AXISYMMETRIC RESPONSE OF CONICAL SHELLS TO BLAST LOAD

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ABSTRACT

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The dynamic response of thin elastic conical shells subject to blast load is formulated according to membrane theory. The solutions are obtained by the technique of separation of variables with Poisson's ratio neglected.

Author

NOMENCLATURE

All symbols are defined in the text where they first appear, and some of the major symbols are listed below:

a_n, b_n	Generalized Fourier coefficients
E	Modulus of elasticity
e_{xx}, $e_{\theta\theta}$	Strain components
h	Thickness of shell
ℓ	Length of shell
$\mathcal{L}$, $\mathcal{D}$	Linear differential operators
m, n	Indices
N_{xx}, $N_{\theta\theta}$	Stress resultants
p	Transient pressure
p_o	Peak overpressure
$r(x)$	Weight function
t	Time
t_d	Duration of transient pressure
v, w	Displacement components
x, θ	Coordinates
α_n, β_n	Separation constants
α	Half apical angle
ρ	Mass density of material
ν	Poisson's ratio
ω_n, $\bar{\omega}_n$	Circular frequencies

INTRODUCTION

A theoretical study of the axisymmetric response of conical shells to blast load is presented in this investigation. The shell is elastic and homogeneous. Two partial differential equations which govern the displacement components in normal and meridianal directions are derived. The solutions are obtained by use of the technique of separation of variables. The governing differential equations are essentially unseparable. To overcome this difficulty, an approximation is made so that the equation of motion for free vibration is satisfied to its mean value along a generator. The results are expressed in series form.

Some progress has been made in the analytic treatment of the response of cylindrical and spherical shells to blast load; it appears that very limited work on the conical shell has been presented. The investigations made by Bluhm [4], Herrmann and Mirsky [8] are good for conical shells with small apical angle.

FORMULATION OF THE PROBLEM

The axisymmetric response of a thin conical shell under blast load will be formulated based upon elastic membrane shell theory. The meridianal lines and parallel circles will be used as the coordinate system (x, θ) as shown in Figure (1). The usual assumptions for thin shells, such as given by Timoshenko [14], are used. Due to symmetry of loading and the geometry of the structure, the displacement or motion is independent of coordinate θ. Furthermore, no shears exist on the meridianal lines.

By summing forces in x and normal directions (Figure (1)), the equations of motion are found to be

$$\frac{\partial N_{xx}}{\partial x} + \frac{1}{x} N_{xx} - \frac{1}{x} N_{\theta\theta} = \rho h \frac{\partial^2 v}{\partial t^2} - p_x \qquad (1)$$

$$\frac{N_{\theta\theta}}{x \tan \alpha} = \rho h \frac{\partial^2 w}{\partial t^2} - p_n \qquad (2)$$

where v and w are respectively the components of displacements in the x and normal directions. p_x and p_n are components of loading in the x and n directions. N_{xx} and $N_{\theta\theta}$ are stress resultants in the x and circumferential directions. ρ is the mass density of the material.

