R-104
Task Order #1

NUMERICAL SOLUTIONS OF THE ONE-DIMENSIONAL
NUCLEON-MESON CASCADE EQUATIONS*

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Abstract

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In shielding calculations for high-energy accelerators it is necessary to solve the nucleon-meson cascade equations numerically for very large distances. For the case of a 10-GeV proton beam and a set of quite special physical assumptions, an analytic solution has been obtained and compared with the numerical solution. The two solutions are shown to be in excellent agreement for thicknesses as large as 30 collision mean free paths ($\sim$2800 g/cm^2).

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[redacted]

$$\Theta(E' - E) = 1 \quad E' \geq E$$

$$= 0 \quad E' < E \quad .$$

These assumptions are, of course, introduced for their simplicity and represent only very approximately a real physical system. In particular, Eq. (3.4) is not very realistic. With appropriate choice of ν the E dependence may be made physically reasonable, but the resulting particle multiplicity varies much too rapidly with E'.

With the assumptions of Eqs. (3.1) to (3.4) a quadrature solution to Eqs. (2.1) to (2.3) can be obtained. The details of obtaining the solution are given in Ref. 5, so only the solution itself will be given here. Furthermore, since only the case of a monoenergetic initial proton flux is of interest here, only the solution appropriate to this case will be given.

Using the initial fluxes

$$\Phi_p(E) = P_o \, \delta(E_o - E) \quad ,$$

$$\Phi_j(E) = 0, \qquad j = N, \pi \quad , \tag{3.5}$$

where P_o = constant and E_o = energy of incident protons, the solution may be written

$$\Phi_{iP}(E,r) = P_o \, \delta(E_o - E - Sr) \, e^{-Qr} \quad , \tag{3.6}$$

$$\Phi_{sP}(E,r) = P_o \, \alpha_P \, \gamma_P \, e^{-Qr} \, K(E_o - E, r) \quad , \tag{3.7}$$

$$\Phi_{s\pi}(E,r) = P_o \, \alpha_\pi \, \gamma_P \, e^{-Qr} \, K(E_o - E, r) \quad , \tag{3.8}$$

I. Introduction

In a series of reports numerical solutions to the equations describing a one-dimensional nucleon-meson cascade have been given for a variety of cases of interest in the shielding of manned space vehicles and high-energy accelerators.[1-4] In the case of accelerator shielding where very thick shields are involved, the numerical calculations are quite extensive and the truncation error could be excessive.

For a very special case an analytic (quadrature) solution to the cascade equations has recently been obtained.[5] In this paper the numerical solution and the analytic solution are compared for the case of a 10-GeV proton beam and are shown to be in excellent agreement after a shield thickness of 30 collision mean free paths.*

In Section II the cascade equations are given. In Section III the assumptions used in obtaining the analytic solution are described and the solution is given. In Section IV the comparison between the analytic and numerical solutions is shown.

II. Cascade Equations

In writing the cascade equations we shall neglect neutral pions since they decay very rapidly into two photons and photons are not included. Furthermore, no distinction will be made between positively and negatively charged pions, and charged pion decay will be neglected.**

Under these circumstances the one-dimensional cascade equations for the particle fluxes may be written[6,7]

$$\Phi_{ij}(E,r) = \Phi_j(E_j)\,\frac{S_j(E_j)}{S_j(E)}\,e^{-\int_E^{E_j}\frac{Q_j(E')}{S_j(E')}dE'} \quad , \quad j = N,P,\pi \quad , \tag{2.1}$$

$$\int_E^{E_j}\frac{dE'}{S_j(E')} = r \quad , \tag{2.2}$$

$$\frac{\partial}{\partial r}\Phi_{sj}(E,r) + Q_j(E)\,\Phi_{sj}(E,r) - \frac{\partial}{\partial E}\left[S_j(E)\,\Phi_{sj}(E,r)\right]$$

$$= \sum_k \int_E^{\infty} F_{jk}(E',E)\,Q_k(E')\left[\Phi_{sk}(E',r) + \Phi_{ik}(E',r)\right] dE' \quad ,$$

$$j,k = N,P,\pi \quad , \tag{2.3}$$

where

N,P,π = neutron, proton, and charged pion, respectively,

$\Phi_j(E)$ = arbitrary initial value flux spectrum,

$\Phi_{ij}(E,r)$ = primary flux per unit energy range of particles of type j,

$\Phi_{sj}(E,r)$ = secondary flux per unit energy range of particles of type j,

E = kinetic energy,

r = dimensionless distance variable defined by the relation $r = \frac{\rho}{\lambda_0} R$,

