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by J. D. Cryer and M. R. Leadbetter

Prepared under Contract No. NASw-905 *by*
RESEARCH TRIANGLE INSTITUTE
Durham, N. C.
for

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PREFACE

This report is the third technical report to be issued under Contract NASw-905 and is devoted specifically to documentation of some additional theoretical results on probabilistic modeling obtained in the basic research effort under the contract. The basic research studies consist mainly of investigations in curve, level and zero crossings by certain normal stochastic processes (both stationary and non-stationary). Such investigations provide measures of the quality of performance and the reliability of certain complex systems.

The text of this report is devoted primarily to the rigorous mathematical development of an expression for the variance of the number of zeros of a stationary random process. The major theoretical results of the study are stated in the first section as a theorem. From the practical viewpoint of computing the desired variance quantity, equation 1.8 is also presented in the first section. The remaining sections are devoted exclusively to the mathematical rigor for proof of the theorem.

CONTENTS

The Variance of the Number of Zeros of Stationary Normal Processes

1. Introduction

Let $\{x(t),\ t \in [0,T]\}$ be a real separable stationary normal stochastic process with $\mathcal{E}[x(t)] = 0$, $\mathrm{Var}[x(t)] = 1$, with covariance function $r(\tau) = \mathcal{E}[x(0)x(\tau)]$ and corresponding (integrated) spectrum $F(\lambda)$, that is,

$$r(\tau) = \int_0^\infty \cos\lambda\tau \; dF(\lambda) \ . \tag{1.1}$$

We consider a random variable N_x defined as the number of zeros of $x(t)$ on the interval $[0,T]$. The importance of N_x and its statistical properties in reliability applications has been discussed previously in the reports of Cramér (1962) and Leadbetter (1963). The mean value of N_x is known even for a large class of non stationary normal process (Leadbetter and Cryer (1964)), the result for the (stationary) case at hand being

$$\mathcal{E}[N_x] = T\,\lambda_2^{1/2}/\pi \tag{1.2}$$

where λ_2 is the second spectral moment, i.e.,

$$\lambda_2 = \int_0^\infty \lambda^2 \; dF(\lambda) \ .$$

Recently, Ylvisaker (1963) has shown that (1.2) is valid under certain weak conditions even if $\lambda_2 = +\infty$. We note here that $\lambda_2 < +\infty$ is equivalent to $r(\tau)$ having a (finite) second derivative at the origin.

Higher moments of N_x have received much less attention in the literature. The formula for $\mathcal{E}[N_x^2]$ is implicit (under certain conditions) in the work of Rice (1944) and is given for a particular normal process by Steinberg, et al (1955). The first derivation for a somewhat general situation seems to occur in a footnote of a paper by Rozanov and Volkonskii (1961) where it is assumed that the sixth spectral moment is finite (or, equivalently, that $r(\tau)$ has a sixth derivative at the origin).

The purpose of this report is to derive the formula for $\mathcal{E}[N_x^2]$ under quite weak conditions, given in the following result.

Theorem: Suppose that the covariance function $r(\tau)$ has a second derivative $r''(\tau)$ which, for all sufficiently small τ, satisfies

$$\lambda_2 + r''(\tau) \leq \Psi(\tau) \quad , \tag{1.4}$$

where $\Psi(\tau)/\tau$ is integrable over $[0,T]$ and $\Psi(\tau)$ decreases monotonely as τ decreases to zero.

Suppose further that the spectral distribution $F(\lambda)$ has a continuous component. Then we have

$$\mathcal{E}[N_x^2] = \mathcal{E}[N_x] + \int_0^T \int_0^T ds\,dt \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |xy| \; p_{t-s}(0,0,x,y)\,dx\,dy \quad , \tag{1.5}$$

where $p_{t-s}(u,v,x,y)$ is the four-variate normal density for the variables $x(t)$, $x(s)$, $x'(t)$, $x'(s)$, and where $x'(t)$, $x'(s)$ denote the (quadratic mean) derivatives of $x(t)$ at t and s, respectively.

The statement of the theorem as it stands is convenient for theoretical purposes. From a practical (computational) standpoint however, the right side of (1.5) may be made somewhat more explicit. Specifically we may write

$$p_{t-s}(u,v,x,y) = (2\pi)^{-2} \; |\Sigma|^{-1/2} \exp\,[-(u,v,x,y)\;\Sigma^{-1}\;(u,v,x,y)'/2] \quad , \tag{1.6}$$

where $\Sigma = \Sigma(\tau)$ is given as

$$\Sigma = \begin{bmatrix} 1 & r(\tau) & 0 & -r'(\tau) \\ r(\tau) & 1 & r'(\tau) & 0 \\ 0 & r'(\tau) & \lambda_2 & -r''(\tau) \\ -r'(\tau) & 0 & -r''(\tau) & \lambda_2 \end{bmatrix} \quad , \; \tau = t-s.$$

Equation (1.5) may then be evaluated (see, for example Rice (1944) or Steinberg, et al (1955)) to yield

$$(1.8)\quad \mathcal{E}[N_x^2] = T\,\lambda_2^{1/2}/\pi + (2/\pi^2)\int_0^T (T-\tau)\,(\Sigma_{33}^2 - \Sigma_{34}^2)^{1/2}(1-r^2(\tau))^{-3/2}[1+\Delta\tan^{-1}\Delta]d\tau,$$

where Σ_{ij} is the (i,j) cofactor of Σ and $\Delta = \Sigma_{34}(\Sigma_{33}^2 - \Sigma_{34}^2)^{-1/2}$, the dependence of Σ_{ij} and Δ on $\tau = t-s$ being suppressed.

The proof of the theorem follows from a series of lemmas given in the next three sections.

2. <u>The process</u> $\{y_n(t)\}$.

There is no loss of generality in taking $T = 1$ and we do so. We use the method of Bulinskaya (1961) in approximating the $x(t)$-process by a sequence of processes defined as follows:

For each positive integer n and each $t \in [0,1]$, let $k = k_n(t)$ be the unique integer such that $k/2^n \leq t < (k+1)/2^n$, (so that $0 \leq k \leq 2^n$). Then define

$$(2.1)\qquad y_n(t) = x(k/2^n) + 2^n(t-k/2^n)[x((k+1)/2^n) - x(k/2^n)],$$

that is, $\{y_n(t):\ 0 \leq t \leq 1\}$ is a new process coinciding with $x(t)$ at points of the form $k/2^n$, and consisting of straight line segments between such points.

Let N_{y_n} denote the number of zeros of the $y_n(t)$- process for $0 \leq t \leq 1$. By definition of $y_n(t)$, clearly $N_{y_n} \leq N_x$.

Ylvisaker (private communication) has shown that the set of sample functions of $x(t)$ which are tangent to the axis somewhere has probability zero. Hence it is easily seen that $N_{y_n} \longrightarrow N_x$, with probability one, as $n \longrightarrow \infty$. Hence also $N_{y_n}^2 \longrightarrow N_x^2$, a.s., and by monotone convergence we have

Lemma 2.1: $\mathcal{E}[N^2_{y_n}] \longrightarrow \mathcal{E}[N^2_x]$, as $n \longrightarrow \infty$.

To evaluate $\mathcal{E}[N^2_{y_n}]$ we use a sequence of functions "approaching a Dirac delta function," viz.,

Definition: A sequence $\{\delta_m(x)\}$ of non negative integrable functions will be called a δ-function sequence if

(2.2)

(i) $\int_{-\infty}^{+\infty} \delta_m(x)\, dx = 1 \qquad$ for all $m = 1,2,\ldots$,

and

(ii) $\int_{-\epsilon}^{\epsilon} \delta_m(x)\, dx \longrightarrow 1$ as $m \longrightarrow \infty$ for any $\epsilon > 0$.

