delta a_y \\ \delta a_z \end{bmatrix} \tag{E-7}$$

Assuming that μ_x, μ_y, μ_z, ν, ρ, τ vary slowly with time, we can Laplace transform Eq. (E-7)

$$\begin{bmatrix} s^2 + \mu_x & -\nu & -\rho \\ -\nu & s^2 + \mu_y & -\tau \\ -\rho & -\tau & s^2 + \mu_z \end{bmatrix} \quad \begin{bmatrix} \delta x(s) \\ \delta y(s) \\ \delta z(s) \end{bmatrix} \quad \begin{bmatrix} \delta a_x \\ \delta a_y \\ \delta a_z \end{bmatrix}$$

(E-8)

Now the stability of the navigation loop is determined by roots of the characteristic equation. Consider first the determinant

$$D(s) \quad = \quad \begin{vmatrix} s^2+\mu_x & -\nu & -\rho \\ -\nu & s^2+\mu_y & -\tau \\ -\rho & -\tau & s^2+\mu_z \end{vmatrix} \tag{E-9}$$

Expanding gives:

$$D(s) = (s^2+\mu_x)\ (s^2+\mu_y)\ (s^2+\mu_z) - \nu\tau\rho - \nu\tau\rho - \rho^2(s^2+\mu_y) - \tau^2(s^2+\mu_x) - \nu^2(s^2+\mu_z$$

or $\quad D(s) = s^6 + (\mu_x+\mu_y+\mu_z)\ s^4 + (\mu_x\,\mu_y + \mu_y\,\mu_z + \mu_z\,\mu_x - \nu^2 - \rho^2 - \tau^2)s^2$

$$- (\mu_x\,\tau^2 + \mu_y\,\rho^2 + \mu_z\,\nu^2 - \mu_x\,\mu_y\,\mu_z + 2\,\nu\rho\tau) \tag{E-10}$$

In view of Eq.'s (D-14), (D-15), (D-16), (D-17) and (D-19), we obtain:

$$\mu_x + \mu_y + \mu_z = 0 \tag{E-11}$$

$$\mu_x\,\mu_y + \mu_y\,\mu_z + \mu_z\,\mu_x - \nu^2 - \rho^2 - \tau^2 = -3\,\frac{g^2}{R^2} \tag{E-12}$$

$$\mu_x\,\tau^2 + \mu_y\,\rho^2 + \mu_z\,\nu^2 - \mu_x\,\mu_y\,\mu_z + 2\,\nu\rho\tau = 2\,\frac{g^3}{R^3} \tag{E-13}$$

Also let $\qquad \dfrac{g}{R} = \omega_3^2 \qquad$ (E-14)

Then Eq. (E-10) can be rewritten as:

$$D(s) = s^6 - 3\,\omega_s^4\,s^2 - 2\,\omega_s^6 \tag{E-15}$$

or

$$D(s) = (s^2 + \omega_s^2)^2\,(s^2 - 2\,\omega_s^2) \tag{E-16}$$

Let

$$\bar{Q} \qquad \begin{bmatrix} s^2+\mu_x & -\nu & -\rho \\ -\nu & s^2+\mu_y & -\tau \\ -\rho & & s^2+\mu_z \end{bmatrix} \tag{E-17}$$

Since $|\bar{Q}| = D(s) \not\equiv 0$, the solution to Eq. (E-8) is

$$\delta \bar{R}(s) = \frac{\text{adj } \bar{Q}(s)}{|\bar{Q}(s)|} \delta \bar{a}(s) \tag{E-18}$$

Now

$$\text{adj } \bar{Q} = \begin{bmatrix} (s^2+\mu_y)(s^2+\mu_z)-\tau^2 & \nu(s^2+\mu_z)+\tau & \nu(s^2+\mu_y)+\nu\tau \\ \nu(s^2+\mu_z)+\rho\tau & (s^2+\mu_x)(s^2+\mu_z)-\rho^2 & \tau(s^2+\mu_x)+\nu\rho \\ \rho(s^2+\mu_y)+\nu\tau & \tau(s^2+\mu_x)+\nu\rho & (s^2+\mu_x)(s^2+\mu_y)-\nu^2 \end{bmatrix} =$$

$$= (s^2+\omega_s^2) \begin{bmatrix} s^2-\mu_x-\omega_s^2 & \nu & \rho \\ \nu & s^2-\mu_y-\omega_s^2 & \tau \\ \rho & \tau & s^2-\mu_z-\omega_s^2 \end{bmatrix} \tag{E-19}$$

and hence,

$$\begin{bmatrix} \delta x(s) \\ \delta y(s) \\ \delta z(s) \end{bmatrix} = \frac{1}{(s^2+\omega_s^2)(s^2-2\omega_s^2)} \begin{bmatrix} s^2-\mu_x-\omega_s^2 & \nu & \rho \\ \nu & s^2-\mu_y-\omega_s^2 & \tau \\ \rho & & s^2-\mu_z-\omega_s^2 \end{bmatrix} \begin{bmatrix} \delta a_x(s) \\ \delta a_y(s) \\ \delta a_z(s) \end{bmatrix} \tag{E-20}$$

Equation (E-20) suggests that the characteristic equation of the system (E-8) is

$$(s^2 + \omega_s^2)\ (s^2 - 2\omega_s^2) = 0 \tag{E-21}$$

The roots of the characteristic equation are $\pm j\omega_s$ and $\pm\sqrt{2}\,\omega_s$. This means that the transient response of the navigation loop consists of a sinusoidal term of bounded amplitude due to the $\pm j\omega$ poles and a hyperbolic cosine term, which increases with time, due to the $\pm\sqrt{2}\,\omega_s$ poles; therefore the system is unstable. Small errors in the initial conditions produce time-increasing error in the output, and the accuracy of the system deteriorates with time.

In order to improve the accuracy of the system we must stabilize the system by damping the errors. Methods of damping are presented in the following section.

E.3 DAMPING THE SYSTEM ERRORS BY MEANS OF REFERENCE TRAJECTORY INFORMATION

The reference trajectory is defined as the ideal predetermined trajectory for a specific mission. During the execution of the mission one does his best to keep the actual trajectory as close to the reference trajectory as possible. Hence, at a given time t the difference between the corresponding positions on the reference and the actual trajectories is small.

The method of damping proposed here is based on the closeness of the two trajectories mentioned above and on the assumption that reference trajectory information is available on board the vehicle; this is usually the case since this information is needed for a number of purposes including guidance. Let

$$\bar{R}_r = \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix}$$ be the position that the vehicle should occupy on the reference trajectory at time t.

$$\delta\bar{R}_r = \bar{R}_r - \bar{R} = \begin{bmatrix} \delta x_r \\ \delta y_r \\ \delta z_r \end{bmatrix}$$ be the position difference between the reference and actual trajectories at time t.

and, anticipating its use, form the difference

$$\bar{R}_r - \bar{s} = \bar{R} + \delta\bar{R}_r - (\bar{R} + \delta\bar{R}) = \delta\bar{R}_r - \delta\bar{R} = \begin{bmatrix} \delta x_r - \delta x \\ \delta y_r - \delta y \\ \delta z_r - \delta z \end{bmatrix} \tag{E-22}$$

Figure E-3 shows the system mechanization of Fig. E-2 with reference trajectory damping added. In this mechanization, the reference trajectory position $\overline{R}_r$ is compared with the calculated position $\overline{s}$ and their difference $\overline{R}_r - \overline{s}$ is modified by the compensation matrix $\overline{H}$. The difference $\overline{R}_r - \overline{s}$ is given by Eq. (E-22); the compensation matrix $\overline{H}$ is to be determined so as to achieve damping.

Now we can write the error equations by inspection of Fig. E-3; they are

$$\ddot{\overline{R}} + \delta\ddot{\overline{R}} = \overline{g} + \delta\overline{g} + \overline{a} + \delta\overline{a} + \overline{H}\,(\delta\overline{R}_r - \delta\overline{R}) \qquad \text{(E-23)}$$

or

$$\delta\ddot{\overline{R}} = \delta\overline{g} + \delta\overline{a} + \overline{H}\,(\delta R_r - \delta R) \qquad \text{(E-24)}$$

since $\ddot{\overline{R}} = \overline{g} + \overline{a}$

Equation (E-24) can be rewritten

$$\delta\ddot{\overline{R}} + \overline{H}\,\delta\overline{R} - \overline{G}\,\delta\overline{R} = \delta\overline{a} + \overline{H}\,\delta\overline{R}_r \qquad \text{(E-25)}$$

Equation (E-25) is the basic error equation of the system in Fig. E-3. The terms $\delta\overline{a} + \overline{H}\,\delta\overline{R}_r$ constitute the forcing functions and hence the error equation of the autonomous system is:

$$\delta\ddot{\overline{R}} + \overline{H}\,\delta\overline{R} - \overline{G}\,\delta\overline{R} = 0 \qquad \text{(E-26)}$$

Equation (E-26) determines the performance of the system. Careful inspection of this equation aids in determining a suitable form for $\overline{H}$. The damping of the system is directly related to the presence of derivative terms $k\,\delta\dot{\overline{R}}$; so $\overline{H}$ must contain terms of the form $k\frac{d}{dt}$ or kp if $p = \frac{d}{dt}$. The speed of response is related to the coefficient of

