NASA TECHNICAL NOTE

NASA TN D-2869

NASA TN D-2869

FACILITY FORM 602

N65-28248
(ACCESSION NUMBER)

(THRU)

37
(PAGES)

1
(CODE)

(NASA CR OR TMX OR AD NUMBER)

21
(CATEGORY)

GPO PRICE $ ______

CFSTI ~~OTS~~ PRICE(S) $ 2.00

Hard copy (HC) ______

Microfiche (MF) .50

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by Helmut J. Horn, Daniel T. Martin, and Doris C. Chandler

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Huntsville, Ala.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JULY 1965

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DEFINITION OF SYMBOLS

Symbol	Definition
t	time
T	time to target
τ	time to complete consumption
χ	thrust angle
F	thrust magnitude
R	value to be extremized
V_{ex}	exhaust velocity
V	velocity
$a, b, c,$ d, K_1, K_2	coefficients of the steering equation
$m, \dot{m}$	mass and flow rate

SUBSCRIPTS

1: V_1	initial or instantaneous
T: V_T	total or final
i: V_i	inertial

SUPERSCRIPTS

~: $\tilde{\chi}$	preliminary value for restricted case
-: $\bar{\chi}$	average

DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
F^*	thrust level producing required distance in remaining flight time
$\bar{\chi}$	an effective average of χ over remaining flight time
P	orbital period
t_{ig}	time of ignition out of orbit
h_a	altitude of aposelenum
h_p	altitude of periselenum

Symbol	Definition
ϕ	central angle to point of origin
ϕ^*	average gravity direction
g	local gravity magnitude
g^*	average gravity magnitude
x-y	space-fixed coordinate system with origin at launch site
ξ, η	rotated system through ϕ_T with origin at body center
$\Delta\dot{\xi}, \Delta\dot{\eta}$	inertial velocity-to-be gained in ξ-η system
χ_ξ	thrust angle related to ξ-η system
χ	thrust angle related to x-y system
δT	time-to-go correction
T'	estimated time-to-go
$\Delta\dot{\xi}', \Delta\dot{\eta}'$	inertial velocity-to-be gained in ξ-η system in time T'
$\Delta V_i'$	total inertial velocity-to-be gained in time T'
ξ-η	rotating system (frozen on each evaluation) with origin at the beginning of the terminal braking phase, the ξ-axis passing through the instantaneous vehicle position (M)
$D, \dot{D}$	slant range and slant range rate
$\theta, \dot{\theta}$	slant range angle with horizontal at lunar landing site, slant range angular rate
θp	angle formed by ξ-axis and local horizontal at terminal braking point
K	proportionality constant of thrust and range
ξ^*	predicted distance traversed during remaining flight time with current thrust magnitude

AN ITERATIVE GUIDANCE SCHEME AND ITS APPLICATION TO LUNAR LANDING

SUMMARY

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A guidance scheme for vehicle flight from lunar orbit to a prescribed point on a spherical, non-rotating moon is presented. The equations of motion have been simplified only to permit a closed solution for the thrust magnitude and thrust direction. The trajectory computations themselves are made under more realistic and accurate assumptions and are not included.

This scheme is another approach to the problem of adaptive guidance mechanization for vacuum flight. The more salient features of the scheme are the limited number of presettings, the closeness to optimization, the closed form solution, and the "homing" feature; i.e., the guidance information is improved as the end point is approached.

The effectiveness of the scheme and the required thrust variations are displayed in Tables I and II. Two flight profiles have been considered: continuous burn of one RL-10 engine for main braking from a 100 km lunar orbit to the beginning of the terminal braking phase at 300 meters altitude; ignition of three RL-10 engines at the 15 km periselenum of a Hohmann transfer ellipse from a 100 n. mi. circular orbit with burn to 300 meters altitude. A description of a timer ignition scheme is given in the appendix.

Author

SECTION I. INTRODUCTION

It has been shown by Fried [1] and Lawden [2] that the optimum thrust direction in a vacuum over a flat surface with a constant gravitational field, constant thrust, and constant specific impulse is

$$\tan \chi = \frac{\partial R/\partial \dot{y}_T + (T - t)\ \partial R/\partial y_T}{\partial R/\partial \dot{x}_T + (T - t)\ \partial R/\partial x_T} = \frac{a + bt}{c + dt} \tag{1}$$

where a, b, c, and d are constants depending on the boundary conditions. For example, if the velocity vector alone is constrained, the direction of this vector can be achieved by

$$\tan \chi = \frac{\partial R/\partial \dot{y}_T}{\partial R/\partial \dot{x}_T} = a' \tag{2}$$

and the magnitude by the burning time t. This relationship can also be seem geometrically in Figure 1.

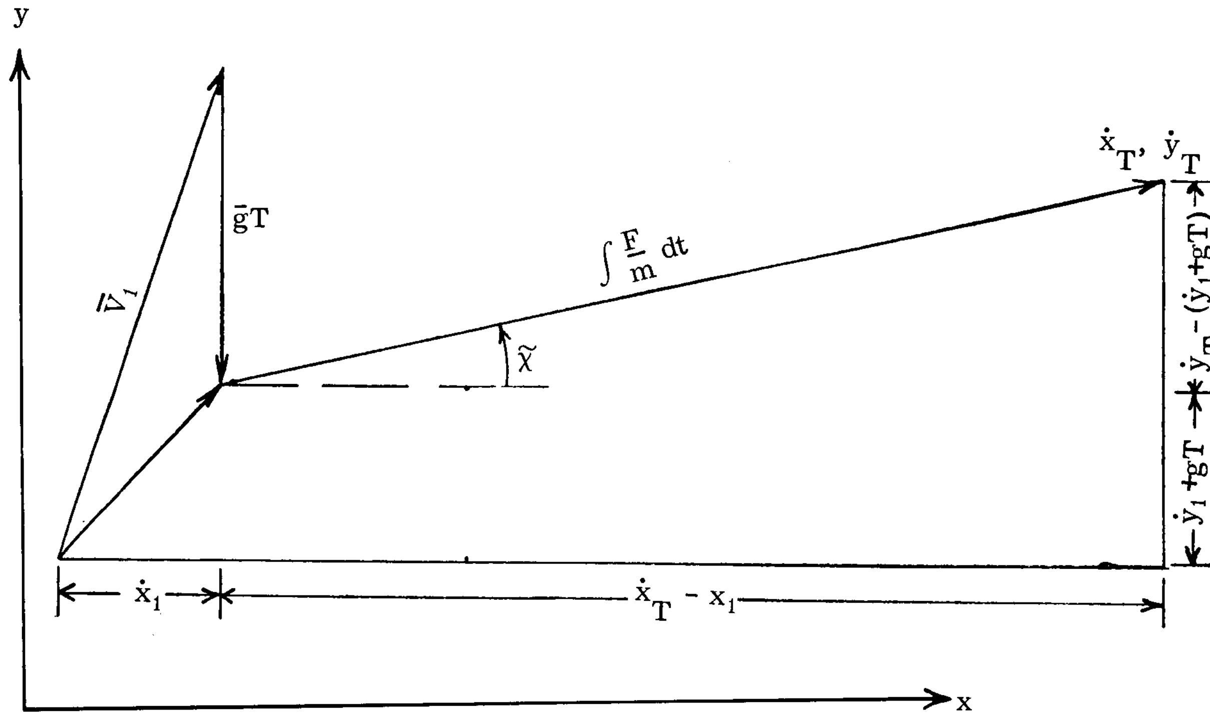

FIGURE 1. THE RELATIONSHIP OF BOUNDARY CONDITIONS

$$\tan \chi = \frac{\dot{y}_i}{\dot{x}_i} = - \frac{(\dot{y}_1 + gT) - \dot{y}_T}{\dot{x}_T - \dot{x}_1} . \tag{3}$$

On the other hand, if the altitude is constrained along with the velocity vector, the thrust attitude direction becomes

$$\tan \chi = \frac{\partial R/\partial \dot{y}_T + (T - t)\ \partial R/\partial y_T}{\partial R/\partial \dot{x}_T} = a_1 + b_1 t . \tag{4}$$

The first order expansion of (2) and (4)

$\chi = a'$

$\chi = a_1 + b_1 t,$

$$\frac{F}{\dot{m}} = -V_{ex}; \frac{m_1}{\dot{m}} = -\tau,$$

where m_1 represents instantaneous mass.