We now evaluate $N^2_{y_n}$ analytically.

Lemma 2.2: If $\{\delta_m(x)\}$ is any δ-function sequence, then, with probability one, we have

(2.3) $$N^2_{y_n} = \lim_{m \to \infty} \int_0^1 \int_0^1 \delta_m[y_n(t)]\, \delta_m[y_n(s)]\, |y_n'(t) y_n'(s)|\, ds dt \quad ,$$

and

(2.4) $$\int_0^1 \int_0^1 \delta_m[y_n(t)]\, \delta_m[y_n(s)]\, |y_n'(t) y_n'(s)|\, ds dt \leq 2^{2n} \quad , \text{ all } m.$$

Proof: The proof follows directly from Lemma 2 of Leadbetter and Cryer (1964) which states that

(2.5)

$$N_{y_n} = \lim_{m \to \infty} \int_0^1 \delta_m[y_n(t)]\, |y_n'(t)|\, dt$$

and

$$\int_0^1 \delta_m[y_n(t)]\, |y_n'(t)|\, dt \leq 2^n \quad .$$

The inequality (2.4) allows us to apply the dominated convergence theorem to (2.3) yielding

Lemma 2.3:

$$\mathcal{E}[N^2_{y_n}] = \lim_{m\to\infty} \int_0^1\int_0^1 \mathcal{E}\{\delta_m[y_n(t)]\, \delta_m[y_n(s)]\,|y_n'(t)\ y_n'(s)|\}dsdt \qquad (2.6)$$

(where the interchange of order of integration is permitted by Fubini's theorem for positive functions).

3. Further preliminary lemmas.

To evaluate the right side of (2.6) we consider the t,s integration over four disjoint regions. Let I_k denote the interval $[k/2^n, (k+1)/2^n)$. For each positive integer n and each $\epsilon > 0$ we define for $0 \le t, s \le 1$ the sets

$S_1 = \{(t,s): \ |t-s| < \epsilon\}$

$S_2 = \{(t,s): \ |t-s| \ge \epsilon$, t and s both in I_k for some k$\}$

$S_3 = \{(t,s): \ |t-s| \ge \epsilon$, for some k, $t \in I_k$ and $s \in I_{k+1}$ or $s \in I_k$ and $t \in I_{k+1}\}$

$S_4 = \{(t,s)$ otherwise, i.e., t and s in separated intervals$\}$

The right side of (2.6) can now be written as

$$\lim_{\epsilon\to 0}\ \lim_{m\to\infty} \left[\iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4}\right] \mathcal{E}\{\delta_m[y_n(t)]\, \delta_m[y_n(s)]\,|y_n'(t)y_n'(s)|\}dsdt \qquad (3.1)$$

and we consider the integrations over the four regions separately.

Lemma 3.1:

$$\lim_{\epsilon\to 0}\ \lim_{m\to\infty} \iint_{S_1} \delta_m[y_n(t)]\, \delta_m[y_n(s)]\,|y_n'(t)y_n'(s)|dsdt = N_{y_n}, \text{ a.s. }, \qquad (3.2)$$

and hence

$$\lim_{\epsilon\to 0}\ \lim_{m\to\infty} \iint_{S_1} \mathcal{E}\{\delta_m[y_n(t)]\, \delta_m[y_n(s)]\,|y_n'(t)y_n'(s)|\}dsdt = \mathcal{E}[N_{y_n}]\ . \qquad (3.3)$$

Proof: Put $\alpha_k = k/2^n$. Let $t_1, t_2,\ldots,t_N$ be the zeros of $y_n(t)$ on [0,1]. Then, with probability one, $\epsilon_o = \frac{1}{2}\min|t_i-k/2^n|$, where the minimum is taken over $i=1,\ldots,N$ and $k=0,1,\ldots,2^n$, is a positive number.

Thus the left side of (3.2) is less than or equal to

$$\lim_{m\to\infty} \sum_{k=o}^{2^n-1} \left[\int_{\alpha_k}^{\alpha_{k+1}} \delta_m[y_n(t)]\,|y_n'(t)|\,dt \right]^2$$

$$+ \lim_{m\to\infty} 2 \sum_{k=1}^{2^n-1} \int_{\alpha_k}^{\alpha_k+\epsilon_o} \delta_m[y_n(t)]\,|y_n(t)|\,dt \;.\; \int_{\alpha_k-\epsilon_o}^{\alpha_k} \delta_m[y_n(s)]\,|y_n(s)|\,ds$$

$$= \lim_{m\to\infty} \sum_{k=o}^{2^n-1} \left[\int_{y_n(\alpha_k)}^{y_n(\alpha_{k+1})} \delta_m(x)\,dx \right]^2$$

$$+ \lim_{m\to\infty} 2 \sum_{k=1}^{2^n-1} \left| \int_{y_n(\alpha_k)}^{y_n(\alpha_k+\epsilon_o)} \delta_m(x)\,dx \right| . \left| \int_{y_n(\alpha_k-\epsilon_o)}^{y_n(\alpha_k)} \delta_m(u)\,du \right|$$

The first term is N_{y_n} as in the proof of lemma 2.2 and the second term is zero since, by definition of ϵ_o, $y_n(\alpha_k+\epsilon_o)$ and $y_n(\alpha_k)$ are of the same sign and similarly $y_n(\alpha_k-\epsilon_o)$ and $y_n(\alpha_k)$ are of the same sign. Hence the left side of (3.2) is less than or equal to N_{y_n}, a.s.

Further, if $\epsilon < \epsilon_o$ then the left side of (3.2) is not less than

$$\lim_{m\to\infty} \sum_{i=1}^{N} \int_{t_i-\epsilon}^{t_i+\epsilon} \int_{t_i-\epsilon}^{t_i+\epsilon} \delta_m[y_n(t)]\; \delta_m[(y_n(s)]\,|y_n'(t)y_n'(s)|\,ds\,dt$$

$$= \lim_{m\to\infty} \sum_{i=1}^{N} \left[\int_{y_n(t_i-\epsilon)}^{y_n(t_i+\epsilon)} \delta_m(u)\,du \right]^2$$

$$= N_{y_n}, \text{ as in the proof of lemma 2.2.}$$

Thus the equality in (3.2) is proved.

Equation (3.3) follows from (3.2) and (2.4) by applying the dominated convergence theorem.

The next result will be needed later.

Lemma 3.2: As $n \longrightarrow \infty$ the following limits hold uniformly for $0 < t,s \leq 1$.

(i) $\quad \mathrm{cov}[y_n(t), y_n(s)] \longrightarrow r(t-s)$,

(ii) $\quad \mathrm{cov}[y_n'(t), y_n(s)] \longrightarrow r'(t-s)$, and

(iii) $\quad \mathrm{cov}[y_n'(t), y_n'(s)] \longrightarrow -r''(t-s)$.

Proof: From (2.1), the definition of $y_n(t)$, we have

$$\mathrm{cov}[y_n(t), y_n(s)]$$

$$= [(1-2^n t+k)(1-2^n s+\ell)+(2^n t-k)(2^n s-\ell)]\ r((k-\ell)/2^n) \tag{3.5}$$
$$+(1-2^n t+k)(2^n s-\ell)\ r((k-\ell-1)/2^n) + (1-2^n s+\ell)(2^n t-k)\ r((k-\ell+1)/2^n)$$

where $k=k_n(t)$ and $\ell=k_n(s)$.