The stress, strain, and displacement relationships are

$$e_{xx} = \frac{\partial v}{\partial x} \qquad (3a)$$

$$e_{\theta\theta} = \frac{1}{x}(v - w \cot \alpha) \qquad (3b)$$

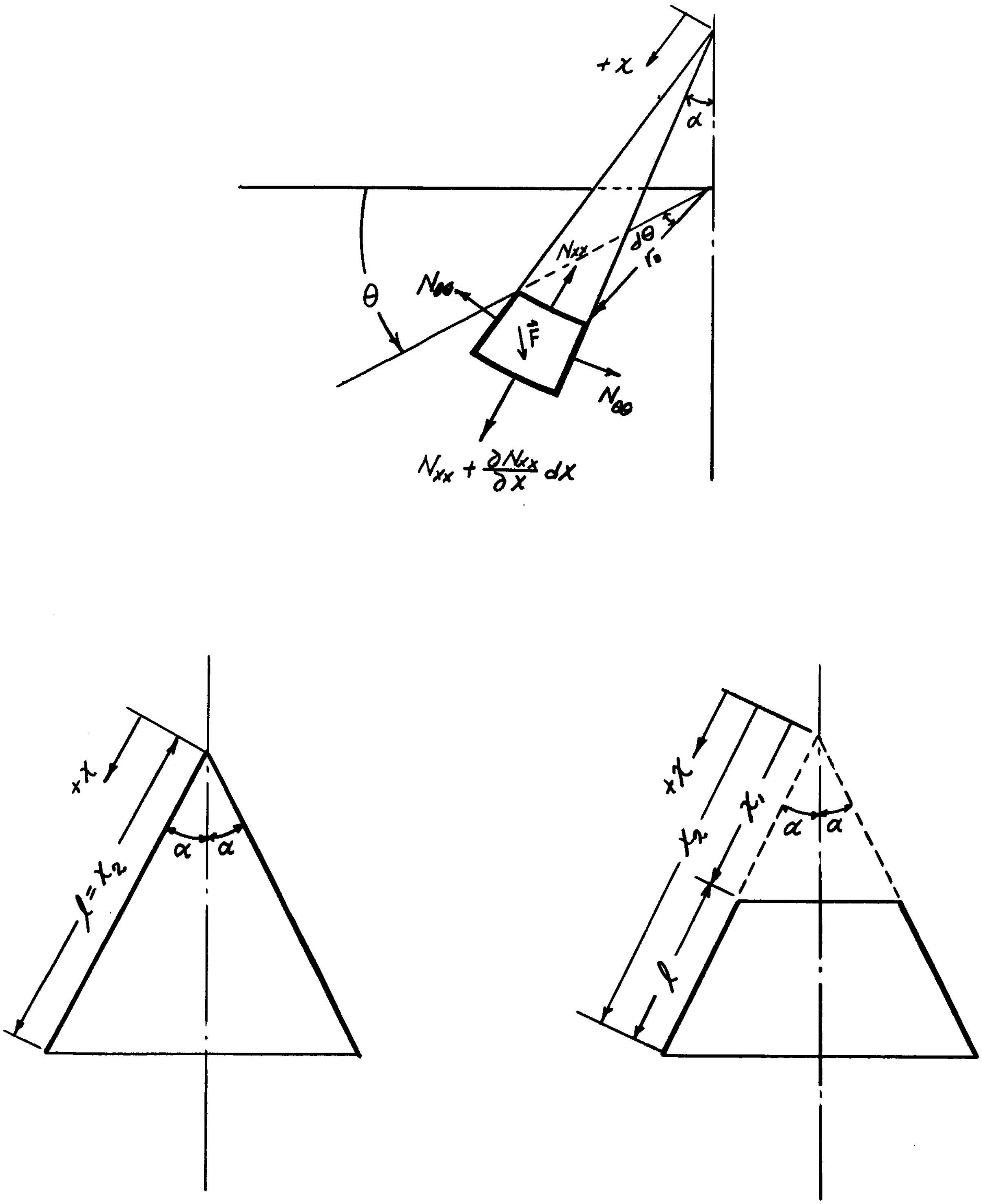

Figure 1. Coordinate System and Symbols.

$$N_{xx} = \frac{Eh}{1 - \nu^2} (e_{xx} + \nu e_{\theta\theta}) = \frac{Eh}{1 - \nu^2}\left[\frac{\partial v}{\partial x} + \frac{\nu}{x} (v - w \cot \alpha)\right] \tag{4a}$$

$$N_{\theta\theta} = \frac{Eh}{1 - \nu^2} (e_{\theta\theta} + \nu e_{xx}) = \frac{Eh}{1 - \nu^2}\left[\frac{1}{x} (v - w \cot \alpha) + \nu \frac{\partial v}{\partial x}\right] \tag{4b}$$

where e_{xx} and $e_{\theta\theta}$ are components of strain in x and θ directions, respectively. E is the modulus of elasticity and ν is the Poisson's ratio.

Substitution of Equations (4a) and (4b) into Equations (1) and (2) yields the following governing differential equations:

$$\frac{\partial^2 v}{\partial x^2} + \frac{1}{x}\frac{\partial v}{\partial x} - \frac{\nu}{x} \cot \alpha \frac{\partial w}{\partial x} - \frac{v}{x^2} + \frac{w}{x^2} \cot \alpha = \frac{\rho(1 - \nu^2)}{E} \frac{\partial^2 v}{\partial t^2} - \frac{p_x(1 - \nu^2)}{Eh} \tag{5a}$$

$$\frac{\nu}{x}\frac{\partial v}{\partial x} + \frac{v}{x^2} - \frac{w}{x^2} \cot \alpha = \frac{\rho(1 - \nu^2)}{E} \tan \alpha \frac{\partial^2 w}{\partial t^2} - \frac{p_n(1 - \nu^2)}{Eh} \tan \alpha \tag{5b}$$

The possible boundary conditions are as follows:

1. <u>Closed Cone:</u>

$v(o, t)$ is finite

$v(x_2, t) = 0$ if supported at $x = x_2$

$\left[\frac{\partial v}{\partial x} + \frac{\nu}{x} (v - w \cot \alpha)\right]_{\substack{x=x_2 \\ t=t}} = 0$ if free at $x = x_2$.

$$v(x, o) = \frac{\partial v}{\partial t} (x, o) = w(x, o) = \frac{\partial w}{\partial t} (x, o) = 0$$

2. Truncated Cone:

Case I. Supported in x-direction at both ends.

(a) With zero initial displacements and velocity

$$v(x_1, t) = v(x_2, t) = v(x, o) = \frac{\partial v}{\partial t}(x, o) = 0$$

$$w(x, o) = \frac{\partial w}{\partial t}(x, o) = 0 \tag{6a}$$

(b) With initial displacements and zero initial velocity

$$v(x_1, t) = v(x_2, t) = \frac{\partial v}{\partial t}(x, o) = \frac{\partial w}{\partial t}(x, o) = 0$$

$$v(x, o) = F(x), \; w(x, o) = G(x) \tag{6b}$$

Case II. Supported in x-direction at $x = x_1$ and free at $x = x_2$

(a) With zero initial displacements and velocity

$$\left[\frac{\partial v}{\partial x} + \frac{\nu}{x}(v - w \cot \alpha)\right]_{\substack{x=x_2 \\ t=t}} = 0$$

$$v(x_2, t) = \frac{\partial v}{\partial t}(x, o) = v(x, o) = 0 \tag{6c}$$

$$w(x, o) = \frac{\partial w}{\partial t}(x, o) = 0$$

(b) With initial displacements and zero initial velocity

$$\left[\frac{\partial v}{\partial x} + \frac{\nu}{x}(v - w \cot \alpha)\right]_{\substack{x=x_2 \\ t=t_1}} = 0$$

$$v(x_2, t) = \frac{\partial v}{\partial t}(x, o) = \frac{\partial w}{\partial t}(x, o) = 0 \tag{6d}$$

$$v(x, o) = F(x)$$

$$w(x, o) = G(x)$$

Since the shock wave front, in general, travels with very high speed, it is reasonable to assume that the blast loading function varies with respect to time only. The actual loading function is shown in Figure (2a). Since the rise time t_r is usually short, the relation between force and time may be approximated by the following continuous functions:

$$p = p_o(1 - \frac{t}{t_d})\, e^{-t/t_d}$$

or

$$p = p_o(1 - t/t_d)$$

which are shown in Figure (2b). t_d is the duration of the load and p_o is the peak overpressure.

