Technical Report No. 32-650 (Part II)

The Rotating Superconductor Part II: The Free Energy

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ABSTRACT

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In this Report, we derive the free energies appropriate for stationary and rotating superconductors with and without an external applied field. A general theorem is deduced that determines the fluxoids of a multiply connected superconductor in its equilibrium state. Also shown is that an isolated stationary superconductor rotates when made superconducting in an external field. This is an effect that has not been noticed previously.

While we derive free energies from the point of view of London theory, which is to treat the superelectrons as having a uniform density, we show that even from the point of view of the Ginzburg–Landau theory, in which this restriction is not imposed, our conclusions are essentially unaltered. In the derivation of the free energies, an expression for the magnetic enthalpy is required. This expression is obtained in a novel way through use of the concept of a "magnetic reservoir."

Author

I. INTRODUCTION

In Part I, we studied the magnetic field present in the bore of a hollow superconducting cylinder when the cylinder rotates in a uniform applied field. The field in the bore was shown to depend on both the fluxoid associated with the bore and the angular velocity of the cylinder.

The fluxoid is determined at the moment the cylinder becomes superconducting. The value of the fluxoid was deduced from a particular model of the superconducting transition. In the model, the transition was assumed to occur first at nucleation sites from which it then spread throughout the cylinder (Ref. 1). It was then shown that, according to this model, the fluxoid must assume a value that allows the field in the bore to be equal to the applied field.

We stated that this value of the fluxoid was also the value for which the free energy of the cylinder is a minimum and that this free energy is essentially the energy of the magnetic field generated by currents in the cylinder. In this Part, these statements are proved, and expressions for various free energies of the superconductor are presented. The free energies are those appropriate to stationary and rotating superconductors with and without an applied external field.

Through use of these expressions, a general theorem is deduced that determines the fluxoids of a multiply connected superconductor in its equilibrium state: *Consider any cut through the superconductor that lowers its connectivity by unity* (such a cut always either links two holes into one or links a hole with the space outside the superconductor); *in equilibrium, the net current through any and all such cuts is zero.*

Also shown is that an isolated stationary superconductor rotates when made superconducting in an external field. This is an effect that has not been noticed previously.

In what follows, we proceed by introducing, in succession, the free energy of the stationary superconductor in zero applied field, its free enegry in an applied field, its free energy when rotating in zero applied field, and finally, its free energy when rotating in an applied field.

While we derive these energies from the point of view of London theory, which is to treat the superelectrons as having a uniform density, we show in Appendix B that even from the point of view of the Ginzburg–Landau theory (Ref. 2), in which this restriction is not imposed, our conclusions are essentially unaltered.

In the derivation of the free energies, an expression for the magnetic enthalpy is required. This expression is obtained in Appendix A in a novel way through use of the concept of a "magnetic reservoir."

II. THE ENERGY OF A STATIONARY SUPERCONDUCTOR IN A ZERO APPLIED FIELD

The total energy of a superconductor in the absence of an external field can be written as

$$E = U_0 + \int_{sup} \left\{ \frac{1}{2} \rho_0 m \mathbf{v}_s^2 + \frac{\mathbf{B}^2}{8\pi} d\tau + \int_{holes+ext} \left\{ \frac{\mathbf{B}^2}{8\pi} \right\} d\tau \qquad (1)$$

The term U_0 is the energy of the superconductor with no supercurrent present, m is the mass of the electron, ρ_0 is the superelectron number density, $\mathbf{v}_s$ is the velocity field of the superelectrons, and $\mathbf{B}$ is the magnetic field (*sup* stands for superconductor and *ext* for exterior). The velocity field $\mathbf{v}_s$ can be eliminated in favor of the magnetic field through the Maxwell equation

$$\nabla \times \mathbf{B} = \frac{4\pi e \rho_0}{c} \mathbf{v}_s \qquad (2)$$

so that Eq. (1) depends on the magnetic field alone:

$$E = U_0 + \int_{sup} \left\{ \lambda^2 \frac{(\nabla \times \mathbf{B})^2}{8\pi} + \frac{\mathbf{B}^2}{8\pi} \right\} d\tau + \int_{holes+ext} \frac{\mathbf{B}^2}{8\pi} d\tau$$

