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THEORETICAL AND EXPERIMENTAL ASPECTS OF THE SHOCK STRUCTURE PROBLEM*

by

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I. INTRODUCTION AND SUMMARY

There does not yet exist a general, reliable theory of the flow of rarefied gases, bridging the gap between simple free molecular flow methods and the Navier-Stokes theory. It is clear today that an essential improvement of the range of applicability of the Navier-Stokes equations cannot be expected from the higher approximations in the Chapman-Enskog procedure but that different methods of finding approximate solutions of the Boltzmann equations are needed. The shock wave structure problem is one of the best test cases for such attempts because it is a realistic case, amenable to experimentation, and simple enough to expect that in the future an exact solution of the Boltzmann equation can be obtained.

The present paper discusses essentially recent results obtained by us from an exact numerical solution of the Bhatnagar-Gross-Krook model. Short discussions of the Navier-Stokes and Mott-Smith theories as well as of recent experimental work are included. The main conclusions reached are the following:

i.) The B-G-K model is capable of describing the essential features of the shock layer and leads to reasonably accurate numerical values for the measurable variables.

ii.) A comparison of the Chapman-Enskog approximation applied to the model and the exact solution shows that useful convergence

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of the C-E series requires stress to pressure ratios, τ/p, less than 0.2, corresponding to shock Mach numbers of less than about 2. iii.) The distribution function within the layer is bimodal, exhibits the gradual change from the molecular beam-like behavior ahead to a Maxwellian distribution behind the shock. The effect of the fast molecules is noticeable even many mean free paths behind the shock. iv.) The flow is nearly locally adiabatic. That is, the total enthalpy is constant to within a few per cent.

II. LOCALLY ADIABATIC FLOW

The general equations of motion for the flow within a shock wave (Fig. 1) can be reduced to a single one, the momentum equation, if one assumes the flow to be "locally adiabatic", i.e., that the heat flux q balances everywhere the work done by the stress τu. For "locally adiabatic flow" the total enthalpy $H = h + \frac{1}{2} u^2$ is constant since the energy equation integrates to

$$H - H_1 = \tau u - q \qquad (1)$$

Within the Navier-Stokes theory one can easily give an upper bound for the maximum total enthalpy variation in terms of the Prandtl number Pr:

$$\left(\frac{H - H_1}{H_1}\right)_{max} < \frac{Pr - 3/4}{\gamma(\gamma + 1) Pr} \qquad (2)$$

In kinetic theory locally adiabatic flow requires a not very stringent relation between the first three moments of the distribution function. Computations for the B-G-K model, shown in Fig. 2 together with the N-S results, demonstrate the comparatively small deviations of H from H_1. The B-G-K model is usally thought

to imply a Prandtl number of unity. This is correct only in the N-S limit, ~~but for high Mach number~~ in general $q > \tau u$!

For locally adiabatic flow with a mass flow m,

$$\tau u = m \frac{\gamma+1}{2\gamma} (u - u_1)(u - u_2) = q \qquad (3)$$

and the characteristic parameter τ/p can be given explicitly (Fig. 3).

A. The Navier-Stokes theory

The Navier-Stokes equations are obtained from the Chapman-Enskog (C-E) approximation if terms proportional to $\frac{\tau}{p}$ are kept. Rapid convergence of the C-E series is thus a sufficient condition for the validity of the N-S equation. The terms of the C-E series ~~are~~ depend on the molecular velocity c however, of the form $\frac{\tau}{p} \underline{c}^2$ and $\frac{\tau}{p} M \underline{c}^3$ and hence there exists always regions in velocity space where the series diverges. Indeed for any Mach number there exist, at large c, regions for which the C-E distribution function is negative. However this impossible region in the distribution function does not necessarily contribute appreciably to the moments like n, u, and T. A comparison of the C-E series applied to the B.G.K model and the exact solution indicates a limit of $\frac{\tau}{p} \doteq 0.2$ for reasonable convergence of the series. A glance at Fig. 3 then shows that one can expect the N-S solution to apply to the complete shock profile for shocks of $M_1 < 2.0$ approximately, however for strong shock waves the N-S theory should be applicable only to a relatively small portion of the shock profile. In our first paper (Ref. Vajn, Narasimha & Chahine 1962) we expected the N-S theory to hold for the whole subsonic portion of the shock profile, i.e., up to $\frac{\tau}{p}$ values of

0.6 or so. The more precise recent computations reduce the expected applicability of the N-S equations and lead to maximum density slope thickness for strong shocks approximately 25% larger than the corresponding N-S value, not only 10% as believed in our first paper.

The precise shape of the distribution function is not given correctly by the C-E expansion for any Mach number. Besides the occurrence of negative regions, the C-E distribution function, due to the non-uniform convergence of the method, fails to account properly for the fast molecules with long free paths. This leads to an asymptotic approach of, e.g., $u(x)$ to its equilibrium value u_1 which, as Liubarsky (ref. 2) has first shown, is of the form $u_2 \sim \exp -x^{2/3}$ rather than $u_{NS} \sim \exp(-x)$. Since $u_2 \doteq u_{NS} \ll u_{NS}$, at least for weak shocks, this behavior is irrelevant for comparison with experiments but of considerable conceptual importance.