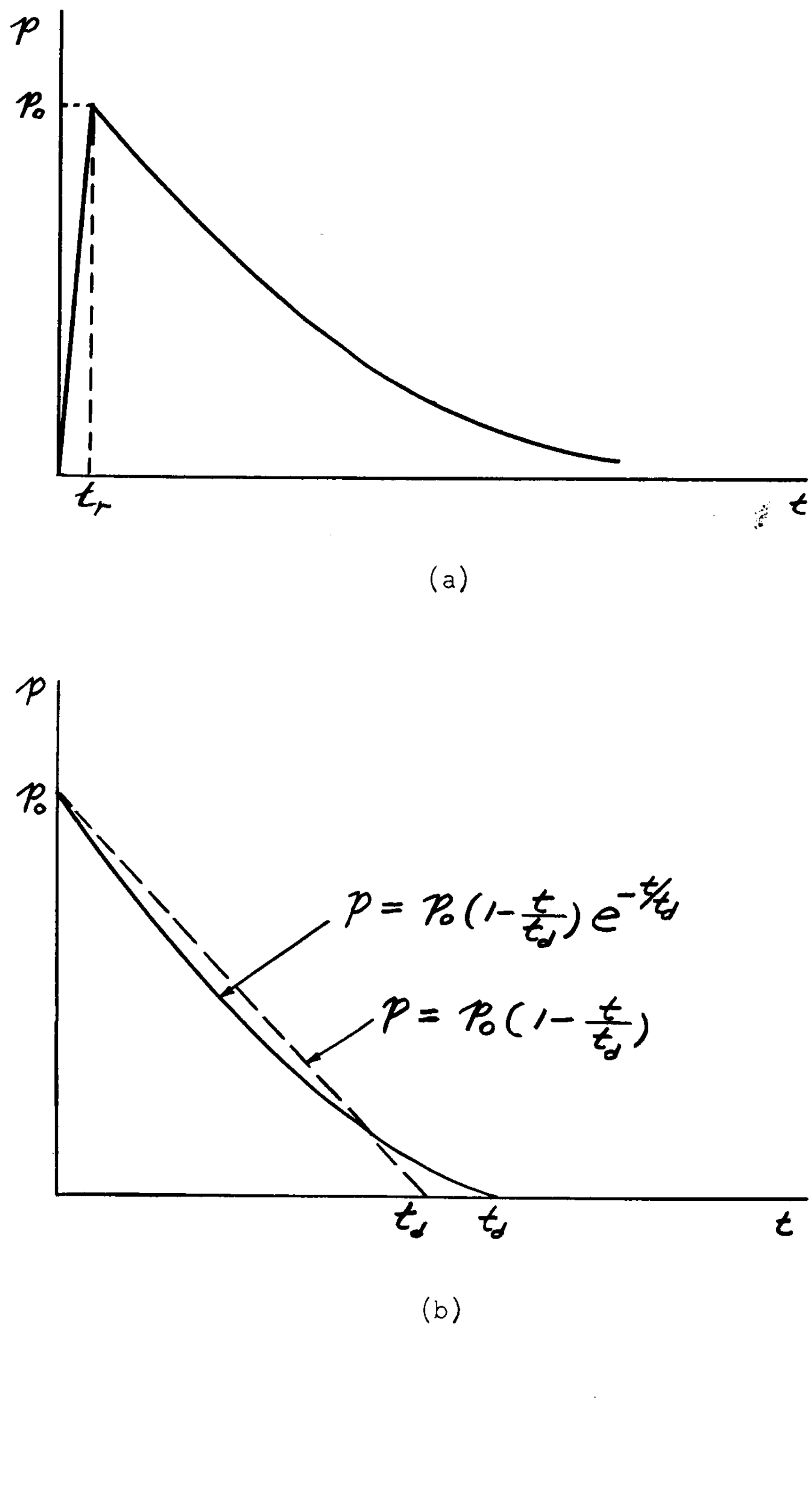

Figure 2. Load-Time Relation.

ANALYSIS

For a simpler case, consider Poisson's ratio to be small and thus negligible and the loading as normal to the surface. The governing differential Equations (5a) and (5b) then reduce to the following form:

$$\frac{\partial^2 v}{\partial x^2} + \frac{1}{x}\frac{\partial v}{\partial x} - \frac{v}{x^2} + \frac{w}{x^2}\cot\alpha = \frac{\rho}{E}\frac{\partial^2 v}{\partial t^2} \tag{7a}$$

$$\frac{v}{x^2} - \frac{w}{x^2}\cot\alpha = -\frac{\rho}{Eh}\tan\alpha + \frac{\rho}{E}\tan\alpha\,\frac{\partial^2 w}{\partial t^2} \tag{7b}$$

The homogeneous solution, neglecting the forcing function, will be sought first. By adding Equations (7a) and (7b), we obtain

$$\frac{\partial^2 v}{\partial x^2} + \frac{1}{x}\frac{\partial v}{\partial x} = \frac{\rho}{E}\frac{\partial^2 v}{\partial t^2} + \frac{\rho}{E}\tan\alpha\,\frac{\partial^2 w}{\partial t^2} \tag{8a}$$