ρ = density of medium, in g/cm^3,

R = distance, in cm,

λ_o = an arbitrary constant with dimensions g/cm^2 which determines the units in which r is measured,

$$Q_j(E) = \frac{\lambda_o N_o}{A} \sigma_j(E),$$

N_o = Avogadro's number,

$\sigma_j(E)$ = nonelastic cross section for particles of type j in the medium being considered,

A = atomic weight of medium being considered,

$$S_j(E) = \frac{\lambda_o N_o}{A} \epsilon_j(E),$$

$\epsilon_j(E)$ = stopping cross section for particles of type j,

$F_{jk}(E',E)\, dE$ = the number of secondary particles of type j in the energy interval E to E + dE produced by the nonelastic collision of a particle of type k with energy E'.

III. Physical Assumptions and Analytic Solution

To reduce the equations to soluble form, we introduce the assumptions

$$Q_j(E) = Q = \text{constant}, \quad j = N,P,\pi \quad , \tag{3.1}$$

$$S_j(E) = S = \text{constant}, \quad j = P,\pi \quad , \tag{3.2}$$

$$S_N(E) = 0 \quad , \tag{3.3}$$

$$F_{ij}(E',E) = \alpha_i \, \gamma_j \, e^{\nu(E'-E)} \, \Theta(E' - E), \quad i,j = N,P,\pi \quad , \tag{3.4}$$

$$\alpha_i, \gamma_j, \nu = \text{constant} \quad ,$$

$$\Phi_{sN}(E,r) = P_o \, \alpha_N \, \gamma_P \, e^{-Qr} \, H(E_o - E, r) \, \Theta(E_o - E) \quad , \tag{3.9}$$

where

$$K(E_o - E, r) = \left[\frac{r}{\beta_c(E_o - E - Sr)}\right]^{\frac{1}{2}} I_1 \left[2\sqrt{\beta_c(E_o - E - Sr)r}\right]$$

$$\cdot \, e^{\nu(E_o - E - Sr)} \, \Theta(E_o - E - Sr)$$

$$+ \int_0^r dr' \, \Theta(E_o - E - Sr + Sr') \left\{ \int_0^{r'} dr'' \left(I_o \left[2\sqrt{\beta_c(E_o - E - Sr + Sr' - Sr'')\, r''}\right] \right.\right.$$

$$\left.\left. \cdot \left[\frac{\beta_N(E_o - E - Sr + Sr' - Sr''}{r' - r''}\right]^{\frac{1}{2}} I_1 \left[2\sqrt{\beta_N(E_o - E - Sr + Sr' - Sr'')(r' - r'')}\right] \right) \right\} \quad ,$$

$$(3.10)$$

$$H(E_o - E, r) = \int_0^r dr' \, I_o \left[2\sqrt{\beta_c(E_o - E - Sr')\, r'}\right]$$

$$\cdot \, I_o \left[2\sqrt{\beta_N(E_o - E - Sr')(r - r')}\right]$$

$$\cdot \, e^{\nu(E_o - E - Sr')} \, \Theta(E' - E - Sr') \quad , \tag{3.11}$$

$$\beta_N = \alpha_N \, \gamma_N \, Q \quad ,$$

$$\beta_c = (\alpha_P \, \gamma_P + \alpha_\pi \, \gamma_\pi) \, Q \quad , \tag{3.12}$$

and I_o and I_1 are the usual hyperbolic Bessel functions.