If only the end velocity vector is specified, it has been shown that a constant thrust attitude direction, $\tilde{\chi}$, will achieve this end condition in a near-optimum fashion. Equations (2.1) can then be integrated over T seconds in closed form:

$$\tilde{\dot{x}}(T) = \dot{x}_1 + (\cos\tilde{\chi})\, V_{ex} \ln\frac{\tau}{\tau - T}$$

$$\tilde{\dot{y}}(T) = \dot{y}_1 + (\sin\tilde{\chi})\, V_{ex} \ln\frac{\tau}{\tau - T} - gT. \tag{2.3}$$

Solution of equations (2.3) for $\tilde{\chi}$ yields

$$\tilde{\chi} = \tan^{-1}\left[\frac{\tilde{\dot{y}}(T) - \dot{y}_1 + gT}{\tilde{\dot{x}}(T) - \dot{x}_1}\right]. \tag{2.4}$$

Inspection of Figure 1 shows the same result geometrically.

The magnitude of the velocity vector is obtained through the burning time T; i.e.,

$$\Delta V_i = \sqrt{(\tilde{\dot{x}}(T) - \dot{x}_1)^2 + [\tilde{\dot{y}}(T) - \dot{y}_1 + gT]^2}, \tag{2.5}$$

which may be set equal to the characteristic velocity

$$\Delta V_i = V_{ex} \ln\frac{\tau}{\tau - T}. \tag{2.6}$$

Equations (2.4), (2.5), and (2.6) can be solved for T and $\tilde{\chi}$ for any combination of current state velocities $\dot{x}_1$, $\dot{y}_1$ and required end velocities $\tilde{\dot{x}}(T)$, $\tilde{\dot{y}}(T)$. Since current state variables are changing, the T-computation and $\tilde{\chi}$-determination proceed stepwise using new $\dot{x}_1$, $\dot{y}_1$ values as they are obtained.

So far, only an end velocity vector condition has been obtained. In order to constrain an end altitude also, it is necessary to introduce a linear thrust attitude law,

$$\chi = \tilde{\chi} - K_1 + K_2 t,$$

which has been justified through experience, is used in developing the scheme. These laws make it possible to generate the function in flight using as information state variables of velocity and displacement, thrust, mass flow, and desired end conditions.

The scheme is modified to approximate a spherical moon assumption by introducing an average gravity direction and magnitude between the instantaneous state and the desired end condition. No longer are a', a_1 and b_1 constant throughout the flight as in the unperturbed flat moon case but constant over the length of time between evaluations. Indeed, one might think of a series of flat moons between the instantaneous point and the end point, an assumption which becomes more valid as the end point is approached. These modifications to approximate a spherical moon do not destroy the ability to solve for a', a_1, b_1, and T in closed form.

Range constraint is achieved by the throttling capability of the engines. The required thrust is inversely proportional to the required range. The change in thrust affects, of course, the solution of χ and T. These quantities are updated, however, to be compatible with the new thrust.

SECTION II. DESCRIPTION

The principles of the scheme can be displayed most clearly by assuming a two-dimensional flat surface as a model. In a subsequent section, the extension to a homogeneous spherical body, still in two dimensions, is presented. Finally the specific application to a lunar landing mode is shown.

The equations of motion relative to a flat surface in a constant gravitational field are

$$\begin{aligned} \ddot{x} &= F/m \cos \chi \\ \ddot{y} &= F/m \sin \chi - g, \end{aligned} \qquad (2.1)$$

where F/m represents acceleration due to a constant thrust and a constant specific impulse, and χ represents the direction of the acceleration vector against the horizontal. Mathematically expressed,

$$\frac{F}{m} = \frac{F}{\dot{m}} \cdot \frac{\dot{m}}{m_1 + \dot{m}t} = V_{ex} \frac{1}{\tau - t} . \qquad (2.2)$$

Since vacuum flight and constant specific impulse are assumed, F and $\dot{m}$ are constant; $\dot{m} < 0$. The following definitions also prevail:

where

$$J = \tau \ln \frac{\tau}{\tau - T} - T$$

$$G = y_1 + \dot{y}_1 T - \sin \tilde{\chi} \, V_{ex} \left[J - T \ln \frac{\tau}{\tau - T} \right] - \frac{gT^2}{2} - y(T) \tag{2.12}$$

$$S = V_{ex} \cos \tilde{\chi} \left[T \, J - \frac{T^2}{2} \ln \left(\frac{\tau}{\tau - T} \right) \right] .$$

SECTION III. EXTENSION OF THE GUIDANCE SCHEME TO A SPHERICAL BODY ASSUMPTION

The theory can be extended to the spherical body assumption by assuming an average value for the gravity direction ϕ^* and magnitude g^*, where

$$\phi^* = \frac{\phi_T - \phi_1}{2} ; \tag{3.1}$$

$$g^* = \frac{g_1 + g_T}{2} . \tag{3.2}$$

Figure 2 depicts the coordinate system used in the guidance computation. The ξ-η system, with origin at the desired injection point, is formed by rotating the x-y system, space-fixed at the launch site through the terminal range angle ϕ_T:

$$\begin{aligned} \xi &= x \cos \phi_T - (R_o + y) \sin \phi_T \\ \eta &= x \sin \phi_T + (R_o + y) \cos \phi_T . \end{aligned} \tag{3.3a}$$

A similar relation holds for the velocity components.

$$\begin{aligned} \dot{\xi} &= \dot{x} \cos \phi_T - \dot{y} \sin \phi_T \\ \dot{\eta} &= \dot{x} \sin \phi_T + \dot{y} \cos \phi_T . \end{aligned} \tag{3.3b}$$