$$\lambda^2 = \frac{mc^2}{4\pi e \rho_0} \tag{3}$$

This functional, when varied with respect to the field **B** within the superconductor, yields as extrema the solutions of

$$\lambda^2 \nabla \times \nabla \times \mathbf{B} + \mathbf{B} = 0 \tag{4}$$

in the superconductor. To fix these solutions, we constrain them to join continuously at the surface to solutions of

$$\nabla \times \mathbf{B} = 0 \tag{5}$$

in the holes and in the exterior. In this way London's equation, Eq. (4), is obtained quite simply as the equation satisfied by the field for which the energy of the superconductor is a minimum (compare Ref. 3). That this is true can, in fact, be thought of as the justification for regarding Eq. (1) as the energy of the superconductor.

If now **B**, or equivalently $\mathbf{v}_s$, depend on other parameters of the superconductor, then the equilibrium values of these parameters are those that make the energy a minimum—or, in other words, in equilibrium, the first variation of the energy with respect to these parameters must vanish and the second variation must be positive. In this way, the parameters that determine the equilibrium solution become completely specified.

A. *The Fluxoid*

We now show how the energy depends on one such parameter—the fluxoid. In showing this dependence, we also demonstrate that a fluxoid exists only for a multiply connected superconductor.

If we perform the variation on Eq. (1), we obtain

$$\delta E = \int_{sup} \left\{ m\rho_0 \mathbf{v}_s \cdot \delta \mathbf{v}_s + \frac{1}{4\pi} \mathbf{B} \cdot \delta \mathbf{B} \right\} d\tau + \int_{holes} \frac{\delta \mathbf{B} \cdot \mathbf{B}}{4\pi} d\tau$$
$$+ \int_{ext} \frac{\delta \mathbf{B} \cdot \mathbf{B}}{4\pi} d\tau + \delta U_0 \tag{6}$$

Now $\mathbf{B} = \nabla \times \mathbf{A}$, so that

$$\delta E = \int_{sup} \rho_0 \mathbf{v}_s \cdot \delta \left(m\mathbf{v}_s + \frac{e}{c}\mathbf{A} \right) d\tau + \oint_{\infty} \hat{\mathbf{n}} \cdot \delta \mathbf{A} \times \mathbf{B} d\sigma + \delta U_0 \tag{7}$$

and as there is no applied field, the last integral vanishes. Since Eq. (2) and Eq. (4) together require that $\nabla \times (m\mathbf{v}_s + e/c\mathbf{A}) = 0$, we see that $m\mathbf{v}_s + e/c\mathbf{A}$ is equal to a gradient which we write as $e/c \nabla \chi$; moreover, $\nabla^2 \chi = 0$ in the superconductor, and we fix the gauge by requiring that the normal derivative of χ, $\partial \chi / \partial n$, vanish on the surface. The existence of the fluxoid is connected with the existence of χ.

We first show that χ vanishes in a simply connected superconductor, so that the fluxoid of such a superconductor also vanishes. Now,

$$\int_{sup} (\nabla \chi)^2 d\tau = \int_{sup} [\nabla \cdot [\chi \nabla \chi] - \nabla^2 \chi] d\tau$$
$$= \int_{sup} \nabla \cdot [\chi \nabla \chi] d\tau$$

In a simply connected superconductor, this can be written as

$$\oint_{ext} \chi \frac{\partial \chi}{\partial n} d\sigma$$

which, of course, vanishes, showing that $\nabla \chi = 0$ in the superconductor and, as a consequence, that

$$\mathbf{v}_s = -\frac{e}{mc}\mathbf{A}$$

Then, for a simply connected superconductor in zero applied field $\delta E = \delta U_0$, and so, in equilibrium the superconductor is devoid of currents and of field.

B. *The Dependence of the Energy on the Fluxoid*

Returning now to the arbitrarily connected superconductor, we show that a fluxoid exists and show at the same time, how it enters into the expression for the energy. We can write δE (if we drop δU_0) entirely in terms of surface integrals:

$$\delta E = \frac{1}{c} \int_{sup} e\rho_0 \mathbf{v}_s \cdot \nabla \delta \chi d\tau$$
$$= \frac{1}{4\pi} \int_{sup} \nabla \cdot (\mathbf{B} \times \nabla \delta \chi) d\tau \tag{8}$$
$$= \frac{1}{4\pi} \oint_{sup} \hat{\mathbf{n}} \cdot (\mathbf{B} \times \nabla \delta \chi) d\sigma$$