IV. Comparison of Numerical and Analytic Solutions

In doing the numerical computations the constants appearing in the equations were chosen to be

$$Q_j = 1, \qquad j = N, P, \pi \ ,$$

$$S_j = 187.6 \text{ MeV}, \qquad j = P, \pi \ ,$$

$$S_N = 0,$$

$$\nu = 7 \times 10^{-4}$$

$$\alpha_j = 10^{-2} \qquad j = N, P, \pi \ ,$$

$$\gamma_j = 1/\nu \qquad j = N, P, \pi \ ,$$

$$\lambda_o = 93.8 \text{ g/cm}^2 \ ,$$

and the constants in the initial flux were chosen to be

$$P_O = 1 \text{ pro./cm}^2 \text{ sec} \ ,$$

$$E_O = 10 \text{ GeV} \ .$$

Before giving the comparison, it is perhaps worth while to make one point. In obtaining the numerical solutions all calculations are done in terms of a lethargy variable, u, defined by

$$u = \log \left[\frac{E_O}{E}\right] \ ,$$

and in terms of lethargy

$$S(u) = \frac{S}{E} = \frac{S}{E_o\, e^{-u}} \ ;$$

i.e., the stopping power is not constant.*** Thus in doing the calculations a variation of the stopping power in lethargy was taken into account.

From Eqs. (2.1) and (3.6) it follows that the calculation of the primary flux in the present instance is quite trivial. The two calculations give for all practical purposes the same answer, and therefore a comparison of the primary flux is not given.

In Fig. 1 the secondary neutron flux as a function of energy for various r values is shown, and in Fig. 2 the secondary proton flux as a function of energy is shown. The solid curves represent the numerical solution, while the plotted points represent the analytic solution. Because of the manner in which the constants are chosen, the comparison of the pion fluxes is exactly the same as the comparison of the proton fluxes and is therefore not shown.

The two solutions are in excellent agreement at all energies and all r values considered. At the very high energies (all curves go to zero at 10 GeV) the numerical solution for the proton flux tends to be slightly higher than the analytic solution, but, since the spectrum is decreasing so rapidly at these energies, the error is of no practical importance.

Acknowledgement

It is a pleasure to thank Dr. F. S. Alsmiller for many helpful discussions concerning both the numerical calculations and the analytic solution.

Footnotes

*The IBM code which was used in doing the numerical calculations reported here is an improved version of the code used before;[1-4] however, it does not give appreciably different results from those obtained previously.

**This decay is neglected here because the analytic solution can be obtained only under this assumption. In general, our numerical solutions include this decay and the resulting muon component.

***The cascade equations written in terms of lethargy are given in Ref. 2.

References

1. R. G. Alsmiller, Jr., and J. E. Murphy, Space Vehicle Shielding Studies: Calculation of the Attenuation of a Model Solar Flare and Monoenergetic Proton Beams by Aluminum Shields, ORNL-3317 (1963); Part II, The Attenuation of Solar Flares by Aluminum Shields, ORNL-3520 (1964); Part III, The Attenuation of a Particular Solar Flare by an Aluminum Shield, ORNL-3549 (1964).

2. R. G. Alsmiller, Jr., F. S. Alsmiller, and J. E. Murphy, Nucleon-Meson Cascade Calculations: Transverse Shielding for a 45-GeV Electron Accelerator, Part I, ORNL-3289 (1962); Part II, ORNL-3365 (1962); Part III, ORNL-3412 (1963).

3. R. G. Alsmiller, Jr., and J. E. Murphy, Nucleon-Meson Cascade Calculations: The Star Density Produced by a 24-GeV Proton Beam in Heavy Concrete, ORNL-3367 (1963).

4. R. G. Alsmiller, Jr., and J. E. Murphy, Nucleon-Meson Cascade Calculations: Shielding Against an 800-MeV Proton Beam, ORNL-3406 (1963).

5. R. G. Alsmiller, Jr., A Solution to the Nucleon-Meson Cascade Equations Under Very Special Conditions, ORNL-3570 (1964).

