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DISTRIBUTED PARAMETER TYPE OF

CONTROL FOR A BILINEAR SYSTEM

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Abstract

A bilinear system is one that is linear in the state vector $X(z,t)$ and also in the control vector $\mu(z,t)$. The following differential equations for a bilinear system has the cross product term $\mu(z,t)X_1(z,t)$ under a spatial operator.

$$\left(1-b\frac{\partial^2}{\partial z^2}\right)\left[\mu(z,t)X_1(z,t)\right] = a_{11}X_1 + a_{12}X_2 \tag{1}$$

$$\frac{\partial}{\partial t}X_2(z,t) = a_{21}X_1 + a_{22}X_2 \tag{2}$$

with boundary conditions,

$$X_1(0,t) = X_1(1,t) = X_2(0,t) = X_2(1,t) = 0 \tag{3}$$

By the separation of variables the reference control is assumed independent of z for the reference solutions. The problem requires that the state variables start at a low level and reach a high level in a minimum time. It is well known from the optimum theory that for a bounded control the optimum process requires the control operating at its extreme values. However, one of the output of this system is found to be discontinuous due to the jump of the control. A second optimization is necessary since discontinuous output is incompatable with a real physical system, and disturbances are to be

compensated for the system. The perturbed equations with the exponential weighting function can be derived.

The second optimization minimizes a double integral with square erros as integrand from the present time t to a final time T_2 and over all the region interested. By application of calculus of variations the Euler-Lagrange equations and proper boundary conditions are obtained. These Euler-Lagrange equations with the original perturbed equations can be solved by using Finite Fourier Sine Transform. The optimum control laws, both spatial and time varying, can be obtained in closed form. If high harmonic disturbances are introduced in the system, they will be subdued by this control. Thus the output will follow closely the reference solution for arbitrary deviation at any time including that of the initial conditions.

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Introduction

Bilinear System occurs in controlling physical processes such as a nuclear reactor. Several papers[1,2] discussed the optimum control for these processes in lumped parameter models. In this paper the optimum control laws are determined analytically for a bilinear system in distributed parameter model.

Process Dynamics

A bilinear system is one that is linear in the state vector $X(z,t)$ and also in the control vector $\mu(z,t)$. However, there are cross product terms of the state and control variables such as $\mu(z,t)\ X(z,t)$. This system is overall linear if the control vector is an explicit function of independent variables. The following partial differential equations has the cross product term $\mu(z,t)X_1(z,t)$ under a spatial operator.

$$\left(1 - b\frac{\partial^2}{\partial z^2}\right) [\mu(z,t)X_1(z,t)] = a_{11}X_1 + a_{12}X_2 \tag{1}$$

$$\frac{\partial}{\partial t}X_2(z,t) = a_{21}X_1 + a_{22}X_2 \tag{2}$$

with the boundary conditions

$$X_1(0,t) = X_1(1,t) = X_2(0,t) = X_2(1,t) = 0 \tag{3}$$

where b and a's are constants. These equations with the boundary conditions can be used as a model for a nuclear reactor where X_1 represents the neutron flux, X_2 the precussor and μ the absorption cross-section. The problem is to start the reactor at very low power (neutron flux) to a very high power (perhaps 6 decades) in minimum time (say 60 seconds) subject to disturbance and arbitrary starting conditions.

The First Optimization Problem--Reference Variables

Since the distribution in space for both state variables X_1 and X_2 must satisfy the boundary conditions (3), the fundamental frequency of the spatial mode of variables X_1 and X_2 can be assumed as sinusoidal in z.

$$X_n(z,t) = Y_n(t)\sin\pi z \qquad n = 1,2 \tag{4}$$

The control variable $\mu(z,t)$ in this case is found later to be a function of time only. Thus,

$$\mu(z,t) = \mu_R(t) \tag{5}$$

Substituting the above equations into Equations (1) and (2), one obtains:

$$Y_1(t) = \frac{a_{12}}{(1+b\pi^2)\mu_R(t) - a_{11}} \; Y_2(t) \qquad (6)$$

and

$$\frac{d}{dt}\left[\, \ell n Y_2(t)\right] = \gamma\ (t) \qquad (7)$$

where

$$\gamma(t) = \frac{a_{21}\ a_{12}}{(1+b\pi^2)\mu_R(t) - a_{11}} + a_{22} \qquad (8)$$

If the control variable $\mu_R(t)$ is bounded for some reason, $\mu_{min} \leqslant \mu_R(t) \leqslant \mu_{max}$, the value of $\gamma(t)$ will be also bounded, $0 \leqslant \gamma(t) \leqslant \gamma$. Thus, the optimization problem for the state variable X_1 to start at a low level and reach a high level in a minimum time can be solved from Equation (7). It is well known from the optimum theory that for a bounded control the optimum process requires the control operating at its extreme value, i.e. at γ or zero (equivalent to μ_{min} or μ_{max}). This is a bang-bang type of control system. Therefore, Equation (4) can be explicitly written for the interval $0 \leqslant t \leqslant T_1$,

$$X_{nR}(z,t) = Y_{no}^{a}\ e^{\gamma t}\ \sin \pi z \qquad n = 1,2 \qquad (9)$$

and for the interval $T_1 \leqslant t \leqslant T_2$

$$X_{nR}(z,t) = Y_{no}^{b}\ e^{\gamma T_1}\ \sin \pi z \qquad n = 1,2 \qquad (10)$$

where Y_{no} are constants; superscripts a and b refer to the time interval $(0, T_1)$ and (T_1, T_2), respectively.

Both Equations (9) and (10) satisfy Equations (1) and (2) but have the ratios of Y_{1o}^a to Y_{2o}^a and Y_{1o}^b to Y_{2o}^b different from each other as given by Equation (6). These different ratios indicate the fact that if a continuity of X_{2_R} is required, i.e. $Y_{2o}^a = Y_{2o}^b$ at $t=T_1$, then X_{1_R} will suffer a jump at $t = T_1$, i.e. $Y_{1o}^a \neq Y_{1o}^b$ when control changes from γ to zero at $t = T_1$.