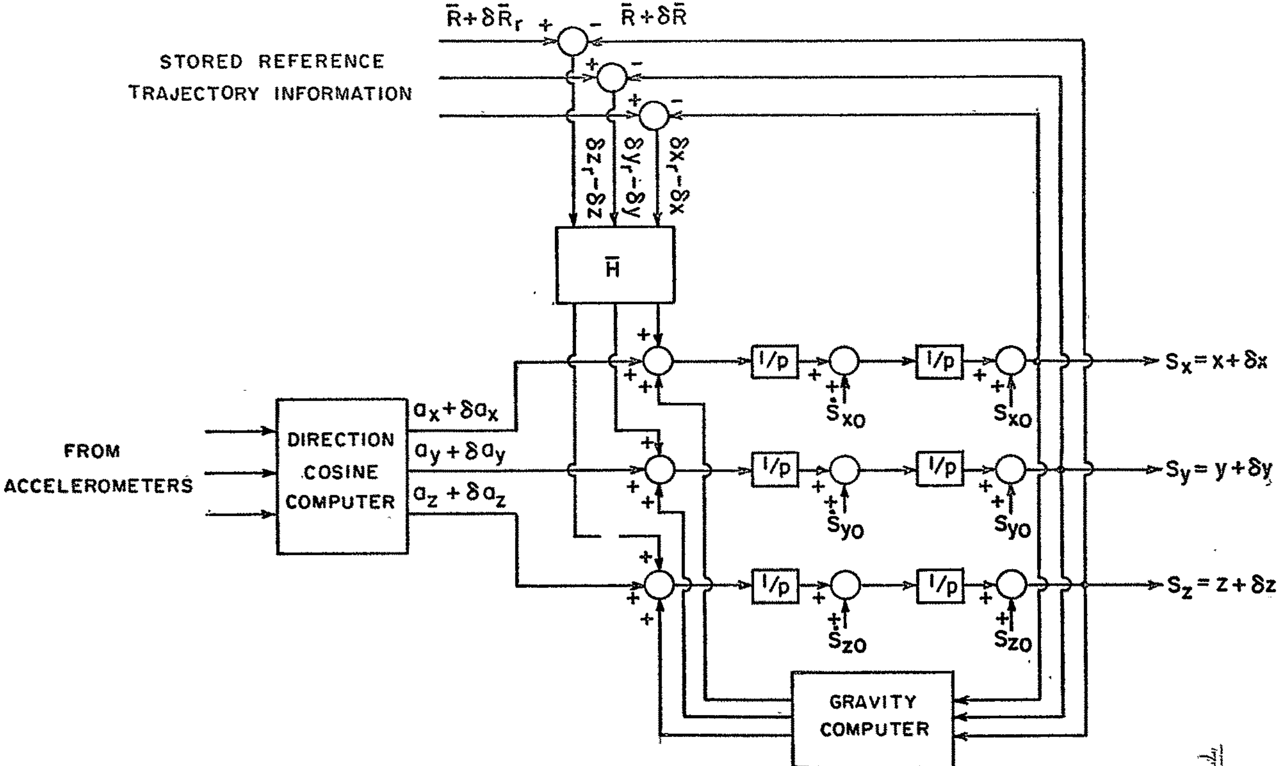

Figure E-3 The Stabilized Navigation System Using Reference Trajectory Information

$\delta\bar{R}$ (i.e. the undamped natural frequency) which we would like to be able to control as much as possible. Since we have no control over $\bar{G}$ we introduce a constant part $\bar{K}$ in $\bar{H}$ so as to obtain the term $(\bar{K}-\bar{G})\delta\bar{R}$ whose coefficient $\bar{K}-\bar{G}$ is adjustable.

With this in mind let us suggest the following form for $\bar{H}$:

$$\bar{H} \quad \begin{bmatrix} (2k_x p + k_x') & k_{xy} & k_{xz} \\ k_{yx} & (2k_y p + k_y') & k_{yz} \\ k_{zx} & k_{zy} & (2k_z p + k_z') \end{bmatrix} \tag{E-27}$$

where all the k's are constants or slowly varying with time.

Recall that $\bar{G}$ is given by Eq. (D-26); introduce both $\bar{G}$ and $\bar{H}$ into Eq. (E-26) take the Laplace transform and expand to obtain:

$$(s^2+2k_x s+k_x'+u_x)\delta x+(k_{xy}-\nu)\delta y+(k_{xz}-\rho)\delta z = \mathcal{L}[\delta a_x+(2k_x p+k_x')\delta x_r+k_{xy}\delta y_r+k_{xz}\delta z_r] \tag{E-28a}$$

$$(k_{yx}-\nu)\delta x+(s^2+2k_y s+k_y'+\mu_x)\delta y+(k_{yz}-\tau)\delta z = \mathcal{L}[\delta a_y+k_{yx}\delta x_r+(2k_y p+k_y')\delta y_r+k_{yz}\delta z_r \tag{E-28b}$$

$$(k_{zx}-\rho)\delta x+(k_{zy}-\tau)\delta y+(s^2+2k_z s+k_z'+\mu_z)\delta z = \mathcal{L}[\delta a_y+k_{zx}\delta x_r+k_{zy}\delta y_r+(2k_z p+k_z')\delta z_r] \tag{E-28c}$$

The characteristic equation of the system (E-28) is:

$$D(s) = \begin{vmatrix} s^2+2k_x s+k_x'+\mu_x & k_{xy} - \nu & k_{xz} - \rho \\ k_{yx} - \nu & s^2+2k_y s+k_y'+\mu_y & k_{yz} - \tau \\ k_{zx} - \rho & k_{zy} - \tau & s^2+2k_z s+k_z' +\mu_z \end{vmatrix} = 0 \tag{E-29}$$

Since ν, ρ, τ are slowly varying terms, it is possible to generate k's such that:

$$k_{xy} = k_{yx} = \nu \tag{E-30a}$$

$$k_{yz} = k_{zy} = \tau \tag{E-30b}$$

$$k_{zx} = k_{xz} = \rho \tag{E-30c}$$

Then the characteristic equation becomes:

$$(s^2+2k_x s+k_x'+\mu_x) \quad (s^2+2k_y s+k_y'+\mu_y) \quad (s^2+2k_z s+k_z'+\mu_z) = 0 \tag{E-31}$$

We like the roots of the characteristic equation to be complex conjugate with negative real parts; hence the following conditions must be satisfied.

$$k_x > 0 \tag{E-32a}$$

$$k_x' + \mu_x > 0 \tag{E-32b}$$

$$k_x^2 - (k_x' + \mu_x) < 0 \tag{E-32c}$$

$$k_y > 0 \tag{E-33a}$$

$$k_y' + \mu_y > 0 \tag{E-33b}$$

$$k_y^2 - (k_y' + \mu_y) < 0 \tag{E-33c}$$

$$k_z > 0 \tag{E-34a}$$

$$k_z' + \mu_z > 0 \tag{E-34b}$$

$$k_z^2 - (k_z' + \mu_z) < 0 \tag{E-34c}$$

To show that these conditions can be satisfied and lead to a desirable situation we recall the condition (D-20); that is:

$$-\frac{2g}{R} \leq \mu_x, \mu_y, \mu_z \leq \frac{g}{R} \tag{D-20}$$

Let us look at the inequalities (E-32), keeping in mind that $\frac{g}{R} = \omega_s^2$ is the undamped frequency of the basic navigation loop without any external information, and that this quantity is very small for an interplanetary flight.

Combining conditions (E-32b) and (E-32c) we obtain

$$0 < k_x^2 < k_x' + \mu_x \tag{E-35}$$

or

$$0 < k_x^2 < k_x' - \frac{2g}{R} \tag{E-36}$$

given that $\min \mu_x = -\frac{2g}{R}$

Clearly the condition (E-36) can be satisfied by a proper choice of k_x'.

Finally the conditions (E-33) and (E-34) can be satisfied in a similar way.

Then the transfer matrix $\overline{H}$ takes the form

$$\overline{H} = \begin{bmatrix} 2k_x p + k_x' & \nu & \rho \\ \nu & 2k_y p + k_y' & \tau \\ \rho & \tau & 2k_y p + k_z' \end{bmatrix} \tag{E-37}$$

where the constant gains satisfy the conditions set above.

From Equations (D-15), (D-16) and (D-17) note that the transfer gains ν, ρ, τ in Eq. (E-37) depend on the true coordinates of the vehicle x, y, z which are not available; instead we generate these gains from the output s_x, s_y, s_z of the system. As a result the generated gains are $\nu+\delta\nu$, $\rho+\delta\rho$, $\tau+\delta\tau$ and when these values are introduced into Eq. (E-38) the terms involving $\delta\nu$, $\delta\rho$, $\delta\tau$ are of higher order and can be dropped from the equations.

Now examine the possibility of avoiding the use of variable gains; a simple choice for $\bar{\bar{H}}$ with this property might be the following:

$$\bar{\bar{H}} = \begin{bmatrix} 2k_x p+k_x' & 0 & 0 \\ 0 & 2k_y p+k_y' & 0 \\ 0 & 0 & 2k_z p+k_z' \end{bmatrix} \qquad \text{(E-38)}$$

Expanding Eq. (E-26) and taking Laplace transforms we obtain:

$$[s^2+2k_x s+k_x'+\mu_x]\delta x - \nu\delta y - \rho\delta z = \mathcal{L}\,[\delta a_x+(2k_x p+k_x')\delta x_r] \qquad \text{(E-39a)}$$

$$-\nu\delta x + [s^2+2k_y s+k_y'+\mu_y]\delta y - \tau\delta z = \mathcal{L}\,[\delta a_y+(2k_y p+k_y')\delta y_r] \qquad \text{(E-39b)}$$

$$-\rho\delta x - \tau\delta y + [s^2+2k_z s+k_z'\mu_z]\delta z = \mathcal{L}\,[\delta a_z+(2k_z p+k_z')\delta z_r] \qquad \text{(E-39c)}$$

The characteristic equation of the system (E-39) is:

$$D(s) = \begin{bmatrix} [s^2+2k_x s+k_x'+\mu_x] & -\nu & -\rho \\ -\nu & [s^2+2k_y s+k_y'+\mu_y] & -\tau \\ -\rho & -\tau & [s^2+2k_z s+k_z'+\mu_z] \end{bmatrix} = 0$$

(E-40)

Now let

$$k_x = k_y = k_z = \zeta\Omega \quad \text{and} \quad k_x' = k_y' = k_z' = \Omega^2 \tag{E-41}$$

and expand Eq. (E-40) as follows:

$$(s^2 + 2\zeta\Omega s + \Omega^2)^3 + (\mu_x + \mu_y + \mu_z)\ (s^2 + 2\zeta\Omega s + \Omega^2)^2$$

$$+ (\mu_x \mu_y + \mu_y \mu_z + \mu_z \mu_x - \nu^2 - \rho^2 - \tau^2)\ (s^2 + 2\zeta\Omega s + \Omega^2)$$

$$+ \mu_x \mu_y \mu_z - \mu_x \tau^2 - \mu_y \rho^2 - \mu_z \nu^2 - 2\nu\rho\tau = 0 \tag{E-42}$$

Recall that the elements of $\overline{G}$ satisfy the following relations:

$$\mu_x + \mu_y + \mu_z = 0 \tag{E-43a}$$

$$\mu_x \mu_y + \mu_y \mu_z + \mu_z \mu_x - \nu^2 - \rho^2 - \tau^2 = -\frac{3g^2}{R^2} \tag{E-43b}$$

$$\mu_x \mu_y \mu_z - \mu_x \tau^2 - \mu_y \rho^2 - \mu_z \nu^2 - 2\nu\rho\tau = -\frac{2g^3}{R^3} \tag{E-43c}$$

where

$$R = (x^2 + y^2 + z^2)^{1/2} \tag{E-43d}$$

$$g = (g_x^2 + g_y^2 + g_z^2)^{1/2} \tag{E-43e}$$

Hence Eq. (E-42) can be rewritten as follows:

$$(s^2 + 2\zeta\Omega s + \Omega^2 + \omega_s^2)^2 (s^2 + 2\zeta\Omega s + \Omega^2 - 2\omega_s^2) = 0 \tag{E-44}$$

where $\omega_s^2 = g/R$

It is desirable that all the roots of Eq. (E-44) have negative real parts for stability reasons and are complex conjugate in order to achieve fast response. It is possible to satisfy both requirements as follows:

Let s_1, s_2 be the roots of

$$s^2 + 2\zeta\Omega s + \Omega^2 + \omega_s^2 = 0$$

and s_3, s_4 be the roots of

$$s^2 + 2\zeta\Omega s + \Omega^2 - 2\omega_s^2 = 0$$

Then

$$s_1, s_2 = -\zeta\Omega \pm [\zeta^2 \Omega^2 - \Omega^2 - \omega_s^2]^{1/2}$$

and

$$s_3, s_4 = -\zeta\Omega \pm [\zeta^2 \Omega^2 - \Omega^2 + 2\omega_s^2]^{1/2}$$

In order to satisfy the requirements on s_1, s_2, s_3 and s_4 we demand that

$$\zeta\Omega > 0 \tag{E-45a}$$

$$\zeta^2\Omega^2 - \Omega^2 - \omega_s^2 < 0 \tag{E-45b}$$

$$\zeta^2\Omega^2 - \Omega^2 + 2\omega_s^2 < 0 \tag{E-45c}$$

Now Eqs. (E-45b) and (E-45c) yield

$$\zeta^2 - 1 < \frac{\omega_s^2}{\Omega^2}$$

and

$$\zeta^2 - 1 < - 2\,\frac{\omega_s^2}{\Omega^2}$$

or combining them

$$- \frac{\omega_s^2}{\Omega^2} < 2\,\frac{\omega_s^2}{\Omega^2} < 1 - \zeta^2 \tag{E-45d}$$

All conditions (E-45) are satisfied if we set

$$\zeta > 0 \tag{E-46a}$$
$$\Omega > 0 \tag{E-46b}$$

and

$$1 - \zeta^2 > 2\,\frac{\omega_s^2}{\Omega^2} \tag{E-46c}$$

A typical value for the damping would be $\zeta = 0.7$; then from (E-46c) we obtain

$$1 - 0.7^2 \;>\; 2\,\frac{\omega_s^2}{\Omega^2}$$

or

$$\Omega \;>\; 2\omega_s$$

Note that the period $T_s = \frac{2\pi}{\omega_s}$ is of the order of several months in the case of interplanetary flight; hence ω_s is small. A large Ω will increase the undamped natural frequencies of the system thus decreasing the period of oscillations considerably. Such an effect is desirable.

From the above discussion it is clear that both the damping ζ and the undamped frequencies can be adjusted with considerable freedom so that a desirable transient response can be achieved.

Looking at the right hand side of Eq. (E-28) we notice that the extra forcing term $\overline{H}\,\delta\overline{R}_r$ was introduced; this results in increased steady-state errors. Given, however, that $\delta\overline{R}_r$ is relatively small the increase in the steady-state error is small.

This method of damping can be used in a time interval during which it is impossible to receive external information for some reason (as, for example, in the case where an observed celestial body is obscured by another). The method has two important features: it damps the navigation system and is completely self-contained. Its accuracy depends on the closeness of the actual and reference trajectories during the time of operation in this mode.

Finally, let us redraw Fig. E-3 for the sake of simplification; this is shown as Fig. E-4. Here, instead of injecting the error signals $2\zeta\Omega s\,[\,\delta\bar{R}_r - \delta\bar{R}\,]$ into the acceleration node of the system, we inject the error signals $2\zeta\Omega\,[\,\delta\bar{R}_r - \delta\bar{R}\,]$ into the velocity node. Obviously this does not change the situation at least as far as the characteristic equation is concerned.

The values of the gains shown in Fig. E-4 are:

$$k_x' = k_y' = k_z' = \Omega^2$$

$$2k_x = 2k_y = 2k_z = 2\zeta\Omega$$

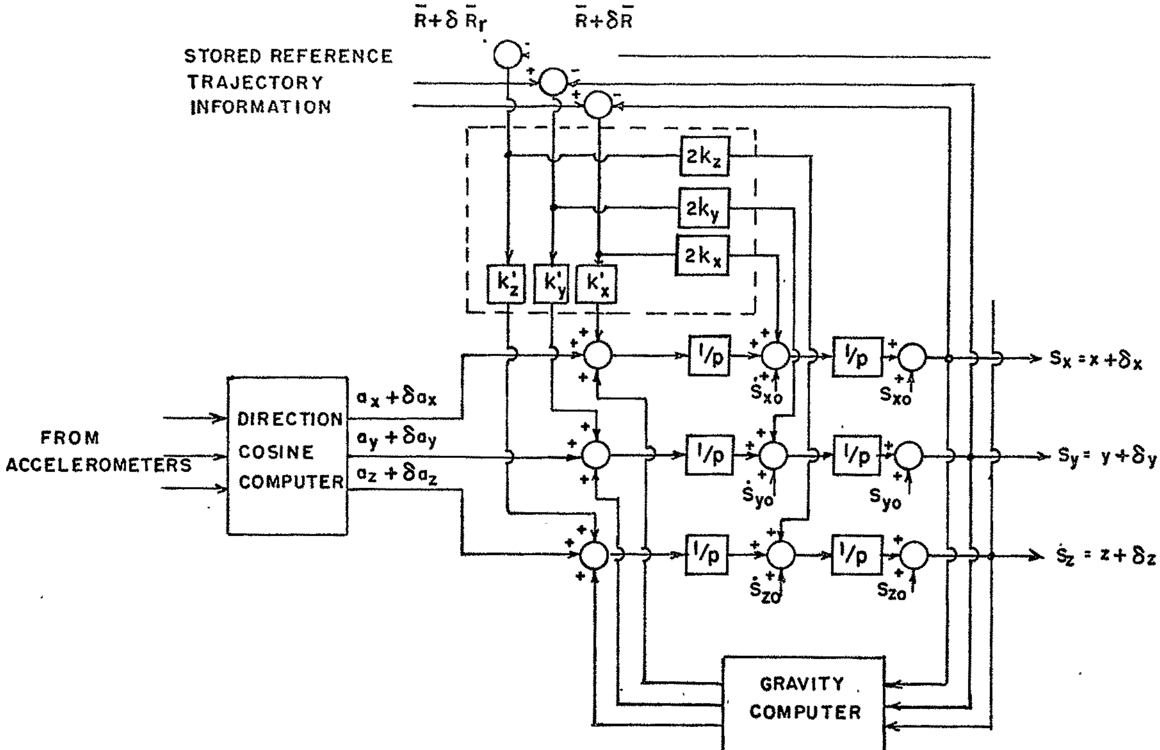

Figure E-4 A Special Case of Reference Trajectory Damping

E.4 DAMPING THE SYSTEM ERRORS BY MEANS OF EXTERNAL VELOCITY INFORMATION

The method of Doppler velocity information for aiding airborne inertial navigation systems is a relatively old one. It has given satisfactory results in the case of terrestial navigation.

In the case of inertial space navigation, the requirements are more severe and radar Doppler is inadequate.
Efforts to cope with the difficulties involved have led to new ideas of which the most important in the area is the optical doppler method. Feasibliity and accuracy studies of optical Doppler(3) led to encouraging results; the test of a breadboard model verified these conclusions.

It is here assumed that such a practical device will be available for use on board the veh·-le. It will measure components of the vehicle velocity in the direction of a fixed star.

Now, define:

$$\overline{v}_d \qquad \begin{bmatrix} v_{dx} \\ v_{dy} \\ v_{dz} \end{bmatrix}$$

the velocity of the vehicle as measured by Doppler devices.

$$\delta\overline{v}_d \qquad \begin{bmatrix} \delta v_{dx} \\ \delta v_{dy} \\ \delta v_{dz} \end{bmatrix}$$

the error in the Doppler velocity measurement

then

$$\bar{v}_d = \dot{\bar{R}} + \delta\bar{v}_d \qquad \begin{bmatrix} \dot{x} + \delta v_{dx} \\ \dot{y} + \delta v_{dy} \\ \dot{z} + \delta v_{dz} \end{bmatrix}$$

We also have

$$\dot{\bar{s}} = \dot{\bar{R}} + \delta\dot{\bar{R}} = \begin{bmatrix} \dot{x} + \delta\dot{x} \\ \dot{y} + \delta\dot{y} \\ \dot{z} + \delta\dot{z} \end{bmatrix} \qquad \text{since } \bar{s} = \bar{R} + \delta\bar{R}$$

and

$$\bar{v}_d - \dot{\bar{s}} = \begin{bmatrix} \delta v_{dx} - \delta\dot{x} \\ \delta v_{dy} - \delta\dot{y} \\ \delta v_{dz} - \delta\dot{z} \end{bmatrix} \qquad \text{(E-47)}$$

The suggested system configuration is shown in Fig. E-5.
The Doppler velocity measurement $\bar{v}_d$ is compared with the velocity $\dot{\bar{s}}$ obtained from the system. The errors $\delta\bar{v}_d - \delta\dot{\bar{R}}$ resulting from this comparison are modified by the compensation matrix $\bar{\bar{H}}$ (as yet specified) and then are fed into the acceleration node of the system.
The elements of the matrix $\bar{\bar{H}}$ will be selected so as to eliminate or reduce the time increasing and oscillating errors in the system.