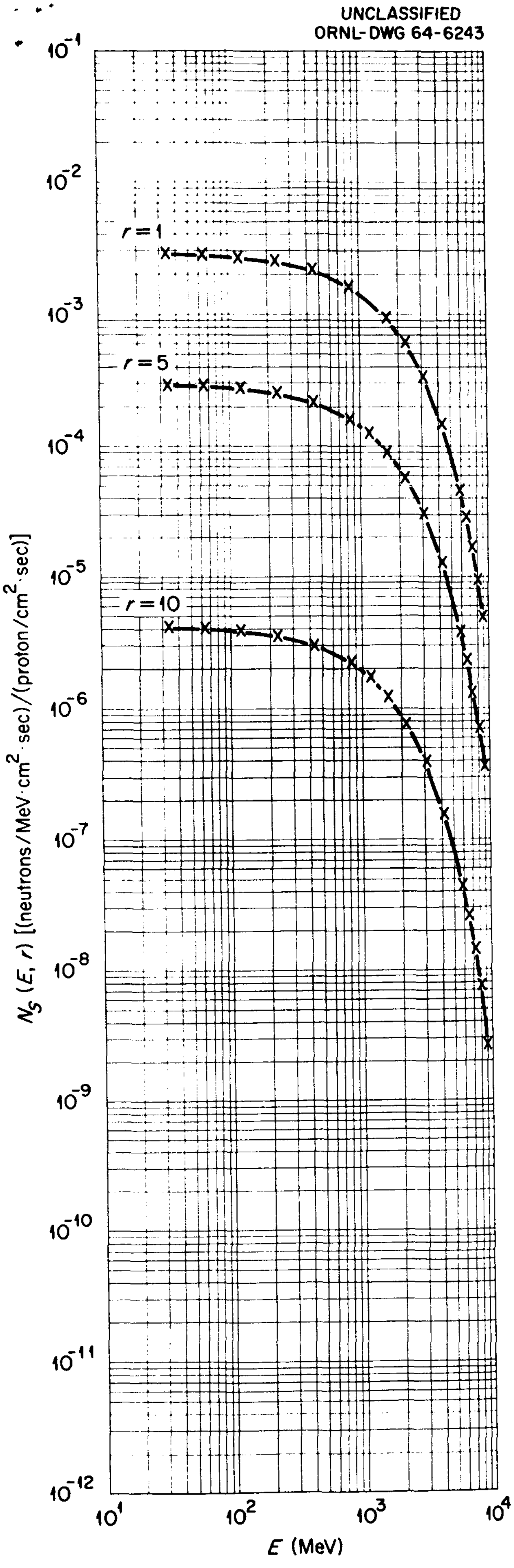
6. H. Messel, Progress in Cosmic Ray Physics, Vol. II, Interscience Publishers, Inc., New York (1954), p. 197.