A more interesting expression for δE is obtained if we write $\mathbf{v}_s \cdot \nabla \delta \chi$ as $\nabla \cdot (\mathbf{v}_s \delta \chi)$. However, as is well known from hydrodynamics (see Ref. 3 and also Refs. 4, 5, and 6), we must now be careful in applying Gauss's theorem to the integral, as $\delta\chi$ may not be single-valued in a non-simply connected domain.* Allowing this, we must render the multiply connected domain simply connected by means of cuts; once this is done, we can then apply Gauss's theorem in the usual way. The integral (8) now becomes

$$\frac{1}{c}\oint_{\substack{ext\ surf\\+hole\ surf}} e\rho_0 \mathbf{v}_s \cdot \hat{\mathbf{n}} \delta \chi d\sigma + \frac{1}{c}\oint_{+cuts} e\rho_0 \mathbf{v}_s \cdot \mathbf{n} \delta \chi d\sigma + \frac{1}{c}\oint_{-cuts} e\rho_0 \mathbf{v}_s \cdot \mathbf{n} \delta \chi d\sigma \qquad (9)$$

the + and − denoting the two different sides of a cut. Since no suppercurrent leaves the superconductor, the first surface integral vanishes, and we are left with

$$\frac{1}{c}\oint_{+cuts} e\rho_0 \mathbf{v}_s \cdot \hat{\mathbf{n}} (\delta \chi_+ - \delta \chi_-)\, d\sigma \qquad (10)$$

*This means that $\nabla \delta \chi$ is singular in the "hole." The simplest example is the function $\tan^{-1}(y/x)$.

As is easily shown (Ref. 4), $\delta\chi_+ - \delta\chi_- = \delta\Phi$ is a constant over the cut, so that

$$\delta E = \frac{1}{c}\sum_{cuts} (J_s)_{cut}\, \delta\Phi_{cut} \qquad (11)$$

where J_s is the supercurrent through the cut. The "period" Φ is, of course, the fluxoid* (Ref. 4), and it can be obtained as

$$\oint \frac{e}{c} \nabla \chi \cdot \hat{\lambda} d\lambda = \oint \left(m\mathbf{v}_s + \frac{e}{c}\mathbf{A} \right) \cdot \hat{\lambda} d\lambda = \frac{e}{c}\Phi$$

where the contour is about a hole.

In equilibrium, then, the fluxoids Φ, whose existence has just been demonstrated, must be such that J_s vanishes through each cut. But this requires that $\mathbf{v}_s$ and $\mathbf{B}$ vanish in the superconductor as we now show. In the same way as Eq. (11) was derived from (7), from Eq. (1) we can derive

$$E = \frac{1}{2c}\sum (J_s)_{cut}\, \Phi_{cut}$$

where the external field is assumed to vanish and we have dropped U_0. Thus, in equilibrium, since $(J_s)_{cut} = 0$, $E = 0$, so that $\mathbf{v}_s^2 = 0$ and $\mathbf{B}^2 = 0$ (Bloch's theorem).

*As shown by London, the real importance of the fluxoid lies in the fact that the fluxoid, once it is fixed in a superconductor, remains fixed so long as the superconductor remains superconducting.

III. THE STATIONARY SUPERCONDUCTOR IN AN EXTERNAL FIELD

We now study the equilibrium of a stationary superconductor in a uniform applied field. In such a field, the appropriate thermodynamic potential for a system is no longer its internal energy but rather its free energy in the field; this energy is also called the magnetic enthalpy. The variation of the enthalpy can be obtained by adding to variations of the internal energy the term

$$-\frac{1}{4\pi}\oint_{ext\ surf} (\delta\mathbf{A} \times \mathbf{B}) \cdot \hat{\mathbf{n}} d\sigma \qquad (12)$$

(see Appendix A). The value of the external field, while not appearing explicitly, enters through the boundary condition to be imposed on the solution of Maxwell's equations at infinity.