From Fig. E-5, the error equation of the system is:

$$\delta\ddot{\bar{R}} = \bar{\bar{G}}\,\delta\bar{R} + \delta\bar{a} + \bar{\bar{H}}\,(\delta\bar{v}_d - \delta\dot{\bar{R}}) \qquad \text{(E-48)}$$

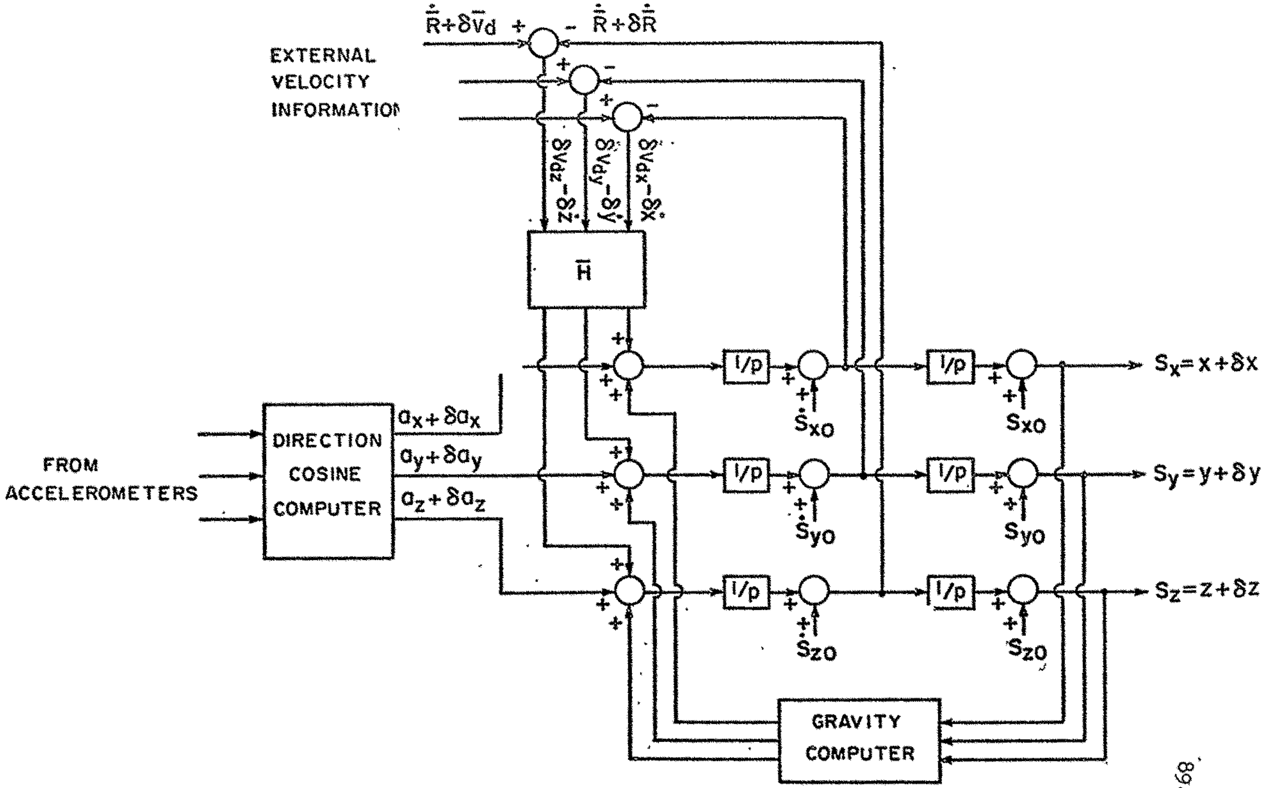

Figure E-5 Velocity Damping

or

$$\delta\ddot{\bar{R}} + \bar{\bar{H}}\,\delta\dot{\bar{R}} - \bar{\bar{G}}\,\delta\bar{R} = \delta\bar{a} + \bar{\bar{H}}\,\delta\bar{v}_d \tag{E-49}$$

compare Eq. (E-49) with Eq. (E-25) of the previous section and note the similarity between them. Let us choose a similar form for $\bar{\bar{H}}$. That is, let

$$\bar{\bar{H}} \quad \begin{bmatrix} (2k_x + \frac{k'_x}{p}) & k_{xy} & k_{xz} \\ k_{yx} & (2k_y + \frac{k'_y}{p}) & k_{yz} \\ k_{zx} & k_{zy} & (2k_z + \frac{k'_z}{p}) \end{bmatrix} \tag{E-50}$$

where all the k's are constants or slowly varying with time.

Introducing both $\bar{\bar{G}}$ and $\bar{\bar{H}}$ into Eq. (E-49) and taking the Laplace transform we obtain.

$$(s^2+2k_x s + k'_x + \mu_x)\delta x + (k_{xy} - \nu)\delta y + (k_{yz} - \rho)\delta z$$
$$= \mathcal{L}\,[\delta a_x + (2k_x + \frac{k'_x}{p})\delta v_{dx} + k_{xy}\delta v_{dy} + k_{xz}\delta v_{dz}$$

(E-51a)

$$(k_{yx} - \nu)\delta x + (s^2 + 2k_y s + k'_y + \mu_y)\delta y + (k_{yz} - \tau)\delta z$$
$$= \mathcal{L}\,[\delta a_y + k_{yx}\delta v_{dx} + (2k_y + \frac{k'_x}{p})\delta v_{dy} + k_{yz}\delta v_{dz}]$$

(E-51b)

$$(k_{zx}-\rho)\delta x + (k_{zy}-\tau)\delta y + (s^2+2k_z s + k_z' + \mu_z)\delta z$$

$$\mathcal{L}[\delta a_z + k_{zx}\delta v_{dx} + k_{zy}\delta v_{dy} + (2k_z + \frac{k_z'}{p})\delta v_{dz}]$$

(E-51c)

The characteristic equation of the system (E-51) is

$$D(s) = \begin{vmatrix} s^2+2k_x s + k_x' + \mu_x & k_{xy} - \nu & k_{xz} - \rho \\ k_{yx} - \nu & s^2+2k_y s + k_y' + \mu_x & k_{yz} - \tau \\ k_{zx} - \rho & k_{zy} - \tau & s^2+2k_z s + k_z' + \mu_z \end{vmatrix} = 0$$

(E-52)

Equation (E-52) is identical to Eq. (E-29). In the discussion of Eq. (E-29) the conditions that k_x, k_y, k_z, k_x', k_y', k_z' must satisfy were indicated and a method of generating k_{xy}, k_{yz}, k_{zx} was suggested.

Then the transfer matrix $\overline{H}$ takes the form:

$$\overline{H} = \begin{bmatrix} 2k_x + \frac{k_x'}{p} & \nu & \rho \\ \nu & 2k_y + \frac{k_y'}{p} & \tau \\ \rho & \tau & 2k_z + \frac{k_z'}{p} \end{bmatrix} \qquad \text{E-53)}$$

The possibility of a solution which does not require variable gains was demonstrated in the previous section. It is possible to apply the same idea here.

Let us choose an $\overline{H}$ of the following form:

$$\overline{H} = \begin{bmatrix} 2k_x + \frac{k'_x}{p} & 0 & 0 \\ 0 & 2k_y + \frac{k'_y}{p} & 0 \\ 0 & 0 & 2k_z + \frac{k'_z}{p} \end{bmatrix} \qquad \text{(E-54)}$$

Introducing this $\overline{H}$ into Eq. (E-49), expand it and taking Laplace transforms we obtain.

$$[s^2 + 2k_x s + k'_x + \mu_x]\delta x - \nu\delta y - \rho\delta z = \mathcal{L}\,[\delta a_x + (2k_x + \frac{k'_x}{p})\delta v_{dx}]$$

(E-55a)

$$-\nu\delta x + [s^2 + 2k_y s + k'_y + \mu_y]\delta y - \tau\delta z = \mathcal{L}\,[\delta a_y + (2k_y + \frac{k'_y}{p})\delta v_{dy}]$$

(E-55b)

$$-\rho\delta x - \tau\delta y + [s^2 + 2k_z s + k'_z + \mu_z]\delta z = \mathcal{L}\,[\delta a_z + (2k_z + \frac{k'_z}{p})\delta v_{dz}]$$

(E-55c)

The characteristic equation of the system (E-55) is:

$$D(s) = \begin{vmatrix} s^2 + 2k_x s + k'_x + \mu_x & -\nu & -\rho \\ -\nu & s^2 + 2k_y s + k'_y + \mu_y & -\tau \\ -\rho & -\tau & s^2 + 2k_z s + k'_z + \mu_z \end{vmatrix} = 0$$

(E-56)

Letting

$$2k_x = 2k_y = 2k_z = 2\zeta\Omega \quad \text{and} \quad k_x' = k_y' = k_z' = \Omega^2 \tag{E-57}$$

we obtain

$$\begin{aligned} &(s^2 + 2\zeta\Omega s + \Omega^2)^3 + (\mu_x + \mu_y + \mu_z)\,(s^2 + 2\zeta\Omega s + \Omega^2)^2 \\ &+(\mu_x \mu_y + \mu_y \mu_z + \mu_z \mu_x - \nu^2 - \rho^2 - \tau^2)\,(s^2 + 2\zeta\Omega s + \Omega^2)^2 \\ &+ \mu_x \mu_y \mu_z - \mu_x \tau^2 - \mu_y \rho^2 - \mu_z \nu^2 - 2\nu\rho\tau = \end{aligned} \tag{E-58}$$

Clearly Eq. (E-58) is identical to Eq. (E-42) of the previous section; hence the stability question concerning the present system has been answered there.

Note, however, that the transfer matrix is different for each case and therefore it operates on $\delta\bar{R}$ and $\delta\bar{v}$, in a different manner; this point may be of some significance depending on the characteristics of $\delta\bar{R}_r$ and $\delta\bar{v}_d$.

E-5 ELIMINATION OF THE DIVERGING SYSTEM ERRORS BY MEANS OF ALTIMETER INFORMATION

The altimeter is a device which can measure the distance of the vehicle from a celestial body.

The barometric altimeter measures the altitude based on atmospheric density measurements.

The radio altimeter operates on the basis of radar principles.

The optical altimeter measures the distance of the vehicle from a planet by measuring the visible angular dimension of the planet.

Both the radio altimeter and the optical altimeter find applications in space navigation. The barometric altimeter can be used for earth bound navigation where the flight takes place inside the atmosphere.

The idea of using an altimeter for stabilizing the navigation system stems from Eq. (D-24) which is:

$$\begin{bmatrix} \delta g_x \\ \delta g_y \\ \delta g_z \end{bmatrix} = \begin{bmatrix} -\frac{g}{R} & 0 & 0 \\ 0 & -\frac{g}{R} & 0 \\ 0 & 0 & -\frac{g}{R} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} + \begin{bmatrix} \mu_x' & \nu & \rho \\ \nu & \mu_y' & \tau \\ \rho & \tau & \mu_z' \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} \tag{D-24}$$

We noted there that the first term in Eq. (D-24) represents the gravity errors due to the position errors included in the numerator $\bar{s}$ of Eq. (D-3) and that the second term of Eq. (D-24) represents the gravity errors due to the position errors included in the denominator of Eq. (D-3).

The altimeter method is based on this observation. The components, s_x, s_y, s_z of $\bar{s}$ are obtained from the output of the system and the altimeter is used to measure the distance of the vehicle from the center of attraction. If the altimeter measurement includes an error δh then the output of the altimeter is $R + \delta h$.

The information $R + \delta h$ and s_x, s_y, s_z is fed into the gravity computer which now simulates the equation

$$\begin{bmatrix} g_{xc} \\ g_{yc} \\ g_{zc} \end{bmatrix} = - \frac{\gamma m}{(R+\delta h)^3} \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} \tag{E-59}$$

as contrasted to the equation

$$\begin{bmatrix} g_{xc} \\ g_{yc} \\ g_{zc} \end{bmatrix} = - \frac{\gamma m}{s^3} \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} \tag{E-60}$$

which is simulated by the gravity computer in all the previous methods of damping. Apparently both computers generate identical gravity components if their inputs are error-free; this, of course, is a basic requirement for all gravity computers.