Perturbation Equations of the Bilinear System

Due to the present of disturbances and errors the state and control variables will cause some deviations about the reference. The actural variables can be expressed by the sum of the reference and their derivations. Thus, the state variables will not be sinusoidal and the control variables not be a function of time only. If these deviations from those unexpected distrubances and errors are small in comparison with the reference values, the cross product terms of the deviations may be neglected. The perturbed equation with exponential weighting function can be derived for the interval $0 \leqslant t \leqslant T_1$ as

$$(1 - b\frac{\partial^2}{\partial z^2})[\mu_R^a x_1^a(z,t)] = a_{11}x_1^a(z,t) + a_{12}\frac{Y_{2o}^a}{Y_{1o}^a}\, x_2^a(z,t) - u^a(z,t) \tag{11}$$

$$\frac{\partial}{\partial t}\, x_2^a(z,t) = a_{21}\,\frac{Y_{1o}^a}{Y_{2o}^a}\, x_1^a(z,t) + (a_{22} - \gamma)x_2^a(z,t) \tag{12}$$

where
$$x_n^a(z,t) = \frac{\Delta X_n(z,t)}{Y_{no}^a e^{\gamma t}} \qquad n = 1,2 \tag{13}$$

$$\mu_R^a = \mu_{min} \tag{14}$$

and $$u^a(z,t) = \frac{1}{Y_{1o}^a e^{\gamma t}}\left(1-b\frac{\partial^2}{\partial z^2}\right)[X_{1_R}\Delta\mu(z,t)] \tag{15}$$

Equation (15) can be inverted by using Green's function, if the quantity $u^a(z,t)$ is determined.[3]

$$\Delta\mu(z,t) = \frac{Y_{1o}^a e^{\gamma t}}{bX_{1_R}} \int_0^1 G(z,\xi)\, u^a(\xi,t)d\xi \tag{16}$$

It is noted that the exponential weighting function $e^{\gamma t}$ is introduced in Equation (13). The state variable X_{nR} of this system is exponential in time, as shown in Equation (9), so is ΔX_n for the same percentage of error. The exponential weighting function will give more uniform distribution of errors at final time as well as at initial time if the percentage of error is kept the same. More over, the weighting function will enlarge the applicable range of the perturbed equations. The perturbed equation for the interval $T_1 \leqslant t \leqslant T_2$ are similar to the Equations (11) and (12), obtained by changing the superscript from a to b, setting the quantity γ equal to zero and also letting

$$x_n^b(z,t) = \frac{\Delta X_n}{Y_{no}^b e^{\gamma T_1}} \qquad n = 1,\ 2 \tag{17}$$

Thus $$\Delta\mu(z,t) = \frac{Y_{1o}^b e^{\gamma T_1}}{b\ X_{1_R}} \int_0^1 G(z,\xi)\, u^b(\xi,t)d\xi \tag{18}$$

The Control System--Second Optimization

If the system starts at the right initial conditions with no disturbances then the control is exactly the same as that of the reference. Under these conditions the state variables of the system are descirbed by Equations (9) and (10). The control variable $\mu(z,t)$ differ from the reference μ_R by an amount of $\Delta\mu(z,t)$ due to the deviation of state variables. From Equations (16) and (18), the quantity $\Delta\mu(z,t)$ can be determined if $u(z,t)$ is known for each time interval. The problem now turns to find the solutions of $u(z,t)$ subjected to any arbitrary initial conditions of $x_1(z,t)$ and $x_2(z,t)$.

The control system requires the actural variables to follow the reference variables as close as possible. Thus, under the ideal conditions those quantities $x_1(z,t)$, $x_2(z,t)$ and $u(z,t)$ are identically zero. A cost functional to be minimized is chosen as

$$e(t)=\int_t^{T_2}\int_0^1\left\{\rho_1(z,\sigma)[x_1(z,\sigma)]^2+\rho_2(z,\sigma)[x_2(z,\sigma)]^2+\rho_3(z,\sigma)[u(z,\sigma)]^2\right\}dzd\sigma \tag{19}$$

where ρ_1, ρ_2, and ρ_3 are weighting functions. The present time t is used as lower limit in Equation (19). This is to minimize the integral from the time to go for the future under any arbitrary present conditions, according to the principle of Dynamic Programing.[4] If one of the values of $x_n(z,t)$ is measurable at present time, the control $u(z,t)$ or $\Delta\mu(z,t)$ can be determined in terms of the measured values. This implies that a feedback loop is introduced for this system. The functional e(t) will also be kept at a minimum for disturbances at any other time. As soon as one of the output deviation can be detected the source of disturbances will be nullified by readjusting the control element to produce $\Delta\mu(z,t)$, in turn, to subdue the output deviation from the reference.

Euler-Lagrange Equations

The method of calculus of variations is employed for minimizing the cost functional e(t) subjected to constraints given in Equations (11) and (12). A modified cost functional J including terms of constraints is constructed for the interval $0 \leq t \leq T_1$

$$J(t)=\int_t^{T_1}\int_0^1\Big\{\rho_1(x_1^a)^2+\rho_2(x_2^a)^2+\rho_3(u^a)^2+\lambda_1^a[(b\frac{\partial^2}{\partial z^2}-1)\mu_R^a x_1^a+a_{11}x_1^a+a_{12}\frac{Y_{2o}^a}{Y_{1o}^a}x_2^a-u^a]$$

$$+\lambda_2^a[a_{21}\frac{Y_{1o}^a}{Y_{2o}^a}x_1^a-(\gamma-a_{22})x_2^a-\frac{\partial}{\partial\sigma}x_2^a]\Big\}\,dz\,d\sigma$$

$$+\int_{T_1}^{T_2}\int_0^1\Big\{\rho_1(x_1^b)^2+\rho_2(x_2^b)^2+\rho_3(u^b)^2+\lambda_1^b[(b\frac{\partial^2}{\partial z^2}-1)\mu_R^b x_1^b+a_{11}x_1^b+a_{12}\frac{Y_{2o}^b}{Y_{1o}^b}x_2^b-u^b]$$

$$+\lambda_2^b[a_{21}\frac{Y_{1o}^b}{Y_{2o}^b}x_1^b+a_{22}x_2^b-\frac{\partial}{\partial\sigma}x_2^b]\Big\}\,dz\,d\sigma \qquad (20)$$

where λ's are Lagrange multipliers. Integrating by parts and collecting various terms for similar variables the first variation of cost functional J is obtained.[5,6]

$$\delta J=\int_{t}^{T_1}\int_{0}^{1}\Big\{[2\rho_1 x_1^a+\mu_R^a b\frac{\partial^2\lambda_1^a}{\partial z^2} - \mu_R^a\lambda_1^a+a_{11}\lambda_1^a+a_{21}\frac{Y_{1o}^a}{Y_{2o}^a}\lambda_2^a]\delta x_1^a$$

$$+[2\rho_2 x_2^a+a_{12}\frac{Y_{2o}^a}{Y_{1o}^a}\lambda_1^a-(\gamma-a_{22})\lambda_2^a+\frac{\partial\lambda_2^a}{\partial\sigma}]\delta x_2^a +[2\rho_3 u^a-\lambda_1^a]\delta u^a\Big\}dz d\sigma$$