Expanding the $r(.)$ terms on the right side of (3.5) into two term Taylor expansions, about the point $t-s$, we obtain

$$\mathrm{cov}[y_n(t),y_n(s)] - r(t-s)$$
$$=[1-2^n t+k)(1-2^n s+\ell)+(2^n s-\ell)(2^n t-k)]\ r'(\theta_n)\ [\frac{k-\ell}{2^n} - (t-s)]$$
$$+(1-2^n t+k)(2^n s-\ell)\ r'(\phi_n)[\frac{-k+\ell+1}{2^n} - (t-s)]$$
$$+ (1-2^n s+\ell)(2^n t-k)\ r'(\Psi_n)\ [\frac{k-\ell+1}{2^n} - (t-s)]$$

where $\quad 0 < |t-s-\theta_n| < |\frac{k-\ell}{2^n} - t+s| < 2\cdot 2^{-n}$,

(3.6) $\quad 0 < |t-s-\phi_n| < |\frac{-k+\ell+1}{2^n} - t+s| < 3\cdot 2^{-n}$,

$\quad 0 < |t-s-\Psi_n| < |\frac{k-\ell+1}{2^n} - t+s| < 3\cdot 2^{-n}$.

By the definition of k and ℓ the quantities $|1-2^n t+k|$, $|1-2^n s+\ell|$, $|2^n t-k|$, and $|2^n s-\ell|$ are all bounded by 1, hence

$$|\text{cov}[y_n(t),y_n(s)] - r(t-s)| \leq (4|r'(\theta_n)|+3|r'(\emptyset_n)|+3|r'(\Psi_n)|)\ 2^{-n}.$$

Since $r'(\tau)$ is uniformly continuous and bounded for $0 \leq \tau \leq 1$ the required uniform limit (i) is obtained.

Again by the definition of $y_n(t)$ we have

$$\text{cov}[y_n'(t),\ y_n(s)]$$

$$= 2^n\{(1-2^n s+\ell)[r((k-\ell+1)/2^n) - r((k-\ell)/2^n)]$$

$$+ (2^n s-\ell)[r((k-\ell)/2^n) - r((k-\ell-1)/2^n)]\}$$

Using three term expansions we find

$$\text{cov}[y_n'(t),\ y_n(s)] - r'(t-s)$$

$$= 2^n\{(1-2^n+\ell)[r''(\emptyset_n)(\frac{-\ell+k+1}{2^n} - t+s)^2 - r''(\theta_n)(\frac{k-\ell}{2^n} - t+s)^2]$$

$$+ (2^n s-\ell)\ [r''(\theta_n)(\frac{k-\ell}{2^n} - t+s)^2 - r''(\Psi_n)\ (\frac{k-\ell-1}{2^n} - t+s)^2]\}$$

where the (new) θ_n, $\emptyset_n$, Ψ_n satisfy (3.6). Hence again

$$|\text{cov}[y_n'(t),\ y_n(s)] - r'(t-s)| \leq (3|r''(\emptyset_n)|+4|r''(\theta_n)|+3|r''(\Psi_n)|)2^{-n}$$

and since $r''(\tau)$ is also uniformly continuous and bounded for $0 \leq \tau \leq 1$ the desired result (ii) holds.

Lastly we have

$$\text{cov}[y_n'(t),\ y_n'(s)] = 2^{2n}[2r((k-\ell)/2^n) - r((k-\ell-1)/2^n) - r((k-\ell+1)/2^n)]$$

$$= -\frac{1}{2}[r''(t-s+h_n-\theta_n) + r''(t-s+h_n+\emptyset_n)]$$

where $\quad h_n = \dfrac{k-\ell}{2^n} - t+s,\ 0 < \theta_n < 2^{-n}$, and $0 < \emptyset_n < 2^{-n}$.

Both $h_n-\theta_n$ and $h_n+\theta_n$ tend uniformly to zero as $n \longrightarrow \infty$ and, by the uniform continuity of $r''(\tau)$, (iii) is proved.

With the help of this lemma we may obtain

Lemma 3.3: $\lim_{m\longrightarrow\infty} \iint_{S_2} \mathcal{E}\{\delta_m[y_n(t)]\,\delta_m[y_n(s)]\,|y_n'(t)y_n'(s)|\}dsdt = 0$

Proof: $S_2 = \bigcup_{k=o}^{2^n-1} \{(t,s):\ |t-s| \geq \epsilon,\ \frac{k}{2^n} \leq t,s < \frac{k+1}{2^n}\} = \bigcup_{k=o}^{2^n-1} W_k$, (say).

For $(t,s) \in W_k$, $y_n(t)$, $y_n(s)$, $y_n'(t)$, $y_n'(s)$ are linearly related so that we have only two non degenerate random variables, and thus

(3.7) $\mathcal{E}\{\delta_m[y_n(t)]\,\delta_m[y_n(s)]\,|y_n'(t)y_n'(s)|\} = \iint_{-\infty}^{\infty} \delta_m(x)\delta_m(y)(\frac{x-y}{t-s})^2\, p_{n,t,s}(x,y)dxdy$

where $p_{n,t,s}(x,y)$ is the bivariate normal density for $(y_n(t),\ y_n(s))$. Taking $\delta_m(x) = \frac{m}{(2\pi)^{1/2}}\, e^{-(mx)^2/2}$ and putting $mx = u$, $my = v$ in (3.7) yields

$$(3.8)\qquad \frac{1}{2\pi m} \iint_{-\infty}^{\infty} e^{-(u^2+v^2)/2}\, (\frac{u-v}{t-s})^2\, p_{n,t,s}(u/m,v/m)dudv$$

Now
$$p_{n,t,s}(x,y) = (2\pi)^{-1}\, D^{-1/2} \exp[-(Cx^2 - 2Bxy + Ay^2)/2D]$$

where
$$A = A_n(t) = \text{var}[y_n(t)], \qquad C = C_n(s) = \text{var}[y_n(s)],$$
$$B = B_n(t,s) = \text{cov}[y_n(t),y_n(s)], \qquad D = D_n(t,s) = AC-B^2$$

By lemma 3.2 these moments tend uniformly to the corresponding moments of the x(t) process. In particular

$$D_n(t,s) \longrightarrow 1-r^2(t-s).$$

Now
$$1-r^2(\tau) = \int_o^\infty (1-\cos\lambda\tau)dF(\lambda) \cdot \int_o^\infty (1+\cos\lambda\tau)dF(\lambda) \geq 0.$$

Equality holds only if $1 = \pm\cos\lambda\tau$ except for a λ-set of $dF(\lambda)$ measure zero. But for $\tau\neq 0$ only countably many λ can satisfy $1 = \pm\cos\lambda\tau$ and hence, since $F(\lambda)$ has a continuous component, the strict inequality holds. Further $r(\tau)$ being continuous implies that $1-r^2(t-s)$ is bounded away from zero when $|t-s| \geq \epsilon > 0$. Hence, for sufficiently large n, the integrand of (3.8) is dominated by

$$\text{const } \epsilon^{-2}(u-v)^2 \, e^{-(u^2+v^2)/2} \quad ,$$

which is integrable in u and v and bounded (constant) in t and s. Hence the integral of (3.7) over S_2 is dominated by a constant multiplied by 1/m which yields the proof of the lemma.