The input-output error relations corresponding to Eq. (E-60) are Eq. (D-24). In the following we develop input-output relations corresponding to Eq. (E-59). These error relations will be different fro

Eq. (D-24) due to the difference in the denominators in Eqs. (E-59) and (E-60). Of course we anticipate that such a change will affect the stability of the entire system in a favorable manner.

Figure E-6 shows the mechanization of a navigation system which uses altimeter information.

Now Eq. (E-59) can be expanded in terms of the errors δx, δy, δz and δh. Consider for example the first of Eq. (E-59), this is:

$$g_{xc} = -\gamma m \frac{s_x}{(R+\delta h)^3} \qquad \gamma m \frac{x+\delta x}{(R+\delta h)^3} \tag{E-61}$$

Also

$$\frac{1}{(R+\delta h)^3} = (R+\delta h)^{-3} = R^{-3} - 3R^{-4}\delta h + \text{higher order terms in } \delta h \tag{E-62}$$

hence

$$\frac{x+\delta x}{(R+\delta h)^3} = (x+\delta x)(R+\delta h)^{-3} = x R^{-3} + \delta x R^{-3} - 3xR^{-4}\delta h + \text{higher order terms} \tag{E-63}$$

Introducing Eq. (E-63) into (Eq. (E-61) we obtain:

$$g_{xc} = -\gamma m \frac{x}{R^3} - \gamma m \frac{\delta x}{R^3} + \gamma m \frac{3x}{R^4}\delta h + \text{higher order terms} \tag{E-64}$$

From Appendix D we have $g_x = -\gamma m \frac{x}{R^3}$ and $\frac{\gamma m}{R^3} = \frac{g}{R} = \omega_s^2$ (E-65)

This means that Eq. (E-60) can be rewritten as follows:

$$g_{xc} = g_x - \omega_s^2\,\delta x + \omega_s^2\,\frac{3x}{R}\,\delta h + \text{higher order terms} \tag{E-66}$$

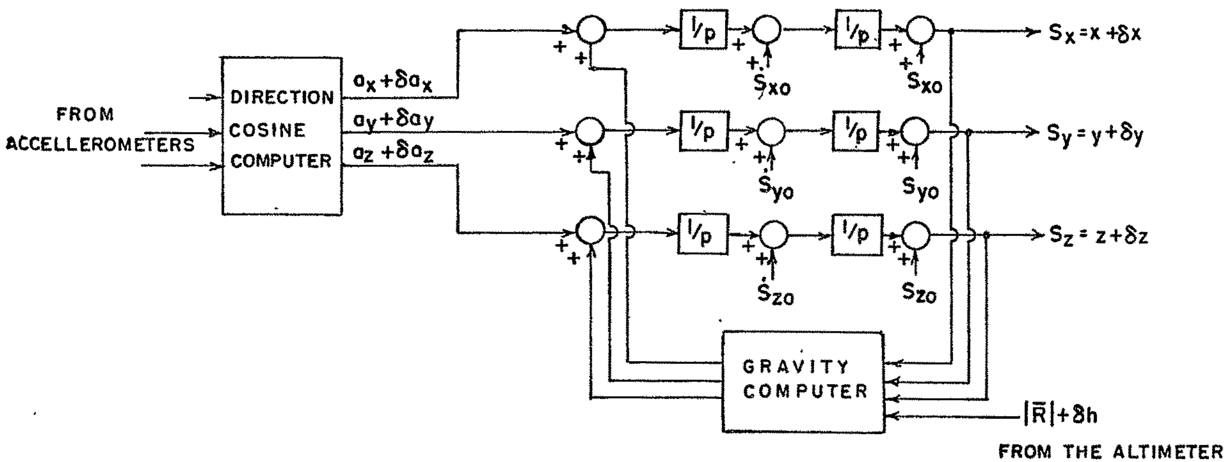

Figure E-6 Altimeter Stabilization

or after dropping higher order terms

$$\delta g_x = g_{xc} - g_x = -\omega_s^2 \delta x + \omega_s^2 \frac{3x}{R} \delta h \tag{E-67}$$

Similarly

$$\delta g_y = g_{yc} - g_y = -\omega_s^2 \delta y + \omega_s^2 \frac{3y}{R} \delta h \tag{E-68}$$

and

$$\delta g_z = g_{zc} - g_z = -\omega_s^2 \delta z + \omega_s^2 \frac{3z}{R} \delta h \tag{E-69}$$

Equations (E-67), (E-68) and (E-69) can be written in matrix form to obtain:

$$\begin{bmatrix} \delta g_x \\ \delta g_y \\ \delta g_z \end{bmatrix} = \begin{bmatrix} -\omega_s^2 & 0 & 0 \\ 0 & -\omega_s^2 & 0 \\ 0 & 0 & -\omega_s^2 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} + 3\omega_s^2 \delta h \begin{bmatrix} \frac{x}{R} \\ \frac{y}{R} \\ \frac{z}{R} \end{bmatrix} \tag{E-70}$$

Equation (E-70) was anticipated from the disucssion of Eq. (D

It is obvious from Eq. (E-70) that there is no coupling in error equations for the gravity computer.

By inspection of Fig. E-6 the error equations of the navigation loop are:

$$\delta \ddot{R} = \delta \bar{a} + \delta \bar{g} \tag{E-71}$$

and introducing (E-70) into (E-71) we obtain:

$$\delta \ddot{x} + \omega_s^2 \delta x = \delta a_x + 3\omega_s^2 \frac{x}{R} \delta h \tag{E-72a}$$

$$\delta \ddot{y} + \omega_s^2 \delta y = \delta a_y + 3\omega_s^2 \frac{y}{R} \delta h \tag{E-72b}$$

$$\delta \ddot{z} + \omega_s^2 \delta z = \delta a_z + 3\omega_s^2 \frac{z}{R} \delta h \tag{E-72c}$$

Assuming that the coefficients of Eq. (E-72) vary slowly we can take Laplace transforms to obtain

$$(s^2 + \omega_s^2)\,\delta x = \mathcal{L}[\delta a_x + 3\omega_s^2 \tfrac{x}{R}\,\delta h] \qquad \text{(E-73a)}$$

$$(s^2 + \omega_s^2)\,\delta y = \mathcal{L}[\delta a_y + 3\omega_s^2 \tfrac{y}{R}\,\delta h] \qquad \text{(E-73b)}$$

$$(s^2 + \omega_s^2)\,\delta z = \mathcal{L}[\delta a_z + 3\omega_s^2 \tfrac{z}{R}\,\delta h] \qquad \text{(E-73c)}$$

The characteristic equation of the system (E-73) is:

$$(s^2 + \omega_s^2) = 0 \qquad \text{(E-74)}$$

Hence the errors vary sinusoidally with bounded amplitude. The error δh of the altimeter measurement acts as a forcing function; its effects on the position errors will be small if it can be made small.

The use of the altimeter did not stabilize the system absolutely; it did, however, eliminate the time increasing errors and this is of considerable value.

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APPENDIX F

DIRECTION COSINE EQUATIONS AND THEIR SIMULATION ON A DIGITAL COMPUTER

Alfred R. Schuler

1. THE DIRECTION COSINE EQUATION

In order to understand the navigation mechanization thoroughly, it is necessary to have a knowledge of the direction cosine equations and how they arise. In the block diagram of Fig. 1, the output quantities are $A_{V_{1c}}$, $A_{V_{2c}}$, $A_{V_{3c}}$, $\Omega_{V_{1c}}$, $\Omega_{V_{2c}}$ and $\Omega_{V_{3c}}$* which are linear accelerations and angular velocities of the vehicle resolved along the vehicular system but measured with respect to the inertial system. The quantities $A_{V_{1c}}$, $A_{V_{2c}}$ and $A_{V_{3c}}$ have included in them the gravity terms. Now in order to determine position with respect to the inertial system, it is necessary to know the orientation of the vehicular system with respect to the inertial system at every instant of time. As discussed by Krishnan[3], it is necessary to relate points in one coordinate system to another rotated arbitrarily with respect to it. Given two sets of axes [I] (with components I_1, I_2 and I_3) and [V] (with components V_1, V_2 and V_3), that are arbitrarily oriented it is possible to specify the components of one in terms of the other. It is done by a series of rotations in a specified order. Then

---

*The c's indicate actual quantities (measured or calculated) at the output of the accelerometer loop transfer function.

$$\begin{pmatrix} \bar{i}_I \\ \bar{j}_I \\ \bar{k}_I \end{pmatrix} = R(\alpha)R(\beta)R(\gamma) \begin{pmatrix} \bar{i}_V \\ \bar{j}_V \\ \bar{k}_V \end{pmatrix} \qquad \text{(F-1)}$$

where $R(\alpha)$, $R(\beta)$ and $R(\gamma)$ are rotation matrices. This equation can be expanded by multiplying the rotation matrices. The rotations specified here are identical to those indicated in Goldstein[1].

$$\begin{pmatrix} \bar{i}_I \\ \bar{j}_I \\ \bar{k}_I \end{pmatrix} = \begin{pmatrix} \cos\psi\cos\phi-\cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi+\cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi-\cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi+\cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \begin{pmatrix} \bar{i}_V \\ \bar{j}_V \\ \bar{k}_V \end{pmatrix}$$

(F-2)

The elements of the matrix product $[D] = R(\alpha)R(\beta)R(\gamma)$ represent the direction cosines of the angles between the three coordinate axes V_-, V_2 and V_3 and the original inertial axes I_1, I_2 and I_3.

$$\begin{pmatrix} \bar{i}_I \\ \bar{j}_I \\ \bar{k}_I \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix} \begin{pmatrix} \bar{i}_V \\ \bar{j}_V \\ \bar{k}_V \end{pmatrix}$$

or $[I] = [D]\,[V]$ (F-3)

Now it is easy to show that

$$R^{-1}(\gamma) = R(\gamma)^t$$

and

$$R^{-1}(\alpha) = R(\alpha)^t \tag{F-4}$$

$$R^{-1}(\beta) = R(\beta)^t$$

That is, the inverse of a rotation matrix is simply equal to its transpose - in other words $R(\alpha)$, $R(\beta)$ and $R(\gamma)$ are orthogonal matrices.