Equations (8a) and the following equation will be used for seeking homogeneous solutions:

$$\frac{v}{x^2} - \frac{w}{x^2}\cot\alpha = \frac{\rho}{E}\tan\alpha\,\frac{\partial^2 w}{\partial t^2} \tag{8b}$$

For possible separation of variables, the homogeneous solutions are assumed in the following form:

$$w_c = \sum_{n=0}^{\infty} W_n(x)\, f_n(t) \tag{9}$$

$$v_c = \sum_{n=0}^{\infty} \bar{V}_n(x)\, \bar{g}_n(t) \tag{10}$$

Substitution of Equations (9) and (10) into Equation (8a) yields

$$\left(\frac{d^2\bar{V}_n}{dx^2} + \frac{1}{x}\frac{d\bar{V}_n}{dx}\right)\bar{g}(t) = \frac{\rho}{E}\,\bar{V}_n\,\frac{d^2g}{dt^2} + \frac{\rho}{E}\tan\alpha\, W_n\,\frac{d^2f}{dt^2} \tag{11}$$

For Equation (15) to be separable, we take

$$\frac{d^2V_n}{dx^2} + \frac{1}{x}\frac{dV_n}{dx} + \beta_n^2\,\frac{\rho}{E}\,V_n = 0 \tag{12}$$

$$\tan\alpha\, W_n = -\,\alpha_n^2\,\bar{V}_n = -\,V_n \tag{13}$$

$$\frac{d^2\bar{g}}{dt^2} + \beta_n^2\,\bar{g}_n = \alpha_n^2\,\frac{d^2f}{dt^2} \tag{14}$$

where β_n and α_n are separation constants. α_n is perfectly arbitrary since we may take

$$g_n = \frac{\bar{g}_n}{\alpha_n^2}$$

and

$$V_n = \alpha_n^2\,\bar{V}_n$$

Equation (10) is thus reduced to

$$v_c = \sum_{n=0}^{\infty} \bar{V}_n(x)\ \bar{g}_n(t) = \sum_{n=0}^{\infty} V_n(x)\ g_n(t) \qquad (10a)$$

It is seen that the condition shown in Equation (13) will not satisfy Equation (8b). This indicates that the governing differential Equations (8a) and (8b) are essentially unseparable. To overcome this difficulty, Equation (8b) may be replaced by an equivalent condition based on a physical point of view. It is known that Equation (8b) represents the equation of motion of an infinitesimal element of the shell in the normal direction. We write the equation of motion for an element along the total length of a generator as the condition equivalent to Equation (8b), i.e.

$$\int_{x_1}^{x_2} \frac{1}{x} (v - w \cot \alpha)\ dx = \int_{x_1}^{x_2} \frac{\rho}{E} \tan \alpha \frac{\partial^2 w}{\partial t^2}\ dx \qquad (15)$$

Substitution of Equations (9), (10a) and (13) into Equation (15) yield

$$(g_n + \cot^2 \alpha\ f_n) \int_{x_1}^{x_2} \frac{V_n(x)}{x}\ dx = - \frac{d^2 f_n}{dt^2} \int_{x_1}^{x_2} \frac{\rho}{E}\ x\ V_n(x)\ dx \qquad (16)$$

We shall use the absolute values of the integrands to evaluate the integrals shown in Equation (16). The integrals when divided by the length of the shell, ℓ, will be interpreted as the mean values of the functions under the integrals. Equation (16) thus reduces to

$$g_n(t) + \cot^2 \alpha\ f_n(t) = - \eta_n \frac{d^2 f_n}{dt^2} \qquad (17)$$

where

$$\eta_n = \frac{\int_{x_1}^{x_2} \frac{\rho}{E} x V_n \operatorname{sgn} V_n \, dx}{\int_{x_1}^{x_2} \frac{V_n}{x} \operatorname{sgn} V_n \, dx} \tag{18}$$

This approximation, Equations (15) through (18), will mean that Equation (8b) is satisfied to its mean value along a generator.

Elimination of $g_n(t)$ between Equations (14) and (17) yields

$$-\cot^2 \alpha \frac{d^2 f_n}{dt^2} - \eta_n \frac{d^4 f_n}{dt^4} - \beta_n^2 \cot^2 \alpha \, f_n - \beta_n^2 \eta_n \frac{d^2 f_n}{dt^2} = \frac{d^2 f_n}{dt^2} \tag{19}$$

or

$$\frac{d^4 f_n}{dt^4} + K_n(\eta_n, \beta_n) \frac{d^2 f_n}{dt^2} + H_n(\eta_n, \beta_n) \, f_n = 0 \tag{20}$$

where

$$K_n = \frac{1}{\eta_n}(\operatorname{cosec}^2 \alpha + \beta_n^2 \eta_n) \tag{21a}$$

$$H_n = \frac{\beta_n^2 \cot^2 \alpha}{\eta_n} \tag{21b}$$

The general solution of Equation (19) becomes

$$f_n = \bar{A}_n \cos \omega_n t + \bar{B}_n \sin \omega_n t + \bar{C}_n \cos \bar{\omega}_n t + \bar{D}_n \sin \bar{\omega}_n t \tag{22}$$

where

$$\omega_n^2 = \frac{+K_n + \sqrt{(K_n)^2 - 4 H_n}}{2} \tag{22a}$$

and

$$\bar{\omega}_n^2 = \frac{+K_n - \sqrt{(K_n)^2 - 4 H_n}}{2} \tag{22b}$$

are two natural frequencies associated with each n. This fact has also appeared in the discussions given by Baker [2] and Lamb [10] for spherical shells.