<u>Lemma 3.4</u>: $\lim\limits_{n\to\infty} \lim\limits_{\epsilon\to o} \lim\limits_{m\to\infty} \iint\limits_{S_3} \mathcal{E}\{\delta_m[y_n(t)]\, \delta_m[y_n(s)]\, |y_n'(t)y_n'(s)|\}dsdt = 0.$

Proof: We can write

$$S_3 = \bigcup_{k=o}^{2^n-2} \{(s,t): \ |t-s| \geq \epsilon, \ \frac{k}{2^n} \leq t < \frac{k+1}{2^n} \leq s < \frac{k+2}{2^n}\} \ \cup$$

$$\bigcup_{k=o}^{2^n-2} \{(s,t): \ |t-s| \geq \epsilon, \ \frac{k}{2^n} \leq s < \frac{k+1}{2^n} \leq t < \frac{k+2}{2^n}\ \}$$

$$= \bigcup_{k=o}^{2^n-2} W_k \cup \bigcup_{k=o}^{2^n-2} W_k' \qquad , \text{ (say)}.$$

For $(t,s) \in W_k$ (or W_k') there are only three non degenerate random variables among the set $y_n(t)$, $y_n(s)$, $y_n'(t)$, $y_n'(s)$. Further by stationarity

$$\iint\limits_{W_k} \mathcal{E}\{\delta_m[y_n(t)]\delta_m[y_n(s)]\,|y_n'(t)y_n'(s)|\} \text{ dsdt}$$

$$= \iint\limits_{W_j} \mathcal{E}\{\delta_m[y_n(t)]\delta_m[y_n(s)]\,|y_n'(t)y_n'(s)|\} \text{ dsdt}$$

and similarly for W_k' and W_j'. Hence the lemma will be proved if we show

$$\text{(3.9)} \quad \lim_{n\to\infty} \lim_{\epsilon\to o} \lim_{m\to\infty} 2^n \int_{2^{-n}}^{2^{-n+1}} \int_{0}^{2^{-n}}\limits_{s-t\geq\epsilon} \mathcal{E}\{\delta_n[y_n(t)]\delta_m[y_n(s)]\,|y_n'(t)y_n'(s)|\}dtds = 0.$$

Define $x_o = x(2^{-n})$, $x_1 = x(0)$, and $x_2 = x(2^{-n+1})$ and for convenience let $t' = 2^{-n} - t$, $s' = s - 2^{-n}$. Then for $0 \leq t < 2^{-n} \leq s < 2^{-n+1}$ we have

$$y_n(t) = 2^n t' x_1 + (1 - 2^n t')x_o \quad , \quad y_n(s) = 2^n s' x_2 + (1 - 2^n s')x_o \quad .$$

Taking again $\delta_m(x) = m(2\pi)^{-1/2} \exp(-m^2x^2/2)$ the integrations in (3.9) may be written

$$-\int_o^{2^{-n}} \int_{2^{-n}}^{o} dtds \iiint_{-\infty}^{\infty} 2^{2n}m^2(2\pi)^{-1}\exp[-\frac{m^2}{2}\{[2^n tx_1 + (1-2^n t)x_o]^2+[2^n sx_2 + (1-2^n s)x_o]^2\}]$$

$$|(x_1-x_o)(x_2-x_o)|p_n(x_o,x_1,x_2)dx_o dx_1 dx_2 \quad , \tag{3.10}$$

where $p_n(x_o,x_1,x_2)$ is the tri-variate normal density for $x(2^{-n})$, $x(0)$, $x(2^{-n+1})$, and we omit the primes on t and s in the remainder of this proof.

We can write the triple integral in (3.10) as the sum of six such integrals over the following regions:

R_1: $x_1 \leq x_o \leq x_2$

R_2: $x_o \leq x_1 \leq x_2$

R_3: $x_2 \leq x_1 \leq x_o$

R_4: $x_2 \leq x_o \leq x_1$

R_5: $x_o \leq x_2 \leq x_1$

R_6: $x_1 \leq x_2 \leq x_o$

From symmetry arguments for x_1 and x_2 we need consider only regions R_1, R_2 and R_3.

Since in region R_1 we have

$$x_1^2 + x_o^2 \leq [2^n tx_1+(1-2^n t)x_o]^2 + [2^n s\ x_2+(1-2^n s)x_o]^2$$

the contribution to (3.10) from region R_1 is not greater than

$$\frac{2^{2n}m^2}{2\pi} \iiint_{-\infty}^{+\infty} e^{-m^2(x_1^2+x_o^2)/2} \ |(x_o-x_1)(x_o-x_2)|p_n(X_o,x_1,x_2)dx_o dx_1 dx_2$$

$$(3.11) \qquad = \frac{2^{2n}}{2\pi m} \iiint_{-\infty}^{\infty} e^{-(u^2+v^2)/2} \ |(u-v)(v/m-x_2)|p_n(u/m,v/m,x_2)|dudvdx_2$$

In more detail we have

$$p_n(x_o,x_1,x_2) = (2\pi)^{-3/2}|V_n|^{-1/2}\exp[-\tfrac{1}{2}(x_o,x_1,x_2)V_n^{-1}(x_o,x_1,x_2)']$$

where

$$V_n = \begin{bmatrix} 1 & r(2^{-n}) & r(2^{-n}) \\ r(2^{-n}) & 1 & r(2^{-n+1}) \\ r(2^{-n}) & r(2^{-n+1}) & 1 \end{bmatrix} .$$

Calculation shown that

$$|V_n| = [1-r(2^{-n+1})][1-2r^2(2^{-n}) + r(2^{-n+1})]$$

By the proof of lemma 3.3 the first factor is strictly positive. Putting $2^{-n} = \alpha$ the second factor may be written

$$1-2[\int_o^{\infty} \cos\lambda\alpha \ dF(\lambda)]^2 + \int_o^{\infty} \cos\ 2\lambda\alpha \ df(\lambda)$$

$$= 2\{\int_o^{\infty} \cos^2\lambda\alpha \ dF(\lambda) - [\int_o^{\infty} \cos\lambda\alpha \ dF(\lambda)]^2\}$$

$$= 2\{\int_o^{\infty} [\cos\lambda\alpha - r(\alpha)]^2 \ dF(\lambda)\}$$

This can be zero only if $\cos\lambda\alpha = r(\alpha)$ a.e. $(F(\lambda))$ which is impossible since $F(\lambda)$ has a continuous component. Therefore $|V_n| > 0$ all n. Hence the integrand of (3.11) is dominated by

$$\text{const } e^{-(u^2+v^2+x_2^2)/2} \ |(u-v)(v-x_2)|$$

(using lemma A.1) which is integrable in u,v,x_2 and const in t and s.

Hence the contribution to (3.10) over regions R_1 and R_4 is dominated by const m^{-1} which tends to zero as $m \longrightarrow \infty$.

For region R_2 we note that $|t-s| > \epsilon$ is equivalent to $t' + s' \geq \epsilon$ and further

$$\{(t,s): \ t + s \geq \epsilon\} \subset \{(t,s): \ t \geq \epsilon, \ s \geq 0\} \cup \{(t,s): \ s \geq \epsilon, \ t \geq 0\}$$

$$= W \cup W' \qquad , \text{(say)}.$$

For $(s,t) \in W$ and x_o, x_1, x_2 in region R_2 we have

$$[\epsilon \ 2^n x_1 + (1+\epsilon 2^n)x_o]^2 + x_o^2 \leq [2^n t x_1 + (1-2^n t)x_o]^2 + [2^n s x_2 + (1-2^n s)x_o]^2$$

An analogous inequality holds for $(s,t) \in W'$. Now

$$\frac{2^{2n}m^2}{2\pi} \iiint_{R_2} \exp\{-\frac{m}{2}[x_o^2 + (\epsilon 2^n x_1 + (1+\epsilon 2^n)x_o)^2]\} |(x_o - x_1)(x_o - x_2)| p_n(x_o, x_1, x_2) dx_o dx_1 dx_2$$

$$\leq \frac{2^{2n}}{2\pi m} \iiint_{-\infty}^{+\infty} \exp\{-\frac{1}{2}[u^2 + (\epsilon 2^n v + (1+\epsilon 2^u)u)^2 + x_2^2]\} |(u-v)(u/m - x_2)| du dv dx_2$$

This integrand, which tends to zero as $m \longrightarrow \infty$, is dominated by the integrable (in u, v, x_2, t, s) function

$$\text{const} \ \exp\{-\frac{1}{2}[u^2 + (\epsilon 2^n v + (1+\epsilon 2^n)u)^2 + x_2^2]\} |(u-v)(u-x_2|$$

and hence again the contribution to (3.10) over R_2 (and R_5) goes to zero like m^{-1}.