B. The Method of Mott-Smith

Mott-Smith (ref. 3) was the first to recognize the bimodal character of the distribution function within a shock wave. He proposed an approximate solution of the Boltzmann equation by assuming f to be of the form

$$f = [1 - \nu(x)] \quad F_1 + \nu(x) F_2 \qquad (4)$$

$\nu(x)$ to be determined by a Galerkin-type technique. Mott-Smith's method was certainly an important step in the right direction; however as he himself clearly recognized, the method is limited by

the arbitrariness in the weighting function used in evaluating $\nu(x)$. Subsequently more elaborate work by Rosen (ref 4), Gustafson (ref 5) and Muckenfuss (ref 6) do not eliminate the arbitrariness in evaluating $\nu(x)$. Sakurai's (ref 7) attempt to prove that the N-S method leads to an asymptotically valid solution of the Boltzmann equation for infinite Mach number, has also not been successful. Today ~~it is nearly possible~~ advances in experimental technique have made it feasible to chose $\nu(x)$ for a best fit with experiments. However this would be sensible only if the Mott-Smith approximation could then be developed into a general method in rarefied gas flows. Such an extension does not seem possible.

C. The Bhatnagar-Gross-Krook theory

The model equation we consider is the one proposed by Bhatnagar, Gross & Krook (ref 8). For one-dimensional flow it can be written

$$v_x \frac{\partial f}{\partial x} = An\,(F - f) \qquad (5)$$

Equation (5) is nonlinear in spite of its appearance, as the parameters in the local Maxwellian F are all moments of the unknown f. We have computed an exact numerical solution of (5) satisfying the correct boundary conditions for a shock wave, for a gas with a realistic viscosity-temperature relation. For the computation it is preferable to convert (5) formally into the integral equation

$$f(x, v_y, v_z, v_x \gtrless 0) = \int_{\pm\infty}^{x} \frac{AnF}{v_x} \exp\left[-\int_{x'}^{x} \frac{An\,dx''}{v_x}\right] dx' \qquad (6)$$

This is now solved by an iterative method whose principles are as follows: A first guess is made at the parameters n, u and T.

(as functions of x) appearing in F on the right hand side; integration with respect to x gives f, and further integration with respect to v gives new values for ρ, u and T. These are now used to generate the next approximation to F, and so on, till there is no sensible variation in the final results. In our computations we used the Navier-Stokes solution for shock structure to provide the first guess. Details of the actual computations, the convergence of the scheme and the results obtained were presented in Chung and Narasimha (1965).

~~The computations reveal the following features of shock structure:~~ For weak shocks (say $M < 2.0$), the exact solution hardly differs from the Navier-Stokes which is thus an adequate approximation. For stronger shocks the profiles deviate more and more from the Navier-Stokes, especially on the cold side, where a long tail develops rapidly. On the hot side the iteration converges relatively slowly, and the departure from Navier-Stokes is less marked. The thickness of the shock, based on the maximum density slope, is about 25% larger than the Navier-Stokes value at a Mach number of 10 (and a viscosity law $\mu \sim T^{\omega}$ with $\omega = .816$). The asymmetry in the temperature profile is very much more pronounced than in the density profile.

We can give a heuristic explanation of these results by considering the form of f as given by (6). Crucial in determining this is the magnitude of what may be called the "sampling distance", $\ell = v_x/An$, appearing in the exponential: As v_x takes all possible values, so does ℓ, but all

velocities v_x do not contribute equally to the moments, in which we are primarily interested. In general, the velocities v_x which contribute most to the first few moments may be expected to be of order u. On the hot side of the shock, the Mach number is always $O(1)$, or $u = O(\bar{c})$. Hence the characteristic sampling distance is $\ell_2 \sim \bar{c}_2/A_2 n_2$, which is proportional to the mean free path Λ_2. ~~Now there is no reason to expect that~~ The gradients within the shock are relatively small on the hot side over a mean free path ~~(see ENSKOG)~~, so if F varies slowly over the distance ℓ_2, we may write, using an obvious asymptotic technique,

$$f(x) = F(x) - \frac{v_x}{An}\frac{\partial F}{\partial x} + \left(\frac{v_x}{An}\right)^2 \frac{\partial^2 F}{\partial x^2} - \ldots \qquad (7)$$

This is, of course, just the Chapman-Enskog series, the second term corresponding to Navier-Stokes, the third to Burnett, etc. Iterating on the Navier-Stokes solution as described earlier, we see that the departures noticed on the hot side could possibly be due to the higher order terms of (7), especially the Burnett terms. The fact that the iteration scheme is essentially evaluating higher derivatives of F is perhaps partly responsible for the relatively slow convergence noticed on the hot side. The typical length scale of the profile ~~(very thin)~~ δ_2 on the hot side of the shock remains Λ_2 to a first approximation.

In a weak shock the Mach number $M = O(1)$ everywhere; and the argument made above for the hot side of a strong shock applies all through a weak shock.

The large deviations noticed on the cold side of strong shocks seem to be due to a different mechanism. Here $u = O(M_1\bar{c}_1)$,

and the characteristic sampling distance $\ell_1 \sim u_1/A_1 n_1 \sim M_1 \Lambda_1$ becomes very large as $M_1 \to \infty$, we see that f is now determined as an integral over the flow and cannot possibly be described in terms of local derivatives (as in (7)). Also we may expect the characteristic length scale on the cold side to become ℓ_1; for, if we should put in for F the Navier-Stokes values (whose scale would be Λ_1/M_1), the attenuation factor multiplying F in (6) would immediately "smudge" it over a distance ℓ_1. We conclude that the characteristic dimension of the 'tail' is $\delta_1 \sim M_1 \Lambda_1$, and not Λ_1/M_1 as in the N-S theory.