$g_n(t)$ may be obtained by substituting Equation (22) into (17); we have

$$g_n(t) = (\eta_n \omega_n^2 - \cot^2 \alpha)(\bar{A}_n \cos \omega_n t + \bar{B}_n \sin \omega_n t)$$

$$+ (\eta_n \bar{\omega}_n^2 - \cot^2 \alpha)(\bar{C}_n \cos \bar{\omega}_n t + \bar{D}_n \sin \bar{\omega}_n t) \tag{23}$$

It is seen that Equation (12) is the standard form for Bessel's equation of zero order with solution

$$V_n = \hat{A}_n J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x) + \hat{B}_n Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x) \tag{24}$$

where J_n and Y_n are Bessel's function of first and second kind, respectively.

If the shell forms a closed cone, $\hat{B}_n$ must be zero since v should be finite at x = 0. The eigenvalues β_n may be generated by applying the geometric boundary condition at $x = x_2$.

Case I. Supported condition, $v(x_2, t) = 0$ from Equation (22)

$$J_o(\beta_n \sqrt{\frac{\rho}{E}} x_2) = 0 \tag{25a}$$

and the roots are

$$\beta_n \sqrt{\frac{\rho}{E}} x_2 = 2.405, 5.520 \ldots.$$

Case II. Free end condition $\frac{\partial v}{\partial x} (x_2,t) = 0$ which leads to

$$J_o'(\beta_n \sqrt{\frac{\rho}{E}} x_2) = -J_1(\beta_n \sqrt{\frac{\rho}{E}} x_2) = 0 \tag{25b}$$

and the roots are

$$\beta_n \sqrt{\frac{\rho}{E}} x_2 = 0, 3.832, 7.016 \ldots.$$

For a truncated conical shell, geometric boundary conditions listed in Equation (6) will be applied to Equation (24) in order to determine β_n and the ratio of $\hat{A}_n$ to $\hat{B}_n$.

Case I. The conditions $v(x_1,t) = v(x_2,t) = 0$ lead to the following homogeneous algebraic equations:

$$\begin{bmatrix} J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_1) & Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_1) \\ J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2) & Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2 \end{bmatrix} \begin{bmatrix} \hat{A}_n \\ \hat{B}_n \end{bmatrix} = 0 \tag{26}$$

For non-trivial solutions, the determinant of the coefficient matrix must vanish. Hence,

$$\frac{J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_1)}{J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2)} = \frac{Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_1)}{Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2)} \tag{27}$$

and

$$\frac{\hat{A}_n}{\hat{B}_n} = -\frac{Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_1)}{J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_1)} = -\frac{Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2)}{J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2)}$$

Case II. The conditions $v(x_2,t) = \frac{\partial v}{\partial x}(x_1,t) = 0$ result in

$$\frac{\hat{A}_n}{\hat{B}_n} = -\frac{Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2)}{J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2)} = -\frac{Y_o'(\beta_n \sqrt{\frac{\rho}{E}}\, x_1)}{J_o'(\beta_n \sqrt{\frac{\rho}{E}}\, x_1)} \tag{28a}$$

and

$$\frac{J_o'(\beta_n \sqrt{\frac{\rho}{E}}\, x_1)}{J_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2)} = \frac{Y_o'(\beta_n \sqrt{\frac{\rho}{E}}\, x_1)}{Y_o(\beta_n \sqrt{\frac{\rho}{E}}\, x_2)} \tag{28b}$$

The solutions of Equations (27) and (28) will give the eigenvalues β_n. The first several roots of Equation (27) and (28) can be found in [1] or [9]. It is a generally known fact that for large x, the Bessel functions involved in the Equation (24) behave very much like trigonometric functions. This phenomenon can be revealed from Equation (12).

$$\frac{d^2V_n}{dx^2} + \frac{1}{x}\frac{dV_n}{dx} + \beta_n^2 \frac{\rho}{E} V_n = 0 \tag{29}$$

By a change of variable of the form

$$U_n = V_n \sqrt{x}$$

Equation (16a) is transformed into

$$\frac{d^2U_n}{dx^2} + (\beta_n^2 \frac{\rho}{E} - \frac{1}{x^2}) U_n = 0 \tag{29a}$$

If the values of x are large we may assume

$$\frac{1}{x^2} << \beta_n^2 \frac{\rho}{E}$$

and neglect it in Equation (29a). The solution of Equation (29a) is seen to be

$$V_n = \frac{U_n}{\sqrt{x}} = \frac{1}{\sqrt{x}} [\hat{A}_n \cos(\beta_n \sqrt{\frac{\rho}{E}} x) + \hat{B}_n \sin(\beta_n \sqrt{\frac{\rho}{E}} x)] \tag{30}$$

The eigenvalues β_n for the two cases shown in Equation (6) are

Case I.

$$\beta_n = \frac{n\pi}{\ell}\sqrt{\frac{E}{\rho}} \tag{31a}$$

and

$$\hat{A}_n = -\tan\left(\beta_n\sqrt{\frac{E}{\rho}}\,x_1\right)\hat{B}_n = -\tan\left(\beta_n\sqrt{\frac{E}{\rho}}\,x_2\right)\hat{B}_n \tag{31b}$$

Case II.

$$\beta_n = \frac{(2n+1)\pi}{2\ell}\sqrt{\frac{E}{\rho}} \tag{31c}$$

and

$$\hat{A}_n = -\tan\left(\beta_n\sqrt{\frac{E}{\rho}}\,x_2\right)\hat{B}_n = \cot\left(\beta_n\sqrt{\frac{E}{\rho}}\,x_1\right)\hat{B}_n \tag{31d}$$

where $\ell = x_2 - x_1$. Equation (30) will be valid for truncated conical shells.

