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TRANSIENT HEAT TRANSFER THROUGH A THIN CIRCULAR PIPE DUE TO UNSTEADY FLOW IN THE PIPE

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ABSTRACT

An analysis of the transient heat transfer between a thin circular tube and the incompressible fluid moving through the tube is made for the case where the temperature of the inlet fluid is kept constant. Both radial conduction of heat in the wall and the heat loss at the outer surface of the cylinder are taken into consideration. It is shown to be especially easy to calculate temperature of both fluid and tube in the initial period by means of the attached figures.

So far it is common to use the temperature of the wall calculated by the assumption that it is constant radially. It is shown that for the insulated tube in the initial period such temperature is equal to the temperature of the outer surface of the tube. The temperature of the inner surface of the tube may be appreciably different from such temperature even for the metallic tube. The difference is extremely large for the tube made from insulating material.

author

NOMENCLATURE

a dimensionless constant, d/R

b dimensionless constant

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c_p	specific heat at constant pressure of inner fluid, kcal/kg deg C
C	heat capacity per unit volume of tube material, kcal/m^3 deg C
d	thickness of the wall
$f_1(\eta), f_2(\eta)$	dimensionless functions defined by equations (24)
$F(\alpha,\gamma;z)$	confluent hyper-geometric function
$F_1(\zeta,\tau), F_2(\zeta,\tau)$	dimensionless functions defined by equations (45a)
$g_1(\xi,\alpha), g_2(\xi,\alpha)$	dimensionless functions
$G = \rho V$	mass flow per unit area, kg/m^2 hr
$G_1(\zeta,\tau), G_2(\zeta,\tau)$	non-dimensional functions defined by equations (45b)
$I_\nu(\eta)$	modified Bessel function of the first kind and of νth order
k	thermal conductivity of tube material, kcal/m hr deg C
k_f	thermal conductivity of fluid kcal/m hr deg C
k_o	dimensionless constant, k/RU_o
k_1	dimensionless constant, k/RU_1
K	dimensionless constant, $\rho c_p k/2CU_o R$
$K_\nu(\eta)$	modified Bessel function of the second kind and of νth order
l	length of the tube, m
L,M,N	dimensionless constants
$L_1(\eta)$	dimensionless function
$N_1(\eta)$	dimensionless function
$\mathcal{L}(\delta)$	Laplace transform of δ
$\mathcal{L}^{-1}(\delta)$	inverse Laplace transform of δ
Nu	Nusselt number, $2RU_o/k_f$, dimensionless
p	variable of Laplace transform
P	pressure of the inner fluid, atm
Pr	Prandtl number, $\mu c_p/k_f$, dimensionless

r	radial distance, m
R	inner radius of the tube, m
Re	Reynolds number, $2RG/\mu$, dimensionless
t	time, hr
$T_1(x,t)$	temperature of inner fluid, deg C
$T_2(x,r,t)$	temperature of tube, deg C
T_i	inlet temperature of inner fluid, deg C
T_s	temperature of outer fluid, constant, deg C
U_o	film coefficient of heat transfer between inner fluid and tube surface, kcal/m^2 hr deg C
U_1	film coefficient of heat transfer between outer fluid and tube surface, kcal/m^2 hr deg C
v	velocity of inner fluid, m/hr
x	axial distance, m
y	$\sqrt{2\zeta\tau}$
α	$\sqrt{p}$
α^*	dimensionless constant, $(1 + a)U_1/U_o$
A	dimensionless constant, $(M/N) - 1$
β	dimensionless constant, $\alpha(1 + a)$
β^*	dimensionless constant, $\alpha^* + 1$
B	dimensionless constant, $A + 1$
γ	dimensionless function defined by equation (14)
δ	$(T_2 - T_s)/(T_i - T_s)$, dimensionless
$\bar{\delta}$	$\mathcal{L}(\delta)$
ζ	$L\xi$, dimensionless
ζ_1	$2LU_o l/RGc_p$, dimensionless
η	r/R, dimensionless

η^*	$2U_o t/a(2 + a)CR$, dimensionless
θ	$(T_1 - T_s)/(T_i - T_s)$, dimensionless
κ	dimensionless constant defined where utilized
μ	viscosity of the inner fluid, kg/m hr
ξ	$2U_o x/RGc_p$, dimensionless
ρ	density of inner fluid, kg/m^3
τ	$N\varphi$, dimensionless
φ	Fourier number, kt/CR^2, dimensionless
ψ	defined by equation (17), dimensionless
$\Psi(\zeta,\tau)$	$e^{-\zeta-\tau} I_o(2\sqrt{\zeta\tau})$, dimensionless
$\Psi^*(\xi,\eta^*)$	$e^{-\xi-\eta^*} I_o(2\sqrt{\xi\eta^*})$, dimensionless

Subscript

s	steady

INTRODUCTION

For any intermittent or blowdown hypersonic wind tunnel it is required to establish an air flow at elevated values of stagnation temperature. Thus, air is preheated in a heater and led to the test section through a settling chamber and nozzle. For the design of such components of the wind tunnel it is required to determine the transient temperature of the wall of each component as well as the temperature of air. This problem may be reduced to determine the transient heat transfer between a circular tube and the incompressible fluid flowing through the tube, since the main flow field is subsonic and the effect of compressibility can be neglected.

Such a problem has been treated already by several investigators. Rizika[1][1] calculated the transient temperature of fluid taking into account the heat loss at the outer surface of the tube. His result is inaccurate and does not contain the analysis of the temperature history of the wall. Judd[2] calculated transient temperature of both fluid and wall for the insulated tube. We extended Judd's calculation so that the effect of heat loss at the outer surface of the tube was included[3]. In these treatments it was assumed that the temperature of the wall is constant radially. The use of such an assumption may be justified if the thermal conductivity of the material of tube is large. Indeed, this was verified experimentally by Judd for a copper tube.

For the high temperature and/or high pressure application such as a hypersonic wind tunnel, however, we are forced to use such material as stainless steel or ceramics. For these materials thermal conductivity is smaller and the validity of the assumption is doubtful. Therefore such an assumption is replaced by a more plausible assumption of the thin wall in this paper.

ANALYSIS

The following assumptions are imposed on the solution:

1. The temperature of inner fluid is function of time and axial distance from the inlet only.

2. The temperature of tube is function of time, axial distance from the inlet, and radial distance from the axis.

3. The temperature of outer fluid is constant.

4. The effect of thermal conduction is negligible in the axial direction.

5. The velocity of inner fluid is axial, constant, and uniform.

6. Both the inner and the outer radii of the tube are constant, and the ratio of the difference between these radii to the inner radius is small.

[1]Numbers in brackets designate references at end of paper.

7. Material constants do not depend upon temperature.

8. There are no energy sources within the tube material itself.

9. The film coefficients of heat transfer between the fluid and tube are uniform and constant over the inner and outer tube surfaces for a constant fluid mass flow rate.

10. The effect of radiation is negligible.

11. The ratio of the thermal capacity per unit volume of air to that of tube material is negligible.

Consider the system shown in Fig. 1. Let an incompressible fluid be flowing in a circular pipe with a flow rate G; the temperature of this fluid at the axial distance x from the inlet and at time t is $T_1(x,t)$. The temperature of the circular pipe is $T_2(x,r,t)$, where r is the radial distance from the axis. The temperature of the fluid outside the pipe is assumed constant and equal to T_s.

Equating the sum of the thermal energy crossing the boundary of an elemental section of fluid gives

$$Gc_p \frac{\partial T_1}{\partial x} + \rho c_p \frac{\partial T_1}{\partial t} - k_f \frac{\partial^2 T_1}{\partial x^2} = - [T_1 - T_2(x,R,t)] \frac{2U_o}{R} \tag{1}$$

Similarly, for an elemental section of the wall, one obtains

$$\frac{\partial^2 T_2}{\partial r^2} + \frac{1}{r} \frac{\partial T_2}{\partial r} + \frac{\partial^2 T_2}{\partial x^2} = \frac{C}{k} \frac{\partial T_2}{\partial t} \tag{2}$$

Now we can neglect the third terms on the left-hand sides of equations (1) and (2) by assumption 4. Adopting the nondimensional variables, these equations are reduced to

$$\frac{\partial \theta}{\partial \xi} + K \frac{\partial \theta}{\partial \varphi} = \delta(\xi,1,\varphi) - \theta \tag{3}$$

$$\frac{\partial^2 \delta}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial \delta}{\partial \eta} = \frac{\partial \delta}{\partial \varphi} \tag{4}$$

where

$$K \equiv \frac{\rho c_p}{C} \frac{k}{2U_oR}$$

Since $k/(2U_oR) = O(1)$, K is negligible by assumption 11. Therefore equation (3) reduces to

$$\frac{\partial \theta}{\partial \xi} + \theta = \delta(\xi, 1, \varphi) \tag{5}$$

Equations (4) and (5) are fundamental equations for this analysis. Initial conditions are given by

$$\theta(\xi, 0) = \delta(\xi, \eta, 0) = 0 \tag{6}$$

Boundary conditions are

$$\left.\begin{aligned} \theta(0,\varphi) &= 1 \\ -k_o \left(\frac{\partial \delta}{\partial \eta}\right)_{\eta=1} &= \theta - \delta(\xi, 1, \varphi) \\ -k_1 \left(\frac{\partial \delta}{\partial \eta}\right)_{\eta=1+a} &= \delta(\xi, 1 + a, \varphi) \end{aligned}\right\} \tag{7}$$

After the following Laplace transformations

$$\bar{\theta}(\xi, p) = \int_0^\infty \theta(\xi, \varphi) e^{-p\varphi} d\varphi$$

$$\bar{\delta}(\xi, \eta, p) = \int_0^\infty \delta(\xi, \eta, \varphi) e^{-p\varphi} d\varphi$$

and using equation (6), equation (4) is reduced to

$$\frac{\partial^2 \bar{\delta}}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial \bar{\delta}}{\partial \eta} - p\bar{\delta} = 0$$

The general solution of this equation is given by

$$\bar{\delta}(\xi, \eta, \alpha) = g_1(\xi, \alpha) I_o(\alpha\eta) + g_2(\xi, \alpha) K_o(\alpha\eta) \tag{8}$$

where $\alpha = \sqrt{p}$, and g_1, and g_2 are dimensionless functions to be determined later. Using this equation, equation (5) is reduced to

$$\frac{\partial\bar{\theta}}{\partial\xi} + \bar{\theta} = I_0(\alpha)g_1(\xi,\alpha) + K_0(\alpha)g_2(\xi,\alpha) \tag{9}$$

Boundary conditions, equations (7), are transformed to

$$\bar{\theta}(0,p) = p^{-1}, \tag{10a}$$

$$-k_0\left(\frac{\partial\bar{\delta}}{\partial\eta}\right)_{\eta=1} = \bar{\theta} - \bar{\delta}(\xi,1,\alpha) \tag{10b}$$

$$-k_1\left(\frac{\partial\bar{\delta}}{\partial\eta}\right)_{\eta=1+\alpha} = \bar{\delta}(\xi,1 + a,\alpha) \tag{10c}$$

Substituting equation (8) into equation (10c) gives

$$g_1(\xi,\alpha) = -\frac{K_0(\beta) - k_1\alpha K_1(\beta)}{I_0(\beta) + k_1\alpha I_1(\beta)} g_2(\xi,\alpha) \tag{11}$$

Using equations (8) and (11), equation (10b) reduces to

$$g_2(\xi,\alpha) = \frac{\bar{\theta}}{k_0\alpha\left\{\frac{K_0(\beta) - k_1\alpha K_1(\beta)}{I_0(\beta) + k_1\alpha I_1(\beta)} I_1(\alpha) + K_1(\alpha)\right\} - \frac{K_0(\beta) - k_1\alpha K_1(\beta)}{I_0(\beta) + k_1\alpha I_1(\beta)} I_0(\alpha) + K_0(\alpha)} \tag{12}$$