A crude measure of the asymmetry of the profile is thus the ratio

$$\frac{\delta_1}{\delta_2} \sim \frac{\ell_1}{\ell_2} \sim M_1 \frac{\Lambda_1}{\Lambda_2}$$

If we take $\mu \sim T^{\omega}$, and define the mean free path as $\Lambda \sim \mu/\rho \bar{c}$ we have

$$\frac{\delta_1}{\delta_2} \sim M_1 \left(\frac{T_1}{T_2}\right)^{\omega - \frac{1}{2}} \sim M_1^{2(1-\omega)} ,$$

recalling that $T_2/T_1 \sim M_1^2$ for large M_1. It is easy to show (from eqn. 3 that the characteristic scales in the Navier-Stokes solution are proportional to the respective viscosities on either side of the shock; hence

$$\left(\frac{\delta_1}{\delta_2}\right)_{N-S} \sim \left(\frac{T_1}{T_2}\right)^{\omega} \sim M_1^{-2\omega} .$$

Since for real gases $1/2 < \omega < 1$, $(\delta_1/\delta_2)_{N\text{-}S} \to 0$, as $M_1 \to \infty$ while for the B-G-K model $(\delta_1/\delta_2) \to \infty$. Thus the Navier-Stokes profile ~~dips suddenly into~~ approaches the upstream ~~side~~ condition abruptly, ~~whereas~~ the B-G-K exact solution reaches the asymptotic state in a long, gradual tail as shown in Fig. 4! It is further interesting to note that the B-G-K tail becomes weaker as $\omega \to 1$, as noted in Figure 6. This result for the ratio of the length scales δ_1/δ_2 may be compared with the ratio of the appropriate free paths $\lambda(u_1)/\lambda(\bar{c}_2)$. For an intermolecular potential of the form $\varphi(r) \sim r^{-s}$, the free path λ depends on the collision velocity v obviously like $\lambda(v) \sim v^{4/s}$. Hence

$$\frac{\lambda(u_1)}{\lambda(\bar{c}_2)} \sim \frac{u_1}{\bar{c}_2}^{4/n} \sim M_1^{2\omega-1} \left(\frac{T_1}{T_2}\right)^{\omega-\frac{1}{2}}$$

and the ratio is thus, for large M_1, nearly independent of M_1, and only weakly dependent on ω. The B-G-K model (except for the case of Maxwell molecules, i.e., $\omega = 1$) is thus likely to exaggerate the tail; the abrupt dip given by the N-S solution on the other hand is certainly unrealistic and hence any solution of the Boltzmann equation can be expected to fall between these two.

A great deal of interesting information is contained in the solution for f of the model equation. When the profiles for the flow quantities within the shock have converged, they may be used to generate the distribution function, by evaluating the integral in (6). Many such computations have been carried out, and some typical results are shown in Figs. 5 and 6.

It is seen that over a fair part of the shock layer the distribution is strongly bimodal. The persistence of the "memory"

of the supersonic stream is remarkable, and can be noticed even at X = 15, where ρ, u and T have reached their downstream asymptotic values to one part in 10^4! This persistence is due to the longer free path of molecules traveling near the upstream velocity u_1, as discussed above; in fact the supersonic stream behaves like a molecular beam being attenuated by the molecules in the rest of velocity space. Qualitatively this behavior is not unexpected and has been used in models for the shock structure, e.g., by Rott and ~~Wittenborg~~ Whittenburg ref. 10 and Broadwell (ref. 11).

There is a similar effect on molecules coming from the subsonic stream, but this is much weaker on account of the smaller free paths.

The Chapman-Enskog distribution function even ~~after~~ for $M_1 < 2$ certainly differs locally considerably from the correct f but does give the lower moments n, u and T with good accuracy.

IV. EXPERIMENTS

Only the density profiles of strong shocks have so far been measured and the accuracy which has been reached is still of the order of $\pm$ 5% at best. Two methods have been used, Hornig's (ref. 12) method, in which the intensity of light refracted at nearly glancing incidence to the shock surface is measured, and the scattering of an electron beam which has been applied in various ways by a number of investigators. The most recent and complete measurements are the ones by Camac (ref. 13) and Russell (ref. 14).

Hornig's method depends on the phase differences in the light reflection from different regions of the shock profile and hence requires shock thicknesses of the order of a wave length of light. It has therefore the advantage of being applicable to thin shock

waves and therefore to small shock tubes operating at high pressure. It has the serious drawback of being very sensitive to the angle of incidence and hence to the shock wave topography, which at high shock tube pressures can be quite complex (e.g., Ref. 15). It is surprising indeed that the measurements are consistent within a few per cent and in good agreement with the other experiments.

The electron beam methods simply use the fact that for sufficiently low beam intensity scattered intensity is proportional to the number of scatterers. The intensity of the scattered beam is either measured directly like in Camac's experiments using a high energy beam or obtained by measuring the "absorption" of a low energy beam like in the earlier measurements of Duff et.al., and the recent ones of Russell. Due to the finite space and time resolution of the electron beam the method is best applied to thick shock waves and hence requires low pressure shock tubes (or wind tunnels) and relatively large ones to keep boundary layer induced curvature effects small.