$$+\int_{T_1}^{T_2}\int_{0}^{1}\Big\{[2\rho_1 x_1^b+\mu_R^b b\frac{\partial^2\lambda_1^b}{\partial z^2} - \mu_R^b\lambda_1^b+a_{11}\lambda_1^b+a_{21}\frac{Y_{1o}^b}{Y_{2o}^b}\lambda_2^b]\delta x_1^b$$

$$+[2\rho_2 x_2^b+a_{12}\frac{Y_{2o}^b}{Y_{1o}^b}\lambda_1^b+a_{22}\lambda_2^b - \frac{\partial\lambda_2^b}{\partial\sigma}]\delta x_2^b +[2\rho_3 u^b-\lambda_1^b]\delta u^b\Big\}dz d\sigma$$

$$+ \text{(T. C.)} \qquad (21)$$

where

$$\text{(T. C.)} = \mu_R^a b\int_{t}^{T_1}[\lambda_1^a\delta\frac{\partial x_1^a}{\partial z}-\frac{\partial\lambda_1^a}{\partial z}\delta x^a]_0^1 d\sigma+\mu_R^b b\int_{T_1}^{T_2}[\lambda_1^b\delta\frac{\partial x_1^b}{\partial z}-\frac{\partial\lambda_1^b}{\partial z}\delta x_1^b]_0^1 d\sigma$$

$$- \int_{0}^{1}\Big\{[\lambda_2^a\delta x_2^a]_{t}^{T_1} + [\lambda_2^b\delta x_2^b]_{T_1}^{T_2}\Big\}dz \qquad (22)$$

Thus, the following Euler-Lagrange equation is obtained by vanishing the integrand of the double integral for $0\leqslant\sigma\leqslant T_1$

$$2\rho_1 x_1^a+\mu_R^a b\frac{\partial^2\lambda_1^a}{\partial z^2} - \mu_R^a\lambda_1^a+a_{11}\lambda_1^a+a_{21}\frac{Y_{1o}^a}{Y_{2o}^a}\lambda_2^a = 0$$

$$2\rho_2 x_2^a+a_{12}\frac{Y_{2o}^a}{Y_{1o}^a}\lambda_1^a -(\gamma-a_{22})\lambda_2^a + \frac{\partial\lambda_2^a}{\partial\sigma} = 0 \qquad (23)$$

$$2\rho_3 u^a - \lambda_1^a = 0$$

Similar equations can be obtained for interval $T_1\leqslant\sigma\leqslant T_2$ by changing superscript from a to b and setting γ equal to zero.

Transversality and Boundary Conditions

In order that the first variation of functional J vanishes the transversality conditions by Equation (22) have to be also zero. This is sufficient if the integrands of each of the integrals in

Equation (22) are zero. For the first integral in Equation (22) one obtains

$$\lambda_1^a \delta \frac{\partial x_1^a}{\partial z}\Big|_{z=1} - \lambda_1^a \delta \frac{\partial x_1^a}{\partial z}\Big|_{z=0} - \frac{\partial \lambda_1^a}{\partial z}\delta x_1^a \Big|_{z=1} + \frac{\partial \lambda_1^a}{\partial z}\delta x_1^a\Big|_{z=0} = 0 \tag{24}$$

The fixed boundary conditions in Equation (3) imply no variation at the boundary points, i.e. $\delta x_1^a\Big|_{z=1} = \delta x_1^a\Big|_{z=0} = 0$

Thus, Equation (24) is simplied as

$$\lambda_1^a \delta \frac{\partial x_1^a}{\partial z}\Big|_{z=1} - \lambda_1^a \delta \frac{\partial x_1^a}{\partial z}\Big|_{z=0} = 0 \tag{25}$$

The above equation can be satisfied if

$$\lambda_1^a(1,\sigma) = \lambda_1^a(0,\sigma) = 0 \tag{26}$$

Similarly, one obtains the following equation from the second integral of Equation (23)

$$\lambda_1^b(1,\sigma) = \lambda_1^b(0,\sigma) = 0 \tag{27}$$

From the third integral one can write as

$$\lambda_2^a \delta x_2^a\Big|_{\sigma=T_1} - \lambda_2^a \delta x_2^a\Big|_{\sigma=t} + \lambda_2^b \delta x_2^b\Big|_{\sigma=T_2} - \lambda_2^b \delta x_2^b\Big|_{\sigma=T_1} = 0 \tag{28}$$

Since the variable x_2^a is known at present time, variation of this quantity is zero at $\sigma=t$ in the interval $t \leqslant \sigma \leqslant T_1$. Thus

$$\delta x_2^a\Big|_{\sigma=t} = 0 \tag{29a}$$

Imposing the natural boundary condition at the final point gives

$$\lambda_2^b\Big|_{\sigma=T_2} = 0 \tag{29b}$$

Equation (28) now reduced to

$$\lambda_2^a \delta x_2^a\Big|_{\sigma=T_1} - \lambda_2^b \delta x_2^b\Big|_{\sigma=T_1} = 0 \tag{30}$$

Now consider the cost functional J in Equation (20) with $t=T_1$. Under any known starting conditions the variation becomes

$$\delta x_2^b\Big|_{\substack{\sigma=t\\ t=T_1}} = 0 \qquad (T_1 \leqslant t \leqslant \sigma \leqslant T_2 \text{ in region b}) \tag{31a}$$

If the state variable X_2 is continuous at T_1 then

$$x_2^a\Big|_{\sigma=T_1} = x_2^b\Big|_{\substack{\sigma=t\\ t=T_1}} \tag{31b}$$

which implies

$$\delta x_2^a\Big|_{\sigma=T_1} = 0 \quad (t \leqslant \sigma \leqslant T_1 \text{ in region a}) \tag{31c}$$

Using the optimum control the future variable $x_2^a(\sigma)$ in this problem will be expressed in terms of $x_2^a(t)$ the present variable in the interval $t \leqslant \sigma \leqslant T_1$. For any given value of $x_2^a(t)$ the future variable $x_2^a(\sigma=T_1)$ can be determined. Thus one of the boundary conditions is

$$x_2^a(\sigma)\Big|_{\sigma=T_1} = x_2^b(\sigma)\Big|_{\substack{\sigma=t\\ t=T_1}} \tag{32}$$

Equations (26), (27), (29a), and (29b) are the transversality conditions. By examining the Euler-Lagrange equation and the perturbed equations, it is found that for unique solution of this system one more boundary condition is needed. This condition can be arbitrarily chosen by imposing a reasonable restriction on the system. A reasonable condition is that continuous state variables of the system are required. This is equivalent to (see Appendix A)

$$\left.\frac{\partial}{\partial\sigma}x_2^a\right|_{\sigma=T_1} + \gamma\sin\pi z + \gamma x_2^a\Big|_{\sigma=T_1} = \left.\frac{\partial}{\partial\sigma}x_2^b\right|_{\substack{\sigma=t\\ t=T_1}} \tag{33}$$

The second variation of functional J is

$$\delta^2 J = \int_t^{T_1}\int_0^1 [\rho_1(\delta x_1^a)^2 + \rho_2(\delta x_2^a)^2 + \rho_3(\delta u^a)^2]\, dz d\sigma + \int_{T_1}^{T_2}\int_0^1 [\rho_1(\delta x_1^b)^2 + \rho_2(\delta x_2^b)^2 + \rho_3(\delta u^b)^2] dz d\sigma \tag{34}$$

If the weighting functions, ρ's, are positive in the integral (t,T_2) function J will have a minimum provided the condition $\delta J=0$ is satisfied.