Substitution of Equations (22) and (24) or (22) and (30) into Equations (10a) and (9), the following general homogeneous solutions are obtained:

$$V_c = \sum_{n=0}^{\infty} X_n(x)\left[(\eta_n\omega_n^2 - \cot^2\alpha)(A_n \cos\omega_n t + B_n \sin\omega_n t)\right.$$

$$\left. + (\eta_n\bar{\omega}_n^2 - \cot^2\alpha)(C_n \cos\omega_n t + D_n \sin\omega_n t)\right] \tag{32a}$$

and

$$W_c = \sum_{n=0}^{\infty} - X_n(x)(A_n \cos \omega_n t + B_n \sin \omega_n t + C_n \cos \bar{\omega}_n t + D_n \sin \bar{\omega}_n t) \quad (32b)$$

where

$$X_n(x) = \lambda_n J_o(\beta_n \sqrt{\tfrac{\rho}{E}}\, x) + Y_o(\beta_n \sqrt{\tfrac{\rho}{E}}\, x) \text{ ... exact} \quad (32c)$$

or

$$X_n(x) = \frac{1}{\sqrt{x}} [\lambda_n \cos(\beta_n \sqrt{\tfrac{\rho}{E}}\, x) + \sin(\beta_n \sqrt{\tfrac{\rho}{E}}\, x)] \text{ ... approx} \quad (32d)$$

in which

$$\lambda_n = \hat{A}_n / \hat{B}_n \quad (32e)$$

The particular solutions will be sought next. Let $\mathcal{L} = \frac{\partial}{\partial t}$ and $\mathcal{D} = \frac{\partial}{\partial x}$ be the linear operators. By eliminating v between Equations (8a) and (8b), the particular solution for w is taken as

$$w_p = \frac{1}{wt\ \alpha \mathcal{D}(x\mathcal{D})} (x)\{1 - F(\mathcal{L},\mathcal{D}) + F^2(\mathcal{L},\mathcal{D})\}$$

$$\{\frac{3 \tan \alpha}{Eh} p - \frac{p \tan \alpha}{E^2 h} x^2 \frac{\partial^2 p}{\partial t^2}\} + C_1 \log x + C_2 \quad (33)$$

where

$$F(\mathcal{L},\mathcal{D}) = \frac{\rho}{E}(3 \tan \alpha - \cot \alpha) \frac{\mathcal{L}^2}{\mathcal{D}(x\mathcal{D})} - (\frac{\rho}{E})^2 \tan \alpha x^2 \frac{\mathcal{L}^4}{\mathcal{D}(x\mathcal{D})} \quad (34)$$

Substitution of Equation (33) into Equation (8b), we obtain the particular solution for v

$$v_p = w_p \cot \alpha - \frac{x^2}{Eh} \tan \alpha p + \frac{\rho x^2}{E} \tan \alpha \frac{\partial^2 w_p}{\partial t^2} \tag{35}$$

C_1 and C_2 will be determined by satisfying the geometric boundary conditions for v at both ends of the shell.

The particular solutions are then expanded into the following series:

$$v_p = \sum_{n=0}^{\infty} a_n X_n(x) \tag{35a}$$

and

$$w_p = \sum_{n=0}^{\infty} b_n \cot \alpha X_n(x) \tag{35b}$$

where

$$a_n = \frac{\int_{x_1}^{x_2} r(x) V_p X_n(x) dx}{\int_{x_1}^{x_2} r(x) X_n^2(x) dx} \tag{36a}$$

$$b_n = \frac{\int_{x_1}^{x_2} r(x) W_p X_n(x) dx}{\int_{x_1}^{x_2} r(x) X_n^2(x) dx} \tag{36b}$$

where r(x) is the weight function. It should be noted that a_n and b_n, the generalized Fourier coefficients, are functions of t.

The final solutions of v and w are obtained by combining Equations (32a) to (35a) and (32b) to (35b), respectively. Or,

$$w = \sum_{n=0}^{\infty} -\cot\alpha\, X_n(x)(A_n \cos\omega_n t + B_n \sin\omega_n t$$

$$+ C_n \cos\bar{\omega}_n t + D_n \sin\bar{\omega}_n t - b_n) \tag{37a}$$

and

$$v = \sum_{n=0}^{\infty} X_n(x)[\eta_n\omega_n^2 - \cot^2\alpha)(A_n \cos\omega_n t + B_n \sin\omega_n t)$$

$$+ (\eta_n\bar{\omega}_n^2 - \cot^2\alpha)(C_n \cos\bar{\omega}_n t + D_n \sin\bar{\omega}_n t) + a_n] \tag{37b}$$

The following initial conditions will be used to determine the unknown coefficients A_n, B_n, C_n, and D_n:

$$v(x,o) = \frac{\partial v}{\partial t}(x,o) = w(x,o) = \frac{\partial w}{\partial t}(x,o) = 0 \tag{38}$$