Putting these expressions for g_1 and g_2 into equation (9) gives

$$\frac{\partial\bar{\theta}}{\partial\xi} = -\gamma\bar{\theta}, \tag{13}$$

where

$$\gamma = \left[1 - \frac{1}{k_0\alpha}\frac{I_0(\alpha)K_0(\beta) - I_0(\beta)K_0(\alpha) - k_1\alpha\{I_0(\alpha)K_1(\beta) + I_1(\beta)K_0(\alpha)\}}{I_1(\alpha)K_0(\beta) + I_0(\beta)K_1(\alpha) - k_1\alpha\{I_1(\alpha)K_1(\beta) - I_1(\beta)K_1(\alpha)\}}\right]^{-1} \tag{14}$$

The solution of equation (13), satisfying the boundary condition (10a), is given by

$$\bar{\theta}(\xi,p) = p^{-1}e^{-\gamma\xi} \tag{15}$$

Substituting equations (11) and (12) into equation (8) gives

$$\bar{\delta}(\xi,\eta,\alpha) = [K_0(\beta)I_0(\alpha\eta) - I_0(\beta)K_0(\alpha\eta) - k_1\alpha\{K_1(\beta)I_0(\alpha\eta) + I_1(\beta)K_0(\alpha\eta)\}]\bar{\theta}/\psi \tag{16}$$

where

$$\psi = K_0(\beta)I_0(\alpha) - I_0(\beta)K_0(\alpha) - k_1\alpha\{K_1(\beta)I_0(\alpha) + I_1(\beta)K_0(\alpha)\}$$
$$-k_0\alpha[I_1(\alpha)K_0(\beta) + I_0(\beta)K_1(\alpha) - k_1\alpha\{I_1(\alpha)K_1(\beta) - I_1(\beta)K_1(\alpha)\}] \tag{17}$$

In general it is not easy to get the inverse Laplace transformations of equations (15) and (16). Therefore we restrict our considerations to the cases where the wall of the tube is thin and the ratio of the thickness of the wall to inner radius of the tube is far less than unity: $a \ll 1$. Since $\beta = (1+a)\alpha$, one has

$$I_\nu(\beta) = (1+a)^\nu \sum_{n=0}^{\infty} \frac{(2a+a^2)^n}{n!}\left(\frac{\alpha}{2}\right)^n I_{\nu+n}(\alpha)$$

$$K_\nu(\beta) = (1+a)^\nu \sum_{n=0}^{\infty} (-1)^n \frac{(2a+a^2)^n}{n!}\left(\frac{\alpha}{2}\right)^n K_{\nu+n}(\alpha)$$

Therefore, neglecting the terms of $O(a^3)$, one obtains

$$\left.\begin{aligned}
I_0(\alpha)K_0(\beta) - I_0(\beta)K_0(\alpha) &= I_1(\alpha)K_1(\beta) - I_1(\beta)K_1(\alpha) = -\frac{a}{2}(2-a)\\
I_0(\alpha)K_1(\beta) + I_1(\beta)K_0(\alpha) &= (1-a)\frac{1}{\alpha} + \left(\frac{1}{\alpha} + \frac{\alpha}{2}\right)a^2\\
I_1(\alpha)K_0(\beta) + I_0(\beta)K_1(\alpha) &= \frac{1}{\alpha} + \frac{\alpha}{2}a^2\\
I_1(\alpha)K_1(\beta) - I_1(\beta)K_1(\alpha) &= -\frac{a}{2}(2-a)
\end{aligned}\right\} \tag{18}$$

Substituting these expressions into equations (14) gives

$$\gamma = L\left(1 - \frac{N}{p+M}\right) \tag{19}$$

where

$$L = \frac{2k_1 + (1 - k_1)a}{2k_1 + \{(k_1/k_o) + 1 - k_1\}a} < 1$$

$$M = \frac{2k_o + 2k_1 + 2(1 - k_1)a - (1 - 2k_1)a^2}{k_o a[2k_1 + \{(k_1/k_o) + 1 - k_1\}a]}$$

$$N = M - \frac{2}{\{2k_1 + (1 - k_1)a\}a}$$

Thus equation (15) is rewritten as

$$\bar{\theta}(\xi, p) = e^{-L\xi} \frac{1}{p} \exp\left(\frac{LN\xi}{p + M}\right) \tag{20}$$

Now one has

$$\mathcal{L}^{-1}\left[\frac{1}{p} \exp\left(\frac{A}{p + M}\right)\right] = e^{-M\varphi} I_o(2\sqrt{A\varphi}) + M\int_o^{\varphi} e^{-M\varphi} I_o(2\sqrt{A\varphi})d\varphi \tag{21}$$

whose derivation is given in appendix. Therefore one obtains

$$\theta(\xi, \varphi) = e^{-L\xi}\left\{e^{-M\varphi} I_o(2\sqrt{LN\xi\varphi}) + M\int_o^{\varphi} e^{-M\varphi} I_o(2\sqrt{LN\xi\varphi})d\varphi\right\} \tag{22}$$

Combining the equations (16) and (18) together with the following equations

$$K_o(\beta)I_o(\alpha\eta) - I_o(\beta)K_o(\alpha\eta) = -\frac{1}{2}\{2a - a^2 - 2(\eta - 1) + (\eta - 1)^2\}$$

$$K_1(\beta)I_o(\alpha\eta) + I_1(\beta)K_o(\alpha\eta) = (1 - a + a^2)\frac{1}{\alpha} + \frac{\alpha}{2}\{a - (\eta - 1)\}^2$$

where the terms of $O(a^3)$ are also neglected gives

$$\bar{\delta}(\xi, \eta, p) = f_1(\eta)\bar{\theta}(\xi, p) + Mf_2(\eta)e^{-L\xi}\frac{1}{p(p + M)} \exp\left(\frac{LN\xi}{p + M}\right) \tag{23}$$

where

$$f_1(\eta) = \frac{\{a - (\eta - 1)\}^2}{(k_o/k_1)a^2 + k_o(2a - a^2) + a^2} \tag{24a}$$

$$f_2(\eta) = 1 - \frac{2k_o + 2(\eta - 1) - (\eta - 1)^2}{2k_o + 2k_1 + 2(1 - k_1)a - (1 - 2k_1)a^2} - f_1(\eta) \tag{24b}$$

Now one has

$$\mathcal{L}^{-1}\left[\frac{1}{p(p + M)} \exp\left(\frac{A}{p + M}\right)\right] = \int_o^{\varphi} e^{-M\varphi} I_o(2\sqrt{A\varphi})d\varphi \tag{25}$$

whose derivation is also given in appendix. Therefore the inverse Laplace transformation of equation (23) is obtained as

$$\delta(\xi,\eta,\varphi) = f_1(\eta)\theta(\xi,\varphi) + Mf_2(\eta)e^{-L\xi}\int_0^{\varphi} e^{-M\varphi}I_0(2\sqrt{LN\xi\varphi})d\varphi \tag{26}$$

Using $\zeta = L\xi$, $\tau = N\varphi$, $A = (M/N) - 1$, $B = A + 1 = M/N$, equations (22) and (26) are reduced to

$$\theta(\zeta,\tau) = e^{-A\tau}\Psi(\zeta,\tau) + B\int_0^{\tau} e^{-A\tau}\Psi(\zeta,\tau)d\tau \tag{27}$$

and

$$\delta(\zeta,\eta,\tau) = f_1(\eta)e^{-A\tau}\Psi(\zeta,\tau) + B\left\{f_1(\eta) + f_2(\eta)\right\}\int_0^{\tau} e^{-A\tau}\Psi(\zeta,\tau)d\tau \tag{28}$$

respectively, where

$$\Psi(\zeta,\tau) = e^{-\zeta-\tau}I_0(2\sqrt{\zeta\tau}) \tag{29}$$

Since $f_1(1) = 1 - L$, $f_1(1 + a) = 0$ and $Bf_2(1) = L$, one obtains

$$\delta(\zeta,1,\tau) = (1 - L)e^{-A\tau}\Psi(\zeta,\tau) + (B + L - BL)\int_0^{\tau} e^{-A\tau}\Psi(\zeta,\tau)d\tau \tag{30}$$

$$\delta(\zeta,1 + a,\tau) = Bf_2(1 + a)\int_0^{\tau} e^{-A\tau}\Psi(\zeta,\tau)d\tau \tag{31}$$

where

$$f_2(1 + a) = \frac{2k_1(1 - a + a^2)}{2k_0 + 2a - a^2 + 2k_1(1 - a + a^2)} \tag{32}$$

Equations (30) and (31) give the temperature of inner and outer surface of the tube wall, respectively. Moreover one gets

$$\delta(0,\eta,\tau) = f_1(\eta) + f_2(\eta)(1 - e^{-B\tau}) \tag{33}$$

which gives the temperature distribution in the tube wall at entrance section.

DISCUSSION

First it will be shown that the solutions obtained in the previous section tend to the steady solutions when τ tends to infinity. Here the steady solutions are the solutions of the equations (4) and (5) where $\partial\delta/\partial\varphi$ is assumed to vanish:

$$\theta_s(\xi) = e^{-\kappa\xi} \tag{34}$$

$$\delta_s(\xi,\eta) = \left(1 - \kappa - \frac{\kappa}{k_o} \ln \eta\right)e^{-\kappa\xi} \tag{35}$$

where

$$\kappa = \frac{k_o}{k_o + \frac{k_1}{1 + a} + \ln (1 + a)}$$

Neglecting the terms of $O(a^3)$ and considering $0 \leq \eta - 1 \leq a \ll 1$ give

$$\kappa\xi = \frac{2k_o\xi}{2k_o + 2k_1 + 2(1 - k_1)a - (1 - 2k_1)a^2} = \left(1 - \frac{1}{B}\right)\zeta$$

and

$$1 - \kappa - \frac{\kappa}{k_o} \ln \eta = 1 - \kappa\left[1 + \frac{1}{k_o}\left\{(\eta - 1) - \frac{1}{2}(\eta - 1)^2\right\}\right] = f_1(\eta) + f_2(\eta)$$

Thus one gets

$$\theta_s = \exp\left\{-\left(1 - \frac{1}{B}\right)\zeta\right\} \tag{36}$$

and

$$\delta_s = \left\{f_1(\eta) + f_2(\eta)\right\} \exp\left\{-\left(1 - \frac{1}{B}\right)\zeta\right\} \tag{37}$$

Now tending τ to infinity in equation (27) gives

$$\theta(\zeta,\infty) = Be^{-\zeta}\int_0^\infty e^{-B\tau}I_o(2\sqrt{\zeta\tau})d\tau$$

Since

$$\int_0^\infty e^{-B\tau}I_o(2\sqrt{\zeta\tau})d\tau = \frac{1}{2\zeta}\int_0^\infty ye^{-b^2y^2}I_o(y)dy$$

where $y = 2\sqrt{\zeta\tau}$ and $b^2 = B/4\zeta$, and

$$\int_0^\infty ye^{-b^2y^2}I_o(y)dy = \frac{1}{2b^2}\exp\left(\frac{1}{4b^2}\right)$$

[see reference 3, equation (35)], this equation is reduced to

$$\theta(\zeta,\infty) = \exp\left\{-\left(1 - \frac{1}{B}\right)\zeta\right\} \tag{38}$$

In a similar manner, from equation (28), one obtains

$$\delta(\zeta,\eta,\infty) = \left\{f_1(\eta) + f_2(\eta)\right\} \exp\left\{-\left(1 - \frac{1}{B}\right)\zeta\right\} \tag{39}$$

Comparing the equations (36) and (37) with equations (38) and (39), it is clear that the present solutions tend to the steady solutions when τ tends to infinity.