The Finite Fourier Sine Transform

In order to determine the optimum control law, it is necessary to solve the equations of auxiliary variables, Equation (23), and the equations of state variables, Equations (11) and (12). These equations can be transformed to a simple form by using the Finite Fourier Sine Transform.[7] Let us define

$$x_{jn}^k(\sigma) = \int_0^1 x_j^k(z,\sigma)\sin n\pi z dz$$

$$\lambda_{jn}^k(\sigma) = \int_0^1 \lambda_j^k(z,\sigma)\sin n\pi z dz$$

and $u_n^k(\sigma) = \int_0^1 u^k(z,\sigma) \sin n\pi z dz$, $k=a,b$; $j=1,2$; and $n=1,2,\ldots$

Applying the above transformation to Equations (11), (12), and (23) with the corresponding boundary conditions given in Equation (3) and transversality conditions from Equations (26) and (27), one obtains

$$-A_n^a x_{1n}^a + A_1^a x_{2n}^a - u_n^a = 0 \tag{35a}$$

$$\frac{\partial x_{2n}^a}{\partial \sigma} = (\gamma - a_{22}) x_{1n}^a - (\gamma - a_{22}) x_{2n}^a \tag{35b}$$

$$2\rho_1 x_{1n}^a - A_n^a \lambda_{1n}^a + (\gamma - a_{22}) \lambda_{2n}^a = 0 \tag{35c}$$

$$\frac{\partial \lambda_{2n}^a}{\partial \sigma} = -A_1^a \lambda_{1n}^a + (\gamma - a_{22}) \lambda_{2n}^a - 2\rho_2 x_{2n}^a \tag{35d}$$

$$2\rho_3 u_n^a - \lambda_{1n}^a = 0 \tag{35e}$$

where

$$A_1^a = \mu_R^a(\pi^2 b+1) - a_{11} = a_{12}\frac{Y_{2o}^a}{Y_{1o}^a};\quad A_n^a = \mu_R^a(n^2\pi^2 b+1) - a_{11},\ n=1,2,\ldots$$

$$\gamma - a_{22} = a_{21}\frac{Y_{1o}^a}{Y_{2o}^a}$$

Since the weighting functions, ρ_1, ρ_2, and ρ_3 are constants, Equation (35) consists of three linear algebraic equation and two first order linear differential equations with constant coefficients. Similarly, equations for interval $T_1 \leqslant \sigma \leqslant T_2$ are obtained by changing superscript from a to b and setting γ equal to zero.

The Optimum Control Law

Solving the above equations, one obtains the optimum control law in transform form for each time interval. By the inversion of the Finite Fourier Sine Transform[7] the optimum control laws are found in Appendix B as

$$u^a(z,t) = 2\sum_{n=1}^{\infty} u_n^a(t)\sin n\pi z \qquad 0 \leqslant t < T_1 \tag{36}$$

$$u^b(z,t) = 2\sum_{n=1}^{\infty} u_n^b(t)\sin n\pi z \qquad T_1 \leqslant t \leqslant T_2 \tag{37}$$

where

$$u_n^a(t) = \frac{A_n^a}{\alpha_n^a}\left[\frac{C_n^a G_n}{H_n} + \rho_1\right] x_{1n}^a(t) - \frac{A_n^a \beta_n^a \omega_n^a}{H_n}\psi_n \tag{38}$$

$$u_n^b(t) = \frac{A_n^b}{\alpha_n^b}\left[-\frac{a_{22}C_n^b}{B_n^b + \omega_n^b \coth\omega_n^b(T_2-t)} + \rho_1\right] x_{1n}^b(t) \tag{39}$$

$$x_{1n}^k(t) = \int_0^1 x_1^k(z,t)\sin n\pi z\,dz \qquad k = a,b$$

$$A_n^a = \mu_{min}(bn^2\pi^2+1) - a_{11}$$

$$A_n^b = \mu_{max}(bn^2\pi^2+1) - a_{11} \tag{40}$$

$$B_n^a = (\gamma - a_{22})\left(1 - \beta_n^a - \frac{\rho_2}{\alpha_n^a\beta_n^a}\right) \qquad B_n^b = -a_{22}\left(1 - \beta_n^b - \frac{\rho_2}{\alpha_n^b\beta_n^b}\right) \tag{41}$$

$$C_n^a = \rho_1\beta_n^a + \rho_2\frac{\rho_1+\alpha_n^a}{\alpha_n^a\beta_n^a} \qquad C_n^b = \rho_1\beta_n^b + \rho_2\frac{\rho_1+\alpha_n^b}{\alpha_n^b\beta_n^b} \tag{42}$$

$$G_n = \left\{\omega_n^a(\gamma - a_{22}) + \frac{\omega_n^a}{C_n^a}(\rho_1+\alpha_n^a)[B_n^a - (\gamma+\omega_n^b\Omega)]\right\}\cosh\omega_n^a(T_1-t)$$

$$+\left\{(\gamma+\omega_n^b\Omega)(\gamma-a_{22}) + \frac{\rho_1+\alpha_n^a}{C_n^a}[B_n^a(\gamma+\omega_n^b\Omega) - (\omega_n^a)^2]\right\}\sinh\omega_n^a(T_1-t)$$

$$H_n = [\omega_n^a(\omega_n^b\Omega + a_{22})]\cosh\omega_n^a(T_1-t) + [(\omega_n^a)^2 - (\gamma+\omega_n^b\Omega)(\gamma-a_{22})]\sinh\omega_n^a(T_1-t)$$

$$\alpha_n^a = \rho_3(A_n^a)^2, \qquad \alpha_n^b = \rho_3(A_n^b)^2 \tag{43}$$

$$\beta_n^a = \frac{A_1^a}{A_n^a}, \qquad \beta_n^b = \frac{A_1^b}{A_n^b} \tag{44}$$

$$\omega_n^b = -a_{22}[1-\frac{\alpha_n^b\beta_n^b(2-\beta_n^b)-\rho_2}{\rho_1+\alpha_n^b}]^{\frac{1}{2}} \qquad \omega_n^a = (\gamma-a_{22})[1-\frac{\alpha_n^a\beta_n^a(2-\beta_n^a)-\rho_2}{\rho_1+\alpha_n^a}]^{\frac{1}{2}} \tag{45}$$