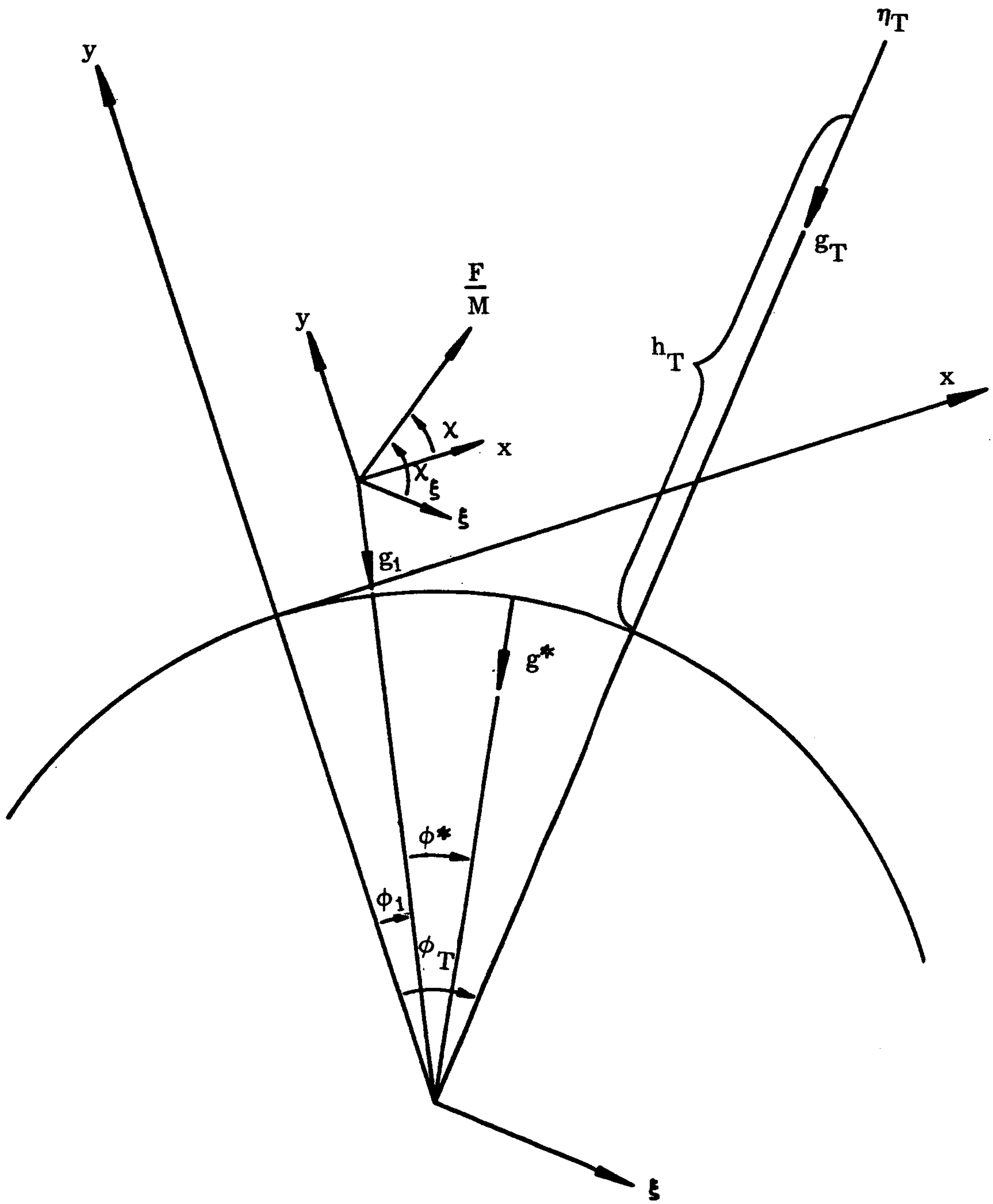

FIGURE 2. FLIGHT GEOMETRY

The steps taken in deriving the thrust attitude angle are the same as those displayed in Section II.

$$\Delta\dot{\xi} = \dot{\xi}_T - \dot{\xi}_1 - g^* T \sin \phi^*$$
$$\Delta\dot{\eta} = \dot{\eta}_T - \dot{\eta}_1 + g^* T \cos \phi^*. \tag{3.4}$$

$$\chi_\xi = \tan^{-1} \frac{\Delta\dot{\eta}}{\Delta\dot{\xi}} . \tag{3.5}$$

$$\Delta V_i = V_{ex} \ln \frac{\tau}{\tau - T} . \tag{3.6}$$

$$\Delta V_i = \sqrt{\Delta\dot{\xi}^2 + \Delta\dot{\eta}^2} \tag{3.7}$$

$$\chi_\xi = \tilde{\chi}_\xi - K_1 + K_2 t \tag{3.8}$$

$$\chi = \tilde{\chi}_\xi - K_1 + K_2 t - \phi_T . \tag{3.9}$$

The equations of motion in the ξ-η coordinate system then becomes

$$\ddot{\xi} = F/m \cos \chi_\xi + g^* \sin \phi^*$$
$$\ddot{\eta} = F/m \sin \chi_\xi - g^* \cos \phi^*.$$

Again the coefficients K_1 and K_2 appearing in (3.8) are determined to preserve the velocity condition imposed by equation (3.5) while enforcing a condition on altitude. Thus,

$$\tilde{\dot{\eta}}(T) = \dot{\eta}_1 - g^* T \cos \phi^* + \sin \tilde{\chi}_\xi V_{ex} \ln \frac{\tau}{\tau - T} \tag{3.10}$$

$$\dot{\eta}(T) = \dot{\eta}_1 - g^{*}T \cos \phi^{*} + \sin \tilde{\chi}_{\xi} V_{ex} \ln \frac{\tau}{\tau - T}$$

$$- K_1 \cos \tilde{\chi}_{\xi} V_{ex} \ln \frac{\tau}{\tau - T} + K_2 V_{ex} \cos \tilde{\chi}_{\xi} \left[-T + \tau \ln \frac{\tau}{\tau - T}\right] \tag{3.11}$$

$$\dot{\eta}(T) - \dot{\tilde{\eta}}(T) = 0 \tag{3.12}$$

$$\eta(T) = \eta_T = \eta_1 + \dot{\eta}_1 T - g^{*} \frac{T^2}{2} \cos \phi^{*} + V_{ex} \sin \tilde{\chi}_{\xi} \left[T - (\tau - T) \ln \frac{\tau}{\tau - T}\right]$$

$$- K_1 V_{ex} \cos \tilde{\chi}_{\xi} \left[T - (\tau - T) \ln \frac{\tau}{\tau - T}\right] \tag{3.13}$$

$$+ V_{ex} K_2 \cos \tilde{\chi}_{\xi} \left\{ - \frac{T^2}{2} - \tau \left[(\tau - T) \ln \frac{\tau}{\tau - T} - T\right] \right\} .$$

Solving equations (3.12) and (3.13) simultaneously for K_1 and K_2,

$$K_1 = \frac{G}{S} J \tag{3.14}$$

$$K_2 = \frac{G}{S} \ln \frac{\tau}{\tau - T} \tag{3.15}$$

where

$$\left.\begin{aligned} J &= \tau \ln \frac{\tau}{\tau - T} - T \\ G &= \eta_1 + \dot{\eta}_1 T - \sin \tilde{\chi}_{\xi} V_{ex} \left[J - T \ln \frac{\tau}{\tau - T}\right] - g^{*} \frac{T^2}{2} \cos \phi^{*} - \eta(T) \\ S &= V_{ex} \cos \tilde{\chi}_{\xi} \left[TJ - \frac{T^2}{2} \ln \left(\frac{\tau}{\tau - T}\right)\right] . \end{aligned}\right\} \tag{3.16}$$

where $\tilde{\chi}$, K_1, and K_2 preserve the velocity condition already obtained, while also enforcing a desired altitude condition. Equations (2.1) become

$$\ddot{x} = F/m \cos(\tilde{\chi} - K_1 + K_2 t)$$

$$\ddot{y} = F/m \sin(\tilde{\chi} - K_1 + K_2 t) - g. \tag{2.7}$$

In order to preserve the velocity condition, K_1 and K_2 must be chosen so that the $\dot{y}$ obtained from the equations (2.7) and $\tilde{\dot{y}}$ are the same. It is assumed that $\cos(-K_1 + K_2 t) \approx 1$ and $\sin(-K_1 + K_2 t) \approx -K_1 + K_2 t$.

$$\dot{y} = \dot{y}_1 + (\sin\tilde{\chi} - K_1 \cos\tilde{\chi})\, V_{ex} \ln\frac{\tau}{\tau - T} + K_2 \cos\tilde{\chi}\, V_{ex}\left[-T + \tau\ln\frac{\tau}{\tau - T}\right] - gT. \tag{2.8}$$

$$\dot{y}(T) - \tilde{\dot{y}} = 0 = -K_1 \cos\tilde{\chi}\, V_{ex} \ln\frac{\tau}{\tau - T} + K_2 \cos\tilde{\chi}\, V_{ex}\left[-T + \tau\ln\frac{\tau}{\tau - T}\right]. \tag{2.9}$$