Since [D] is the product of three orthogonal matrices, it too is orthogonal, that is:

$$[D]^{-1} = [D]^t = R(\gamma)^t R(\beta)^t R(\alpha)^t$$

$$\begin{pmatrix} d_{11} & d_{21} & d_{31} \\ d_{12} & d_{22} & d_{32} \\ d_{13} & d_{23} & d_{33} \end{pmatrix} \tag{F-5}$$

Let R_V be a column matrix whose elements are the components of $\bar{R}$ in the orthogonal coordinate frame V. Components of this same vector $\bar{R}$ in another orthogonal but rotated frame I are related to R_V by

$$R_I = D_V^I R_V \tag{F-6}$$

D_V^I is the direction cosine matrix of the V axes referred to the I axes. It will normally be used synonmously with D. Also, D_I^V will represent D^t, the transpose of D.

Differentiating Eq. (F-6) with respect to time yields:

$$\dot{R}_I = D_V^I \dot{R}_V + \dot{D}_V^I R_V$$

$$= D \dot{R}_V + \dot{D} R_V \tag{F-7}$$

In general, both D and $\bar{R}$ are functions of time.

The time rate of change of a vector $\bar{R}$ in the inertial frame is related to the time rate of change in the vehicular frame by the Coriolis operator equation:[1]

$$\left(\frac{d\bar{R}}{dt}\right)_I = \left(\frac{d\bar{R}}{dt}\right)_V + \bar{\omega}_V \times \bar{R}_V \qquad \text{(F-8)}$$

Now define the matrix $[\omega_V]$ to be a nine component symmetric matrix whose elements are the components of angular velocity along the axes of the V frame and with respect to the I frame.

$$[\omega_V] \triangleq \begin{bmatrix} 0 & -\omega_{V_3} & \omega_{V_2} \\ \omega_{V_3} & 0 & -\omega_{V_1} \\ -\omega_{V_2} & \omega_{V_1} & 0 \end{bmatrix} \qquad \text{(F-9)}$$

The Coriolis equation can then be written in matrix form:

$$\dot{R}_I = \dot{R}_V + [\omega_V]R_V \qquad \text{(F-10)}$$

Let the unit vecotrs in the two coordinate systems be

$$\bar{\theta}_I = \bar{i}_I + \bar{j}_I + \bar{k}_I$$

$$\bar{\theta}_V = \bar{i}_V + \bar{j}_V + \bar{k}_V$$

Then θ_I and θ_V are the associated column matrices of their components

Then using Eq. (F-7)

$$\dot{\theta}_I = 0 = D\dot{\theta}_V + \dot{D}\theta_V ,$$

since the unit vectors in the inertial frame are non-rotating constants.

Also using Eq. (F-10)

$$\dot{\theta}_I = 0 = \dot{\theta}_V + [\omega_V]\theta_V$$

Combining these two equations gives

$$D[\omega_V]\theta_V = \dot{D}\theta_V$$

or

$$\dot{D} = D[\omega_V] \qquad \text{(F-11)}$$

Expanding the terms in the matrices gives

$$\begin{bmatrix} \dot{d}_{11} & \dot{d}_{12} & \dot{d}_{13} \\ \dot{d}_{21} & \dot{d}_{22} & \dot{d}_{23} \\ \dot{d}_{31} & \dot{d}_{32} & \dot{d}_{33} \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{bmatrix} 0 & -\omega_{V_3} & \omega_{V_2} \\ \omega_{V_3} & 0 & -\omega_{V_1} \\ -\omega_{V_2} & \omega_{V_1} & 0 \end{bmatrix}$$

Equating components:

$$\left.\begin{aligned} \dot{d}_{11} &= d_{12}\,\omega_{V_3} - d_{13}\,\omega_{V_2} \\ \dot{d}_{12} &= d_{13}\,\omega_{V_1} - d_{11}\,\omega_{V_3} \\ \dot{d}_{13} &= d_{11}\,\omega_{V_2} - d_{12}\,\omega_{V_1} \end{aligned}\right\} \qquad \text{(F-12)}$$

$$\left.\begin{aligned} \dot{d}_{21} &= d_{22}\,\omega_{V_3} - d_{23}\,\omega_{V_2} \\ \dot{d}_{22} &= d_{23}\,\omega_{V_1} - d_{21}\,\omega_{V_3} \\ \dot{d}_{23} &= d_{21}\,\omega_{V_2} - d_{22}\,\omega_{V_1} \end{aligned}\right\} \qquad \text{(F-13)}$$

$$\left.\begin{aligned} \dot{d}_{31} &= d_{32}\,\omega_{V_3} - d_{33}\,\omega_{V_2} \\ \dot{d}_{32} &= d_{33}\,\omega_{V_1} - d_{31}\,\omega_{V_3} \\ \dot{d}_{33} &= d_{31}\,\omega_{V_2} - d_{32}\,\omega_{V_1} \end{aligned}\right\} \qquad \text{(F-14)}$$

These equations can be implemented and [D] can be evaluated as a function of time. In a digital computing system $[\omega_V]$ is not continuously available since the accelerometers are sampled at discrete instants of time which are separated by an interval ΔT. Between samplings the [V] frame will rotate and since the angles are non-commutative, computational errors will arise.

2. SOLUTION OF DIRECTION COSINE EQUATIONS USING DIFFERENCE EQUATIONS

The digital solution is based upon the replacement of the differential equations by difference equations.

For the first set of differential equations (F-12) we have:

$$
\begin{aligned}
d_{11}(i+1) &= [d_{12}(i)\,\omega_{V_3}(i) - d_{13}(i)\,\omega_{V_2}(i)]\,\Delta T + d_{11}(i) \\
d_{12}(i+1) &= [d_{13}(i)\,\omega_{V_1}(i) - d_{11}(i)\,\omega_{V_3}(i)]\,\Delta T + d_{12}(i) \qquad \text{(F-15)} \\
d_{13}(i+1) &= [d_{11}(i)\,\omega_{V_2}(i) - d_{12}(i)\,\omega_{V}\,(i)]\,\Delta T + d_{13}(i)
\end{aligned}
$$

The arguments i and i+1 imply samples at times T_i and T_{i+1} respectively. Also, $T_{i+1} - T_i = \Delta T$.

Given $d_{11}(i)$, $d_{12}(i)$, $d_{13}(i)$, $\omega_{V_1}(i)$, $\omega_{V_2}(i)$ and $\omega_{V_3}(i)$, these equations can be solved for $d_{11}(i+1)$, $d_{12}(i+1)$, and $d_{13}(i+1)$. It is assumed that the inertial package provides the necessary angular information at each sampling interval i.

Rearranging slightly and putting into matrix form gives:

$$\begin{bmatrix} d_{11}(i+1) \\ d_{12}(i+1) \\ d_{13}(i+1) \end{bmatrix} = \begin{bmatrix} 1 & \omega_{V_3}(i)\Delta T & -\omega_{V_2}(i)\Delta T \\ -\omega_{V_3}(i)\Delta T & & \omega_{V_1}(i)\Delta T \\ \omega_{V_2}(i)\Delta T & -\omega_{V_1}(i)\Delta T & 1 \end{bmatrix} \begin{bmatrix} d_{11}(i) \\ d_{12}(i) \\ d_{13}(i) \end{bmatrix} \tag{F-16}$$

or

$$\begin{bmatrix} d_{11}(i+1) \\ d_{12}(i+1) \\ d_{13}(i+1) \end{bmatrix} = [L] \begin{bmatrix} d_{11}(i) \\ d_{12}(i) \\ d_{13}(i) \end{bmatrix} \quad \text{where } L = \begin{bmatrix} 1 & \omega_{V_3}(i)\Delta T & -\omega_{V_2}(i)\Delta T \\ -\omega_{V_3}(i)\Delta T & 1 & \omega_{V_1}(i)\Delta T \\ \omega_{V_2}(i)\Delta T & -\omega_{V_1}(i)\Delta T & 1 \end{bmatrix}$$

The difference equation form for Eq. (F-13) is:

$$\begin{aligned} d_{21}(i+1) &= [d_{22}(i)\,\omega_{V_3}(i) - d_{23}(i)\,\omega_{V_2}(i)]\,\Delta T + d_{21}(i) \\ d_{22}(i+1) &= [d_{23}(i)\,\omega_{V_1}(i) - d_{21}(i)\,\omega_{V_3}(i)]\,\Delta T + d_{22}(i) \\ d_{23}(i+1) &= [d_{21}(i)\,\omega_{V_2}(i) - d_{22}(i)\,\omega_{V_1}(i)]\,\Delta T + d_{23}(i) \end{aligned} \tag{F-17}$$

Put into matrix form, this becomes:

$$\begin{bmatrix} d_{21}(i+1) \\ d_{22}(i+1) \\ d_{23}(i+1) \end{bmatrix} = [L] \begin{bmatrix} d_{21}(i) \\ d_{22}(i) \\ d_{23}(i) \end{bmatrix} \tag{F-18}$$

Likewise, the difference equation for the differential equations (F-14) are

$$d_{31}(i+1) = [d_{32}(i)\,\omega_{V_3}(i) - d_{33}(i)\,\omega_{V_2}(i)]\,\Delta T + d_{31}(i)$$

$$d_{32}(i+1) = [d_{33}(i)\,\omega_{V_1}(i) - d_{31}(i)\,\omega_{V_3}(i)]\,\Delta T + d_{32}(i) \qquad \text{(F-19)}$$

$$d_{33}(i+1) = [d_{31}(i)\,\omega_{V_2}(i) - d_{32}(i)\,\omega_{V_1}(i)]\,\Delta T + d_{33}(i)$$

In matrix form these equations appear as:

$$\begin{bmatrix} d_{31}(i+1) \\ d_{32}(i+1) \\ d_{33}(i+1) \end{bmatrix} \quad [L] \quad \begin{bmatrix} d_{31}(i) \\ d_{32}(i) \\ d_{33}(i) \end{bmatrix} \qquad \text{(F-20)}$$

It is to be noted that [L] is not an orthogonal matrix. This is true even for the case in which rotation is about only one axis. Since the fundamental requirement for a coordinate transformation is that the transformation matrix be orthogonal, we see that Eqs. (F-16), (F-18) and (F-20) will deviate from their desired true values with the passage of time. No mention has yet been made regarding the effect of errors in the measurement of $\bar{\omega}_V$. Errors in this quantity, which will be considered as our study advances, will also contribute to misorientating the coördinate system.

The difference equation approach introduces not only misorientation errors but it also results in a transformed coordinate frame for which the "unit" vectors are neither mutually orthogonal nor of unit length.