By applying conditions listed in Equation (38) to Equations (37a) and (37b), we obtain

$$A_n = \frac{-b_n(0)(\eta_n\bar{\omega}_n^2 - \cot^2\alpha) - a_n(0)}{\eta_n(\omega_n^2 - \bar{\omega}_n^2)} \tag{39a}$$

$$C_n = \frac{b_n(0)(\eta_n\omega_n^2 - \cot^2\alpha) + a_n(0)}{\eta_n(\omega_n^2 - \bar{\omega}_n^2)} \qquad (39b)$$

$$B_n = \frac{-\frac{\partial b_n}{\partial t}(0)(\eta\bar{\omega}_n^2 - \omega t^2\alpha) - \frac{\partial a_n}{\partial t}(0)}{\omega_n\eta_n(\omega_n^2 - \bar{\omega}_n^2)} \qquad (39c)$$

$$D_n = \frac{\frac{\partial b_n}{\partial t}(0)(\eta\omega_n^2 - \cot^2\alpha) + \frac{\partial a_n}{\partial t}(0)}{\bar{\omega}_n\eta_n(\omega_n^2 - \bar{\omega}_n^2)} \qquad (39d)$$

From Equation (22a) and (22b), it is seen that

$$\omega_n^2 - \bar{\omega}_n^2 = \sqrt{K_n^2 - 4\,H_n} \qquad (40)$$

The response, after the shock wave has passed the structure, will be

$$w = \sum_{n=0}^{\infty} -\cot\alpha\, X_n(x)(A_n' \cos\omega_n t + B_n' \sin\omega_n t$$

$$+ C_n' \cos\bar{\omega}_n t + D_n' \sin\bar{\omega}_n t) \qquad (41a)$$

and

$$v = \sum_{n=0}^{\infty} X_n(x)[\eta_n\omega_n^2 - \cot^2\alpha)(A_n' \cos\omega_n t + B_n' \sin\omega_n t)$$

$$+ (\eta\bar{\omega}_n^2 - \cot^2\alpha)(C_n' \cos\bar{\omega}_n t + D_n' \sin\omega_n t)] \qquad (41b)$$

The unknown coefficients will be determined by using the known conditions

$$v(x,t_d),\ \frac{\partial v}{\partial t}(x,t_d),\ w(x,t_d) \text{ and } \frac{\partial w}{\partial t}(x,t_d)$$

which are evaluated from Equations (37a) and (37b).

EXAMPLE

For illustrative purposes, an example for a truncated conical shell with both ends supported is presented here. The boundary conditions shown in Equation (6a) are thus used. The following data are considered:

$$x_2 = 1.5\, x_1, \quad \alpha = \pi/4$$

$$p = p_o(1 - t/t_d) \tag{42}$$

The solutions of Equation (27) tabulated in [9] indicate that

$$\beta_n^2 \frac{\rho}{E} \geq \frac{1/2(6.27)^2}{\ell^2} \gg \frac{1}{\ell^2}$$

where $\ell = x_2 - x_1$. Therefore, the homogeneous solution shown in Equation (32d) will be used.

The particular solutions for this case will be taken as

$$w_p = \frac{3}{Eh}\left(\frac{x^2}{4} + \bar{C}_1 \log x + \bar{C}_2\right)p_o(1 - t/t_d) \tag{43a}$$

and

$$v_p = \frac{3}{Eh}\left(-\frac{x^2}{12} + \bar{C}_1 \log x + \bar{C}_2\right)p_o(1 - t/t_d) \tag{43b}$$

where

$$\bar{C}_1 = \frac{x_1^2 - x_2^2}{12 \log \frac{x_1}{x_2}} \tag{44a}$$

and

$$\bar{C}_2 = \frac{x_2^2 \log x_1 - x_1^2 \log x_2}{12 \log \frac{x_1}{x_2}} \tag{44b}$$

It is seen from Equations (43a) and (43b) that

$$\frac{\partial a_n(0)}{\partial t} = - \frac{a_n(0)}{t_d} \tag{45a}$$

and

$$\frac{\partial b_n(0)}{\partial t} = - \frac{b_n(0)}{t_d} \tag{45b}$$

From Equation (18), we have

$$\eta_n = \frac{\frac{\rho}{E} \int_{x_1}^{x_2} x \sin \frac{n\pi x}{\ell} \operatorname{sgn}(\sin \frac{n\pi x}{\ell})\, dx}{\int_{x_1}^{x_2} \frac{1}{x} \sin \frac{n\pi x}{\ell} \operatorname{sgn}(\sin \frac{n\pi x}{\ell})\, dx} \tag{46a}$$

in which the integral involved in the denominator will be taken in the following form:

$$\int_{\xi_1}^{\xi_2} \frac{1}{\xi} \sin \frac{n\pi\xi}{\ell}\, d\xi = \sum_{m=1}^{\infty} m! \left(\frac{\ell}{n\pi}\right)^{2m-1} \left[\frac{\cos \frac{n\pi\xi_1}{\ell}}{(\xi_1)^{2m-1}} - \frac{\cos \frac{n\pi\xi_2}{\ell}}{(\xi_2)^{2m-1}}\right] \tag{46b}$$

The series shown in Equation (46b) converges rapidly and if only the first term is considered, Equation (46a) is approximately equal to

$$\eta_n = \frac{5n\ell \frac{\rho}{E}}{\frac{5}{6} + \sum_{j=1}^{n-1} \frac{2n}{(2n + j)}} \tag{46c}$$