The coefficients K_1 and K_2 must be chosen also to satisfy the altitude condition $y(T)$:

$$y(T) = y_1 + \dot{y}_1 T + (\sin\tilde{\chi} - K_1 \cos\tilde{\chi})\, V_{ex}\left[T - (\tau - T)\ln\frac{\tau}{\tau - T}\right]$$

$$+ K_2 \cos\tilde{\chi}\, V_{ex}\left\{-\frac{T^2}{2} - \tau\left[(\tau - T)\ln\frac{\tau}{\tau - T} - T\right]\right\} - \frac{gT^2}{2}. \tag{2.10}$$

Equations (2.9) and (2.10) can be solved simultaneously for K_1 and K_2:

$$K_1 = \frac{G}{S} J$$

$$K_2 = \frac{G}{S} \ln\left(\frac{\tau}{\tau - T}\right)$$

It may be helpful to go into some detail on the computation of T, the time-to-go, which is determined from equations (3.6) and (3.7), since their form would seem to indicate the need for an iterative procedure to solve them. However, an iteration computation in flight is undesirable since there is no guarantee of convergence in the allotted time. Therefore, the following method is employed: Let T' be a guess of the time-to-go so that

$$T = T' + \delta T. \tag{3.17}$$

Then

$$\begin{aligned} \Delta\dot{\xi}' &= \dot{\xi}_T - \dot{\xi}_1 - g^* T' \sin\phi^* \\ \Delta\dot{\eta}' &= \dot{\eta}_T - \dot{\eta}_1 + g^* T' \cos\phi^*, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \Delta V_i'^2 = &\left[\dot{\xi}_T - \dot{\xi}_1 - g^* T' \sin\phi^*\right]^2 + \\ &\left[\dot{\eta}_T - \dot{\eta}_1 + g^* T' \cos\phi^*\right]^2 . \end{aligned} \tag{3.19}$$

Now, referring to equations (3.4), (3.7) and (3.17),

$$\begin{aligned} \Delta V_i^2 = &\left[\dot{\xi}_T - \dot{\xi}_1 - g^*(T' + \delta T)\sin\phi^*\right]^2 + \\ &\left[\dot{\eta}_T - \dot{\eta}_1 + g^*(T' + \delta T)\cos\phi^*\right]^2 , \end{aligned} \tag{3.20}$$

so that

$$\Delta V_i^2 - \Delta V_i'^2 = 2\Delta\dot{\eta}' g^* \cos\phi^* - 2\Delta\dot{\xi}' g^* \sin\phi^* + g^{*2}(\delta T)^2. \tag{3.21}$$

For convenience of expression, the following relation is introduced:

$$\lambda = (\Delta\dot{\eta}' g^* \cos\phi^* - \Delta\dot{\xi}' g^* \sin\phi^*), \tag{3.22}$$

so that

$$\Delta V_i^2 \approx g^{*2}(\delta T)^2 + 2\lambda\delta T + \Delta V_i'^2. \tag{3.23}$$

The total final inertial velocity in the guessed time T' is

$$V_{T_i} = V_{1i} + V_{ex} \ln \frac{\tau}{\tau - T} , \tag{3.24}$$

or

$$\Delta V_i = V_{ex} \ln \frac{\tau}{\tau - T} . \tag{3.25}$$

Expressing (3.25) as a series approximation,

$$\Delta V_i \approx - V_{ex} \ln \left(1 - \frac{T'}{\tau}\right) - V_{ex} \left[\frac{- \delta T}{\tau - T'} + \frac{1}{2} \left(\frac{- \delta T}{\tau - T'}\right)^2 + \ldots\right] . \tag{3.26}$$

Let

$$L = V_{ex} \ln \frac{\tau}{\tau - T'}$$

$$K = \frac{V_{ex}}{\tau - T'} ;$$

then equation (3.26) becomes

$$\Delta V_i \approx L + K\delta T + \frac{1}{2}K \frac{(\delta T)^2}{\tau - T'} \approx L + K\delta T, \tag{3.27}$$

if second order terms are discarded. Squaring equation (3.27) and setting it equal to equation (3.23) gives

$$g^{*2} (\delta T)^2 + 2\lambda\delta T + \Delta V'^2 = L^2 + 2KL\delta T + K^2(\delta T)^2. \tag{3.28}$$

Collecting coefficients of like terms results in

$$- a(\delta T)^2 + 2b\delta T + c = 0 \tag{3.29}$$

where

$$\begin{aligned} a &= (K^2 - g^{*2}) \\ b &= \lambda - KL \\ c &= \Delta V_i'^2 - L^2. \end{aligned} \tag{3.30}$$

Solving the quadratic for δT,

$$\delta T = \frac{b + \sqrt{b^2 + ac}}{a} \tag{3.31}$$

The positive root is chosen in equation (3.31), since b is negative, and δT should be small. The time to go, T, may now be determined and the correct velocity components become

$$\begin{aligned} \Delta\dot{\xi} &= \Delta\dot{\xi}' - g^* \delta T \sin \phi^* \\ \Delta\dot{\eta} &= \Delta\dot{\eta}' + g^* \delta T \cos \phi^*. \end{aligned} \tag{3.32}$$

The spherical Earth assumption assumes a knowledge of the terminal range angle ϕ_T. Presetting a range angle based on previous experience is one approach; however, an on-board computation of ϕ_T is more desirable since it will change accordingly with perturbations which may be encountered. Therefore, the following approach is taken.

The horizontal distance obtained from a linear acceleration on a flat surface in time T* is

$$x_T = \int_0^{T^*} (V_1 + V_{ex} \ln \frac{\tau}{\tau - T})\, dt, \tag{3.33}$$

or

$$x_T = x_1 + T^* \left[V_1 + V_{ex} + V_{ex} \frac{\tau - T^*}{T^*} \ln \frac{\tau - T^*}{\tau} \right]. \tag{3.34}$$

Dividing by the radius vector r_T,

$$\phi_T = \phi_1 + \frac{T^*}{r_T} \left[V_1 + V_{ex} + V_{ex} \frac{\tau - T^*}{T^*} \ln \frac{\tau - T^*}{\tau} \right]. \tag{3.35}$$

Referring to equation (3.1),

$$\phi^* = \frac{T^*}{2r_T} \left[V_1 + V_{ex} + V_{ex} \frac{\tau - T^*}{T^*} \ln \frac{\tau - T^*}{\tau} \right]. \tag{3.36}$$

This approximation has proved quite sufficient and has the characteristic of improving in accuracy as the end condition is approached.

Briefly, in retrospect, the values preset are final altitude, h_T; final velocity components, $\dot{\xi}_T$ and $\dot{\eta}_T$ and physical constants of the central body; values obtained in flight are instantaneous state variables x_1, y_1, $\dot{x}_1$, $\dot{y}_1$, and some measurement of the specific impulse or fuel flow rate.

SECTION IV. APPLICATION OF THE GUIDANCE SCHEME TO LUNAR LANDING

The guidance equations developed in the previous sections for ascent are virtually identical to those needed for lunar descent. The two essential modifications are a change in coordinate system due to the way the instantaneous state information is obtained and the use of range prediction to vary thrust, since it is desired to arrive at a prescribed spot on or above the lunar surface.

The potential presenoe of a beacon lends itself to the coordinate systems shown in Figure 3. The ξ-η system has its origin at the beginning of the terminal braking phase; the ξ-axis passes through the instantaneous vehicle position (M). The beacon provides slant range, D, and slant range rate, $\dot{D}$. It is also assumed that the angle θ and its angular rate $\dot{\theta}$ are known or can be computed.