Next it will be shown that the present solutions tend to the solutions obtained by assuming that $k = \infty$[3] when k tends to infinity. When k tends to infinity one obtains

$$\begin{aligned}
L &= 1 \\
B &= \frac{1}{1 - a + a^2}\frac{U_1}{U_0} + 1 \simeq (1 + a)\frac{U_1}{U_0} + 1 = \beta^* \\
A &\simeq \alpha^* \\
\tau &= \frac{(1 - a + a^2)2U_0t}{a(2 - a)CR} \simeq \frac{2U_0t}{a(2 + a)CR} = \eta^*
\end{aligned} \tag{40}$$

$$f_1(\eta) = 0$$

$$Bf_2(\eta) = 1$$

where $*$ denotes the symbol used in reference 3. Thus it is proved that the present solutions coincide with the previous results [equations (25) and (26) of reference 3], that is

$$\theta = e^{-\alpha^*\eta^*}\Psi^* + \beta^*\int_0^{\eta^*} e^{-\alpha^*\eta^*}\Psi^* d\eta^*$$

$$\delta = \int_0^{\eta^*} e^{-\alpha^*\eta^*}\Psi^* d\eta^*$$

where $\Psi^* = e^{-\xi-\eta^*}I_0(2\sqrt{\xi\eta^*})$, to the order of a.

INITIAL SOLUTIONS

Solutions for $A\tau \ll 1$ will be called as "initial solutions" in this paper. Neglecting the terms of $O(A^2\tau^2)$, equations (27), (28), (30), and (31) are reduced to

$$\theta(\zeta,\tau) = F_1(\zeta,\tau) - AF_2(\zeta,\tau) - A^2G_2(\zeta,\tau) \tag{41}$$

$$\begin{aligned}\delta(\zeta,\eta,\tau) = &\{f_1(\eta)F_1(\zeta,\tau) + f_2(\eta)G_1(\zeta,\tau)\} \\ &- A[f_1(\eta)F_2(\zeta,\tau) - f_2(\eta)\{G_1(\zeta,\tau) \\ &- G_2(\zeta,\tau)\}] - A^2\{f_1(\eta) + f_2(\eta)\}G_2(\zeta,\tau)\end{aligned} \tag{42}$$

$$\begin{aligned}\delta(\zeta,1,\tau) = &\{(1 - L)F_1(\zeta,\tau) + LG_1(\zeta,\tau)\} \\ &- A[(1 - L)F_2(\zeta,\tau) + LG_2(\zeta,\tau)] \\ &- A^2(1 - L)G_2(\zeta,\tau)\end{aligned} \tag{43}$$

$$\begin{aligned}\delta(\zeta,1 + a,\tau) = &f_2(1 + a)[G_1(\zeta,\tau) + A\{G_1(\zeta,\tau) \\ &- G_2(\zeta,\tau)\} - A^2 G_2(\zeta,\tau)]\end{aligned} \tag{44}$$

respectively,[2] where

$$\left.\begin{aligned}F_1(\zeta,\tau) &= \Psi + \int_0^\tau \Psi d\tau \\ F_2(\zeta,\tau) &= \tau\Psi - \int_0^\tau (1 - \tau)\Psi d\tau\end{aligned}\right\} \tag{45a}$$

$$\left.\begin{aligned}G_1(\zeta,\tau) &= \int_0^\tau \Psi d\tau \\ G_2(\zeta,\tau) &= \int_0^\tau \tau\Psi d\tau\end{aligned}\right\} \tag{45b}$$

In the previous report [3] it was shown that F_1 and G_1 are temperature of inner fluid and of tube wall, respectively, assuming that the latter is constant radially and that there is no heat loss at the outer surface of the tube.

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[2]If $\tau < 1$, it is easily shown that $A^2G_2(\zeta,\tau) = O(A^2\tau^2)$, so that terms proportional to $A^2G_2(\zeta,\tau)$ can be eliminated from these equations.

Graphs of the functions F_1, F_2, G_1, and G_2 have been published in that report and presented here also (as Figs. 2(a) through 2(h)) for the reader's convenience. Each term of the solutions can be easily calculated by means of these figures.

If there is no heat loss at the outer surface of the thin tube, one obtains $U_1 = 0$ and $k_1 = \infty$, so that $A = 0$,[3] $f_2(1 + a) = 1$ and

$$L = [1 + a/\{k_0(2 - a)\}]^{-1} \tag{46}$$

Therefore temperatures of inner fluid and of outer surface of wall of insulated thin tube are equal to those of inner fluid and of tube wall, respectively, which are calculated by the assumption of radially constant wall temperature. Temperature of inner surface of wall of insulated thin tube is given by the linear combination of these temperatures:

$$\delta(\zeta,1,\tau) = (1 - L)F_1(\zeta,\tau) + LG_1(\zeta,\tau) \tag{47}$$

where L is given by equation (46).

The first order effect of heat loss at the outer surface of the wall might be easily taken into consideration for many cases of practical importance. For this purpose, equation (42) is rewritten as

$$\delta(\zeta,\eta,\tau) = \{f_1(\eta)F_1(\zeta,\tau) + Bf_2(\eta)G_1(\zeta,\tau)\}$$
$$- A\{f_1(\eta)F_2(\zeta,\tau) + f_2(\eta)G_2(\zeta,\tau)\} - A^2\{f_1(\eta) + f_2(\eta)\}G_2(\zeta,\tau)$$

Usually the second and the third terms are far smaller than the first term. In such cases this equation is reduced to

$$\delta(\zeta,\eta,\tau) = f_1(\eta)F_1(\zeta,\tau) + Bf_2(\eta)G_1(\zeta,\tau) \tag{48}$$

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[3] It follows that each first term of equations (41) through (44) represents the solution for the insulated tube, and the second and the third terms represent the correction for the heat loss.

In a similar manner one obtains

$$\theta(\zeta,\tau) = F_1(\zeta,\tau) \tag{49}$$

$$\delta(\zeta,1,\tau) = (1 - L)F_1(\zeta,\tau) + LG_1(\zeta,\tau) \tag{50}$$

$$\delta(\zeta,1 + a,\tau) = Bf_2(1 + a)G_1(\zeta,\tau) \tag{51}$$

Thus there is no appreciable effect of heat loss on the temperature of inner fluid. If a and a/k_1 are small quantities, $a/k_o = O(1)$ and the second order small quantities are neglected, one gets

$$Bf_2(1 + a) = 1 - a/(2k_1) \tag{52}$$

$$L = \frac{2}{2 + a/k_o} + \frac{1 - k_1}{(2 + a/k_o)^2} \frac{a^2}{k_o k_1} \tag{53}$$

SAMPLE CALCULATION

As examples temperature distributions of both wall material and air have been calculated for a settling chamber of a hypersonic wind tunnel. Here three kinds of material are considered as possible tube material for such a wind tunnel: stainless steel (18Cr-8Ni), alumina brick and alumina castable. Dimensions of the tube are:

length of the tube	$l = 3$ m
inner radius of the tube	$R = 0.325$ m
thickness of the wall	$d = 0.041$ m

so that $a = d/R = 0.126$. U_o was calculated from the empirical relation[4]

$$Nu = 0.023\, Re^{0.8} Pr^{0.4}$$

for the fully developed turbulent flow in a circular pipe where $Nu = 2RU_o/k_f$, $Re = 2RG/\mu$, $Pr = \mu c_p/k_f$ and μ is the viscosity of air. U_1 was calculated by means of the empirical formula [4]

$$U_1 = 0.929 \left(\frac{\langle T_2 \rangle - T_s}{R + d} \right)^{0.25} , \text{ kal/m}^2 \text{ hr deg C}$$

where $\langle T_2 \rangle$ is the mean value of the temperature of the outer surface of tube. Since $\langle T_2 \rangle$ was not known a priori, it was determined by a trial and error procedure. Constants for inner air are given by Table 1, where M is design Mach number and P is pressure. Constants for the settling chamber are given by Table 2, where ζ/x, τ/t, and all dimensionless quantities are the mean values of four which correspond to the four cases in Table. 1. The variation for A, L, and $f_2(1 + a)$ for these four cases is less than 7 percent. Calculated temperatures of inner air and of inner and outer surfaces of tube wall are shown in Table 3 for both inlet ($\zeta = 0$) and exit ($\zeta = \zeta_1$) except the temperature of inner air at inlet which is equal to unity. Since $A\tau < 0.021$, initial solution was used for the calculation. The temperature of tube wall is highest at inlet. This temperature was also calculated by the former method[3] where the temperature of tube wall was assumed constant radially. The results are included in Table 3 as $\delta(0,\tau)$ and $\delta(0,\eta^*)$. From equation (40) one obtains

$$\eta^* = \frac{2 - a}{(2 + a)(1 - a + a^2)} \tau$$

so that the difference between η^* and τ is $O(a^2)$. It may be clear from Table 3 that $\delta(\tau)$ is greater than $\delta(\eta^*)$ in the amount of $O(a^2)$. Thus it could be concluded that the thin-wall approximation estimates the temperature of tube wall higher than the accurate value. From the designing point of view this overestimate is in safety side.

The temperature of inner air was also calculated by the former method for $\zeta = \zeta_1$. This agrees with the present result in three figures for all cases. This confirms equation (49). It might be seen that Tables 2 and 3 also confirm equations (50) through (53).

The temperature distributions in the tube wall are presented in Figs. 3 through 5. It will be seen in the refractory materials temperature decreases rapidly near the inner surface and then decreases gradually to the temperature of the outer surface.

CONCLUSIONS

There is an appreciable temperature gradient in tube wall even if the material of the tube is such metal as stainless steel. This temperature gradient is very large if the material is such insulating refractory materials as alumina brick and alumina castable, in which case temperature decreases rapidly near the inner surface and then decreases gradually to that of outer surface.

Temperature of tube wall calculated by the assumption that this temperature is constant radially is equal to the temperature of outer surface of the tube in the initial period, provided $a/k_1 \ll 1$. It might be dangerous to use such temperature for the design since the temperature difference between inner and outer surfaces is sometimes very large.

Temperature of inner fluid in the initial period calculated by the thin-wall approximation is the same as that calculated by the assumption of radially constant wall temperature.

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4. Heat Transmission, W. H. McAdams, McGraw-Hill Book Company, Inc., New York, third edition, 1954.

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APPENDIX

1. Derivation of equation (21)

One has, from reference 5, page 244,

$$e^{A/p} - 1 = \mathcal{L}\left[\sqrt{\frac{A}{\varphi}}\, I_1(2\sqrt{A\varphi})\right]$$

and from reference 5, page 129,

$$F(p + M) = \mathcal{L}\left[e^{-M\varphi} f(\varphi)\right] \tag{54}$$

where $F(p) = \mathcal{L}[f(\varphi)]$. Combining these equations gives

$$\exp\left(\frac{A}{p + M}\right) - 1 = \mathcal{L}\left[\sqrt{\frac{A}{\varphi}}\, e^{-M\varphi} I_1(2\sqrt{A\varphi})\right]$$

By the convolution theorem, one obtains

$$\mathcal{L}^{-1}\left[\frac{1}{p}\left\{\exp\left(\frac{A}{p + M}\right) - 1\right\}\right] = e^{-M\varphi} I_0(2\sqrt{A\varphi}) - 1 + M\int_0^{\varphi} e^{-M\varphi} I_0(2\sqrt{A\varphi})\, d\varphi$$

Since $1/p = \mathcal{L}[1]$ and

$$\frac{1}{p}\exp\left(\frac{A}{p + M}\right) = \frac{1}{p}\left\{\exp\left(\frac{A}{p + M}\right) - 1\right\} + \frac{1}{p}$$

one gets equation(21).

2. Derivation of equation (25)

One has, from reference 5, page 245,

$$p^{-1} e^{A/p} = \mathcal{L}[I_0(2\sqrt{A\varphi})]$$

Combining this and equation (54) gives

$$\frac{1}{p + M}\exp\left(\frac{A}{p + M}\right) = \mathcal{L}\left[e^{-M\varphi} I_0(2\sqrt{A\varphi})\right]$$

By the convolution theorem equation (25) is derived.