In region R_3 $\quad x_1^2 + x_2^2 \leq [2^n t x_1 + (1-2^n t)x_o]^2 +]2^n s x_2 + (1-2^n s)x_o]^2$,

and

$$-\int_o^{2^{-n}} \int_{2^{-n}}^{o} \frac{2^{2n}m^2}{2\pi} \iiint_{-\infty}^{\infty} \exp[-\frac{m^2}{2}(x_1^2 + x_2^2)] |(x_o - x_1)(x_o - x_2)| p_n(x_o, x_1, x_2) dx_o dx_1 dx_2 \ dsdt$$

$$= \frac{1}{2\pi} \iiint_{-\infty}^{\infty} \exp[-\frac{1}{2}(u^2 + v^2)] |(x_o - u/m)(x_o - v/m)| p_n(x_o, u/m, v/m) dx_o du dv$$

As in the previous cases, by dominated convergence, this tends to (as $m \longrightarrow \infty$)

$$\frac{1}{2\pi} \iiint_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(u^2+v^2)]\ x_o^2\ p_n(x_o,0,0)\,du\,dv\,dx_o$$

$$= \int_{-\infty}^{\infty} x_o^2\ p_n(x_o,0,0)\,dx_o$$

From the explicit form of $p_n(x_o,x_1,x_2)$ this can be written

$$(2\pi)^{-3/2}|V_n|^{-1/2} \int_{-\infty}^{\infty} x^2\ \exp[-(1-r^2(2^{-n+1}))x^2/(2|V_n|)]dx$$

(3.12) $$= K|V_n|[1-r(2^{-n+1})]^{-3/2} \quad ,$$

where K is a constant not depending on n.

We find that as $n \longrightarrow \infty$ $|V_n| = o(2^{-4n})$, since $r''(\tau)$ exists and is continuous at $\tau=0$.

Further

$$1-r^2(2^{-n+1}) = -2^{-2n+2}r''(\theta) + o(2^{-2n}), \text{ where } 0 < \theta < 2^{-n+1} .$$

Hence

$$2^n|V_n|[1-r^2(2^{-n+1})]^{-3/2}$$

$$= 2^n\ o(2^{-4n})/[-2^{-2n+2}r''(\theta)+o(2^{-2n})]^{3/2}$$

$$= o(1)/[-r''(\theta)+o(1)]^{3/2}$$

$$\longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty .$$

Therefore the contribution to (3.9) over regions R_3 and R_6 is zero proving the desired result.

Before considering the t,s region S_4 we need to ensure that the joint distribution of $y_n(t)$, $y_n(s)$, $y_n'(t)$, $y_n'(s)$ is non singular for $(t,s) \in S_4$. This result is provided by the next lemma.

Lemma 3.5: For $(t,s,) \in S_4$ the covariance matrix $\Sigma_n(t,s)$ of $y_n(t)$, $y_n(s)$, $y_n'(t)$, $y_n'(s)$ is non-singular for every n.

Proof: From the definition of the $y_n(t)$ process it is easy to see that for $(t,s) \in S_4$ $\Sigma_n(t,s)$ is non-singular if the covariance matrix of $x(0)$, $x(2^{-n})$, $x(j2^{-n})$, $x((j+1)2^{-n})$ is non-singular for $j > 1$. Writing r_m for $r(m2^{-n})$ this covariance matrix is

$$
\begin{bmatrix} 1 & r_1 & r_j & r_{j+1} \\ r_1 & 1 & r_{j-1} & r_j \\ r_j & r_{j-1} & 1 & r_1 \\ r_{j+1} & r_j & r_1 & 1 \end{bmatrix} \tag{3.13}
$$

Some calculation shows that the determinant of (3.13) can be written as

$$
[(1-r_{j-1})(1-r_{j+1}) - (r_1-r_j)^2]\,[(1+r_{j-1})(1+r_{j+1}) - (r_1+r_j)^2] \quad . \tag{3.14}
$$

Consider the first factor. We can write (putting $2^{-n} = \alpha$)

$$
\begin{aligned}
(r_1-r_j)^2 &= [\int_0^\infty (\cos\alpha\lambda - \cos\alpha j\lambda)\, dF(\lambda)]^2 \\
&= [\,2\int_0^\infty (\sin\alpha(j+1)\lambda/2)(\sin\alpha(j-1)\lambda/2)\, dF(\lambda)]^2 \\
&\leq \int_0^\infty 2\sin^2\alpha(j+1)\lambda/2\, dF(\lambda) \cdot \int_0^\infty 2\sin^2\alpha(j-1)\lambda/2\, dF(\lambda) \\
&= \int_0^\infty [1-\cos\alpha(j+1)\lambda]\, dF(\lambda) \cdot \int_0^\infty [1-\cos\alpha(j-1)\lambda]\, dF(\lambda) \\
&= (1 - r_{j+1})(1 - r_{j-1}) \quad .
\end{aligned} \tag{3.15}
$$

The inequality (3.15) is Schwarz' inequality. Equality holds only if

$$
\sin\alpha(j-1)\lambda/2 = \text{const } \sin\alpha(j+1)\lambda/2 \tag{3.16}
$$

except on a λ-set of $F(\lambda)$ measure zero. For α and $j > 1$ fixed (3.16) can be satisfied for only countably many λ values. Hence, since $F(\lambda)$ contains a continuous component, the inequality at (3.15) is strict and so the first factor of (3.14) is strictly positive. An analogous argument shows that the second factor is strictly positive and hence the lemma.

Now we can show

<u>Lemma 3.6</u>:

$$(3.17) \quad \mathcal{E}[N_{y_n}^2] = \mathcal{E}[N_{y_n}] + \int\int_{S_4^*} dsdt \iint_{-\infty}^{\infty} |xy| p_{n,t,s}(0,0,x,y)dxdy + o(1), \quad \text{as } n \longrightarrow \infty,$$

where $p_{n,t,s}(u,v,x,y)$ is the four-variate normal density for $y_n(t)$, $y_n(s)$, $y_n'(t)$, $y_n'(s)$, and $S_4^* = \lim_{\epsilon \longrightarrow 0} S_4$.

Proof: We first appeal to lemma 2.3 and equation (3.1). The term $\mathcal{E}[N_{y_n}]$ in (3.17) is given by lemma 3.1, while the o(1) term follows from lemmas 3.3 and 3.4. Hence (3.17) will hold if we can show

$$(3.18) \quad \lim_{\epsilon \longrightarrow 0} \lim_{m \longrightarrow \infty} \int\int_{S_4} \mathcal{E}\{\delta_m[y_n(t)]\delta_m[y_n(s)]\,|y_n'(t)y_n'(s)|\}dsdt$$

$$= \int\int_{S_4^*} dsdt \iint_{-\infty}^{\infty} |xy| p_{n,t,s}(0,0,x,y)dxdy.$$

Explicitly we have

$$p_{n,t,s}(u,v,x,y) = (2\pi)^{-2}|\Sigma_n|^{-1/2}\exp[-(u,v,x,y)\Sigma_n^{-1}(u,v,x,y)'/2]$$

where $\Sigma_n = \Sigma_n(t,s) = \mathrm{cov}(y_n(t), y_n(s), y_n'(t), y_n'(s))$.