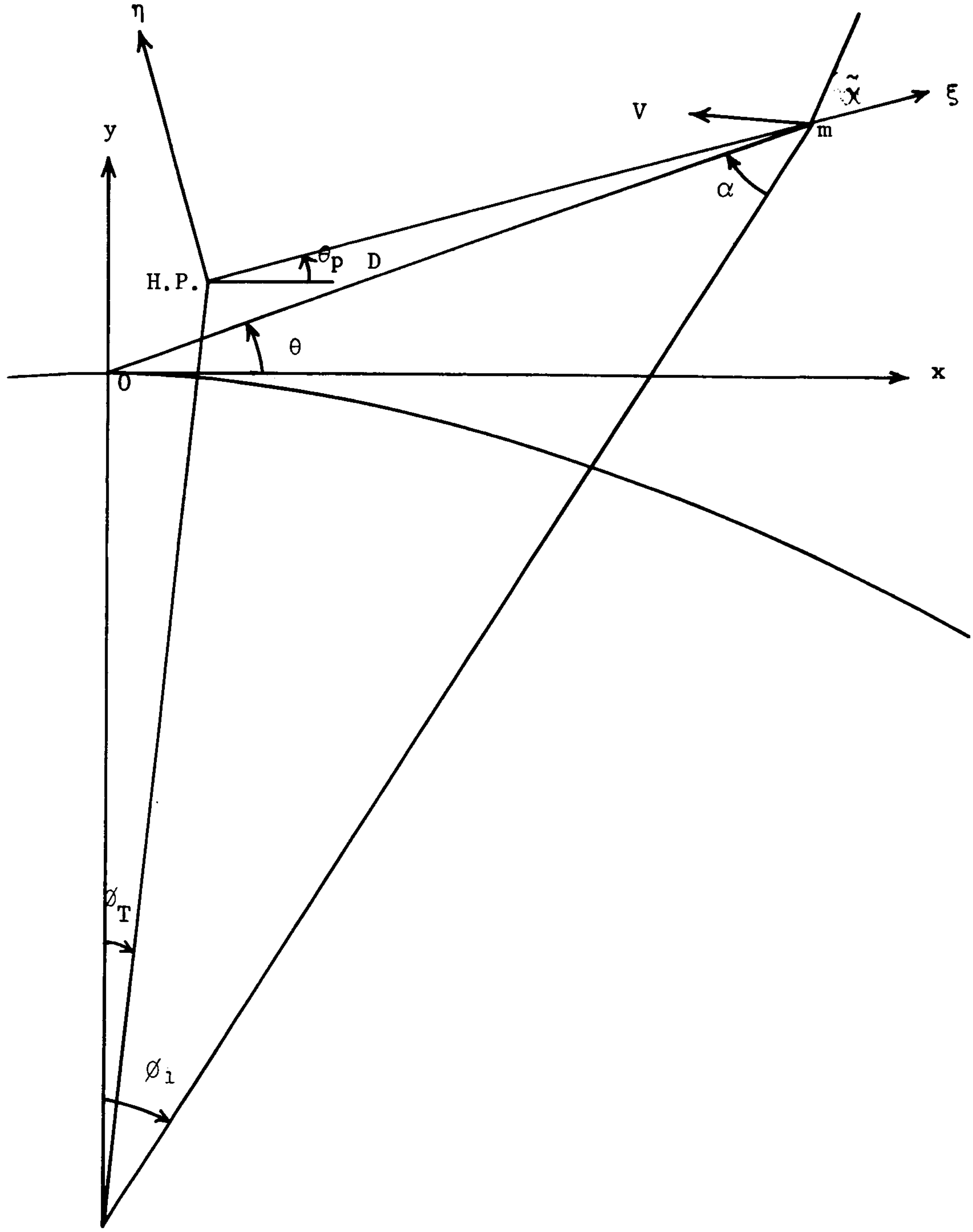

FIGURE 3. LUNAR FLIGHT GEOMETRY

The beacon is located at the origin of a space-fixed system with its y-axis parallel to the radius vector to that point. The coordinates of the predesignated terminal braking point and the vehicle velocity at that point in the space-fixed system are $(x_T, y_T, \dot{x}_T, \dot{y}_T)$.

The vector extending from the terminal braking point to the vehicle is

$$\vec{\xi} = \vec{i}\,(D \cos \theta - x_T) + \vec{j}\,(D \sin \theta - y_T) \tag{4.1}$$

$$\vec{\xi} = \vec{i}\,\xi_1 \cos \theta p + \vec{j}\,\xi_1 \sin \theta p. \tag{4.1a}$$

Using these relationships,

$$\xi_1 = (\vec{\xi} \cdot \vec{\xi})^{\frac{1}{2}} = [(D \cos \theta - x_T)^2 + (D \sin \theta - y_T)^2]^{\frac{1}{2}} , \tag{4.2}$$

$$\cos \theta p = \frac{D \cos \theta - x_T}{\xi_1} , \qquad \sin \theta p = \frac{D \sin \theta - y_T}{\xi_1} , \tag{4.3}$$

$$\tan \theta p = \frac{D \sin \theta - y_T}{D \cos \theta - x_T} . \tag{4.4}$$

The velocity components in the D-θ system are transferred into the space-fixed system:

$$\dot{x}_1 = \dot{D} \cos \theta + D\,\dot{\theta} \sin \theta \qquad y_1 = - \dot{D} \sin \theta + D\dot{\theta} \cos \theta. \tag{4.5}$$

The instantaneous values are obtained by transferring to the ξ-η coordinate system:

$$\dot{\xi}_1 = \dot{x}_1 \cos \theta p - \dot{y}_1 \sin \theta p \qquad \dot{\eta}_1 = \dot{x}_1 \sin \theta p + \dot{y}_1 \cos \theta p. \tag{4.6}$$

Similarly, for the terminal velocity components,

$$\dot{\xi}_T = \dot{x}_T \cos \theta p - \dot{y}_T \sin \theta p$$
$$\dot{\eta}_T = \dot{x}_T \sin \theta p + \dot{y}_T \cos \theta p. \tag{4.7}$$

A significant change necessary is the determination of the average gravity direction ϕ^*, since the η-axis is not parallel to the local vertical at the terminal braking point.

$$\phi^* = -\frac{1}{2}(\phi_T + \phi_1) - \theta p. \tag{4.8}$$

Equation (4.8) can be more readily understood by considering first the η-axis parallel to the local vertical at the terminal braking point which yields equation (3.1) and then subtracting the angle $(\theta p + \phi_T)$ necessary to rotate the local vertical into the η-axis.

The guidance equations are identical to those developed in the previous two sections. The form of equations (3.14) and (3.15) is the same; however, the terms η_1 and $\eta(T)$ which appeared in equations (3.16) are dropped, since in the ξ-η coordinate system for lunar landing, they are zero. Thus,

$$J = \tau \ln \frac{\tau}{\tau - T} - T \tag{4.9}$$

$$G = \dot{\eta}_1 T - \frac{1}{2} g^* \cos \phi^* T^2 - V_{ex} \sin \tilde{\chi} \left[J - T \ln \frac{\tau}{\tau - T} \right]$$

$$S = V_{ex} \cos \tilde{\chi} \left[T J - \frac{T^2}{2} \ln \left(\frac{\tau}{\tau - T} \right) \right].$$

Equation (3.9) becomes the following to relate the thrust altitude angle to the space-fixed x-y coordinate system:

$$\chi = \tilde{\chi} - K_1 + K_2 t + \theta p. \tag{4.10}$$

Thrust Control

In the previous sections, no constraint has been placed on range. For lunar landing, however, it is desired to control range since a specific point on the lunar surface should be reached. To accomplish this task, variable thrust is superimposed on the guidance scheme.