$$\Omega = \frac{\omega_n^b(\rho_1+\alpha_n^b)\tanh\omega_n^b(T_2-T_1)+[B_n^b(\rho_1+\alpha_n^b)-a_{22}C_n^b]}{[B_n^b(\rho_1+\alpha_n^b)-a_{22}C_n^b]\tanh\omega_n^b(T_2-T_1)+\omega_n^b(\rho_1+\alpha_n^b)]}$$

$$\cong 1 \text{ for } T_2-T_1 \geqslant 60 \text{ sec. i.e. } \tanh\omega_n^b(T_2-T_1) \cong 1 \tag{46}$$

$$\psi_n = \frac{\gamma}{2}, \; n = 1; \quad \psi_n = 0, \; n \neq 1$$

The block diagram for implementing the optimum control law are shown in Figure 1. The numerical values from the above equations have also been computed for the optimum control of reactivity during nuclear rocket start-up. However, it is not given here due to the limited length of the paper.

Optimum Responses of the State Variables

By knowing the optimum control law, the responses of $x_1(z,t)$ and $x_2(z,t)$ can be obtained from the perturbed equations in terms of its Finite Fourier Sine Transforms:[7]

$$x_j^k(z,t) = 2\sum_{n=1}^{\infty} x_{jn}^k(t) \sin n\pi z; \quad k=a,b \text{ and } n=1,2,\ldots \tag{47}$$

The following equations are the responses of $x_{1n}(t)$ and $x_{2n}(t)$ for the interval $0 \leqslant t \leqslant T_1$ obtained from Equations (35) and (38):

$$x^a_{1n}(t) = g(t)x^a_{2n}(t) + f(t), \quad n = 1, 2, .. \tag{48a}$$

$$\frac{dx^a_{2n}}{dt} = (\gamma-a_{22})[g(t)-1]x^a_{2n} + (\gamma-a_{22})f(t),$$

$$x^a_{2n}(t) = [\int_0^t (\gamma-a_{22})f(t)\, e^{-(\gamma-a_{22})\int_0^t[g(t)-1]dt}\, dt$$

$$+ x^a_{2n}(0)]\, e^{(\gamma-a_{22})\int_0^t[g(t)-1]dt} \tag{48b}$$

where

$$f(t) = \frac{\omega^a_n\psi^a_n}{H_n}\, g(t)\ , \qquad g(t) = \frac{\beta^a_n\alpha^a_n}{\alpha^a_n+\rho_1+\frac{C^a_nG_n}{H_n}} \tag{49}$$

Similarly, the response for the interval $T_1 \leqslant t \leqslant T_2$ are

$$x^b_{1n}(t) = h(t)x^b_{2n}(t), \qquad n = 1, 2, .. \tag{50a}$$

$$\frac{dx^b_{2n}}{dt} = -a_{22}[h(t)-1]x^b_{2n}\ ,$$

$$x^b_{2n}(t) = x^b_{2n}(T_1)\, e^{-a_{22}\int_{T_1}^t [h(t)-1]dt} \tag{50b}$$

where

$$h(t) = \frac{\alpha^b_n\beta^b_n}{\alpha^b_n - \frac{a_{22}C^b_n}{B^b_n+\omega^b_n\coth\omega^b_n(T_2-t)}} \tag{51}$$

The response curves are characterized by the properties of function h(t), g(t) and f(t). It can be seen from Equation (50b) that if $a_{22} < 0$ (reactor decay constant $= \lambda = -a_{22} > 0$) and $h(t) < 1$, the response x^b_{2n} will decrease with time, so does the response of x^b_{1n}. Thus, the sufficient conditions for the weighting function ρ_1 and ρ_2 to have the decreasing responses as shown in Appendix C are

$$\rho_3 = 1 \ , \qquad \rho_1 \gg \alpha_n^b \qquad \text{and} \qquad \alpha_n^a \beta_n^a \ (2 - \beta_n^a) > \rho_2 \gtrsim 0 \tag{52}$$

Conclusion

Attempts are made to control the bilinear system in both the time and space domains. Two consecutive optimization procedures are applied for achieving this purpose. The bang-bang control is used for selecting the reference variables and the optimum feedback control is employed to adjust the system to approximately follow the reference variables under arbitrary starting conditions and disturbances.

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Appendix A Condition for continuous derivative of X_2 at T_1

By definition of Equations (13) and (17)

$$x_2^a(z,\sigma) = \frac{\Delta X_2^a(z,\sigma)}{Y_{2o}^a e^{\gamma\sigma}} = \frac{X_2^a(z,\sigma) - Y_{2o}^a e^{\gamma\sigma}\sin\pi z}{Y_{2o}^a e^{\gamma\sigma}} = \frac{X_2^a(z,\sigma)}{Y_{2o}^a\ e^{\gamma\sigma}} - \sin\pi z \quad \text{(A1)}$$

$$x_2^b(z,\sigma) = \frac{\Delta X_2^b(z,\sigma)}{Y_{2o}^b\ e^{\gamma T_1}} = \frac{X_2^b(z,\sigma) - Y_{2o}^b e^{\gamma T_1}\sin\pi z}{Y_{2o}^b e^{\gamma T_1}} = \frac{X_2^b(z,\sigma)}{Y_{2o}^b e^{\gamma T_1}} - \sin\pi z \quad \text{(A2)}$$

thus,

$$\frac{\partial}{\partial\sigma}x_2^a\Big|_{\sigma=T_1} - \frac{\partial}{\partial\sigma}x_2^b\Big|_{\substack{\sigma=t\\ t=T_1}} = \frac{\partial}{\partial\sigma}\left[\frac{X_2^a(z,\sigma)}{Y_{2o}^a\ e^{\gamma\sigma}}\right]\Big|_{\sigma=T_1} - \frac{\partial}{\partial\sigma}\left[\frac{X_2^b(z,\sigma)}{Y_{2o}^b e^{\gamma T_1}}\right]\Big|_{\substack{\sigma=t\\ t=T_1}}$$

$$= \frac{\frac{\partial}{\partial\sigma}X_2^a(z,\sigma)}{Y_{2o}^a\ e^{\gamma\sigma}}\Big|_{\sigma=T_1} - \gamma\frac{X_2^a(z,\sigma)}{Y_{2o}^a\ e^{\gamma\sigma}}\Big|_{\sigma=T_1} - \frac{\frac{\partial}{\partial\sigma}X_2^b(z,\sigma)}{Y_{2o}^b e^{\gamma T_1}}\Big|_{\substack{\sigma=t\\ t=T_1}} \quad \text{(A3)}$$