A typical plot of the maximum density slope thickness versus M_1 is given in Fig. 7. As a comparison computations based on the N-S approximation, B-G-K model and the Mott-Smith methods are shown. The agreement of the measurements with the B-G-K model computation is as good as can be expected. The agreement with the Mott-Smith computations is better, but then this is true for only one particular and arbitrary way of applying the method.

The most characteristic trend in the B-G-K profiles, the "tail" of the profile is very small for the density and probably anyway somewhat exaggerated by the model, consequently more detailed comparison of theory and experiment requires temperature profile or distribution function measurements.

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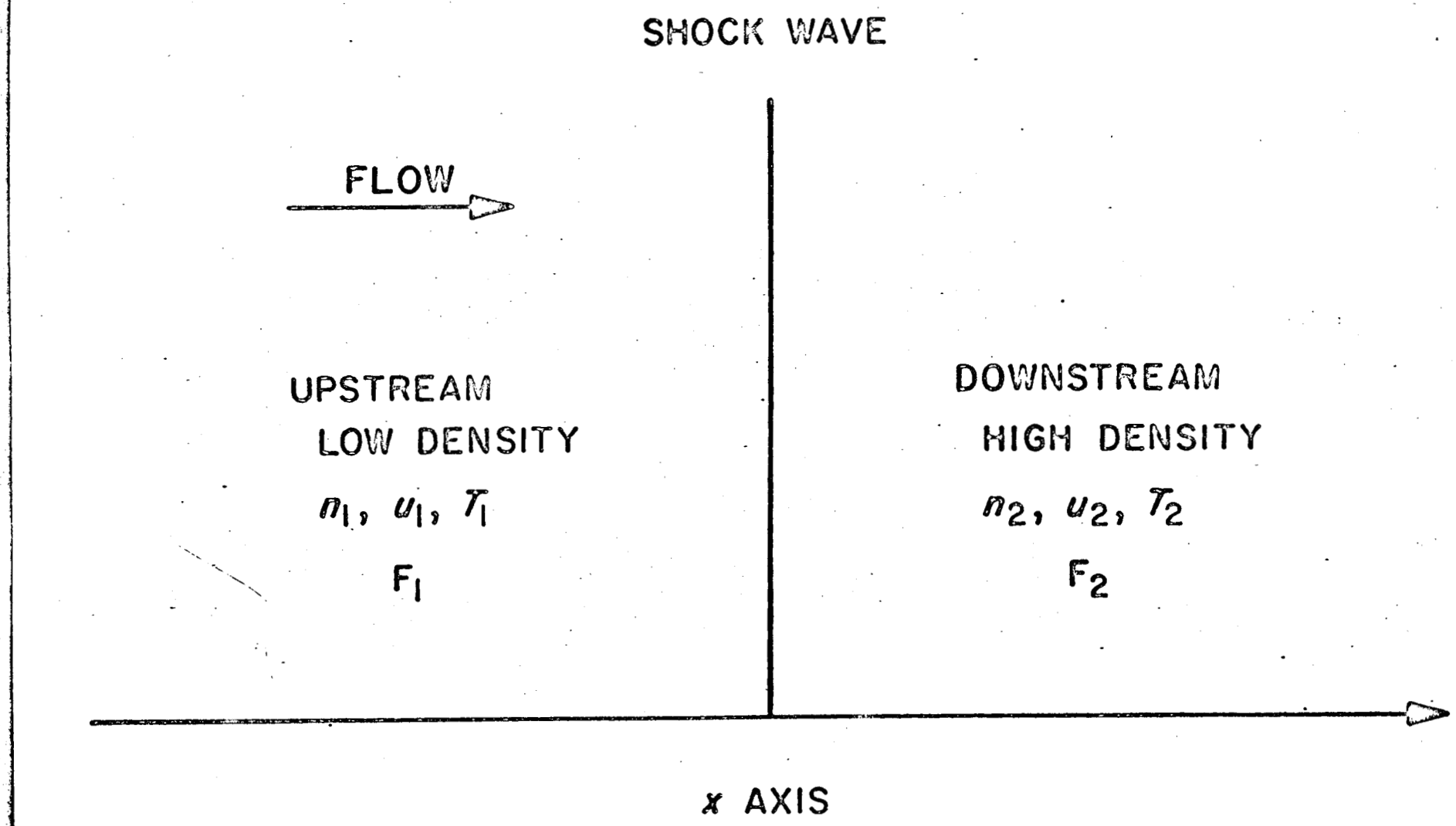
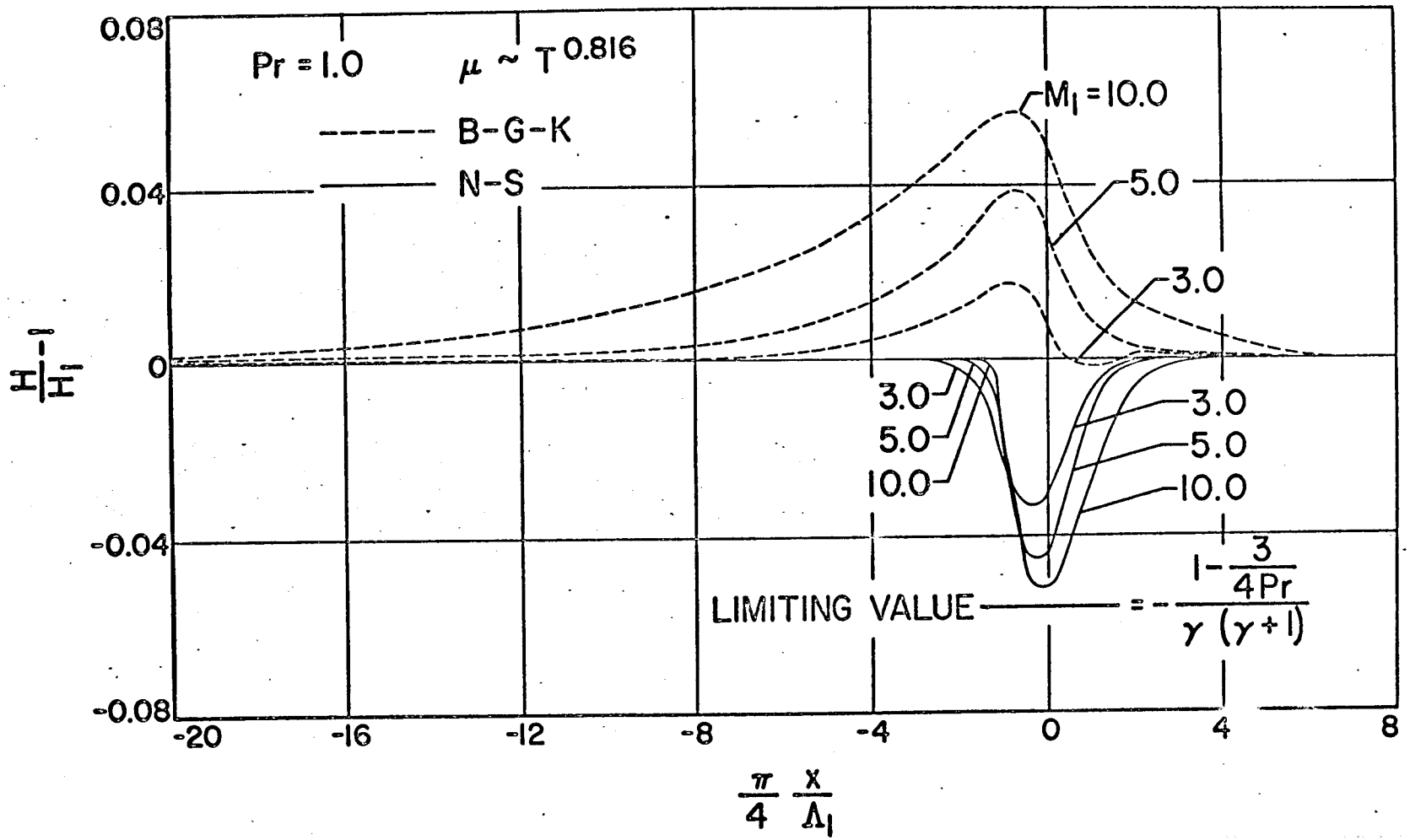
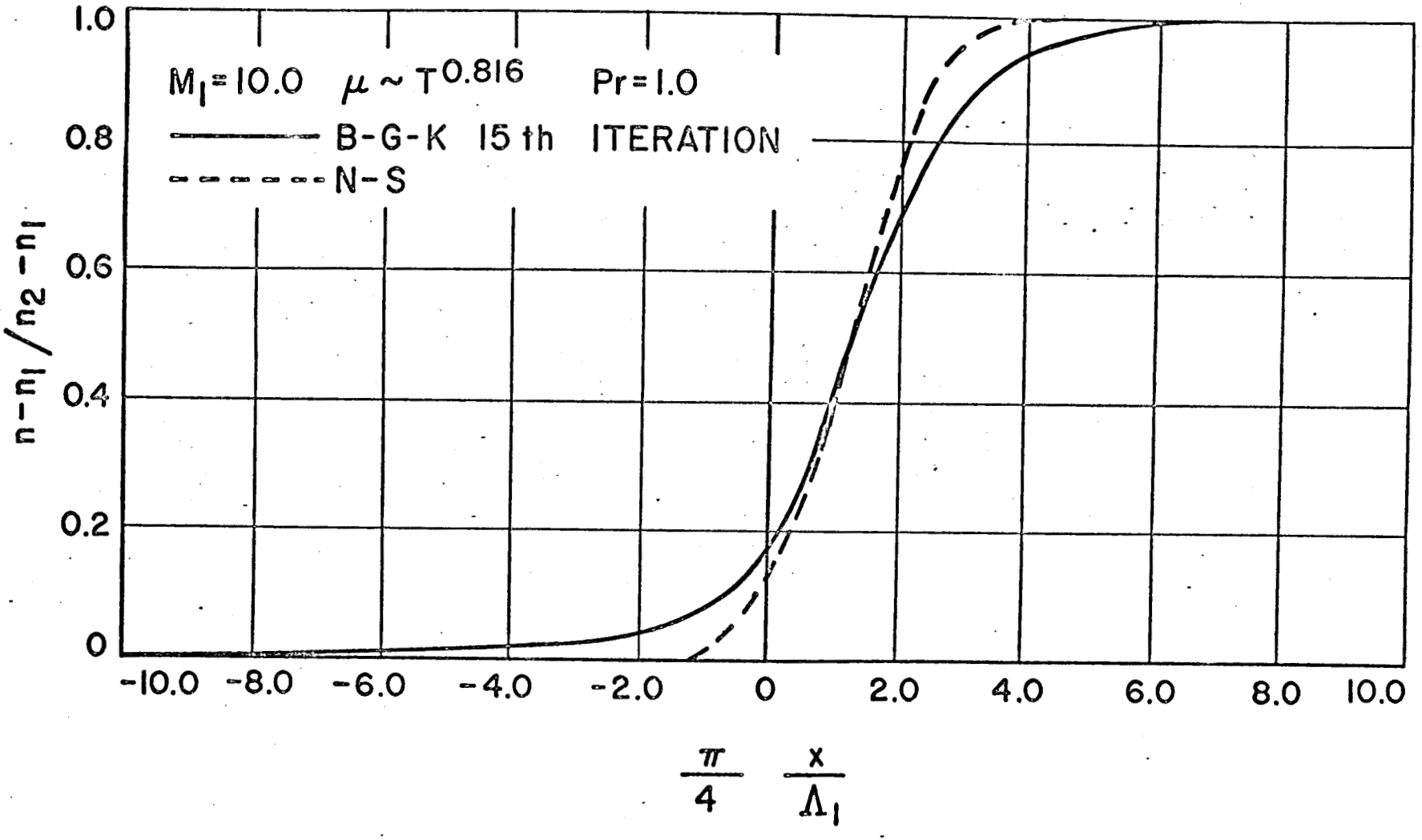
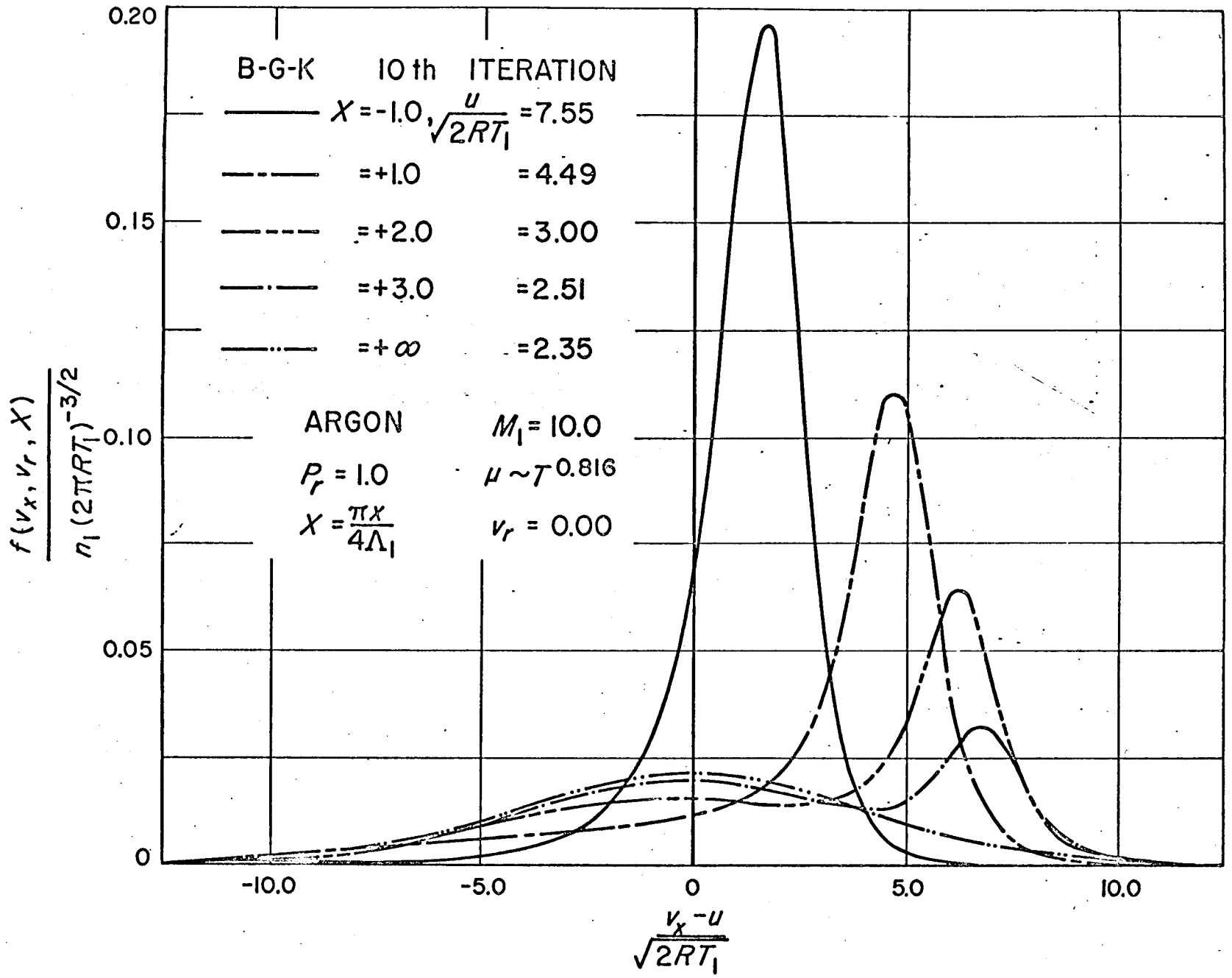
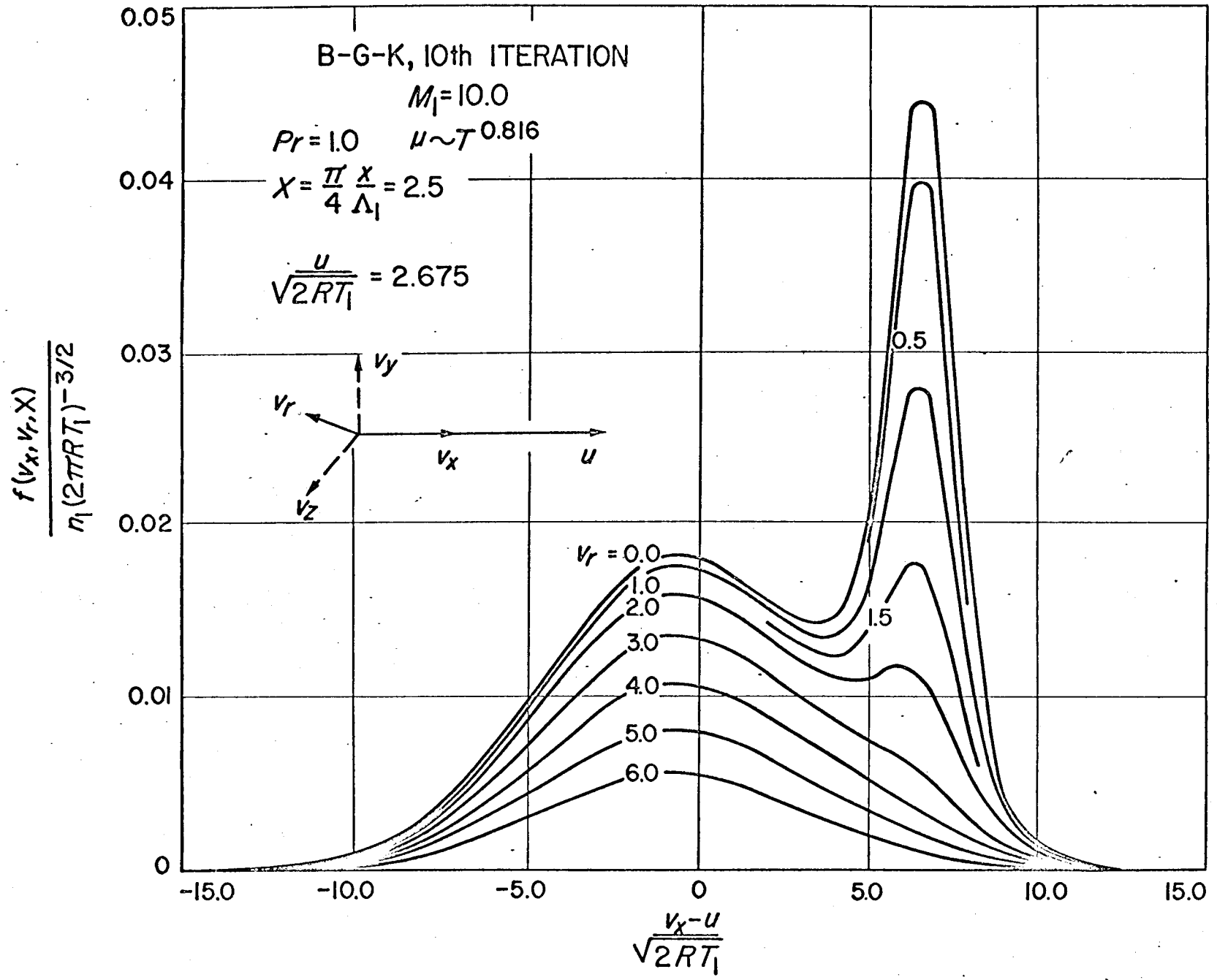
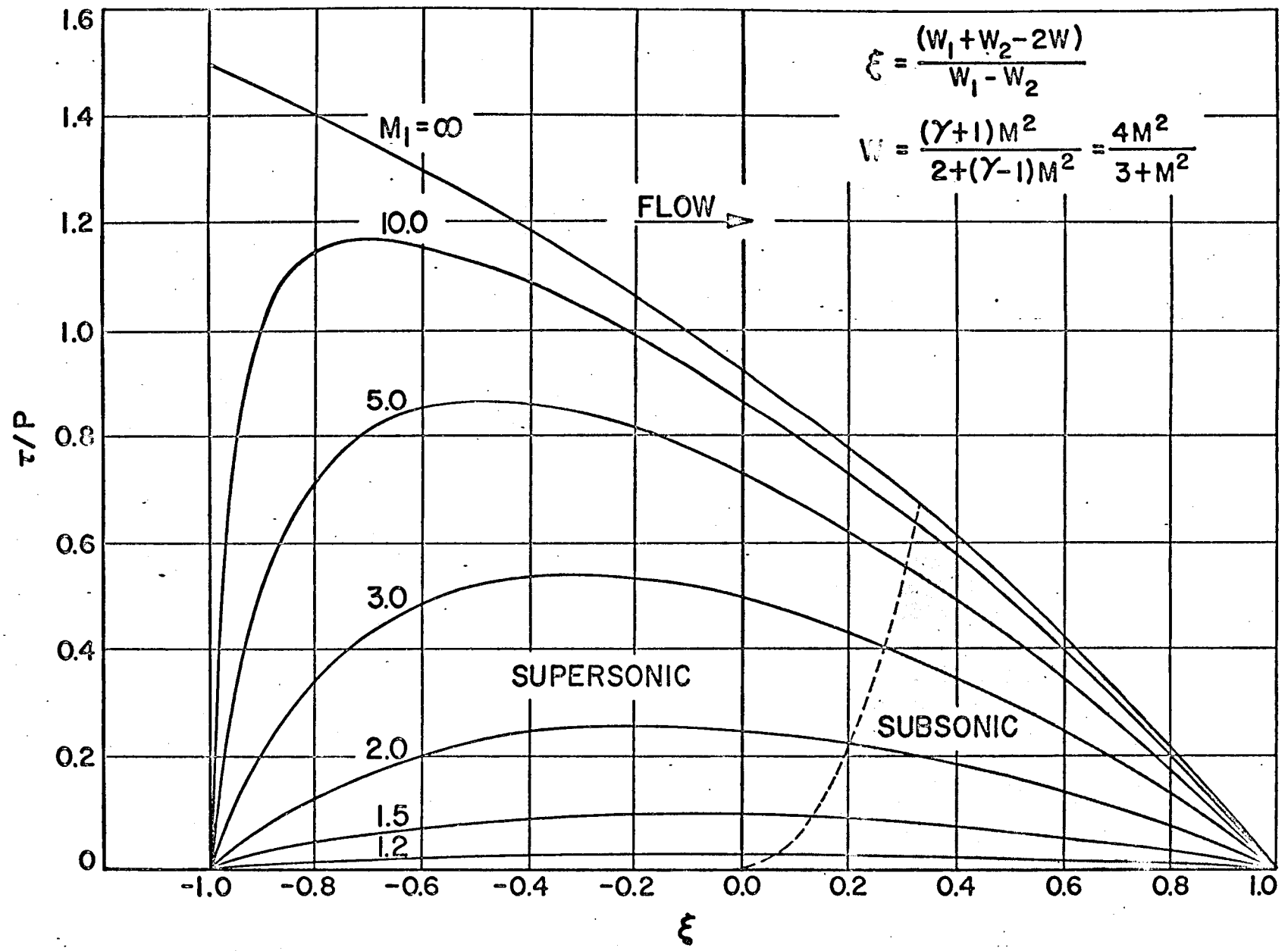