The variation of the enthalpy* becomes

$$\delta\epsilon = \int_{sup} \left[m\rho_0 \mathbf{v}_s \cdot \delta\mathbf{v}_s + \frac{\nabla \times \mathbf{B}}{4\pi} \cdot \delta\mathbf{A} \right] d\tau$$

$$= \int_{sup} \rho_0 \mathbf{v}_s \cdot \delta\left(m\mathbf{v}_s + \frac{e}{c}\mathbf{A} \right) d\tau$$

and so,

$$\delta\epsilon = \frac{1}{c} \sum_{cuts} (J_s)_{cut}\, \delta\Phi_{cut} \qquad (13)$$

Again we see that equilibrium demands that the fluxoids be such as to make the current through each cut vanish. However, this does not imply that the fluxoids vanish as it did when no external field was present.

*The enthalpy can be expressed in terms of surface integrals just as the internal energy was expressed in terms of surface integrals. In fact, the expression is the same, the reason being that the surface integral at infinity can be ignored since it is constant in any variation.

IV. THE ROTATING SUPERCONDUCTOR

We now study the equilibrium of a rotating superconductor in zero applied field; but first, we discuss the internal energy of this system. The energy of a rotating superconductor is easily obtained from that of a stationary superconductor merely by adding to it the kinetic energy of the lattice.*,**

The expressions for the energy of the rotating and stationary superconductor appear very similar except when $\mathbf{v}_s$ is written in terms of $\mathbf{B}$, for then

$$\mathbf{v}_s = \mathbf{v}_l + \frac{c}{4\pi\rho_0 e} \nabla \times \mathbf{B} \qquad (14)$$

where $\mathbf{v}_l$ is the velocity field of the lattice. This difference shows up in the London equation. Variation of the energy with respect to $\mathbf{B}$ now yields a London equation of the form

$$\lambda^2 \nabla \times \nabla \times \mathbf{B} + \mathbf{B} + \frac{mc}{e} \nabla \times \mathbf{v}_l = 0^\dagger \qquad (15)$$

However, the energy we have just discussed is not the thermodynamic potential relevant for systems that

*We regard the lattice as rigid and use the term not only for the actual lattice of the superconductor but for the lattice plus any rigid body that is rigidly attached to it.

**If we wish, we can again express the energy in terms of surface integrals. The expression is (8) except for the addition of

$$(KE)_{lattice} + \frac{1}{2} m\rho_0 e \int_{sup} \mathbf{v}_l^2\, d\tau$$

$$\frac{mc}{8\pi e} \oint_\infty \hat{\mathbf{n}} \cdot [(\boldsymbol{\omega} \times \mathbf{v}_l) \times \mathbf{B} + 2\boldsymbol{\omega} \times \mathbf{A}]\, d\sigma$$

Here $(KE)_{lattice}$ denotes the kinetic energy of the lattice, which is rotating at the angular velocity $\boldsymbol{\omega}$.

†In writing (15), we have assumed that time-independent solutions of the Maxwell equations exist for the rotating superconductor. This is certainly true as long as we assume the frequency of rotation ω to be low enough to disregard the presence of normal electrons; that is to say, we require $\omega \ll c^2/(4\pi\sigma\lambda^2)$ (where σ is the normal conductivity).

rotate at a constant angular velocity. Processes that take place at constant angular velocity are much the same as processes that take place at constant pressure or at constant field. The appropriate potential for these processes is not the energy but the corresponding enthalpy. Similarly, for processes that take place at a constant angular velocity, the appropriate potential is again an enthalpy.

The essential difference between an enthalpy and an energy is that an enthalpy includes work done on the reservoir, the reservoir being any system that serves to keep the appropriate parameter (pressure or field) fixed. A reservoir that serves to keep the angular velocity fixed is a large flywheel on which the system is rigidly mounted.

We now apply these ideas to the rotating superconductor. If the lattice is deformable, we must regard the entire superconductor as our system rather than just the superelectrons alone, and so we have to imagine the superconductor to be rigidly fastened to a flywheel. However, if the lattice is assumed rigid, we can ignore the distinction between flywheel and lattice and lump them together calling their combination "the lattice." Consequently, we regard the superelectrons alone as our system.