For continuous slope of $X_2(z,\sigma)$ at $\sigma = T_1$, we have

$$\frac{\partial X_2^a(z,\sigma)}{\partial\sigma}\Big|_{\sigma=T_1} = \frac{\partial X_2^b(z,\sigma)}{\partial\sigma}\Big|_{\substack{\sigma=t\\ t=T_1}} \quad \text{(A4)}$$

Substituting into Equation (A3) gives

$$\frac{\partial}{\partial\sigma}x_2^a(z,\sigma)\Big|_{\sigma=T_1} - \frac{\partial}{\partial\sigma}x_2^b(z,\sigma)\Big|_{\substack{\sigma=t\\ t=T_1}} = -\gamma x_2^a(z,\sigma)\Big|_{\sigma=T_1} - \gamma\sin\pi z \quad \text{(A5)}$$

or

$$\frac{\partial}{\partial\sigma}x_2^a(z,\sigma)\Big|_{\sigma=T_1} + \gamma x_2^a(z,\sigma)\Big|_{\sigma=T_1} + \gamma\sin\pi z = \frac{\partial}{\partial\sigma}x_2^b(z,\sigma)\Big|_{\substack{\sigma=t\\ t=T_1}} \quad \text{(A6)}$$

The Finite Fourier Sine Transform of the above equation is

$$\frac{\partial}{\partial\sigma}x_{2n}^a(\sigma)\Big|_{\sigma=T_1} + \gamma x_{2n}^a(\sigma)\Big|_{\sigma=T_1} + \psi_n = \frac{\partial}{\partial\sigma}x_{2n}^b(\sigma)\Big|_{\substack{\sigma=t\\ t=T_1}} \quad \text{(A7)}$$

where $\psi_n = \frac{\gamma}{2}$, $n = 1$; $\psi_n = 0$, $n \neq 1$

Appendix B Solution of optimum control law

Equation (35) is used here for solving the optimum control law for the interval $0 \leqslant t < T_1$. Eliminating λ^a_{1n} from Equations (35e) and (35c), one obtains

$$u^a_n(\sigma) = \frac{A^a_n}{2\alpha^a_n}[(\gamma - a_{22})\lambda^a_{2n} + 2\rho_1 x^a_{1n}] \tag{B1}$$

where

$$\alpha^a_n = \rho_3(A^a_n)^2$$

Substituting Equation (B1) into Equation (35a) gives

$$x^a_{2n} = \frac{1}{2\alpha^a_n\beta^a_n}[(\gamma - a_{22})\lambda^a_{2n} + 2(\rho_1 + \alpha^a_n)x^a_{1n}] \tag{B2}$$

The following equation can be obtained by substituting the above equation into Equation (35b):

$$[\frac{d}{d\sigma} + (\gamma - a_{22})(1 - \frac{\alpha^a_n\beta^a_n}{\rho_1 + \alpha^a_n})]x^a_{1n} + \frac{\gamma - a_{22}}{2(\rho_1 + \alpha^a_n)}[\frac{d}{d\sigma} + (\gamma - a_{22})]\lambda^a_{2n} = 0 \tag{B3}$$

where

$$\beta^a_n = \frac{A^a_1}{A^a_n}$$

Eliminating λ^a_{1n} and x^a_{2n} from Equations (35c), (35d) and (B2), we have

$$2C^a_n x^a_{1n} + (\frac{d}{d\sigma} - B^a_n)\lambda^a_{2n} = 0 \tag{B4}$$

where

$$B^a_n = (\gamma - a_{22})(1 - \beta^a_n - \frac{\rho_2}{\alpha^a_n\beta^a_n}), \quad C^a_n = \rho_1\beta^a_n + \rho_2\frac{\rho_1 + \alpha^a_n}{\alpha^a_n\beta^a_n}$$

Solutions for λ^a_{2n} and x^a_{1n} from Equations (B3) and (B4) are

$$\lambda^a_{2n} = E_1(t)\cosh\omega^a_n(\sigma - t) + E_2(t)\sinh\omega^a_n(\sigma - t) \tag{B5}$$

$$x_{1n}^{a} = -\frac{1}{2C_n^a}\Big\{E_1(t)[\omega_n^a \sinh\omega_n^a(\sigma-t) - B_n^a \cosh\omega_n^a(\sigma-t)]$$

$$+ E_2(t)[\omega_n^a \cosh\omega_n^a(\sigma-t) - B_n^a \sinh\omega_n^a(\sigma-t)]\Big\} \qquad \text{(B6)}$$

where

$$\omega_n^a = (\gamma-a_{22})\sqrt{1 - \frac{\alpha_n^a\beta_n^a(2-\beta_n^a) - \rho_2}{\rho_1 + \alpha_n^a}} \qquad \text{(B7)}$$

t = a quantity carried as parameter

From Equation (B2) one obtains

$$x_{2n}^{a} = \frac{1}{2\alpha_n^a\beta_n^a}\Big\{E_1(t)[M_n^a \cosh\omega_n^a(\sigma-t) - N_n^a \sinh\omega_n^a(\sigma-t)]$$

$$+ E_2(t)[M_n^a \sinh\omega_n^a(\sigma-t) - N_n^a \cosh\omega_n^a(\sigma-t)]\Big\} \qquad \text{(B8)}$$

where

$$M_n^a = (\gamma-a_{22}) + \frac{B_n^a}{C_n^a}(\rho_1+\alpha_n^a), \qquad N_n^a = \frac{\omega_n^a}{C_n^a}(\rho_1+\alpha_n^a)$$

The similar solution for the interval $T_1 \leqslant \sigma \leqslant T_2$ can be obtained by changing superscripts from a to b and arbitrary constants from E to F.

$$\lambda_{2n}^{b} = F_1(t)\cosh\omega_n^b(\sigma-t) + F_2(t)\sinh\omega_n^b(\sigma-t) \qquad \text{(B9)}$$

$$, \quad x_{1n}^{b} = -\frac{1}{2C_n^b}\Big\{F_1(t)[\omega_n^b \sinh\omega_n^b(\sigma-t) - B_n^b \cosh\omega_n^b(\sigma-t)]$$

$$+ F_2(t)[\omega_n^b \cosh\omega_n^b(\sigma-t) - B_n^b \sinh\omega_n^b(\sigma-t)]\Big\} \qquad \text{(B10)}$$

$$x_{2n}^{b} = \frac{1}{2\alpha_n^b\beta_n^b}\Big\{F_1(t)[M_n^b \cosh\omega_n^b(\sigma-t) - N_n^b \sinh\omega_n^b(\sigma-t)]$$

$$+ F_2(t)[M_n^b \sinh\omega_n^b(\sigma-t) - N_n^b \cosh\omega_n^b(\sigma-t)]\Big\} \qquad \text{(B11)}$$