In other words, a right handed cartesian coordinate system must satisfy the following mathematical constraints.

$$\bar{i}_I \cdot \bar{i}_I = 1 \qquad \bar{i}_I \cdot \bar{j}_I = 0$$
$$\bar{j}_I \cdot \bar{j}_I = 1 \qquad \bar{i}_I \cdot \bar{k}_I = 0 \qquad \text{(F-21)}$$
$$\bar{k}_I \cdot \bar{k}_I = 1 \qquad \bar{j}_I \cdot \bar{k}_I = 0$$

$$\bar{i}_V \cdot \bar{i}_V = 1 \qquad \bar{i}_V \cdot \bar{j}_V = 0$$
$$\bar{j}_V \cdot \bar{j}_V = 1 \qquad \bar{i}_V \cdot \bar{k}_V = 0 \qquad \text{(F-22)}$$
$$\bar{k}_V \cdot \bar{k}_V = 1 \qquad \bar{j}_V \cdot \bar{k}_V = 0$$

$\bar{i}_I$, $\bar{j}_I$ and $\bar{k}_I$ are the unit vectors in the inertial frame, $\bar{i}_V$, $\bar{j}_V$ and $\bar{k}_V$ are the unit vectors in the vehicular frame. Expanding Eq. (F-21) gives:

$$d_{11}^2 + d_{12}^2 + d_{13}^2 = 1 \qquad d_{11}d_{21} + d_{12}d_{22} + d_{13}d_{23} = 0$$
$$d_{21}^2 + d_{22}^2 + d_{23}^2 = 1 \qquad d_{11}d_{31} + d_{12}d_{32} + d_{13}d_{33} = 0$$
$$d_{31}^2 + d_{32}^2 + d_{33}^2 = 1 \qquad d_{21}d_{31} + d_{22}d_{32} + d_{23}d_{33} = 0$$

(F-23)

Expanding Eqs. (F-22) yields:

$$
\begin{aligned}
d_{11}^2 + d_{21}^2 + d_{31}^2 &= 1 \qquad & d_{11}d_{12} + d_{21}d_{22} + d_{31}d_{32} &= 0 \\
d_{12}^2 + d_{22}^2 + d_{32}^2 &= 1 \qquad & d_{11}d_{13} + d_{21}\, d_{23} + d_{31}d_{33} &= 0 \\
d_{13}^2 + d_{23}^2 + d_{33}^2 &= 1 \qquad & d_{12}d_{13} + d_{22}d_{23} + d_{32}d_{33}. &= 0
\end{aligned}
\qquad \text{(F-24)}
$$

All of the twelve equations (F-23 and F-24) must be satisfied by any real orthogonal transformation matrix.

A usual procedure for evaluation of the direction cosines is to compute six of them using, for example, Eqs. (F-16) and (F-18). Using Eq. (F-16), the three components of $\bar{i}_I$ are computed in terms of $\bar{i}_V$, $\bar{j}_V$ and $\bar{k}_V$. Likewise, using Eq. (F-18), the three components of $\bar{j}_I$ are computed in terms of the vehicular frame unit vectors. Then the following three equations, taken from Eq. (F-24) are necessary and sufficient to insure that $\bar{i}_I$ and $\bar{j}_I$ are othrogonal and each of unit length:

$$
\begin{aligned}
d_{11}^2 + d_{21}^2 + d_{31}^2 &= 1 \\
d_{12}^2 + d_{22}^2 + d_{32}^2 &= 1 \\
d_{11}d_{12} + d_{21}d_{22} + d_{31}d_{32} &= 0
\end{aligned}
\qquad \text{(F-25)}
$$

The computer must evaluate the left hand sides of Eq. (F-25). If the results do not coincide with the desired values, the first equation is normalized to unity.

An algorithm is then provided for satisfying the second two equations utilizing the values of d_{11}, d_{21} and d_{31} just computed. The three components of $\bar{k}_I$ are then evaluated through the relationship

$$\bar{k}_I = \bar{i}_I \times \bar{j}_I$$

$$= (d_{11}\,\bar{i}_V + d_{12}\,\bar{j}_V + d_{13}\,\bar{k}_V) \times (d_{21}\,\bar{i}_V + d_{22}\,\bar{j}_V + \bar{d}_{23}\,k_V)$$

$$= (d_{12}d_{23} - d_{22}d_{13})\,\bar{i}_V + (d_{13}d_{21} - d_{23}d_{11})\,\bar{j}_V$$

$$+ (d_{11}d_{22} - d_{21}d_{12})\,\bar{k}_V \qquad \text{(F-26)}$$

3. SOLUTION OF DIRECTION COSINE EQUATIONS BASED UPON A TAYLOR SERIES EXPANSION

Kosmola[2] has suggested a solution in the form of a Taylor series. Supposing the elements of ω_V to be continuous functions of time, the direction cosine matrix D is expanded into a Taylor series about T_i:

$$D(T_i + \Delta T) = D(T_i) + \dot{D}(T_i)\,\Delta T + \ddot{D}(T_i)\,\frac{(\Delta T)^2}{2!} + \dddot{D}(T_i)\,\frac{(\Delta T)^3}{3!} + \ldots$$

(F-27)

From Eq. (F-11)

$$\dot{D} = D[\omega_V]$$

$$\ddot{D} = \dot{D}[\omega_V] + D[\dot{\omega}_V] = D[\omega_V]^2 + D[\dot{\omega}_V] = D([\omega_V]^2 + [\dot{\omega}_V])$$

$$\dddot{D} = D([\omega_V]^3 + 3[\omega_V]\,[\dot{\omega}_V] + [\ddot{\omega}_V]) \tag{F-28}$$

etc.

Thus, we can write Eq. (F-28) as:

$$D(T_i + \Delta T) = D(T_i) \left(1 + \Delta T\,[\omega_V(T_i)] + \frac{(\Delta T)^2}{2!} \{[\omega_V(T_i)]^2 + [\dot{\omega}_V(T_i)]\} + \frac{(\Delta T)^3}{3!} \{[\omega_V(T_i)]^3 + 3\,[\omega_V(T_i)]\,[\dot{\omega}_V(T_i)] + [\omega_V(T_i)]\} + \ldots\ldots \right) \tag{F-29}$$

Thus, the new transformation matrix $D(T_i + \Delta T) = D(T_{i+1})$ can be derived from $D(T_i)$ by an infinite series of matrix operations.

We now make a fundamental assumption that makes the problem of finding an exact direction cosine matrix solvable. The assumption is that ω_V remains constant during each sampling interval ΔT. Under these conditions all derivatives of ω_V become zero and Eq. (F-29) can be written:

$$D(T_i + \Delta T) = D(T_i) \{ 1 + [\omega_V(T_i)]\,\Delta T + [\omega_V(T_i)]^2 \frac{(\Delta T)^2}{2!} + [\omega_V(T_i)]^3 \frac{(\Delta T)^3}{3!} + [\omega_V(T_i)]^4 \frac{(\Delta T)^4}{4!} + \ldots \}$$

$$= D(T_i)\, e^{[\omega_V(T_i)]\Delta T} \tag{F-30}$$

Let

$$[\omega_V(T_i)]\Delta T = [\Theta_V(T_i)] = \int_{T_i}^{T_i + \Delta T} [\omega_V(T_i)]\, dt \qquad \text{(F-31)}$$

and rewrite Eq. (F-16) as

$$D(T_i + \Delta T) = D(T_i)\; e^{[\Theta_V(T_i)]} \qquad \text{(F-32)}$$

where $[\Theta_V]$ is a skew symmetric matrix since $[\omega_V]$ is a skew symmetric matrix.

$$[\Theta_V] = \begin{bmatrix} 0 & -\Theta_{V_3} & \Theta_{V_2} \\ \Theta_{V_3} & 0 & -\Theta_{V_1} \\ -\Theta_{V_2} & \Theta_{V_1} & 0 \end{bmatrix} \qquad \text{(F-33)}$$

Let $\quad \ell^2 = \Theta_{V_1}^2 + \Theta_{V_2}^2 + \Theta_{V_3}^2 \qquad \text{(F-34)}$

and note the following two properties of the skew symmetric matrix $[\Theta_V]$:

$$[\Theta_V]^{2m+2} = (-1)^m\, [\Theta_V]^2\, \ell^{2m} \qquad \text{(F-35)}$$

$$[\Theta_V]^{2m+1} = (-1)^m\, [\Theta_V]\, \ell^{2m} \qquad m = 1, 2, 3 \ldots \qquad \text{(F-36)}$$

Using these two recurrence relations in Eq. (F-32) yield:

$$
\begin{aligned}
D[T_i + \Delta T] = D(T_i) \{ & 1 + [\Theta_V(T_i)] + [\Theta_V(T_i)]^2 \; \frac{}{2!} \\
& - [\Theta_V(T_i)] \; \ell^2 \; \frac{1}{3!} - [\Theta_V(T_i)]^2 \; \frac{\ell^2}{4!} + [\Theta_V(T_i)] \; \frac{\ell^4}{5!} \\
& + [\Theta_V(T_i)]^2 \; \frac{\ell^4}{6!} - [\Theta_V(T_i)] \; \frac{\ell^6}{7!} - [\Theta_V(T_i)]^2 \; \frac{\ell^6}{8!} \\
& + \ldots + \} \\
= \; & D(T_i) \\
& + D(T_i) \; [\Theta_V(T_i)] \; \{ 1 - \frac{\ell^2}{3!} + \frac{\ell^4}{5!} - \frac{\ell^6}{7!} + \cdots \} \\
& + D(T_i) \; [\Theta_V(T_i)]^2 \; \{ \frac{1}{2!} - \frac{\ell^2}{4!} + \frac{\ell^4}{6!} - \frac{\ell^6}{8!} + \cdots \} \\
= \; & D(T_i) \\
& + \frac{D(T_i) \; [\Theta_V(T_i)]}{\ell} \; \{ \ell - \frac{\ell^3}{3!} + \frac{\ell^5}{5!} - \frac{\ell^7}{7!} + \cdots \} \\
& - \frac{D(T_i) \; [\Theta_V(T_i)]^2}{\ell^2} \; \{ 1 - \frac{\ell^2}{2!} + \frac{\ell^4}{4!} - \frac{\ell^6}{6!} + \frac{\ell^8}{8!} \; \cdots \} \\
& + \frac{D(T_i)[\Theta_V(T_i)]^2}{\ell^2} \\
= \; & D(T_i) \; [1 + \frac{[\Theta_V(T_i)]}{\ell} \sin \ell + \frac{[\Theta_V(T_i)]^2}{\ell^2} \{ 1 - \cos \ell \}]
\end{aligned}
\qquad (F\text{-}37)
$$