We now derive what amounts to the rotational enthalpy for the system of the superelectrons. The energy of the total system—superelectrons, magnetic field, and lattice—is written as

$$E + \frac{\mathbf{L}_l^2}{2I_l} \tag{16}$$

The energy of the electrons and the energy of the field are represented by the first term; the second term is the kinetic energy of the lattice written in terms of its angular momentum $\mathbf{L}_l$ and its moment of inertia I_l. Variation of (16) yields

$$\delta E + \frac{\mathbf{L}_l \cdot \delta \mathbf{L}_l}{I_l} \tag{17}$$

Evidently, this variation of the energy vanishes in equilibrium as must the variation of the energy of any isolated system in equilibrium. But if the system is isolated, its total angular momentum is constant, so that in (17), we can replace $\delta\mathbf{L}_l$ by $-\delta\mathbf{L}_{sup}$, which is the negative of the angular momentum change of the superelectrons. The angular momentum of the superelectrons can be written as

$$\int_{sup} m\rho_0 \mathbf{r} \times \mathbf{v}_s d\tau \tag{18}$$

Since we can write $\mathbf{L}_l/I_l$ as $\boldsymbol{\omega}$, the fixed angular velocity, expression (17), becomes

$$\delta E - \int m\rho_0 \mathbf{v}_l \cdot \delta \mathbf{v}_s d\tau \tag{19}$$

where we have replaced $\boldsymbol{\omega} \times \mathbf{r}$ by $\mathbf{v}_l$. The variation of the rotational enthalpy is thus

$$\int_{sup} m\rho_0 (\mathbf{v}_s - \mathbf{v}_l) \cdot \delta \mathbf{v}_s d\tau + \delta \int_{\substack{all \\ space}} \frac{\mathbf{B}^2}{8\pi} d\tau^* \tag{20}$$

If we perform the now familiar transformations, we obtain

$$\int_{sup} m\rho_0 (\mathbf{v}_s - \mathbf{v}_l) \cdot \frac{e}{c} \nabla \delta \chi d\tau = \frac{1}{c} \sum_{cuts} J_{cut}\, \delta\Phi_{cut} \tag{21}$$

for the variation of the enthalpy, and we find that, in equilibrium, the fluxoids of a rotating superconductor are such as to cause the *total* current through any cut to be zero.

*The enthalpy can be reduced to surface integrals. The enthalpy can be written as (8) plus

$$(KE)_{lattice} + \frac{1}{2} m\rho_0 e \int_{sup} \mathbf{v}_l^2\, d\tau + \frac{mc}{8\pi e} \oint_{\infty} \hat{\mathbf{n}} \cdot \{(\boldsymbol{\omega} \times \mathbf{r}) \times \mathbf{B} + 2\boldsymbol{\omega} \times \mathbf{A}\} d\sigma$$

V. THE SUPERCONDUCTOR ROTATING UNIFORMLY IN A UNIFORM APPLIED FIELD

To obtain the thermodynamic potential for a superconductor rotating at a uniform angular velocity in a uniform field, we need only add the term $1/4\pi \oint_{ext} \hat{\mathbf{n}} \cdot \mathbf{B} \times \delta\mathbf{A}\, d\sigma$ to the variation of the rotational enthalpy (see Appendix A). The result is again

$$\oint m\rho_0 (\mathbf{v}_s - \mathbf{v}_l) \cdot \frac{e}{c}\, \delta\nabla\chi\, d\sigma = \frac{1}{c} \sum J_{cut}\, \delta\Phi_{cut} \tag{22}$$

and once more we see that equilibrium requires that the net current through any cut vanish.

VI. THE ISOLATED SUPERCONDUCTOR IN AN EXTERNAL FIELD

For an isolated superconductor, the total angular momentum—that of superelectrons plus that of the lattice—must vanish. Considerations similar to those above indicate once more that for the superconductor to be in equilibrium, the fluxoids must be such that the total current through any cut vanishes. This condition, together with the condition that the total angular momentum vanish, determines both the fluxoid and the angular velocity of the lattice. If the superconductor is simply connected, the fluxoid must vanish, and there is no condition on the current. The angular velocity is determined only by the condition that the total angular momentum vanish.

In either case, it is plain to see that an isolated superconductor placed in a uniform field should rotate once it is made superconducting. This effect would be extremely difficult to demonstrate experimentally.

VII. DISCUSSION

The calculations of the enthalpies given above is somewhat idealized. For example, we have assumed the lattice to be rigid. Consequently, we have neglected contributions to the energy of terms that couple the strain field to other fields that are present—the magnetic, supercurrent, and centrifugal fields.* Nevertheless, this assumption is convenient since it allows us to convert any one of the enthalpies obtained above to the corresponding Gibbs free energy by merely adding to the enthalpy the free energy that the stationary superconductor has in the absence of an applied field.