TABLE 1.- CONSTANTS FOR INNER AIR

M	P (atm)	G (kg/m^2 hr)	T_i (deg C)	ρ (kg/m^3)	μ (kg/m hr)	k_f (kcal/m hr deg C)	c_p (kcal/kg deg C)	Re	Pr	Nu
5	22.7	2.50×10^5	600	9.11	0.138	0.0494	0.267	1.17×10^6	0.749	1466
5	22.7	2.50×10^5	1200	5.41	.190	.0741	.293	8.53×10^5	.753	1141
7	96.7	2.33×10^5	600	37.9	.140	.0494	.270	1.09×10^6	.763	1395
7	96.7	2.33×10^5	1200	22.7	.190	.0741	.294	7.97×10^5	.755	1078

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TABLE 2.- CONSTANTS FOR THE SETTLING CHAMBER

Tube material	C (kcal/m^3 deg C)	k (kcal/m hr deg C)	k/C (m^2/hr)	k_o	k_1	L	M	N	A	$f_2(1 + a)$	ζ/x (1/m)	τ/t (1/hr)
Stainless steel	923	14.0	0.0152	0.369	6.99	0.847	18.5	17.3	0.0696	0.927	0.00902	2.48
Alumina brick	824	2.10	.00255	.0553	1.05	.466	70.7	63.1	.120	.843	.00497	1.52
Alumina castable	494	1.00	.00202	.0263	.499	.307	103	87.7	.170	.754	.00328	1.68

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TABLE 3.- TEMPERATURE DISTRIBUTIONS

Material	τ	$\delta(0,1,\tau)$	$\delta(0,1+a,\tau)$	$\delta(0,\tau)$	$\delta(0,\eta^*)$	$\theta(\zeta_1,\tau)$	$\delta(\zeta_1,1,\tau)$	$\delta(\zeta_1,1+a,\tau)$
Stainless steel	0.04	0.186	0.039	0.039	0.039	0.972	0.181	0.038
	.08	.218	.076	.077	.076	.973	.212	.074
	.12	.248	.112	.113	.112	.974	.242	.109
	.16	.278	.146	.147	.146	.975	.271	.142
Alumina brick	.025	.545	.024	.025	.025	.985	.537	.023
	.050	.556	.046	.049	.049	.986	.549	.045
	.075	.567	.068	.072	.071	.986	.559	.067
	.100	.578	.089	.095	.094	.987	.570	.088
Alumina castable	.03	.701	.026	.030	.030	.990	.694	.026
	.06	.709	.051	.058	.057	.991	.702	.051
	.09	.716	.075	.086	.085	.991	.710	.074
	.12	.724	.099	.112	.111	.991	.717	.098

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FIGURE CAPTIONS

Fig. 1 Thin circular tube.

Fig. 2(a) Temperature functions at $\xi = 0$: $F_1 = 1$, $F_2 = 0$, $G_1 = 1 - e^{-\zeta}$, $G_2 = 1 - (1 + \zeta)e^{-\zeta}$.

Fig. 2(b) Temperature functions at $\xi = 0.01$: $F_1(0.01,0) = 0.99005$.

Fig. 2(c) Temperature functions at $\xi = 0.03$: $F_1(0.03,0) = 0.97045$.

Fig. 2(d) Temperature functions at $\xi = 0.1$: $F_1(0.1,0) = 0.90484$.

Fig. 2(e) Temperature functions at $\xi = 0.3$: $F_1(0.3,0) = 0.74082$.

Fig. 2(f) Temperature functions at $\xi = 1$: $F_1(1,0) = 0.36788$.

Fig. 2(g) Temperature functions at $\xi = 3$: $F_1(3,0) = 0.049787$.

Fig. 2(h) Temperature functions at $\xi = 10$: $F_1(10,0) = 0.000045$.

Fig. 3(a) Wall temperature distributions for stainless steel tube at inlet.

Fig. 3(b) Wall temperature distributions for stainless steel tube at exit.

Fig. 4(a) Wall temperature distributions for alumina brick tube at inlet.

Fig. 4(b) Wall temperature distributions for alumina brick tube at exit.

Fig. 5(a) Wall temperature distributions for alumina castable tube at inlet.

Fig. 5(b) Wall temperature distributions for alumina castable tube at exit.

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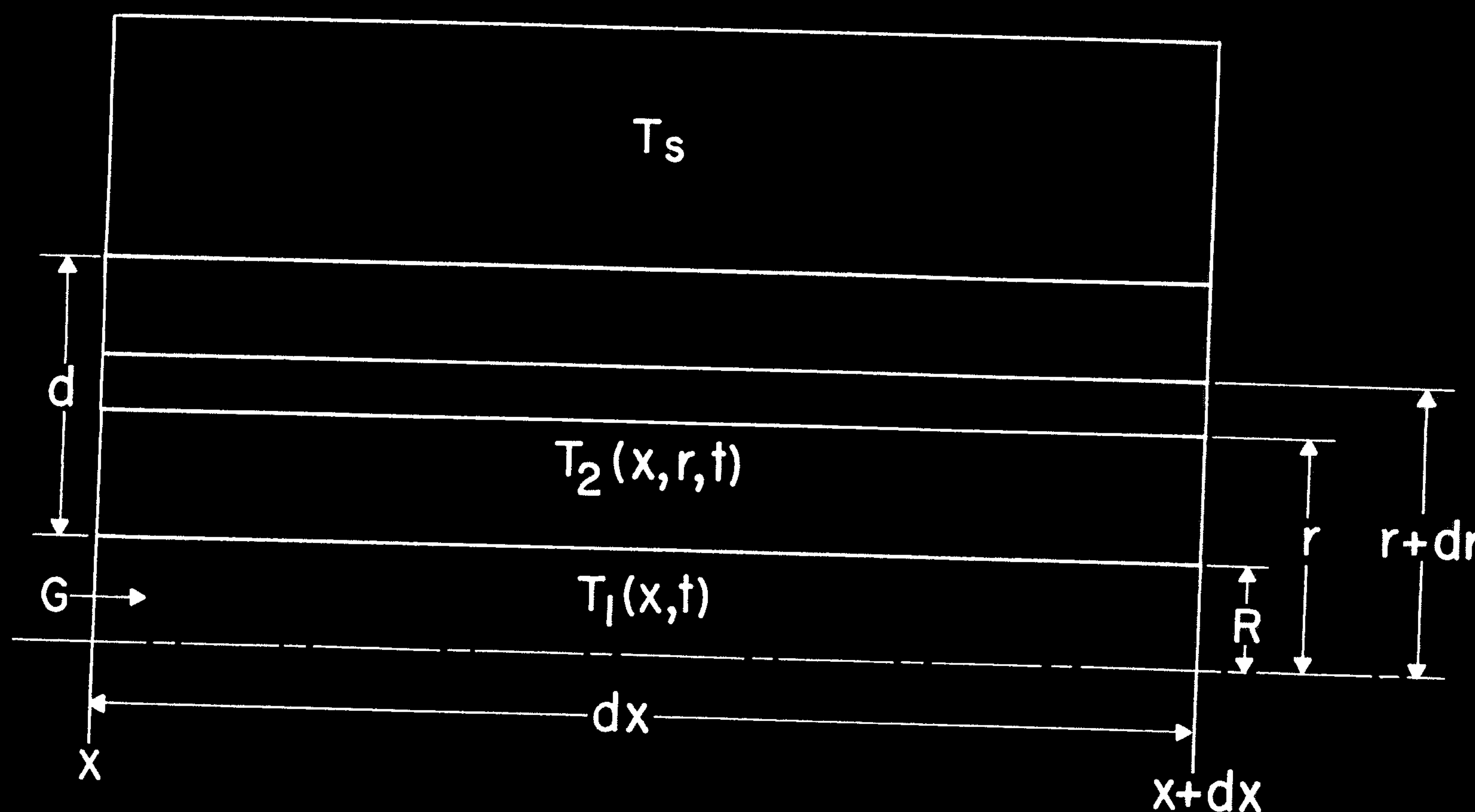

Figure 1.

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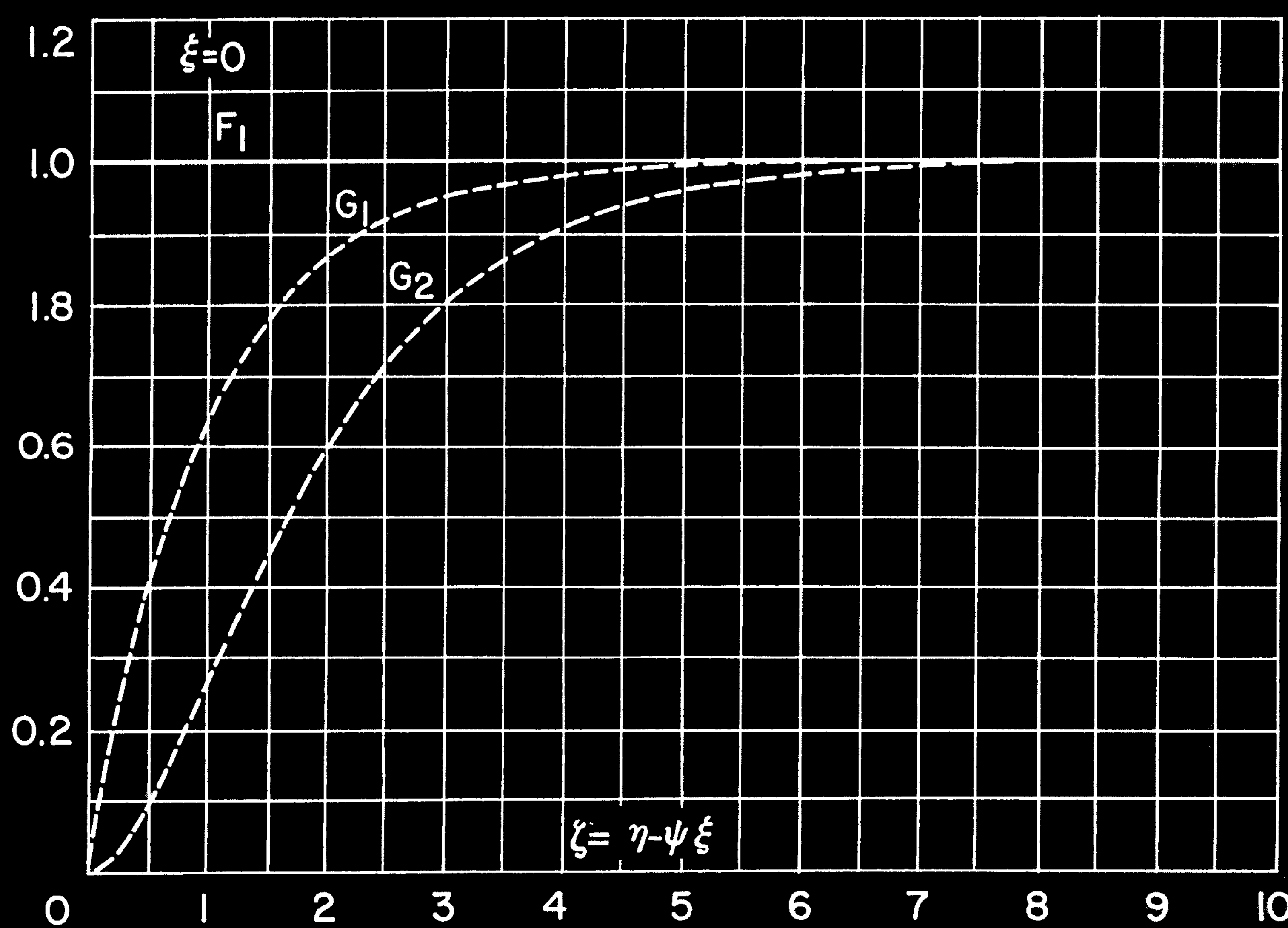

Figure 2(a)