Now

$$[\theta_V(T_i)] \quad \begin{bmatrix} 0 & -\omega_{V_3}\Delta T & \omega_{V_2}\Delta T \\ \omega_{V_3}\Delta T & 0 & -\omega_{V_1}\Delta T \\ -\omega_{V_2}\Delta T & \omega_{V_1}\Delta T & 0 \end{bmatrix} \qquad \text{(F-38)}$$

$$[\theta_V(T_i)]^2 = \begin{bmatrix} -\omega_{V_3}^2(\Delta T)^2 - \omega_{V_2}^2(\Delta T)^2 & \omega_{V_1}\omega_{V_2}(\Delta T)^2 & \omega_{V_1}\omega_{V_3}(\Delta T)^2 \\ +\omega_{V_1}\omega_{V_2}(\Delta T)^2 & -\omega_{V_3}^2(\Delta T)^2 - \omega_{V_1}^2(\Delta T)^2 & \omega_{V_2}\omega_{V_3}(\Delta T) \\ \omega_{V_1}\omega_{V_3}(\Delta T)^2 & \omega_{V_2}\omega_{V_3}(\Delta T)^2 & -\omega_{V_2}^2(\Delta T)^2 - \omega_{V_1}^2(\Delta T)^2 \end{bmatrix}$$

(F-39)

where it is understood that the ω's are evaluated at time $t = T_i$

Also from Eq. (F-34)

$$\ell = (\omega_{V_1}^2 + \omega_{V_2}^2 + \omega_{V_3}^2)^{1/2} \Delta T \qquad \text{(F-40)}$$

Expanding Eq. (F-37) in terms of its components yields:

$$\begin{bmatrix} d_{11}(T_{i+1}) & d_{12}(T_{i+1}) & d_{13}(T_{i+1}) \\ d_{21}(T_{i+1}) & d_{22}(T_{i+1}) & d_{23}(T_{i+1}) \\ d_{31}(T_{i+1}) & d_{32}(T_{i+1}) & d_{33}(T_{i+1}) \end{bmatrix} =$$

$$\begin{bmatrix} d_{11}(T_i) & d_{12}(T_i) & d_{13}(T_i) \\ d_{21}(T_i) & d_{22}(T_i) & d_{23}(T_i) \\ d_{31}(T_i) & d_{32}(T_i) & d_{33}(T_i) \end{bmatrix} \times \begin{bmatrix} a & -d+e & f+g \\ d+e & b & -h+i \\ -f+g & h+i & c \end{bmatrix} \tag{F-41}$$

where

$$a = 1 - x_1 M \qquad f = \omega_{V_2} N$$

$$b = 1 - x_2 M \qquad g = \omega_b M$$

$$c = 1 - x_3 M \qquad h = \omega_{V_1} N$$

$$d = \omega_{V_3} N \qquad i = \omega_c M$$

$$e = \omega_a M$$

$$x_3 = \omega_{V_1}^2 + \omega_{V_2}^2 \qquad \omega_a = \omega_{V_1} \omega_{V_2}$$

$$x_2 = \omega_{V_1}^2 + \omega_{V_3}^2 \qquad \omega_b = \omega_{V_1} \omega_{V_3}$$

$$x_1 = \omega_{V_2}^2 + \omega_{V_3}^2 \qquad \omega_c = \omega_{V_2} \omega_{V_3}$$

$$M = (1 - \cos \ell) \frac{(\Delta T)^2}{\ell} = \frac{(\Delta T)^2 (1 - \cos \ell)}{(\omega_{V_1}^2 + \omega_{V_2}^2 + \omega_{V_3}^2)(\Delta T)^2} = \frac{(1 - \cos \ell)}{\omega_V^2}$$

$$N = \frac{\Delta T \sin \ell}{\ell} = \frac{\sin \ell}{|\omega_V|}$$

Equation (F-41) can be solved on a computer to give the exact direction cosine matrix at the future time, T_{i+1}. Effort is presently being directed towards a) an error analysis for the case in which $\bar{\omega}_V$ is not known precisely but has a specified probability density function and b) an extension of the above results for the case of non zero angular acceleration between samples.

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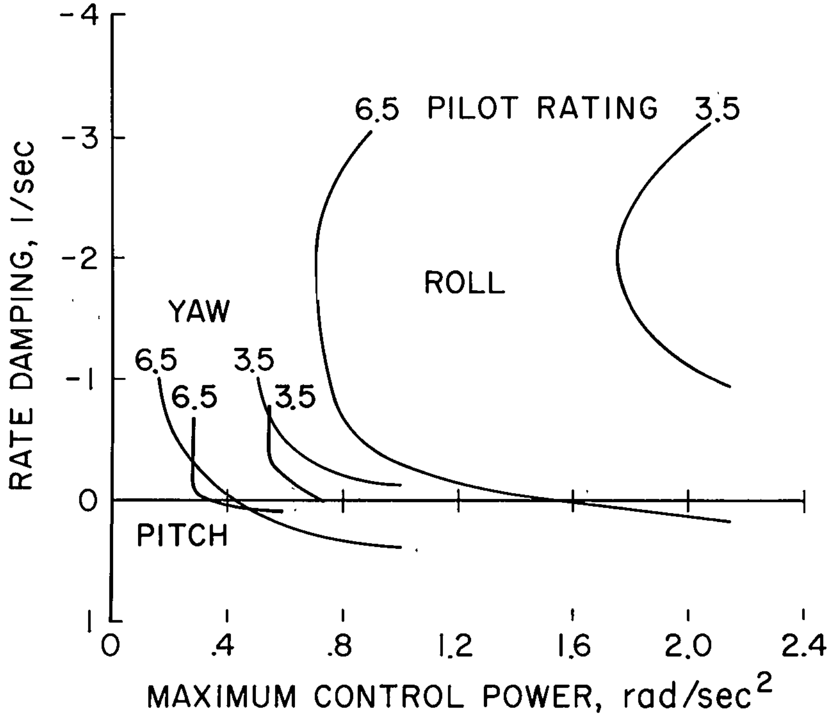
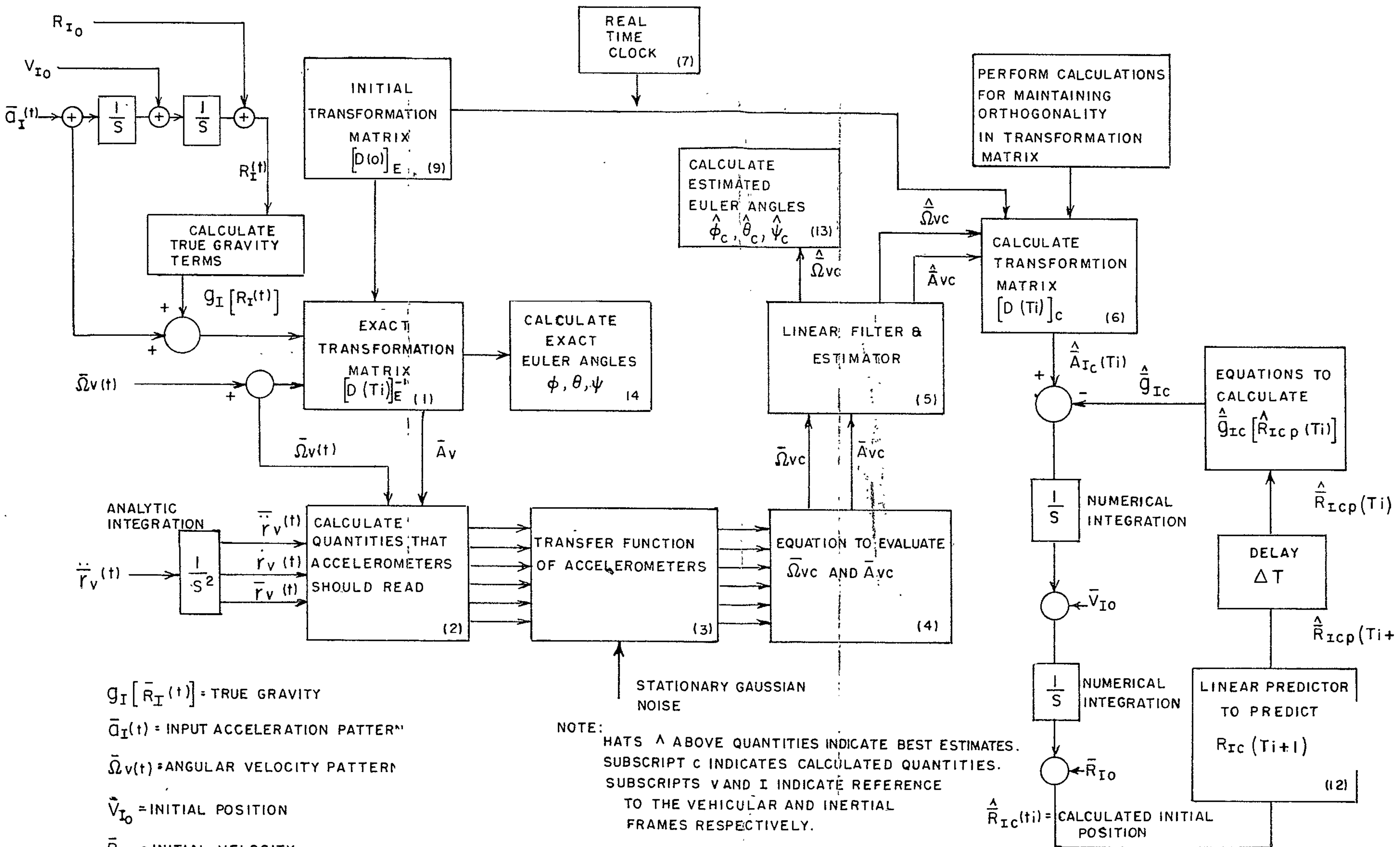
CONTROL BOUNDARIES





$g_I[\bar{R}_I(t)]$ = TRUE GRAVITY

$\bar{a}_I(t)$ = INPUT ACCELERATION PATTERN

$\bar{\Omega}v(t)$ = ANGULAR VELOCITY PATTERN

$\bar{V}_{I_0}$ = INITIAL POSITION

$\bar{R}_{I_0}$ = INITIAL VELOCITY

$\ddot{\bar{r}}_v(t), \dot{\bar{r}}_v(t), \bar{r}_v(t)$ = ACCELERATION, VELOCITY AND POSITION RESPECTIVELY OF MOVING (m) REFERENCE FRAME WITH RESPECT TO THE (v) FRAME.

NOTE:
HATS ∧ ABOVE QUANTITIES INDICATE BEST ESTIMATES.
SUBSCRIPT c INDICATES CALCULATED QUANTITIES.
SUBSCRIPTS V AND I INDICATE REFERENCE TO THE VEHICULAR AND INERTIAL FRAMES RESPECTIVELY.

FIGURE I BLOCK DIAGRAM OF SIMULATION

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