The thrust (F) is inversely proportional to the distance ξ^*, the predicted distance corresponding to the thrust (F).

$$F = \frac{K}{\xi^*}, \tag{4.11}$$

where K is a proportionality constant. There will exist a desired thrust level (F^*) such that the required distance (ξ) is covered during the remaining time.

$$F^* = \frac{K}{\xi_1}. \tag{4.12}$$

Solving yields

$$F^* = \frac{\xi^*}{\xi_1} F. \tag{4.13}$$

Equation (4.13) is the thrust control law for selecting the thrust level. The displacement ξ^* is the predicted distance that will be traversed during the remaining flight at the thrust level (F):

$$\xi^* = \xi_1 - \xi(T). \tag{4.14}$$

The total average acceleration in the ξ-direction is

$$\ddot{\xi}(t) = \frac{F}{m_1 + \dot{m}t} \cos\overline{\chi} + g^* \sin\phi^*. \tag{4.15}$$

The value of $\cos\overline{\chi}$ is an effective average value of $\cos\chi_\xi$ over the time T. The positive sign on the gravitation component is due to the negative ϕ^*.

Using the fundamental laws of calculus,

$$\int_0^T d\xi(t) = \xi(T) - \xi_1 = \int_0^T \dot{\xi}(t)\, dt \tag{4.16}$$

$$\xi(T) = \xi_1 + \int_0^T \dot{\xi}(t)\, dt$$

$$\int_0^t d\dot{\xi}(t) = \dot{\xi}(t) - \dot{\xi}_1 = \int_0^t \ddot{\xi}(t)\, dt$$

$$(4.17)$$

$$\dot{\xi}(T) = \dot{\xi}_1 + \int_0^T \ddot{\xi}(t)\, dt.$$

Combining equations (4.15), (4.16), and (4.17) and placing into (4.14),

$$\xi^* = -\int_0^T \left\{ \left[\dot{\xi}_1 + \int_0^t \left[\frac{F}{m_1 + \dot{m}t} \cos \overline{\chi} + g^* \sin \phi^*\right] dt\right] \right\} dt. \tag{4.18}$$

Performing the designated integration yields

$$\xi^* = -\dot{\xi}_1 T - V_{ex} \cos \overline{\chi} \left[T - (\tau - T) \ln \frac{\tau}{\tau - T}\right] - g^* \sin \phi^* \frac{T^2}{2}. \tag{4.19}$$

Using the defining equation (3.5) for $\tilde{\chi} = \overline{\chi}$ and the defining equation (3.6) for ΔV, the relation

$$\Delta V = \frac{\Delta \dot{\xi}}{\cos \overline{\chi}} = V_{ex} \ln \frac{\tau}{\tau - T} \tag{4.20}$$

is obtained.

Placing equation (4.20) into equation (4.19) yields

$$\xi^* = -\dot{\xi}_1 T - V_{ex} \cos\overline{\chi}\, T + (\tau - T)\, \Delta\dot{\xi} - g^* \sin\phi^* \frac{T^2}{2} \,. \tag{4.21}$$

Using

$$\Delta\dot{\xi} = \dot{\xi}_T - \dot{\xi}_1 - g^* T \sin\phi^* \tag{4.22}$$

gives

$$\xi^* = \tau\Delta\dot{\xi} - T(\dot{\xi}_T - g^* T \sin\phi^*) - V_{ex} \cos\overline{\chi}\, T - g^* \sin\phi^* \frac{T^2}{2} \,. \tag{4.23}$$

Rewriting equation (2.6) and substituting in equation (4.20) yields

$$T = \tau \left(1 - e^{-\frac{\Delta\dot{\xi}}{V_{ex} \cos\overline{\chi}}} \right) \tag{4.24}$$

For convenience, let

$$Z = \frac{\Delta\dot{\xi}}{V_{ex} \cos\overline{\chi}} \,. \tag{4.25}$$

For computational purposes it is worthwhile not to have the exponential in (4.24); therefore, equation (4.24) is expressed in a Maclaurin series:

$$T = \tau - \tau \left(1 - Z + \frac{Z^2}{2!} - \frac{Z^3}{3!} + \frac{Z^4}{4!} - \frac{Z^5}{5!} + \ldots \right) , \tag{4.26}$$

or

$$T = \tau Z - \tau \frac{Z}{2} \left\{ 1 - \frac{Z}{3} \left[1 - \frac{Z}{4} \left(1 - \frac{Z}{5} \right) \right] \right\} \,. \tag{4.27}$$

Placing (4.27) into (4.23), and using (4.25), gives

$$\xi^* = -T \left(\dot{\xi}_T - \frac{1}{2} g^* \sin\phi^* T \right) + \frac{1}{2} \tau (\Delta\dot{\xi})\, Z \left\{ 1 - \frac{Z}{3} \left[1 - \frac{Z}{4} \left(1 - \frac{Z}{5} \right) \right] \right\} \,. \tag{4.28}$$

Hence, the value of ξ^* needed to select the proper thrust magnitude using equations (4.13) is obtained.

The task remains to determine $\cos \overline{\chi}$ so that equation (4.25) may be evaluated. Let $\cos \overline{\chi}$ be defined as follows:

$$\cos \overline{\chi} = \frac{2}{T^2} \int_0^T \int_0^t \cos \chi_\xi \, dt^2, \tag{4.29}$$

where $\chi_\xi = \tilde{\chi}_\xi - K_1 + K_2 t$. Performing the designated integration,

$$\cos \overline{\chi} = \frac{2}{T^2} \int_0^T \frac{1}{K^2} \left[\sin \chi - \sin (\tilde{\chi} - K_1) \right] dt, \tag{4.30}$$

$$\cos \overline{\chi} = \frac{2}{K_2 T^2} \left\{ - \frac{1}{K_2} \left[\cos \chi - \cos (\chi - K_1) \right] - T \sin (\tilde{\chi} - K_1) \right\} . \tag{4.31}$$

Expanding the cos χ term and rearranging,

$$\cos \overline{\chi} = \frac{2}{K_2^2 T^2} \left\{ \cos (\tilde{\chi} - K_1) \; [1 - \cos K_2 T] - \sin (\tilde{\chi} - K_1) \; [K_2 T - \sin K_2 T] \right\} . \tag{4.32}$$

From the computer viewpoint it is most desirable to eliminate as many trigonometric functions as possible. Although $\cos \overline{\chi}$ can immediately be obtained from equation (4.32) the following steps are taken to obtain a more suitable form for the computer. Using the trigonometric identity,

$$\cos \chi = A \cos (\tilde{\chi} - K_1 + \omega) = A \cos (\tilde{\chi} - K_1) \cos \omega - A \sin (\tilde{\chi} - K_1) \sin \omega. \tag{4.33}$$

Equating terms with 4.32,

$$\begin{aligned} A \cos \omega &= \frac{2}{K_2^2 T^2} [1 - \cos K_2 T]. \\ A \sin \omega &= \frac{2}{K_2^2 T^2} [K_2 T - \sin K_2 T]. \end{aligned} \tag{4.34}$$

Squaring (4.34), adding, and expanding into series, we obtain

$$A^2 = \frac{4}{K_2^4 T^4} \left\{ \left[\frac{(K_2T)^2}{2} - \frac{(K_2T)^4}{24} + \dots \right]^2 + \left[\frac{(K_2T)^3}{6} - \dots \right]^2 \right\}$$

$$A^2 = \frac{4}{K_2^4 T^4} \left\{ \left[\frac{(K_2T)^4}{4} - \frac{2(K_2T)^6}{48} + \dots \right] + \left[\frac{(K_2T)^6}{36} - \dots \right] \right\} \tag{4.35}$$

$$A = \sqrt{1 - \left(\frac{1}{6} - \frac{1}{9}\right) (K_2T)^2}$$

Using the binomial expansion gives

$$A = 1 - \frac{(K_2T)^2}{36}, \tag{4.36}$$

and expanding (4.34) in a series gives

$$\omega = \sin^{-1} \frac{2}{A\, K_2^2 T^2} \left[\frac{(K_2T)^3}{6} - \dots \right] \approx \frac{K_2T}{3}. \tag{4.37}$$

A further simplification can be made if it is recalled that the enforcement of equation (3.12) implies approximately that

$$\int_0^T \tilde{\chi}\, dt \approx \int_0^T (\tilde{\chi} - K_1 + K_2 t)\, dt$$

$$\tilde{\chi} T \approx (\tilde{\chi} - K_1 + K_2 \frac{T}{2}) T. \tag{4.38}$$

Thus,

$$K_2 T \approx 2K_1. \tag{4.39}$$

Figure 4 shows this approximate relationship.