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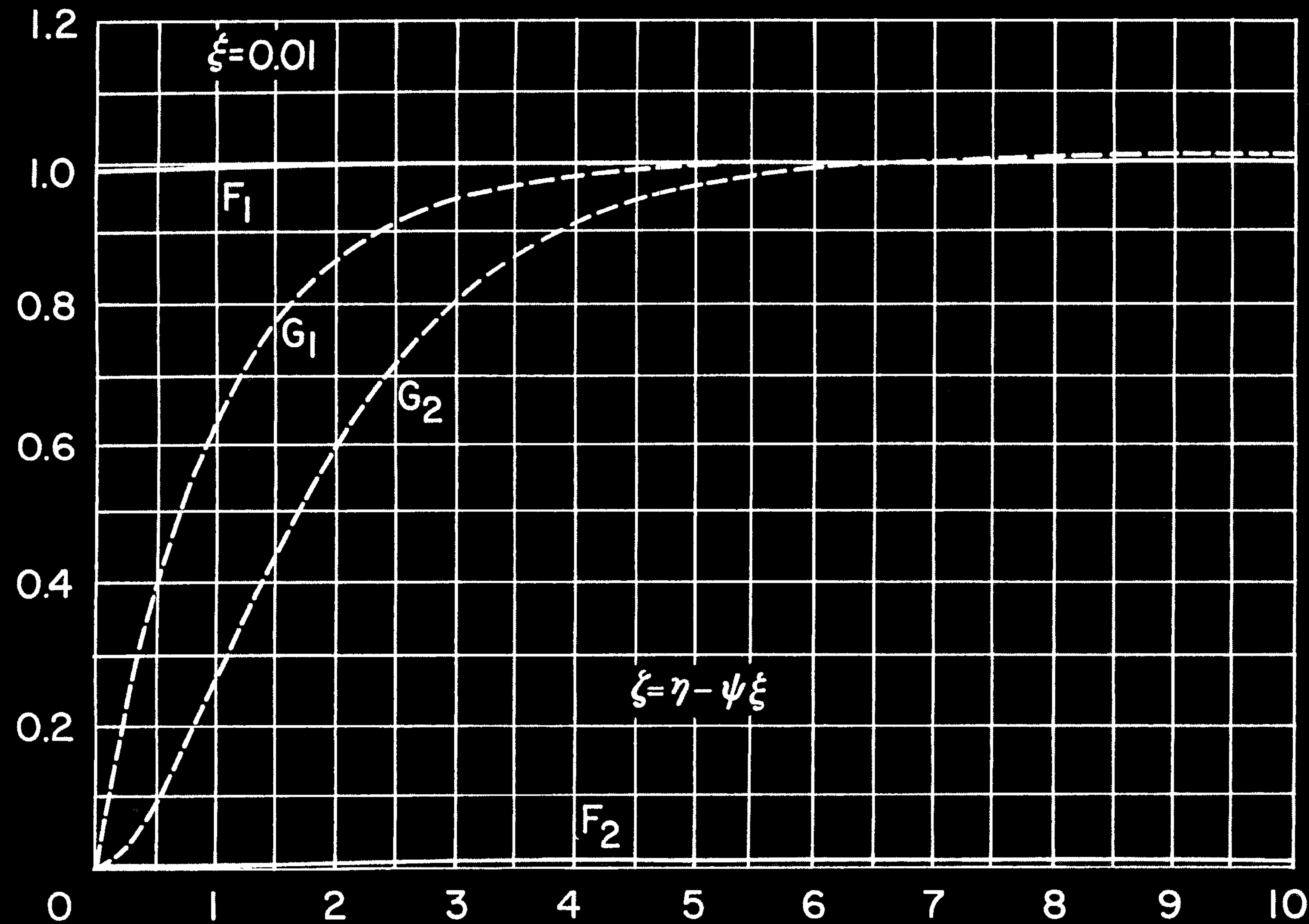

Figure 2(b)

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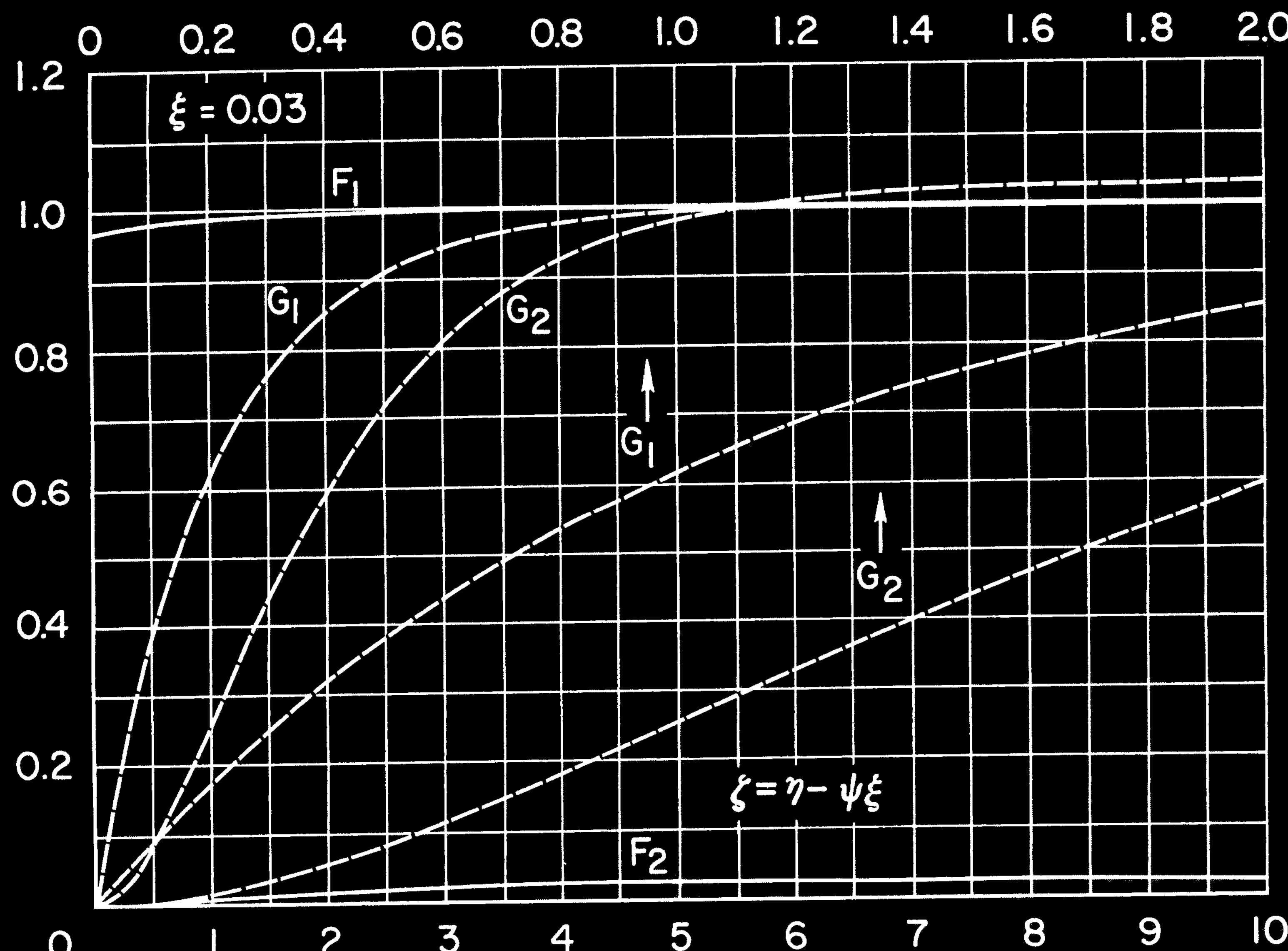

Figure 2(c)

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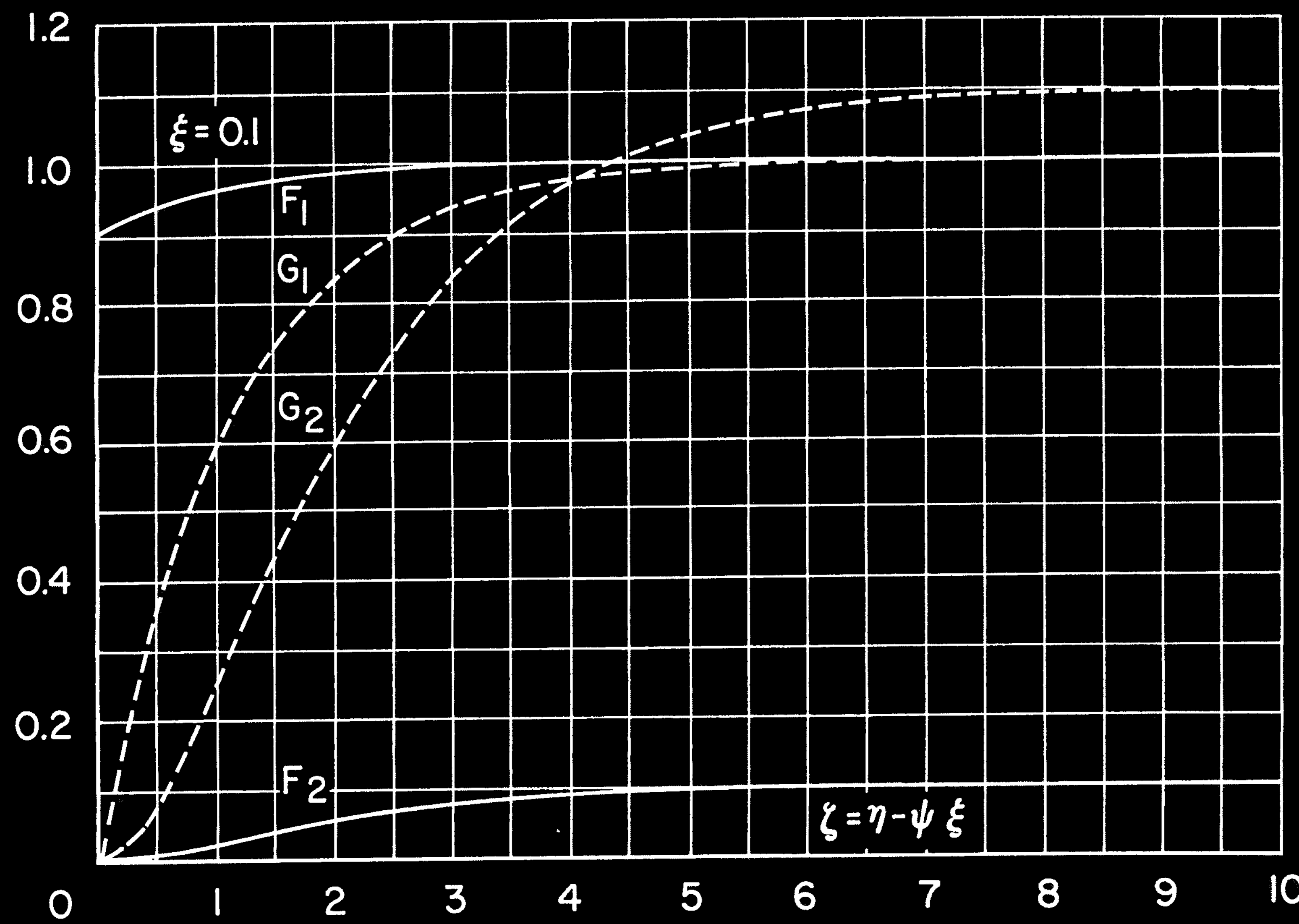

Figure 2(d)