In deriving the free energy in this way, however, we also have neglected the effect of magnetic field and supercurrent on the superconducting condensation energy. The spatial variation of field and supercurrent serve to make this energy also a quantity with spatial variation, so that it is not enough to take the condensation energy into account merely by including it as it appears in the free energy of the isolated stationary superconductor. However, if we limit ourselves to fields and currents that never exceed critical values, we need not consider transition to the normal state, so that it never becomes crucial to consider either the condensation energy or its dependence on the field and supercurrent as they explicitly appear in the free energy. That is not to say that these variations of the condensation energy do not contribute to the free energy at all, but rather that their effect—modification of the critical temperature—is irrelevant as long as the superconductor stays superconducting (see Appendix B).

*Nevertheless, investigation of the coupling of the strain field to the fluxoid may lead to interesting results. In fact, application of Le Chatelier's principle would suggest that there is an effect. Because supercurrents push against the surface of a superconductor, we might expect that when a superconductor (non-simply connected, of course) is squeezed and then cooled, supercurrents would be generated to oppose the squeeze.

Disregarding possible effects of the strain field, we can characterize our main result by saying that in equilibrium, the fluxoid is such as to minimize magnetic pressure difference between a hole in the superconductor and the exterior of the superconductor. For an infinitely long hollow cylinder, the theorem becomes quite simple: in equilibrium, the field in the bore is the same as the field outside the cylinder. This holds regardless of the radial dimensions of the cylinder, but we must assume that the current density never exceeds its critical value; otherwise, the cylinder would go normal and the theorem would not apply.

It should be noted that while the magnetic pressure drop across a superconducting wall tends to vanish, the pressure drop from an external surface to the interior of the superconductor is, of course, quite large. This pressure drop is maintained by the flow of supercurrent next to the external surfaces in a "skin" with a thickness of the order of a penetration depth. The currents, however, are maintained at the expense of their kinetic energy. Nevertheless, the theorem proves what the elementary calculations for the example of the hollow cylinder show: namely, that for configurations in which no current is allowed to flow about a hole, the resulting decrease in the internal energy of the superconductor is more than made up for by the work that must be done on the magnetic reservoir to keep the field in the hole different from the field outside the superconductor; therefore, these configurations are not equilibrium configurations.

APPENDIX A

The Magnetic Enthalpy

In this Appendix, we derive anew the thermodynamic potential that is appropriate for a system subject to a uniform applied magnetic field; for such a system, it is this potential—the magnetic enthalpy—that is a minimum when the system is in equilibrium. Our derivation of this potential, we feel, has some advantages over the usual derivations, not the least of which is a better "physical feel" for the quantity usually described as "the work done on the field."

In the usual derivations, a current flowing in a long solenoid serves as the source of the uniform field. An immediate objection arises in using such a source because the flow of current constitutes an irreversible process, while the potential we wish to derive is to apply to reversible processes only. To meet this objection, however, we need only imagine the resistance in the wire to be vanishingly small.

A stronger objection to the use of a solenoid as source, however, is that it puts the derivation of the magnetic enthalpy on a different footing from the derivation of other enthalpies. To show this difference most clearly, it is convenient first to review the orthodox derivation of enthalpies. The usual procedure in deriving an enthalpy is to couple the system of interest (*SI*) to another system called a reservoir. This reservoir serves to keep the quantity of interest, call it *P*—which may be the pressure, the applied electric field, the angular velocity (see above), or as it is here, the applied magnetic field—constant. This reservoir is defined as any system characterized by the property that any variation of the state of the system results in a change in *P* that is second order in the variation. The total system, made up of the system *SI* plus the reservoir, is considered to be isolated; consequently, the equilibrium states of the total system may be determined by the thermodynamic principle of minimum energy. This principle states that for a state of an isolated system to be an equilibrium state, the energy of the system as a function of the state of the system must be a minimum. The energy of the total system is the sum of the energies of *SI* and of the reservoir. For the total energy to be a minimum, as we require, its variation with respect to the variables of *SI* must vanish. While the energy of the reservoir does not explicitly depend on the variables of *SI*, it is made to depend on these variables through constraints. Any change of the energy of the reservoir is the work done on it in a virtual process by the system *SI*. This change of energy then comes to depend on changes of the state of the system *SI*, through imposed constraints that couple the system to the reservoir. The variation of the energy of the total system is then expressible solely in variations of the variables of the system *SI* alone. This variation of the total energy so expressed constitutes the variation of the enthalpy, and so ultimately defines the enthalpy itself.