Taking again $\delta_m(x) = m(2\pi)^{-1/2} \exp(-m^2x^2/2)$, we may write the expectation in (3.18) as

$$\iiiint_{-\infty}^{\infty} (2\pi)^{-3}m^2|\Sigma_n|^{-1/2}|xy|\exp[-(m^2u^2+m^2v^2+(u,v,x,y)\Sigma_n^{-1}(u,v,x,y)')/2]dudvdxdy$$

$$(3.19) \quad = \iiiint_{-\infty}^{\infty} (2\pi)^{-3}|\Sigma_n|^{-1/2}|xy|\exp[-(u^2+v^2+(u/m,v/m,x,y)\Sigma_n^{-1}(u/m,v/m,x,y)')/2]dudvdxdy.$$

By lemma 3.5 we have that $|\Sigma_n(t,s)|$ is bounded away from zero uniformly for $(t,s) \in S_4$. Further an application of lemma A.1 show that

$$\exp\,[-(u,v,x,y)\,\Sigma_n^{-1}\,(u,v,x,y)'/2]$$

$$\leq \exp\,[-(x,y)A^{-1}\,(x,y)'/2]$$

where A is the 2 x 2 covariance matrix of $y_n'(t)$, $y_n'(s)$. As a corollary to lemma 3.5 A is non-singular for $(t,s) \in S_4$ and indeed by the calculations of lemma 3.2 the elements of A^{-1} are bounded functions of t and s, the diagonal elements being bounded away from zero for $(t,s) \in S_4$.

Hence, by dominated convergence, the limit as $m \longrightarrow \infty$ of (3.19) may be taken under the integrations to yield

$$\lim_{\epsilon \longrightarrow 0} \int_{S_4}\int dsdt \int_{-\infty}^{\infty}\int |xy|p_{n,t,s}(0,0,x,y)dxdy$$

$$= \int_{S_4^*}\int dsdt \int_{-\infty}^{\infty}\int |xy|p_{n,t,s}(0,0,x,y)dxdy \qquad ,$$

the last expression being justified by monotone convergence since S_4 increases to S_4^* as $\epsilon \longrightarrow 0$.

4. The remaining limit

To evaluate the limit of (3.20) as $n \longrightarrow \infty$ we again appeal to the dominated convergence theorem. That $\iint_{-\infty}^{\infty} |xy|p_{n,t,s}(0,0,x,y)dxdy$ can be dominated by a function of t and s which is integrable over $0 \leq t,\ s, \leq 1$ is provided by the following lemmas.

Lemma 4.1:

$$(4.1) \qquad \iint_{-\infty}^{\infty} |xy| p_{n,t,s}(0,0,x,y)dxdy \leq (\sigma_{33}\ \sigma_{34})^{1/2}/(2\pi|\Sigma^{(1)}|^{3/2}) ,$$

where $\Sigma_n(t,s) = \begin{bmatrix} \Sigma^{(1)} & \Sigma^{(2)} \\ \Sigma^{(2)'} & \Sigma^{(3)} \end{bmatrix}$, i.e., $\Sigma^{(1)}$ is the upper left 2 x 2 corner of $\Sigma_n(t,s)$ (defined at (3.18)), and σ_{ij} is the (i,j) cofactor of $\Sigma_n(t,s)$.

Proof: Partition Σ_n^{-1} into 2 x 2 sub-matrices as $\begin{bmatrix} B_1 & B_2 \\ B_2' & B_3 \end{bmatrix}$, then

$$(4.2) \qquad \iint_{-\infty}^{\infty} |xy| p_{n,t,s}(0,0,x,y)dxdy = (2\pi)^{-2}\ |\Sigma|^{-1/2} \iint_{-\infty}^{\infty} |xy| \exp[-\tfrac{1}{2}(x,y)B_3(x,y)']dxdy$$

By Schwarz' inequality this is less than or equal to

$$(2\pi)^{-1}|\Sigma_n|^{-1/2}[(2\pi)^{-1/2}|B_3|^{1/2} \iint_{-\infty}^{\infty} x^2\ \exp(-\tfrac{1}{2}(x,y)B_3(x,y)')dxdy]^{1/2}$$

$$\cdot\ [(2\pi)^{-1/2}|B_3|^{1/2} \iint_{-\infty}^{\infty} y^2\ \exp(-\tfrac{1}{2}(x,y)B_3(x,y)')dxdy]^{1/2}$$

$$(4.3) \qquad = (2\pi)^{-1}|\Sigma_n|^{-1/2}|B_3|^{-1/2}[(B_3^{-1})_{11}(B_3^{-1})_{22}]^{1/2}$$

where $(B_3^{-1})_{ij}$ represents the (i,j) element of B_3^{-1} .

Now $|B_3| = |\Sigma^{(1)}|/|\Sigma_n|$ (see Anderson (1958) p. 42 for example) and so $[|\Sigma_n|\cdot|B_3|]^{-1/2} = |\Sigma^{(1)}|^{-1/2}$. Further

$$B_3 = |\Sigma_n|^{-1} \begin{bmatrix} \sigma_{33} & \sigma_{43} \\ \sigma_{34} & \sigma_{44} \end{bmatrix} ,$$

and therefore

$$B_3^{-1} = [|\Sigma_n|\cdot|B_3|]^{-1}\begin{bmatrix}\sigma_{33} & -\sigma_{34}\\ -\sigma_{43} & \sigma_{44}\end{bmatrix} = |\Sigma^{(1)}|^{-1}\begin{bmatrix}\sigma_{33} & -\sigma_{34}\\ -\sigma_{43} & \sigma_{44}\end{bmatrix}.$$

Hence $(B_3^{-1})_{11} = |\Sigma^{(1)}|^{-1}\sigma_{33}$ and $(B_3^{-1}) = |\Sigma^{(1)}|^{-1}\sigma_{44}$. Putting these values into (4.3) yields the desired result.

In order to dominate the right side of (4.1) by an integrable function of t and s for $0 \leq t,\ s \leq 1$ we need to consider the behavior of the quantities σ_{33}, σ_{44} and $|\Sigma^{(1)}|$ for $\tau = t-s$ near zero. The required results are provided in the next two lemmas.

Lemma 4.2: Let $\chi_n = \chi_n(t,s)$ be the indicator function for the set S_4^*, i.e.,

$$\chi_n = \begin{cases} 1 & (t,s) \in S_4^* \\ 0 & (t,s) \notin S_4^* \end{cases}$$

Then

$$\chi_n(t,s)\cdot|\Sigma^{(1)}|^{-1} \leq [\lambda_2\tau^2 + o(\tau^2)]^{-1}$$

as $\tau = t-s \longrightarrow 0$ and the o-term is uniform in n.

Proof: We note that for $(t,s) \in S_4^*$ we have $|t-s| \geq 2^{-n}$. This is the only property of S_4^* which is used here.

For convenience of proof in this and the following lemma we let

$$(4.5)\qquad \Sigma_n(t,s) = \begin{bmatrix} A & F & B & G\\ F & D & H & E\\ B & H & C & J\\ G & H & J & C\end{bmatrix},$$

$$\mu = 2^n t - k_n(t),\quad \lambda = 2^n s - k_n(s),$$

and for integer m, $r_m = r(m/2^n)$. With this notation we have $\Sigma^{(1)} = AD - F^2$ where

$$A = 1-2\mu(1-\mu)(1-r_1),\qquad D = 1-2\lambda(1-\lambda)(1-r_1),$$

and $F = [(1-\mu)(1-\lambda) + \mu\lambda]r_j + (1-\mu)\lambda r_{j-1} + (1-\lambda)\mu r_{j+1}$, where $j = k_n(t) - k_n(s)$.

Now $r_1 = 1-2^{-2n-1}(\lambda_2-\phi)$ where $\phi = \lambda_2 + r''(\theta_1)$, $0 \leq \theta_1 \leq 2^{-n}$, and $r_m = 1-\lambda_2 2^{-2n-1} m^2 + \Psi_m 2^{-1}$ $(m = j-1, j, j+1)$, where $\Psi_m = [\lambda_2 + r''(\xi_m)]2^{-2n-1}m^2$ with $0 \leq \xi_m \leq 2^{-n}m$.