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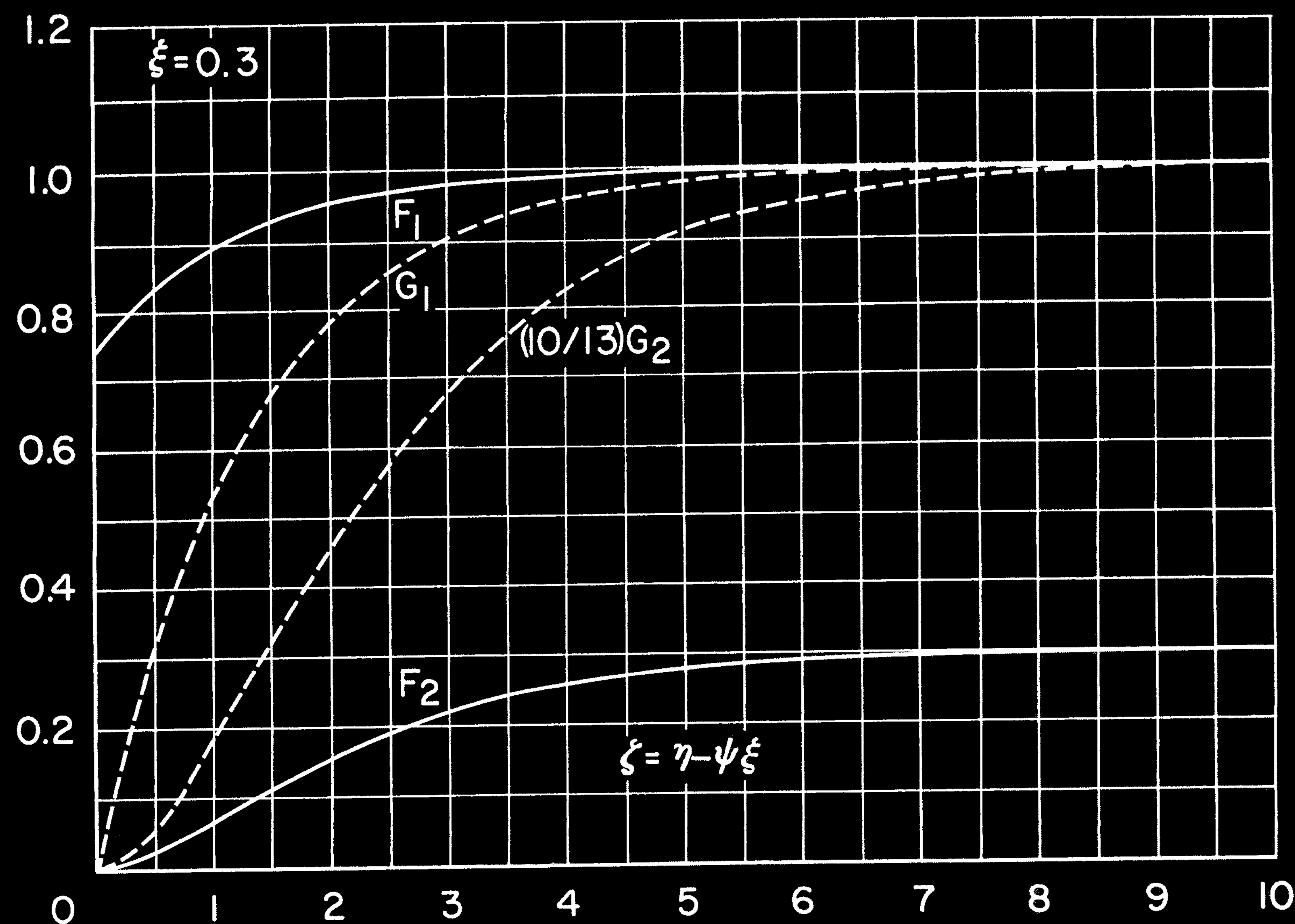

Figure 2(e)

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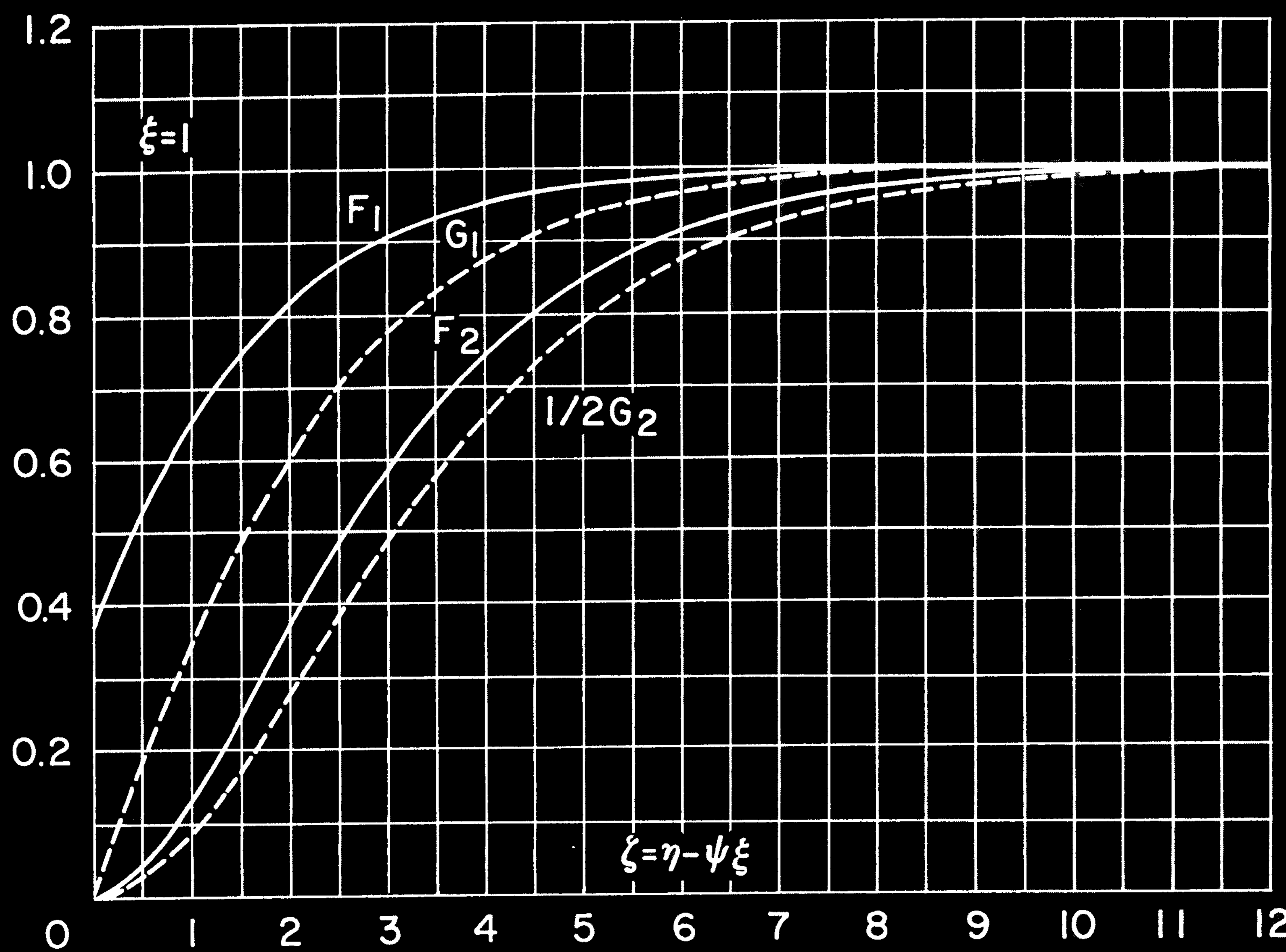

Figure 2(f)

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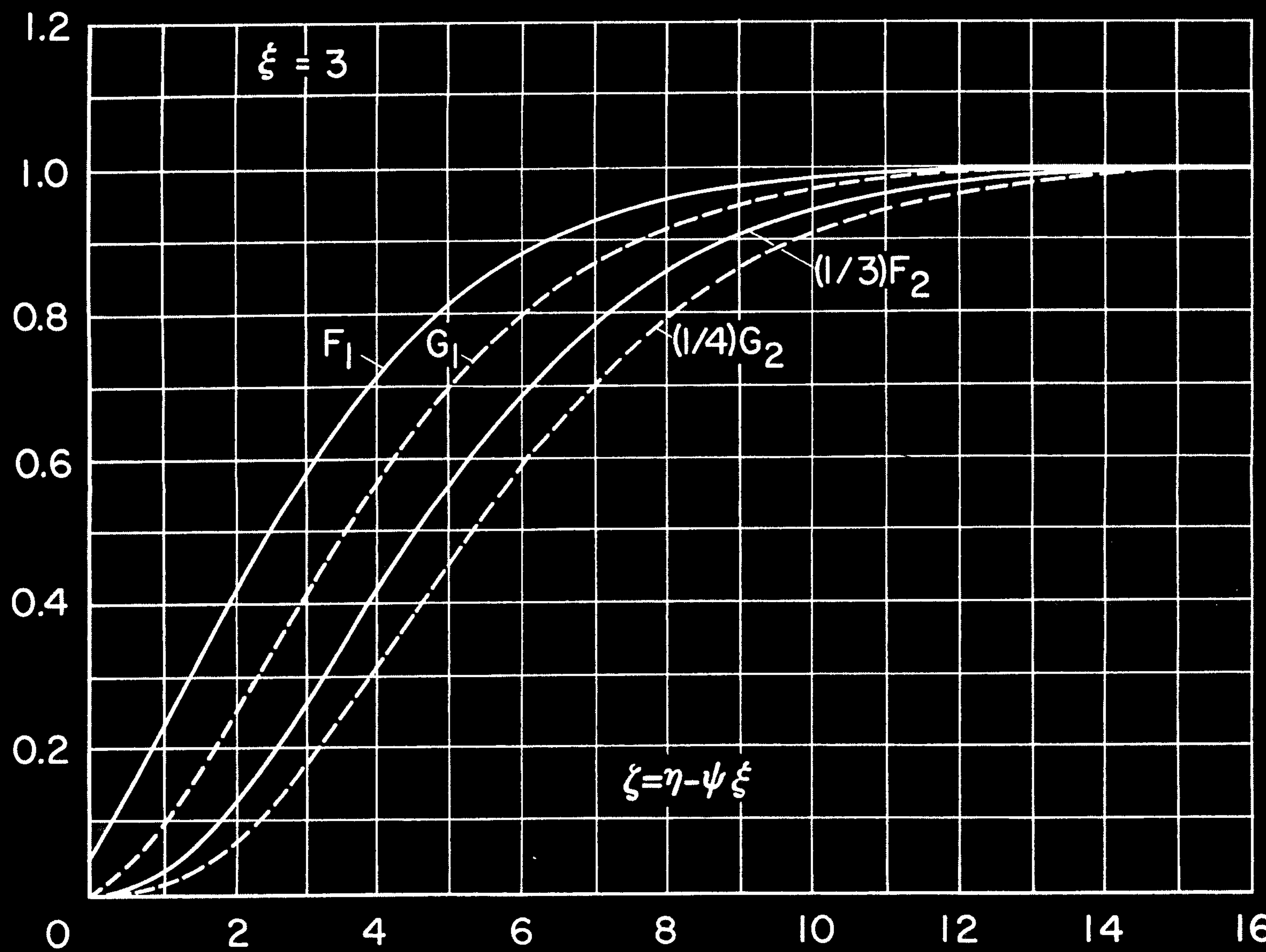

Figure 2(g)