SHOCK WAVE
FLOW
UPSTREAM
LOW DENSITY
n_1, u_1, T_1
F_1
DOWNSTREAM
HIGH DENSITY
n_2, u_2, T_2
F_2
x AXIS


Pr = 1.0
$\mu \sim T^{0.816}$
B-G-K
N-S
$M_1 = 10.0$
5.0
3.0
3.0
5.0
10.0
3.0
5.0
10.0
LIMITING VALUE $= -\frac{1-\frac{3}{4Pr}}{\gamma(\gamma+1)}$
0.08
0.04
0
-0.04
-0.08
-20
-16
-12
-8
-4
0
4
8
$\frac{H-\bar{H}}{H_1}$
$\frac{\pi}{4}\frac{x}{\Lambda_1}$


$M_1 = 10.0$ $\mu \sim T^{0.816}$ $Pr = 1.0$
B-G-K 15 th ITERATION
N-S
$n - n_1 / n_2 - n_1$
1.0
0.8
0.6
0.4
0.2
0
-10.0 -8.0 -6.0 -4.0 -2.0 0 2.0 4.0 6.0 8.0 10.0
$\frac{\pi}{4}$ $\frac{x}{\Lambda_1}$


B-G-K 10 th ITERATION
X = -1.0, u/√(2RT1) = 7.55
= +1.0 = 4.49
= +2.0 = 3.00
= +3.0 = 2.51
= +∞ = 2.35
ARGON M1 = 10.0
Pr = 1.0 μ ~ T^0.816
X = πx/4Λ1 vr = 0.00
f(vx, vr, X) / n1 (2πRT1)^-3/2
0.20
0.15
0.10
0.05
0
-10.0
-5.0
0
5.0
10.0
(vx - u)/√(2RT1)


B-G-K, 10th ITERATION
$M_1 = 10.0$
$Pr = 1.0$ $\mu \sim T^{0.816}$
$X = \frac{\pi}{4}\frac{x}{\Lambda_1} = 2.5$
$\frac{u}{\sqrt{2RT_1}} = 2.675$
v_y
v_r
v_x
u
v_z
$v_r = 0.0$
1.0
2.0
3.0
4.0
5.0
6.0
0.5
1.5
0.05
0.04
0.03
0.02
0.01
0
-15.0
-10.0
-5.0
0
5.0
10.0
15.0
$\frac{f(v_x, v_r, X)}{n_1(2\pi RT_1)^{-3/2}}$
$\frac{v_x - u}{\sqrt{2RT_1}}$


$\xi = \frac{(W_1 + W_2 - 2W)}{W_1 - W_2}$
$W = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2} = \frac{4M^2}{3+M^2}$
$M_1 = \infty$
10.0
5.0
3.0
2.0
1.5
1.2
FLOW
SUPERSONIC
SUBSONIC
τ/p
ξ
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
-1.0
-0.8
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0