From Equations (21) and (22), we have

$$K_n = \frac{1}{\eta_n}\left[2 + \left(\frac{n\pi}{\ell}\right)^2 \frac{E}{\rho}\, \eta_n\right] \tag{46d}$$

$$H_n = \frac{1}{\eta_n}\left(\frac{n\pi}{\ell}\right)^2 \frac{E}{\rho} \tag{46e}$$

and

$$\begin{Bmatrix} \omega_n^2 \\ \bar{\omega}_n^2 \end{Bmatrix} = \frac{2 + \left(\frac{n\pi}{\ell}\right)^2 \frac{E}{\rho}\, \eta_n}{2\eta_n} \begin{Bmatrix} + \\ - \end{Bmatrix} \frac{1}{2}\sqrt{\left[\frac{2 + \left(\frac{n\pi}{\ell}\right)^2 \frac{E}{\rho}\, \eta_n}{\eta_n}\right]^2 - \frac{4}{\eta_n}\left(\frac{n\pi}{\ell}\right)^2 \frac{E}{\rho}} \tag{46f}$$

By use of Equations (32d), (36) and (43), the Fourier coefficients a_n and b_n become

$$a_n = \frac{6P_o}{Eh\ell}\left(1 - \frac{t\iota}{t_d}\right) \int_{x_1}^{x_2} \left(- \frac{x^2}{12} + \bar{C}_1 \log x + \bar{C}_2\right) \sqrt{x} \sin \frac{n\pi x}{\ell}\, dx \tag{47a}$$

$$b_n = \frac{6p_o}{Eh\ell}\left(1 - \frac{t}{t_d}\right)\int_{x_1}^{x_2}\left(\frac{x^2}{4} + \bar{C}_1 \log x + \bar{C}_2\right)\sqrt{x}\,\sin\frac{n\pi x}{\ell}\,dx \tag{47b}$$

or

$$a_n = \frac{6p_o}{Eh}\,\ell^{5/2}\left(1 - \frac{t}{t_d}\right)\left\{-\frac{1}{12}h_n + \frac{\bar{C}_2}{\ell^2}k_n + \frac{\bar{C}_1}{\ell^{7/2}}\int_{x_1}^{x_2} x^{1/2}\log x\,\sin\frac{n\pi x}{\ell}dx\right\} \tag{48a}$$

$$b_n = \frac{6p_o}{Eh}\,\ell^{5/2}\left(1 - \frac{t}{t_d}\right)\left\{\frac{1}{4}h_n + \frac{\bar{C}_2}{\ell^2}k_n + \frac{\bar{C}_1}{\ell^{7/2}}\int_{x_1}^{x_2} x^{1/2}\log x\,\sin\frac{n\pi x}{\ell}\,dx\right\} \tag{48b}$$

where

$$h_n = \left(\frac{1}{\ell}\right)^{7/2}\int_{x_1}^{x_2} x^{5/2}\sin\frac{n\pi x}{\ell}\,dx$$

$$= \left(\frac{1}{n\pi}\right)^{7/2}\Big[-(3n\pi)^{5/2}\cos 3n\pi + (2n\pi)^{5/2}\cos 2n\pi$$

$$- \frac{15}{8}\Big\{-\sqrt{3n\pi}\cos 3n\pi + \sqrt{2n\pi}\cos 2n\pi$$

$$+ \sqrt{\frac{\pi}{2}}\,C(\sqrt{6n}) - \sqrt{\frac{\pi}{2}}\,C(\sqrt{4n})\Big\}\Big]$$

$$k_n = \left(\frac{1}{\ell}\right)^{3/2}\int_{x_1}^{x_2} x^{1/2}\sin\frac{n\pi x}{\ell}\,dx$$

$$= \left(\frac{1}{n\pi}\right)^{3/2}\Big[-\sqrt{3n\pi}\cos 3n\pi + \sqrt{2n\pi}\cos 2n\pi$$

$$+ \quad \frac{\pi}{2}\left\{C(\sqrt{6n}) - C(\sqrt{4n})\right\}\Big]$$

in which C is the Fresnel's integral and is tabulated in [9]. Substituting $t = 0$ into Equation (48) we have

$$a_n(0) = 6\ell^{5/2} \frac{p_o}{Eh}\left[- \frac{1}{12} h_n + \frac{\bar{C}_2}{\ell^2} k_n + \frac{\bar{C}_1}{\ell^{7/2}} \int_{x_1}^{x_2} x^{1/2} \log x \sin \frac{n\pi x}{\ell} dx\right] \tag{49a}$$

$$b_n(0) = 6 \frac{p_o \ell^{5/2}}{Eh}\left[\frac{1}{4} h_n + \frac{\bar{C}_2}{\ell^2} k_n + \frac{\bar{C}_1}{\ell^{7/2}} \int_{x_1}^{x_2} x^{1/2} \log x \sin \frac{n\pi x}{\ell} dx\right] \tag{49b}$$

The transient response is then determined by substituting Equations (44) through (49) into Equation (39) and then (37). The response for $t > t_d$ can readily be determined by use of Equation (41).

CONCLUDING REMARKS

The governing differential equations were made separable with approximation. The natural frequencies ω_n and $\bar{\omega}_n$ corresponding to specific modes are therefore not exact. It is of interest to note that there exists two natural frequencies corresponding to each mode. This fact is true for the extensional vibration for spherical shells as discussed in [2] and [10]. No exact solution or experimental data seemed to be available at the present time in order to evaluate the error, due to the approximation made in the analysis, involved in the present solution. The method of solution presented in this study is straightforward. However, the solutions expressed in series form appear to be lengthy when numerical results are needed. High speed electronic computers may be efficiently employed for this purpose. Since the response of one case may look entirely different from the other case even for two identical shells, if the durations of the load are different; hence, no numerical example is presented in this paper.

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