Thus

$$AD = \{1-2\mu(1-\mu)[2^{-2n-1}(\lambda_2-\phi)]\}\{1-2\lambda(1-\lambda)[2^{-2n-1}(\lambda_2-\phi)]\}$$

$$= 1-[\mu(1-\mu) + \lambda(1-\lambda)]\, 2^{-2n}(\lambda_2-\phi) + \mu\lambda(1-\mu)(1-\lambda)\, 2^{-4n}(\lambda_2-\phi)^2$$

By definition $0 \leq \mu, \lambda \leq 1$ for all t,s and n. Further $\phi = \lambda_2 + r''(\theta_1) \leq \Psi(\theta_1)$ where Ψ is given by the theorem. By assumption $\Psi(\tau)$ decreases as τ decreases to zero. Thus, for $\tau \geq 2^{-n}$,

$$[\mu(1-\mu) + \lambda(1-\lambda)]\, 2^{-2n}\, \phi \leq \text{const } \tau^2\, \Psi(\theta_1) \leq \text{const } \tau^2\Psi(\tau)$$

and $2^{-4n}(\lambda_2-\phi)^2 \leq \text{const } \tau^4$. Hence for $\tau \geq 2^{-n}$ we have

$$AD = 1 - [\mu(1-\mu) + \lambda(1-\lambda)]\, 2^{-2n}\, \lambda_2 + o(\tau^3), \quad \text{as } \tau \longrightarrow 0,$$

where $o(\tau^3)$ is uniform in n.

Now

$$F = 1 - [j^2+2j(\mu-\lambda)+\mu+\lambda-2\mu\lambda]\lambda_2 2^{-2n-1} + [(1-\mu)(1-\lambda) + \mu\lambda]\Psi_j/2$$

$$+ (1-\mu)\lambda\, \Psi_{j-1}/2 + (1-\lambda)\mu\, \Psi_{j+1}/2$$

For $\tau \geq 2^{-n}$ we have $2^{-n}(j+1) \leq 2\tau$ and thus $(1-\lambda)\mu\Psi_{j+1} \leq \text{const } \tau^2\Psi(\xi_{j+1}) \leq \text{const } \tau^2\Psi(2\tau)$

and similarly for the other Ψ_m terms. Hence, subject to $\tau \geq 2^{-n}$,

$$F^2 = 1 - [j^2+2j(\mu-\lambda)+\mu+\lambda-2\mu\lambda]\, \lambda_2\, 2^{-2n} + o(\tau^2)$$

where $o(\tau^2)$ is uniform in n.

Therefore

$$\chi_n(t,s)|\Sigma_n^{(1)}(t,s)|^{-1} \leq [(j+\mu-\lambda)^2\, \lambda_2 2^{-2n} + o(\tau^2)]^{-1}$$

$$= [\lambda_2\tau^2 + o(\tau^2)]^{-1}, \quad \text{as } \tau \longrightarrow 0,$$

where $o(\tau^2)$ is uniform in n as required.

<u>Lemma 4.3</u>:

(4.6) $$\chi_n(t,s)\,(\sigma_{33}\ \sigma_{44})^{1/2} \leq K\,\tau^2\,\Psi(2\tau) + o(\tau^3) \quad , \text{ as } \quad \tau \longrightarrow 0 \ ,$$

where the o-term is uniform in n and K is a constant.

Proof: In the notation or (4.5) we can write

(4.7) $$\sigma_{33} = C(AD-F^2) + E(2FG-AE) - DG^2 \qquad ,$$

and we consider the three terms separately.

From the proof of lemma (4.2) we have

$$\chi_n\cdot(AD-F^2) \leq (j+\mu-\lambda)^2\lambda_2\ 2^{-2n} + K\,\tau^2\Psi(2\tau) + o(\tau^3)$$

where $o(\tau^3)$ is uniform in n. Further $C = 2^{2n+1}(1-r_1) = \lambda_2-\phi$ and hence since

$\chi_n\cdot\phi \leq \Psi(\tau)$

(4.8) $$\chi_n\cdot C(AD-F^2) \leq (j+\mu-\lambda)^2\ \lambda_2^2\ 2^{-2n} + K\ \ \tau^2\Psi(2\tau) + o(\tau^3)$$

For the second term we find

$$E = 2^n(2\lambda-1)(1-r_1) = 2^{-n}(\lambda-1/2)(\lambda_2-\phi)$$

so that

$$AE = 2^{-n}(\lambda-1/2)(\lambda_2-\phi) - 2^{-3n}\mu(1-\mu)(\lambda-1/2)(\lambda_2-\phi)^2 \qquad .$$

Also

$$G = 2^n[(1-\mu)(r_{j-1}-r_j) + \mu(r_j-r_{j+1})]$$

$$= 2^{n-1}[\lambda_2 2^{-2n}(2j+2\mu-1) + (2\mu-1)\Psi_{j-1} - \mu\Psi_{j+1}]$$

and hence

$$\chi_n\cdot 2FG = 2^n[\lambda_2 2^{-2n}(2j+2\mu-1) + (2\mu-1)\Psi_j + (1-\mu)\ \Psi_{j-1} - \mu\Psi_{j+1}]$$

$$\cdot[1-(j^2+2j(\mu-\lambda)+\mu+\lambda-2\mu\lambda)\lambda_2\ 2^{-2n-1} + o(\tau^2)]$$

$$= 2^n[\lambda_2\ 2^{-2n}(2j+2\mu-1) + (2\mu-1)\Psi_j + (1-\mu)\Psi_{j-1} - \mu\Psi_{j+1} + o(\tau^3)]$$

with $o(\tau^3)$ uniform in n.

The second term is therefore

$$(4.9)\quad \chi_n\cdot E(2FG-AE) = \lambda_2^2\ 2^{-2n}(\lambda-1/2)(2j+2\mu-\lambda-1/2) + 2^{-2n}\phi[(2\lambda_2+\phi)-\lambda_2(2j+2\mu-1)]$$

$$+ (\lambda-1/2)(\lambda_2-\phi)[(2\mu-1)\Psi_j + (1-\mu)\Psi_{j-1} - \mu\Psi_{j+1}] + o(\tau^3) \quad ,$$

where $o(\tau^3)$ is uniform in n.

For the last term we have

$$D = 1-2\lambda(1-\lambda)(1-r_1) = 1-\lambda(1-\lambda)\ 2^{-2n}(\lambda_2-\phi) \quad .$$

For G we expand r_{j-1} and r_{j+1} around j, i.e.,

$$r_{j-1} = r_j - 2^{-n}r_j' + 2^{-2n-1}r''(\xi_1) \quad , \quad 2^{-n}(j-1) \leq 2^{-n}j \quad ,$$

and $\quad r_{j+1} = r_j + 2^{-n}r_j' + 2^{-2n-1}\ r''(\xi_2), \quad 2^{-n}j \leq \xi_2 \leq 2^{-n}(j+1) \ ;$

then

$$G = 2^n[(1-\mu)(r_{j-1}-r_j) + \mu(r_j-r_{j+1})]$$

$$= -r_j' + 2^{-n-1}[(1-\mu)\ r''(\xi_1) - \mu r'(\xi_2)] \quad .$$

Now writing $r_j' = j2^{-n}r''(\xi_3),\ 0 \leq \xi_3 \leq j2^{-n}$, yields

$$G = 2^{-n}\lambda_2(j+\mu-1/2) - j2^{-n}\phi_3 + (1-\mu)2^{-n-1}\phi_1 - \mu 2^{-n-1}\phi_2 \quad ,$$

where $\quad \phi_i = \lambda_2 + r''(\xi_i) \quad , \quad i = 1,2,3.$

Hence

$$G^2 \geq 2^{-2n}\ \lambda_2^2\ (j+\mu-1/2)^2 + 2^{-2n}\lambda_2(j+\mu-1/2)\ [-2j\phi_3 + (1-\mu)\phi_1 - \mu\phi_2]$$

and therefore

$$(4.10)\quad -\chi_n\cdot DG^2 \leq -2^{-2n}\lambda_2^2(j+\mu-1/2)^2 - 2^{-2n}\lambda_2(j+\mu-1/2)[-2j\phi_3+(1-\mu)\phi_1-\mu\phi_2] + o(\tau^3),$$

$$\text{as } \tau \longrightarrow 0 \quad ,$$

where $o(\tau^3)$ is again uniform in n. The $o(\tau^3)$ comes from terms like $2^{-4n}j^2$, $2^{-4n}j^2\phi_3$, and $2^{-4n}j\phi_1$ which are all dominated by const. τ^4 if $\tau \geq 2^{-n}$.