Now the important point is that all the variations are virtual. That is to say that they are arbitrary variations of mathematical variables not connected by any equation of motion, variations that need not even be in accord with physical laws. However, in the derivations of the magnetic enthalpy, as they are usually given, it is essential that the time dependence of the variation of flux through the system must be taken into account. Such a variation is not a virtual one. This makes it appear as if somehow the general procedure for obtaining enthalpies just outlined must be modified for a magnetic system. Yet, in the usual derivations, the time dependence of the flux variation is vital. The time dependence of the flux is responsible for the back emf in the solenoid, and consideration of this back emf is certainly essential, since it is responsible for the "work done on the field." It is the work that must be done against the back emf to keep the current in the solenoid constant. This work, however, is not supplied by the system of interest but is work to be supplied by an additional system required to keep the current constant. In the usual derivations, the need for such a system is ignored. In fact, this system is the "magnetic reservoir;" but we see that to use a solenoid carrying a constant current as the source of the field—aside from the other difficulties—ultimately leads to a rather elaborate system.

Here, we bypass all the difficulties associated with making a solenoid the source of the applied field by introducing a simple magnetic field reservoir. The reservoir is the flux contained in a hollow, infinitely long, perfectly conducting cylinder having a large bore. (If we wish, we may even think of the cylinder as being a superconductor.) Present in the bore is *SI* and some amount of trapped magnetic flux. In the absence of *SI*, the field in the bore is, of course, uniform. When *SI* is present, any changes in it may cause the lines of force to distort, but no change can change the total number of lines trapped in the bore. This, then, is the constraint that couples the field to our *SI*.

If the bore is large enough, any bending of the lines still leaves the field essentially uniform and unchanged through most of the bore. Moreover, for such a large bore it is also true that any changes in the field lead to changes in the kinetic energy of current in the walls of the cylinder that are negligible compared to the changes in energy of the field itself, so that the currents in the walls can be neglected in our considerations.*

We now calculate the variation of the total energy. The energy of the system composed of *SI* and the field in the bore can be written as

$$U_{SI}(\mathbf{B}_\infty) + \int_{\substack{\text{bore minus}\\ \text{syst}}} \frac{\mathbf{B}^2}{8\pi}\, d\tau \tag{A-1}$$

Here, $U_{SI}(\mathbf{B}_\infty)$ is the energy of the system in a field that is uniform at ∞, the field there having the value $\mathbf{B}_\infty$. Any variation of the total energy can be written as

$$\delta U_{SI}(\mathbf{B}_\infty) + \int \frac{\mathbf{B}\cdot\delta\mathbf{B}}{4\pi}\, d\tau \tag{A-2}$$

The integral can be transformed into two surface integrals, one at the outer surface of the system, the other at the inner surface of the cylinder. We have, in fact, if $\mathbf{B} = \nabla \times \mathbf{A}$,

$$\oint \mathbf{B}\cdot\delta\mathbf{B}\, d\tau = -\oint_{\text{syst}\rightarrow\text{bore}} \hat{\mathbf{n}}\cdot\mathbf{B}\times\delta\mathbf{A}\, d\sigma + \oint_{\text{bore}\rightarrow\text{cyl}} \hat{\mathbf{n}}\cdot\mathbf{B}\times\delta\mathbf{A}\, d\sigma \tag{A-3}$$

As we now show, the last integral must vanish because of the constraint that the flux in the bore be constant. At the wall, $\mathbf{B}$ is uniform and so may be taken out of the integral; the integral then becomes proportional to the change of flux throughout the bore, which, of course, must be zero.

Thus, we find that the variation of the magnetic enthalpy is given by

$$\delta U_{SI}(\mathbf{B}_\infty) - \frac{1}{4\pi}\oint_{\text{syst}} \hat{\mathbf{n}}\cdot(\mathbf{B}\times\delta\mathbf{A})\, d\sigma \tag{A-4}$$

where the entropy of the system is to be held constant, as is $\mathbf{B}_\infty$, the field at the wall (the applied field). The vector potential $\mathbf{A}$ at the surface is to be expressed in terms of the variables of *SI* alone.