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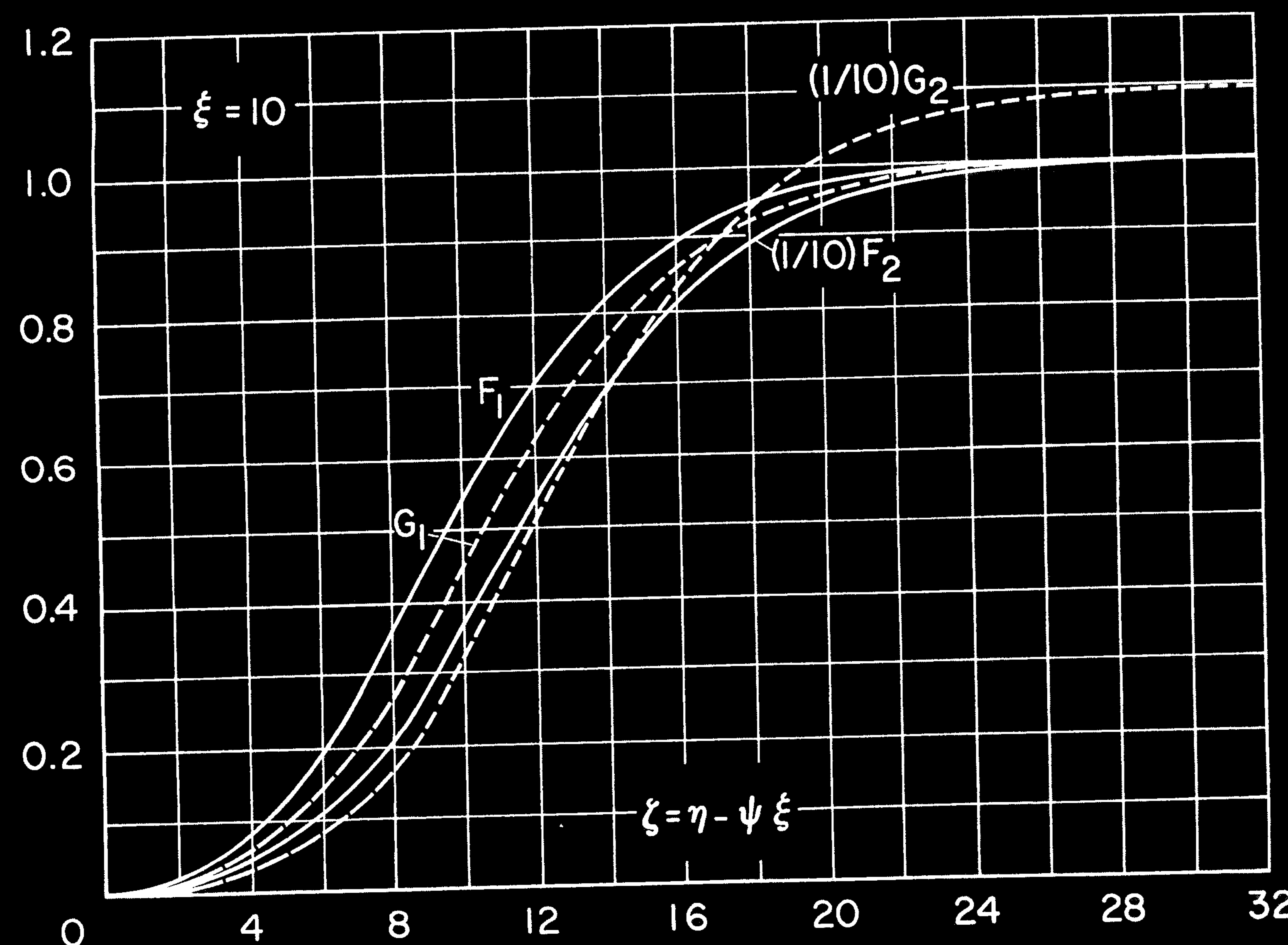

Figure 2(h)

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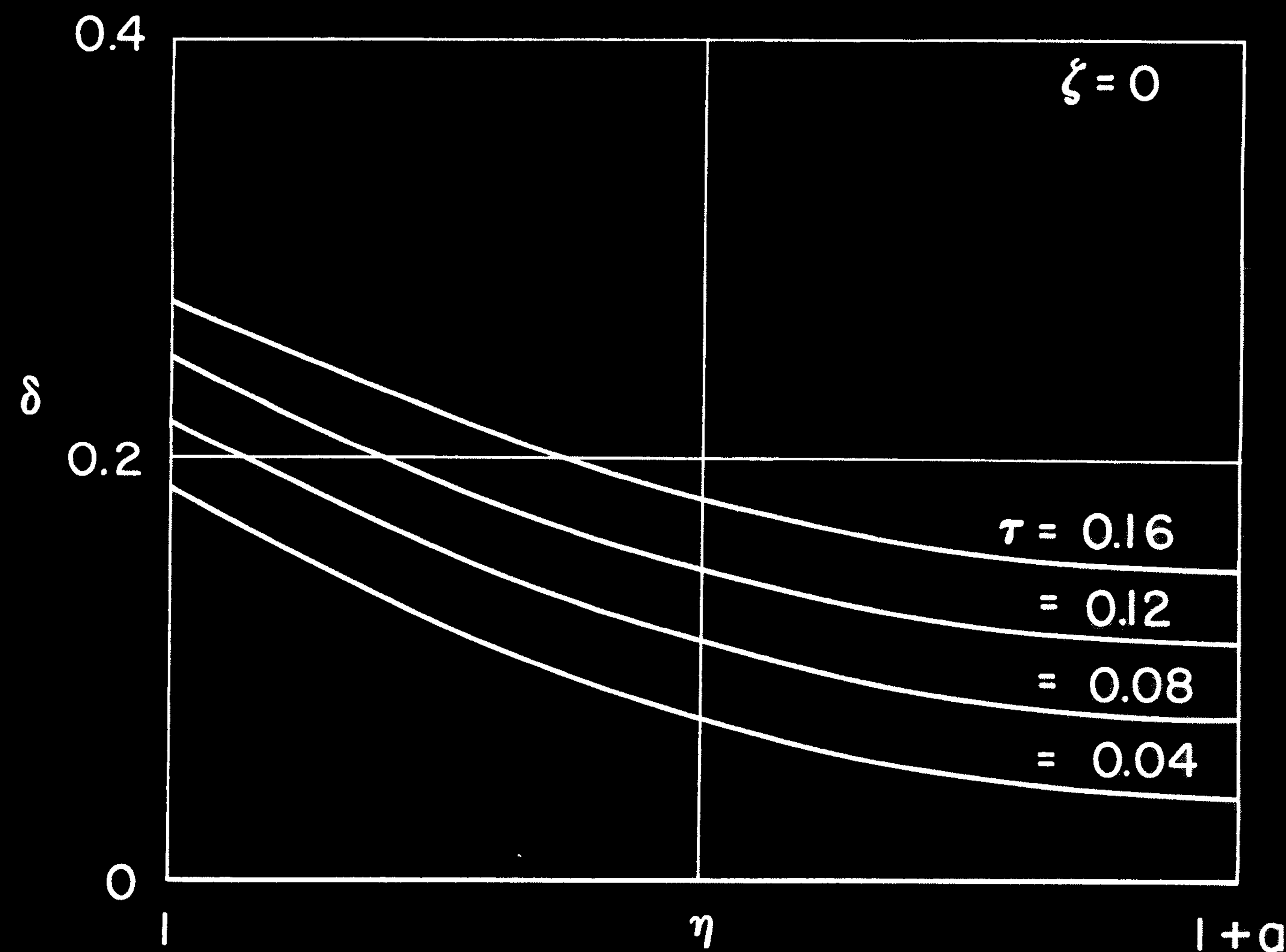

Figure 3(a)

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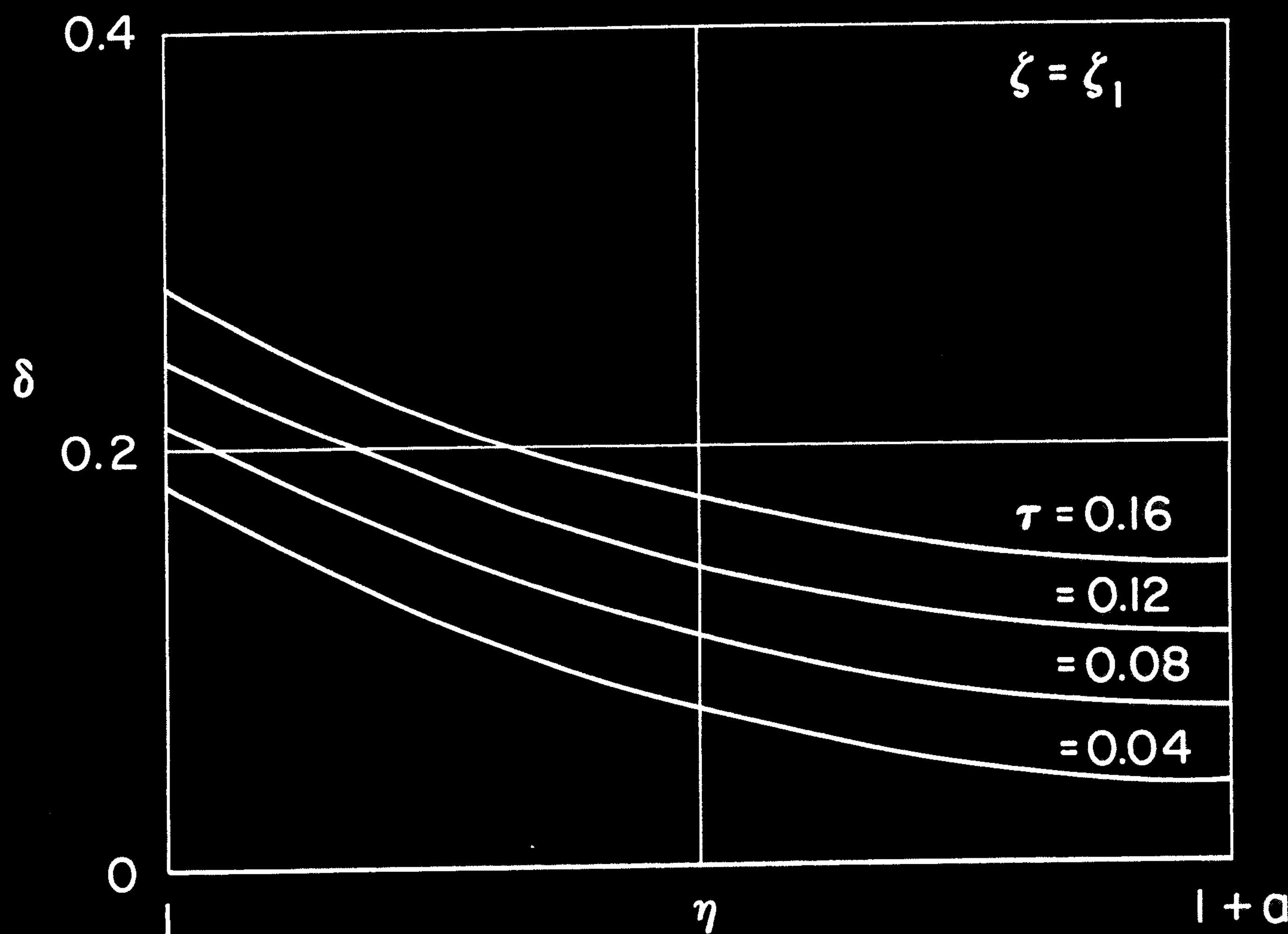

Figure 3(b)

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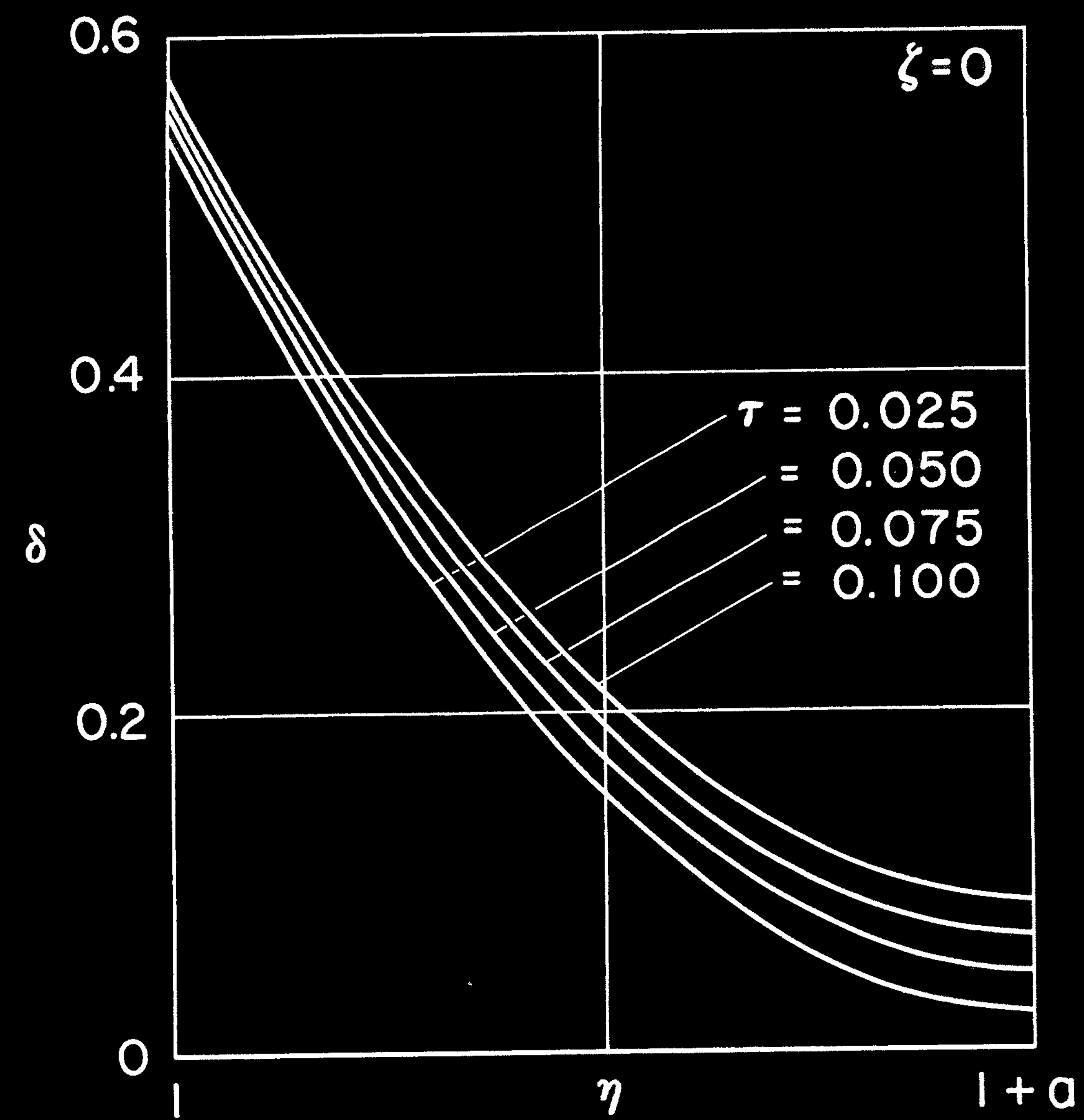