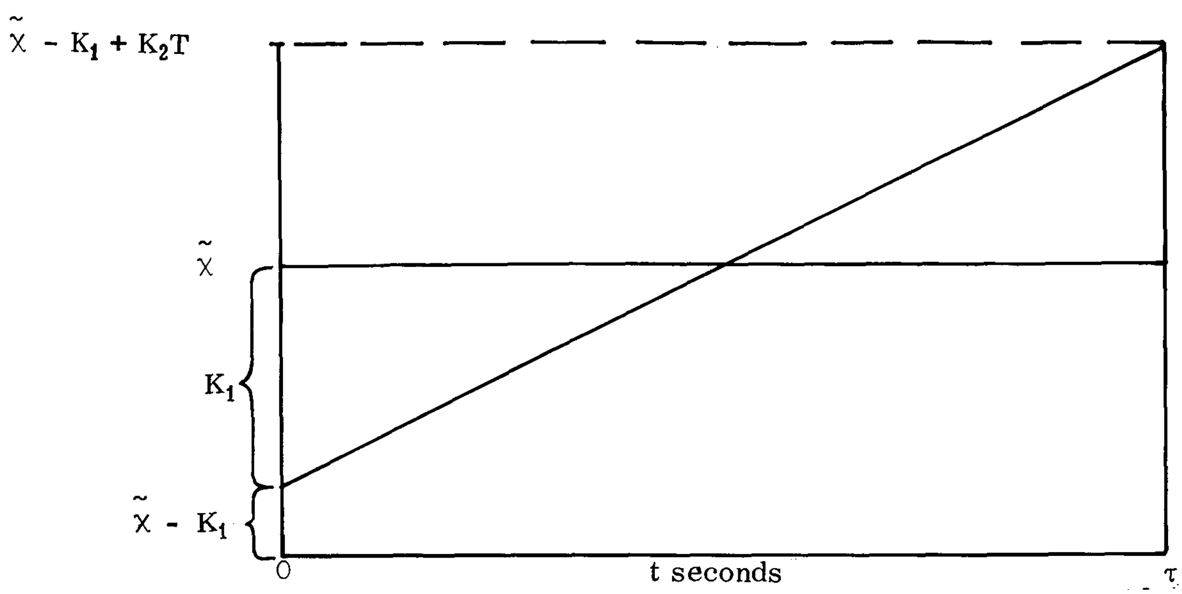

FIGURE 4. THE APPROXIMATE RELATIONSHIP OF $K_2T \approx 2K_1$

Evaluating (4.33), using (4.36), (4.37), and (4.39), yields

$$\cos \bar{\chi} = \left(1 - \frac{K_1^2}{9}\right) \cos \left(\tilde{\chi} - \frac{K_1}{3}\right). \tag{4.40}$$

With these relationships, the required thrust can be computed to fulfill the range constraint. It should be noted that changing the thrust magnitude affects the solution of K_1, K_2, $\tilde{\chi}$, and T; however, these quantities are updated with the new thrust evaluation. The following block diagram may be helpful in showing the input and the mechanization of the scheme itself.

$$\left.\begin{array}{l} h = h_T \\ y = y_T \\ V = V_T \end{array}\right\} \quad \tilde{\chi},\ K_1,\ K_2,\ T$$

$$\left. x = x_T \right\} \quad F^* \text{ which affects control quantities above}$$

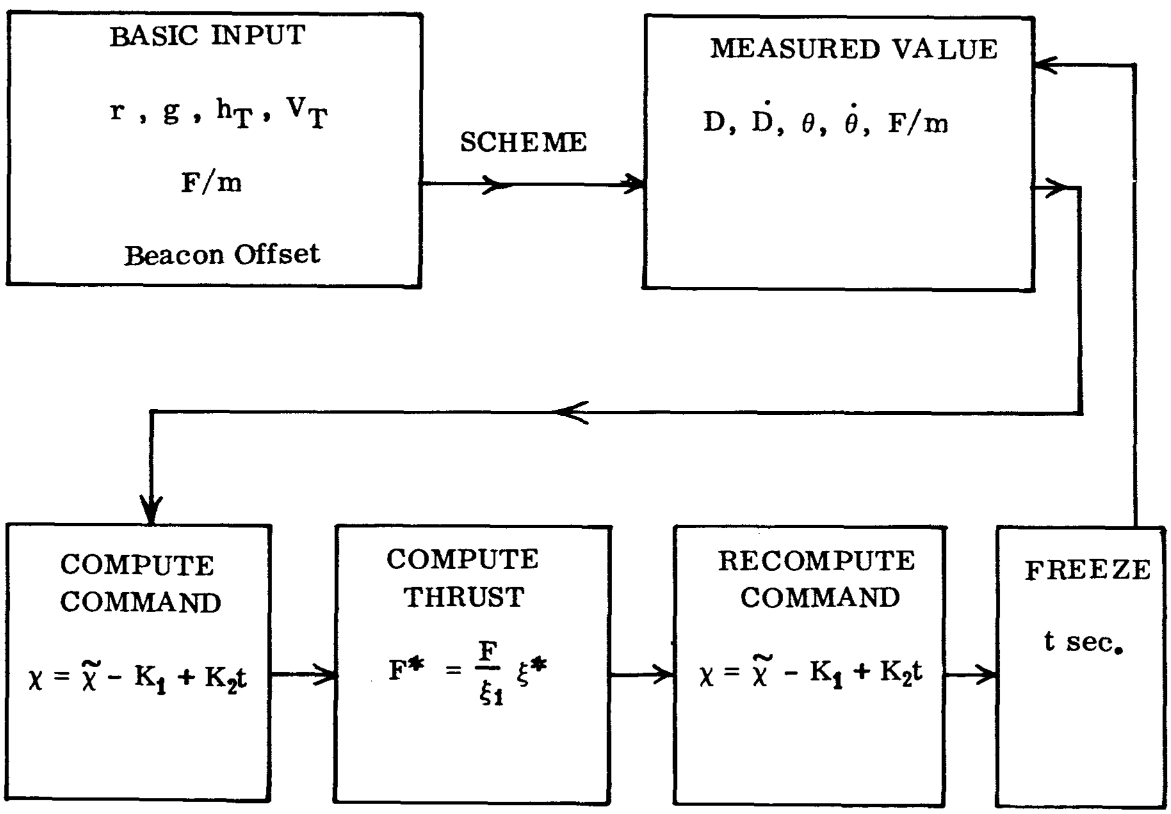

FIGURE 5. BLOCK DIAGRAM SHOWING MECHANIZATION OF SCHEME

SECTION V. DISCUSSION OF RESULTS

Two rather typical flight profiles for lunar landing to exemplify the effectiveness of the guidance scheme are (1) continuous burn of one RL-10 engine, producing a nominal thrust of 15,000 lbs, from a 100 kilometer circular lunar orbit to a terminal braking point of 300 meters altitude, and (2) main braking by three RL-10 engines, producing a nominal thrust of 42,000 lbs, from the 15 kilometer periselenum of a Hohmann transfer ellipse to a terminal braking point of 300 meters altitude.