The optimum control law for the interval $0 \leqslant t < T_1$ is obtained by setting $\sigma = t$ in Equation (B1) and (B5)

$$u_n^a(t) = \frac{A_n^a}{2\alpha_n^a}[(\gamma-a_{22})E_1(t) + 2\rho_1 x_{1n}^a(t)] \tag{B12}$$

The arbitrary constant $E_1(t)$ is to be determined for obtaining the optimum control law. Substituting Equation (B11) into Equation (A7) leads to

$$\frac{\partial}{\partial\sigma}x_{2n}^a\Big|_{\sigma=T_1} + \gamma x_{2n}^a\Big|_{\sigma=T_1} + \psi_n = \frac{\omega_n^b}{2\alpha_n^b\beta_n^b}[-F_1(T_1)N_n^b + F_2(T_1)M_n^b] \tag{B13}$$

By using Equations (B9)and (29b) and setting $t = T_1$ we have

$$F_1(T_1) = -F_2(T_1)\tanh\omega_n^b(T_2-T_1) \tag{B14}$$

Substituting Equation (B11) into Equation (32a) and letting $t = T_1$ gives

$$x_{2n}^a\Big|_{\sigma=T_1} = \frac{1}{2\alpha_n^b\beta_n^b}[M_n^b F_1(T_1) - N_n^b F_2(T_1)] \tag{B15}$$

Eliminating $F_1(T_1)$ and $F_2(T_1)$ from Equations (B13), (B14) and (B15) one obtains

$$\frac{\partial}{\partial\sigma}x_{2n}^a\Big|_{\sigma=T_1} + (\gamma+\omega_n^b\Omega)x_{2n}^a\Big|_{\sigma=T_1} + \psi_n = 0 \tag{B16}$$

where

$$\Omega = \frac{N_n^b\tanh\omega_n^b(T_2-T_1) + M_n^b}{M_n^b\tanh\omega_n^b(T_2-T_1) + N_n^b}$$

The following equation is obtained by substituting Equation (B8) into Equation (B16)

$$E_1(t)L_n + E_2(t)G_n + 2\alpha_n^a\beta_n^a\psi_n = 0 \tag{B17}$$

where

$$G_n = [\omega_n^a(\gamma-a_{22}) + \frac{\omega_n^a}{C_n^a}(\rho_1+\alpha_n^a)(B_n^a-\gamma-\omega_n^b\Omega)]\cosh\omega_n^a(T_1-t)$$
$$+ \left\{(\gamma+\omega_n^b\Omega)(\gamma-a_{22}) + \frac{\rho_1+\alpha_n^a}{C_n^a}[B_n^a(\gamma+\omega_n^b\Omega)-(\omega_n^a)^2]\right\}\sinh\omega_n^a(T_1-t)$$

$$L_n = [\omega_n^a(\gamma-a_{22})+ \frac{\omega_n^a}{C_n^a}(\rho_1+\alpha_n^a)(B_n^a-\gamma-\omega_n^b\Omega)]\sinh\omega_n^a(T_1-t)$$

$$+\left\{(\gamma+\omega_n^b\Omega)(\gamma-a_{22})+ \frac{\rho_1+\alpha_n^a}{C_n^a}[B_n^a(\gamma+\omega_n^b\Omega)-(\omega_n^a)^2]\right\}\cosh\omega_n^a(T_1-t)$$

Since the quantity $x_{1n}^a(t)$ is measurable, from Equation (B6) one obtains

$$x_{1n}^a(t) = \frac{1}{2C_n^a}[B_n^a E_1(t) - \omega_n^a E_2(t)] \tag{B18}$$

Solving $E_1(t)$ from Equations (B17) and (B18) and substituting into Equation (B12), we have the optimum control law

$$u_n^a(t) = \frac{A_n^a}{\alpha_n^a}[\frac{C_n^a G_n}{H_n} + \rho_1]x_{1n}^a(t) - \frac{A_n^a \beta_n^a \omega_n^a}{H_n}\psi_n \tag{B19}$$

where

$$H_n = [\omega_n^a(\omega_n^b\Omega+a_{22})]\cosh\omega_n^a(T_1-t)$$
$$+ [(\omega_n^a)^2- (\gamma+\omega_n^b\Omega)(\gamma-a_{22})]\sinh\omega_n^a(T_1-t)$$

The optimum control law for the interval $T_1 \leqslant t \leqslant T_2$ can be obtained by the similar procedures.

Appendix C Sufficient condition for decreasing responses

The weighting functions ρ_1, ρ_2 and ρ_3 in Equation (19) is relative. One can assume that ρ_3 is unity without losing any generality. Thus

$$\rho_3 = 1 > 0 \tag{C1}$$

The quantities A_n, C_n, α_n and β_n in Equations (40), (42), (43) and (44), respectively, are positive for a nuclear reactor. Also the quantity β_n^b given by Equation (44) is between zero and unity. Thus the function of h(t) in Equation (51) will be less than unity if

$$B_n^b + \omega_n^b \coth\omega_n^b(T_2-t) > 0 \tag{C2}$$

Because the value of $\coth\omega_n^b(T_2-t)$ is larger than unity and $\omega_n^b > 0$

we can write

$$B_n^b + \omega_n^b \coth\omega_n^b(T_2-t) \geqslant B_n^b + \omega_n^b \tag{C3}$$

Substituting the value of B_n^b and ω_n^b into the above equation gives

$$B_n^b+\omega_n^b\coth\omega_n^b(T_2-t) \geqslant B_n^b+\omega_n^b \cong -a_{22}[2-\beta_n^b- \frac{\rho_2}{\alpha_n^b\beta_n^b} - \frac{\alpha_n^b\beta_n^b(2-\beta_n^b)-\rho_2}{2(\rho_1+\alpha_n^b)}] \tag{C4}$$

Therefore, Equation (C2) can be satisfied if

$$\alpha_n^b\beta_n^b(2 - \beta_n^b) > \rho_2 \geqslant 0 \tag{C5}$$

and

$$\rho_1 \gg \alpha_n^b \tag{C6}$$

This conditions are sufficient for the weighting functions ρ_1, and ρ_2 to have ensure the decreasing response in the interval $T_1 \leqslant t \leqslant T_2$.