If we combine (4.8), (4.9), and (4.10) as required in (4.7) we find that the terms not multiplied by ϕ_i or Ψ_i, i.e., the first term in each case, all cancel. From (4.9) the terms that remain are like

$$2^{-2n}j\phi = 2^{-2n}j[\lambda_2 + r''(\theta_1)] \text{ and } \mu\Psi_{j+1} = \mu 2^{-2n-1}(j+1)^2[\lambda_2 + r''(\xi_{j+1})]$$

which subject to $\tau \geq 2^{-n}$, are all dominated in absolute value by const. $\tau^2\Psi(2\tau)$. Similarly the corresponding terms in (4.10) are like $2^{-2n}\ j^2\ \phi_3$ which is also dominated by const. $\tau^2\Psi(2\tau)$. Hence in combining these results we have that

$$\sigma_{33} \leq K\ \tau^2\ \Psi(2\tau) + o(\tau^3) \quad , \quad \text{as} \quad \tau \longrightarrow 0 \quad , \tag{4.11}$$

where $o(\tau^3)$ is uniform in n, and K is a constant. From the definition of σ_{33} and σ_{44} we see that they differ only in that t and s are interchanged. Thus the bound given in (4.11) will also hold for σ_{44} which yields the proof of the lemma.

<u>Proof of the theorem:</u> By lemmas 4.2 and 4.3 we have

$$\chi_n(t,s)(\sigma_{33}\ \sigma_{44})^{1/2}/|\Sigma^{(1)}|^{3/2} \leq [K\ \tau^2\ \Psi(2\tau) + o(\tau^3)]/[\lambda\ \tau^2 + o(\tau^2)]^{3/2}$$

$$= [K\ \Psi(2\tau)/\tau + o(1)]/[\lambda_2 + o(1)]^{3/2}$$

Hence

$$\chi_n(t,s)(\sigma_{33}\ \sigma_{44})^{1/2}/|\Sigma^{(1)}|^{3/2} \leq K_1\ \Psi(2\tau)/\tau + K_2 \tag{4.12}$$

where K_1 and K_2 are (absolute) constants. In terms of χ_n equation (3.17) may be written

$$\mathcal{E}[N^2_{y_n}] = \mathcal{E}[N_{y_n}] + \int_0^1\int_0^1 \chi_n(t,s) \iint_{-\infty}^{\infty}|xy|p_{n,t,s}(0,0,x,y)dxdydsdt + o(1) \quad , \tag{4.13}$$

$$\text{as } n \longrightarrow \infty.$$

By lemma 4.1 and inequality (4.12) the integrand of the t,s integration is dominated by $K_1\ \Psi(2\tau)/\tau + K_2$ which, by the assumptions of the theorem on Ψ, is integrable for

$0 \leq \tau \leq 1$. Hence we may take the limit of both sides of (4.13) as $n \longrightarrow \infty$ to obtain (using lemma 2.1).

$$(4.14) \qquad \mathcal{E}[N_x^2] = \mathcal{E}[N_x] + \int_0^1\int_0^1 ds\,dt \int_{-\infty}^{\infty}\int |xy| \; p_{t-s}(0,0,x,y)\; dx\,dy \quad ,$$

the interchange of the limit and the x,y integration being justified by the explicit form given in (1.8) and by lemma (3.2). This is the desired result.

Appendix

Lemma A.1: If $\underline{x} = (x_1,\ldots,x_n)$, $\underline{y} = (y_{n+1},\ldots,y_m)$ are real row vectors and A is an mxm symmetric positive definite matrix then

$$\text{(A.1)} \qquad \min_{\underline{x}}[(\underline{x},\underline{y})\, A^{-1}(\underline{x},\underline{y})'] = \underline{y}A_3^{-1}\underline{y}' \quad ,$$

where A_3 is obtained by partitioning as $A = \begin{bmatrix} A_1 & A_2 \\ A_2' & A_3 \end{bmatrix}$ corresponding to the dimensions of $\underline{x}$ and $\underline{y}$.

Proof: Let A^{-1} also be partitioned as $\begin{bmatrix} P_1 & P_2 \\ P_2' & P_3 \end{bmatrix}$, then

$$(\underline{x},\underline{y})A^{-1}(\underline{x},\underline{y})' = \underline{x}P_1\underline{x}' + 2\underline{x}P_2\underline{y}' + \underline{y}P_3\underline{y}' \quad .$$

Hence

$$\frac{d}{d\underline{x}}[(\underline{x},\underline{y})A^{-1}(\underline{x},\underline{y})'] = 2P_1\underline{x}' + 2P_2\underline{y}' = 0$$

implies

$$\text{(A.2)} \qquad \underline{x}' = -P_1^{-1}P_2\underline{y}' \quad .$$

Note that $\frac{d^2}{d\underline{x}d\underline{x}'}[(\underline{x},\underline{y})A^{-1}(\underline{x},\underline{y})'] = 2P_1$ which, by virtue of A being positive definite, has positive eigenvalues and therefore ensures that (A.2) yields a minimum. The minimum value is

$$\text{(A.3)} \qquad \underline{y}(P_3 - P_2'P_1^{-1}P_2)\underline{y}'$$

which gives the final result since $P_3 - P_2'P_1^{-1}P_2 = A_3^{-1}$; see Anderson (1958) p. 42, for example.

REFERENCES

Anderson, T. W. (1958), An Introduction to Multivariate Statistical Analysis, John Wiley and Sons, New York.

Bulinskaya, E. C. (1961), "On the mean number of crossings of a level by a stationary Gaussian process," Theor. Probability Appl., 6, pp. 435-438.

Cramér, H. (1962), "Notes on extreme values of normal stationary processes," Research Triangle Institute Working Paper GU-68 No. 5.

Leadbetter, M. R. (1963), "On crossings of arbitrary curves by certain Gaussian processes," Research Triangle Institute Technical Report GU-68 No. 5.

Leadbetter, M. R. and Cryer, J. D. (1964), "On the mean number of curve crossings by non-stationary normal processes," Research Triangle Institute Technical Report SU-181 No. 1.

Rice, S. O. (1944), "Mathematical analysis of random noise," Bell System Tech. J., 23, pp. 282-332; 24, pp. 46-156.

Rozanov, Yu. A. and Volkonskii, V. A. (1961), "Some limit theorems for random functions, II.," Theor. Probability Appl., 6, pp. 186-199

Steinberg, H., Schultheiss, P. M., Wogrin, C. A., and Zweig, F. (1955). "Short-time frequency measurements of narrow-band random signals by means of zero counting process," J. Appl. Phys., 26, pp. 195-201.

Ylvisaker, N. D. (1963), "On level crossings of stationary Gaussian processes," Technical Report No. 2 on Contract NSFG-25211, Mathematics Dept., University of Washington.

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