Figure 4(a)

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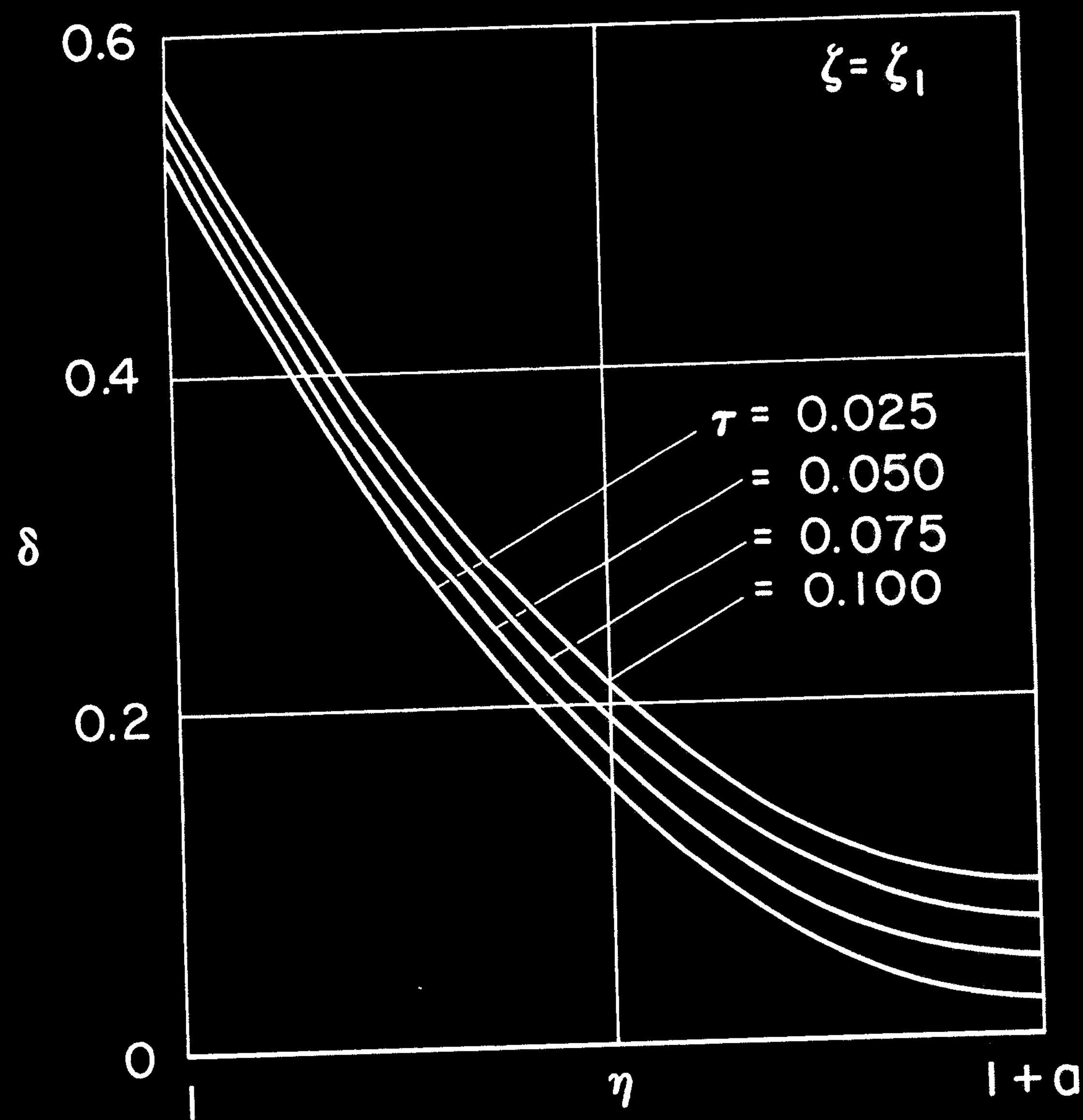

Figure 4(b)

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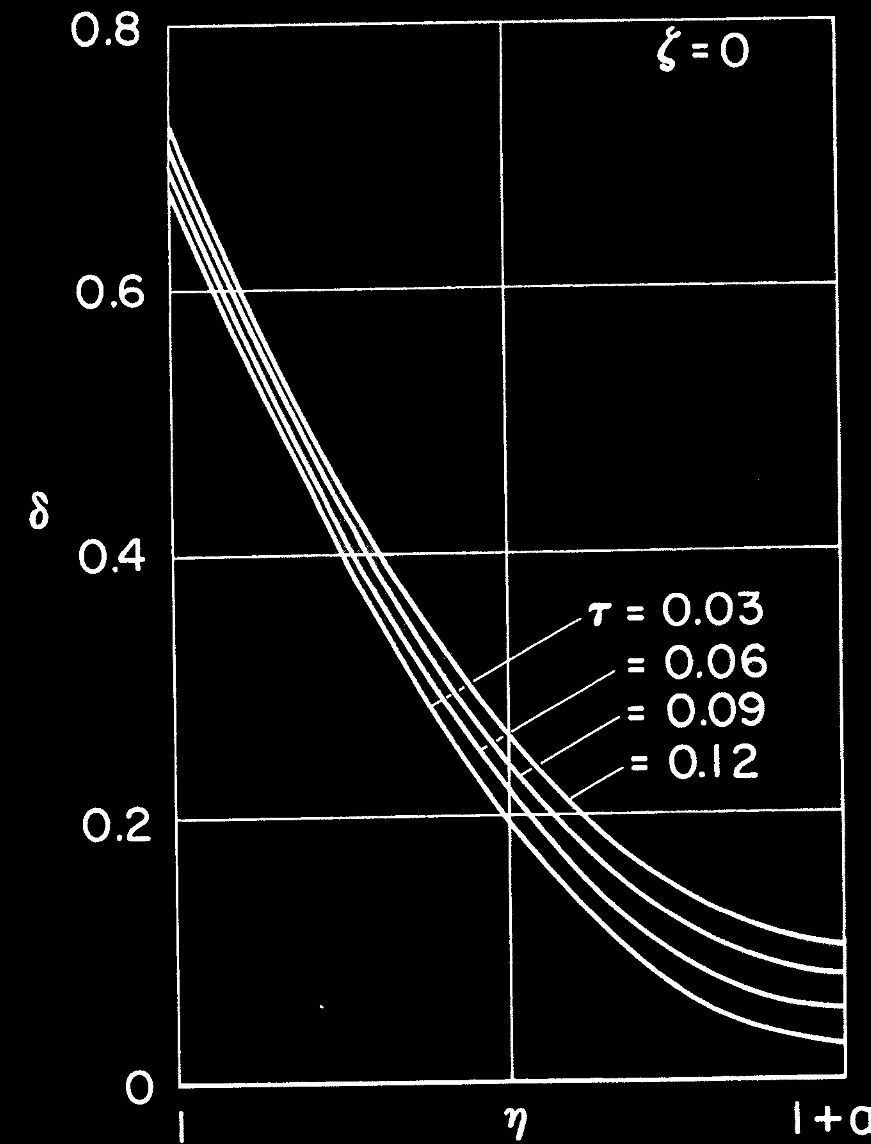

Figure 5(a)

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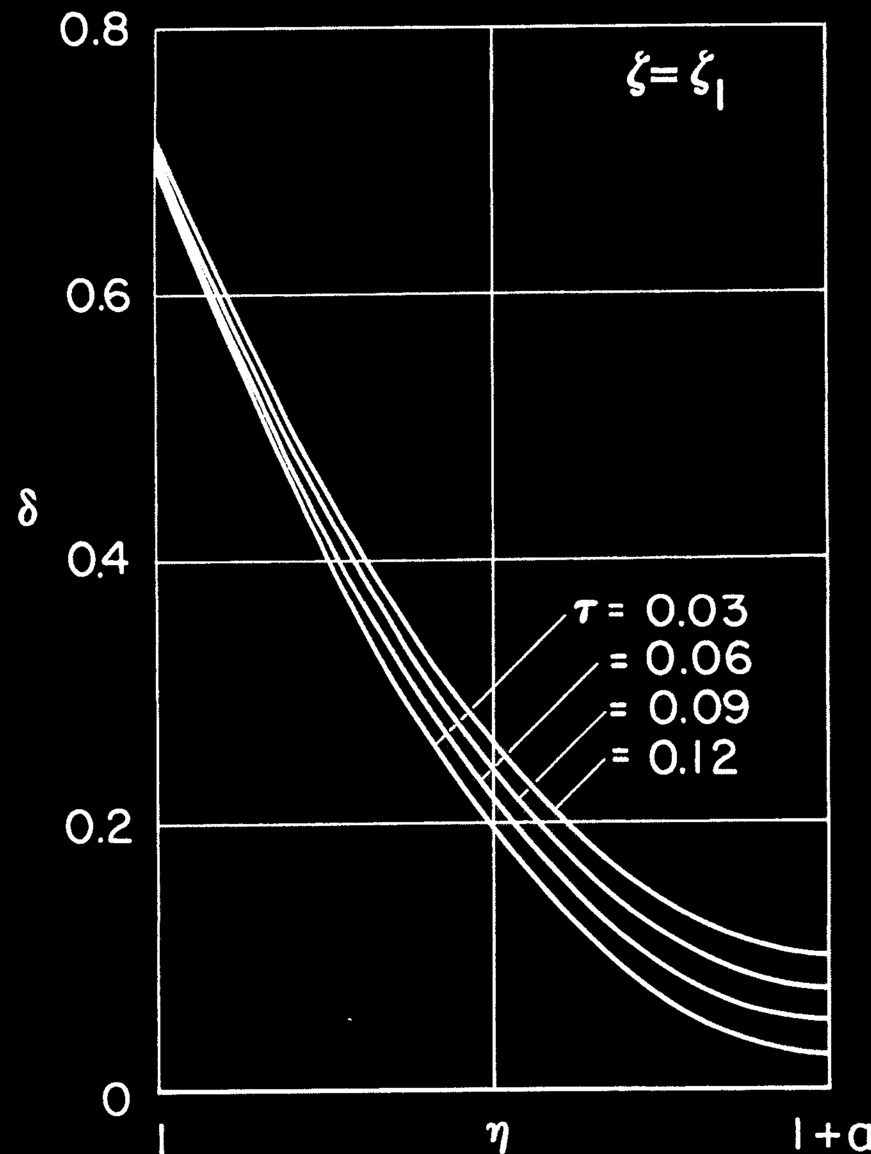

Figure 5(b)

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