The significance of the surface integral is clear; it is the work done on the field by the system. This is the work that comes about through the compression of the flux lines in the bore when flux is made to leave the system *SI*. When flux moves from the bore into *SI*, the work is counted as negative.

It is to be noticed that unless *SI* is an infinite cylinder, $\mathbf{B}$ at the surface of the system *SI* is not equal to $\mathbf{B}_\infty$. In general, then, $\mathbf{B}$ must be calculated from the magnetostatic Maxwell equation, together with the constitutive equations of *SI*.

*For a superconductor, the kinetic energy of these currents is roughly the same as the energy of the field that has penetrated into the wall. For a cylinder with a large bore, either of these energies is negligible compared to the energy of the field in the bore.

APPENDIX B

The Fluxoid in the Ginzburg–Landau Theory

We choose to write the Ginzburg–Landau (Ref. 2) (G-L) expression for the free energy of a superconductor as

$$E = \int \left\{ \frac{1}{2} m\rho \mathbf{v}_s^2 + \frac{\mathbf{B}^2}{8\pi} + F(\rho) \right\} d\tau \qquad \text{(B-1)}$$

By writing the energy in this form, we have done away with the effective wave function Ψ introduced by G-L to describe the electron superfluid. In the form (B-1), the G-L energy is seen to be quite similar to the London energy (Eq. 1) differing from it only by the term $F(\rho)$. As we show, this difference leaves unchanged the condition ($J_{cut} = 0$) satisfied by current (J_{cut}) passing through a cut in a multiply connected superconductor in thermodynamic equilibrium.

To examine the consequences of Eq. (B-1), very much the same procedure is used as was used to examine the consequences of Eq. (1). We again use Eq. (2), but with ρ_0 now equal to ρ, and we obtain as Euler equations

$$\nabla \times \left[\frac{\nabla \times \mathbf{B}}{\rho} \right] + \frac{4\pi e^2}{mc^2} \mathbf{B} = 0 \qquad \text{(B-2)}$$

and

$$\frac{\delta F(\rho)}{\delta \rho} + \left(\frac{mc}{4\pi e} \right)^2 \frac{(\nabla \times \mathbf{B})^2}{m\rho^2} = 0 \qquad \text{(B-3)}$$

The first of these is again London's first equation

$$\nabla \times \mathbf{v}_s + \frac{e}{mc} \mathbf{B} = 0 \qquad \text{(B-4)}$$

so that we can again write $m\mathbf{v}_s + e/c\mathbf{A} = e/c\nabla\chi$ and are again led to the fluxoid of London theory,

$$\oint_{hole} (\nabla\chi \cdot \hat{\boldsymbol{\lambda}} d\lambda) = \Phi \qquad \text{(B-5)}$$

The real difference between London theory and G-L theory is the second equation (B-3). This equation allows (under conditions we need not go into here) the superelectron density to vanish throughout regions of the superconductor (Ref. 7); these regions are called vortices. Though such a hole is not a hole in the superconductor, it is nevertheless a hole in the superelectron fluid itself, and since either hole constitutes an absence of superfluid, either hole raises the connectivity of the superfluid, so that either hole will have a fluxoid* associated with it.

Since London's first equation is still true, we can still write

$$\delta E = \frac{1}{c} \sum_{cuts} J_{cut}\, \delta\Phi_{cut}$$

where E is the appropriate enthalpy. The only change from London theory is that now in making cuts, the vortices count as holes.

*Since a vortex, when introduced into a simply connected superconductor, changes the fluxoid from a zero to a non-zero value, we may well ask what happens to the law of conservation of the fluxoid. This apparent contradiction is resolved as soon as we remember that conservation of the fluxoid holds only if no part of the superconductor goes normal; in the formation of a vortex however, the portion of the superfluid destined to become the core of the vortex does go normal, and thus the full theorem is not violated. We should mention that the vortices are most likely to be formed at the surface of the superconductor, where the current density is largest. Once the vortices are formed there, the magnetic pressure gradient that exists tends to push them into the interior.

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*This reference was inadvertently omitted from Part I of Technical Report No. 32-650. It should have been cited there as Ref. 2 in place of the second Hildebrandt reference.

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