The sufficient condition for the interval $0 \leqslant t \leqslant T_1$ can be obtained as follows. By using the definition of C_n^a, H_n and G_n in Equation (49) one obtains

$$g(t)= - \frac{\omega_n^a(\omega_n^b\Omega+a_{22})\cosh\omega_n^a(T_1-t)+[(\omega_n^a)^2-(\gamma+\omega_n^b\Omega)(\gamma-a_{22})]\sinh\omega_n^a(T_1-t)}{[(\gamma-a_{22})+(\omega_n^b\Omega+a_{22})\frac{1-\omega_n^a}{\beta_n^a}]\cosh\omega_n^a(T_1-t)+(\gamma+\omega_n^b\Omega)(\gamma-a_{22})\sinh\omega_n^a(T_1-t)} \tag{C7}$$

With the conditions in Equations (46) and (C6) we have the following approximations

$$\omega_n^a(\omega_n^b\Omega+a_{22}) \cong a_{22}(\gamma-a_{22})\frac{\alpha_n^b\beta_n^b(2 - \beta_n^b) - \rho_2}{2(\rho_1 + \alpha_n^b)} \tag{C8}$$

$$(\omega_n^a)^2 - (\gamma+\omega_n^b\Omega)(\gamma-a_{22}) \cong - (\gamma-a_{22})[(\gamma-a_{22}) \frac{\alpha_n^a\beta_n^a(2-\beta_n^a)-\rho_2}{\rho_1 + \alpha_n^a} + a_{22} \frac{\alpha_n^b\beta_n^b(2-\beta_n^b)-\rho_2}{2(\rho_1+\alpha_n^b)}] \tag{C9}$$

$$(\gamma + \omega_n^b\Omega)(\gamma - a_{22}) \cong (\gamma - a_{22})^2 \tag{C10}$$

$$(\gamma-a_{22}) + (\omega_n^b\Omega+a_{22}) \frac{1-\omega_n^a}{\beta_n^a} \cong (\gamma- a_{22}) + a_{22}\frac{\alpha_n^b\beta_n^b(2-\beta_n^b)-\rho_2}{2(\rho_1+\alpha_n^b)} \tag{C11}$$

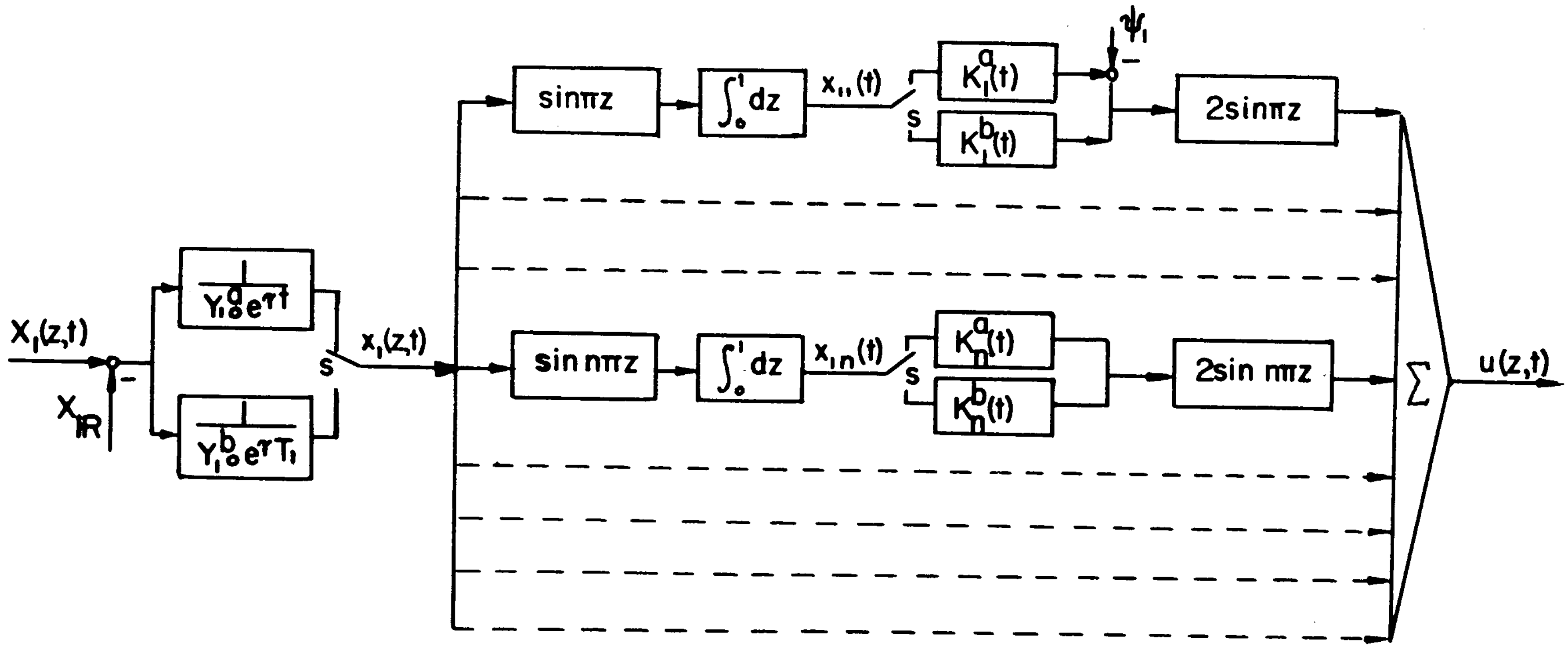
For the nuclear reactor we know that $\alpha_n^b > \alpha_n^a$, $\beta_n^b \cong \beta_n^a$, $0<\lambda<1$ and $0<\gamma<1$. If we let $\alpha_n^a\beta_n^a(2-\beta_n^a) > \rho_2 \gtrsim 0$, it is concluded that

$\omega_n^a(\omega_n^b\Omega+a_{22})<0$, $(\gamma+\omega_n^b\Omega)(\gamma-a_{22})>0$, $(\gamma-a_{22})+(\omega_n^b\Omega+a_{22})\frac{1-\omega_n^a}{\beta_n^a} >0$ and

$$\left|(\gamma-a_{22})+(\omega_n^b\Omega+a_{22})\frac{1-\omega_n^a}{\beta_n^a}\right| > \left|(\gamma+\omega_n^b\Omega)(\gamma-a_{22})\right| > \left|\omega_n^a(\omega_n^b\Omega+a_{22})\right| > \left|(\omega_n^a)^2-(\gamma+\omega_n^b\Omega)(\gamma-a_{22})\right|$$

From the above equations we conclude that Equation (C7) is positive and less than unity, i.e.

$$0 < g(t) < 1$$



$$K_n^a(t) = \frac{A_n^a}{\alpha_n^a}\left(\frac{C_n^a G_n}{H_n} + \rho_1\right), \qquad K_n^b(t) = \frac{A_n^b}{\alpha_n^b}\left(\frac{-a_{22}C_n^b}{B_n^b + \omega_n^b \coth \omega_n^b (T_2 - t)} + \rho_1\right), \qquad \psi_1 = \frac{A_1^a \beta_1^a \omega_1^a \gamma}{2H_1}.$$

Switches s will switch at $t = T_1$

Fig I Block Diagram For Control Variable u