Tables I and II show the absolute value of the thrust variation and the displacement and velocity dispersions evaluated at the 300 meter terminal braking point for various initial dispersions in state variables and engine performance, and for radar errors emanating from the beacon. The guidance parameters were determined every ten seconds. The trajectories generated by the scheme adhere closely to those generated by variational calculus procedures. The maximum payload difference between the two methods is approximately 50 lbs.

Since it is believed that the choice of initial errors is conservative, it can be seen that the scheme itself has virtually no error and is as accurate as the source of information it requires. Indeed, the scheme can be made as accurate as desired by increasing the number of evaluations limited, however, as the desired end point is reached, by the possibility of control parameters becoming indeterminate.

A crude analysis has been made of computer requirements to implement the scheme. These requirements are listed in detail in Table III. It should be borne in mind that the computer time estimates are based only on the program used in checking out the scheme. It may be of interest to note that the guidance evaluation takes ten seconds for one step on the RPC 4000, which means, for the cases in Tables I and II, that the coefficients would be obtained in real time. On the GE-225 digital computer the evaluation can be made in 0.5 seconds. The scheme requires 1300 words of storage; however, it is estimated that with optimum programming, this requirement can be reduced by approximately 50 percent.

The single RL-10 case is somewhat unrealistic in that 15,000 lb nominal thrust has been assumed, and thrust increases for some perturbations cannot be realized with the existing engine. The case does show, when compared with the 3-engine case, that the percent thrust variation required of the single engine is considerably higher than that required for the 3-engine case for nearly identical disturbances. In view of this trend, and because of long powered flight times and ranges required of the single engine flight profile, the single engine descent cannot be highly recommended.

TABLE I

SCHEME AND RADAR ERROR ANALYSIS

Continuous Burn of One Engine from 100 km Orbit

F_{NOM} = 15,000 lb

W_1 = 71,000 lb

W_T = 43,655 lb

Perturbation	$\lvert\Delta F\rvert$ lb	Δx m	Δh m	$\Delta\dot{x}$ m/s	$\Delta\dot{y}$ m/s	ΔWgt -lb
0	736	.05	-.04	-.001	-.001	0
Δx = 25 km	709	.03	-.02	-.002	-.002	-287
Δy = 25 km	1064	.07	-.03	.000	-.010	-606
$\Delta\dot{x}$ = -50 m/s	730	.07	-.05	.001	-.013	-194
$\Delta\dot{y}$ = 50 m/s	791	.06	-.03	.001	-.011	-277
ΔI_{sp} = -4.25	1826	2.57	2.54	.254	.566	-144
ΔWgt = 500 lb	1704	3.21	2.74	.285	.640	-155
Radar Errors						
1% Slant Range	1288	-2.86	-5.37	-.001	-.015	184
1% Range Rate + 1% Slant Range	502	-.29	-2.22	.101	.259	6
10% Angular Rate of Slant Range	1185	10.81	-5.51	1.036	-1.253	265
1° Δ Slant Range Angle	1372	-1.08	-3.93	.414	- .595	233
RSS of Scheme Error + Radar Error	3850	12	10	1.2	1.7	875

TABLE II

DESCENT FROM 15 km PERISELENUM OF HOHMANN TRANSFER ELLIPSE WITH 3 ENGINES

W_1 = 71,000 lb

F_{NOM} = 42,000 lb

W_T = 46,543 lb

Perturbation	$\lvert\Delta F\rvert$ lb	Δx m	Δh m	$\Delta\dot{x}$ m/s	$\Delta\dot{y}$ m/s	ΔWgt -lb
0	64	-.01	.01	-.013	-.005	0
Δx = -25 km	193	.00	.00	-.020	-.009	52
Δy = 10 km	196	-.02	.03	-.028	-.013	-391
$\Delta\dot{x}$ = -50 m/s	59	.00	-.01	-.018	-.012	-523
$\Delta\dot{y}$ = 50 m/s	135	-.02	.03	-.020	-.005	-194
ΔI_{sp} = - 4.25	2383	13.66	5.43	1.481	.666	-31
ΔWgt = 500 lb	2194	13.84	5.48	1.530	.690	-31
Radar Errors						
1% Slant Range	3139	-7.76	-4.91	-.016	.002	12
1% Range Rate + 1% Slant Range	1895	10.51	4.38	1.317	.603	-28
10% Angular Rate of Slant Range	689	18.78	-43.76	1.105	-2.726	-6
1° Δ Slant Range Angle	1457	-1.30	-17.60	.311	-2.320	9
RSS Scheme + Radar	5200	30	48	2.8	3.8	690

TABLE III

COMPUTER REQUIREMENTS FOR GUIDANCE SCHEME

I. Built in addition, subtraction, multiplication and division.

II. Subroutines

A. sin

B. cos

C. $\sqrt{\ }$

D. log

E. $\tan^{-1}$

F. integration for gravity computer, if required

III. Storage Requirements

A.	integration	16 (erasable)
B.	storage and program	910
C.	subroutines	250
	TOTAL ≈	1300

IV. Time Requirements per Step, no printing

A.	GE 225, internal floating point	0.5 sec
B.	RPC 4000, programmed floating point	10.0 sec

APPENDIX

IGNITION CRITERION FOR MAIN BRAKING

It is possible that some cases of main braking may require ignition before the beacon has been acquired. For this reason, the timer approach is attractive both for simplicity and accuracy. The following assumptions are made:

1. The period of the actual lunar orbit in which the vehicle flies prior to braking is known - measured from the point at which the slant range rate, $\dot{D}$, is zero.

2. The ignition time for the nominal case, $\bar{t}_{ig}$, is known and is available aboard the vehicle.

3. The thrust attitude program and the thrust magnitude for the nominal main braking case are precalculated and are available aboard the vehicle.

The time of ignition for any case, t_{ig}, is then defined as

$$t_{ig} = P - \bar{P} + \bar{t}_{ig} \, . \tag{a}$$

At this time the vehicle begins to fly, with the nominal thrust attitude program and the nominal thrust magnitude, until the beacon is acquired, when the guidance scheme produces the necessary thrust vector to accomplish the mission.

The following table displays a typical example, based on a nominal main braking phase from the 20 kilometer periselenum of a Hohmann transfer ellipse, the aposelenum of which is at 100 kilometers. It is assumed that three RL-10 engines with a nominal thrust of 40,000 pounds are available and that acquisition of the beacon occurs sixty seconds after ignition.

The table shows the change in thrust magnitude and direction called for when the beacon is acquired due to errors in the original ellipse, and errors resulting from flying the nominal thrust magnitude and direction in the region where the beacon is not available. The errors in state variables at this point can easily be handled by the guidance scheme.

ERROR ANALYSIS OF PERTURBED ORBITS
FOR TIMED IGNITION

Nominal Thrust = 40,000 lb

Guidance Scheme Becomes Effective 60 Seconds After Ignition

h_a km	hp km		ΔF60 lb	Δx km	Δy km	$\Delta\dot{x}$ m/s	$\Delta\dot{y}$ m/s	$\Delta\chi60$	ΔWgt lb
100	20		0	0	0	0	0	0	0
100	10		135	-.98	-10.23	6.79	3.11	13.02	-59
100	30		642	1.41	10.26	-6.73	-3.47	-13.51	-66
90	20		-91	.28	.04	-2.20	-.55	.10	0
110	20		73	-.29	-.04	2.18	.56	-.11	0

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