7. B. Rossi, High-Energy Particles, Prentice-Hall, Inc., New Jersey (1956), p. 232.

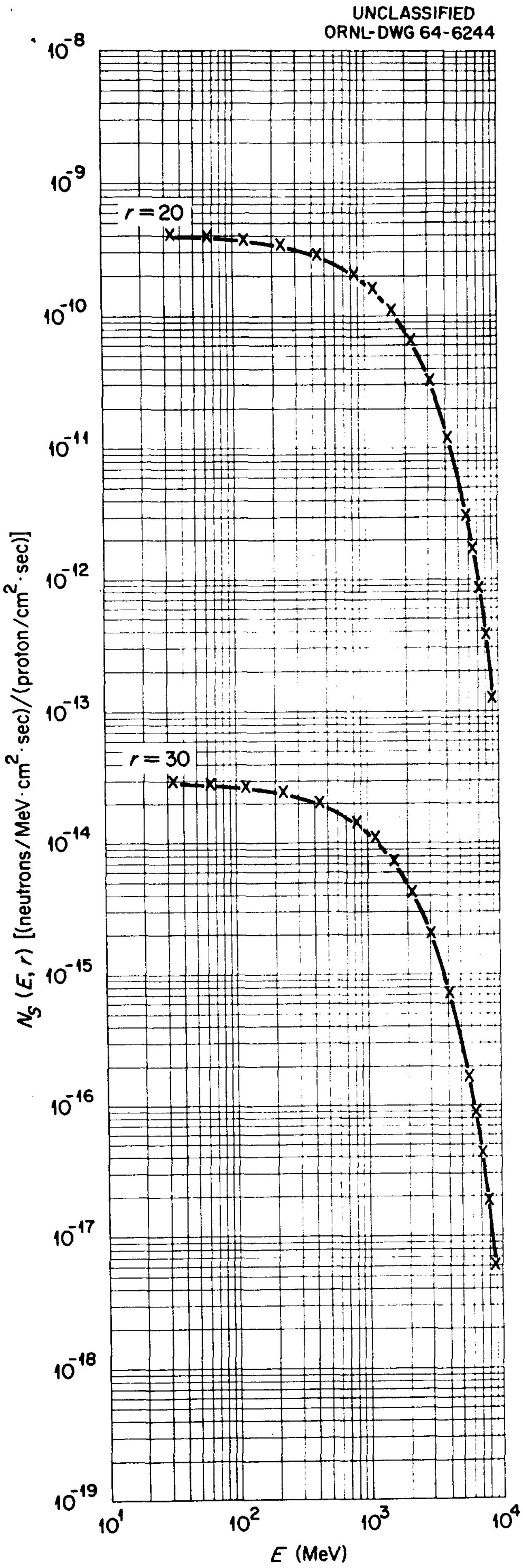
Figure Captions

Fig. 1 Secondary Neutron Flux vs. Energy (E_o = 10 GeV). — Numerical solution; X analytic solution; r is measured in collision lengths (= 93.8 g/cm^2).

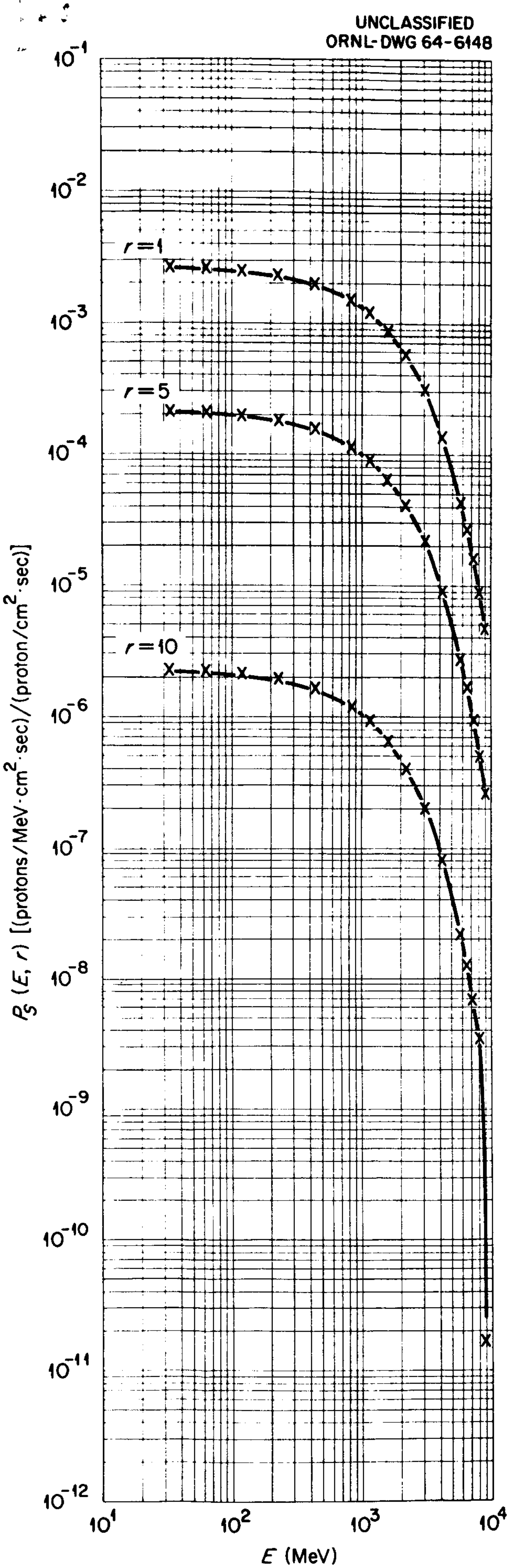
Fig. 2 Secondary Proton Flux vs. Energy (E_o = 10 GeV). — Numerical solution; X analytic solution; r is measured in collision lengths (= 93.8 g/cm^2).

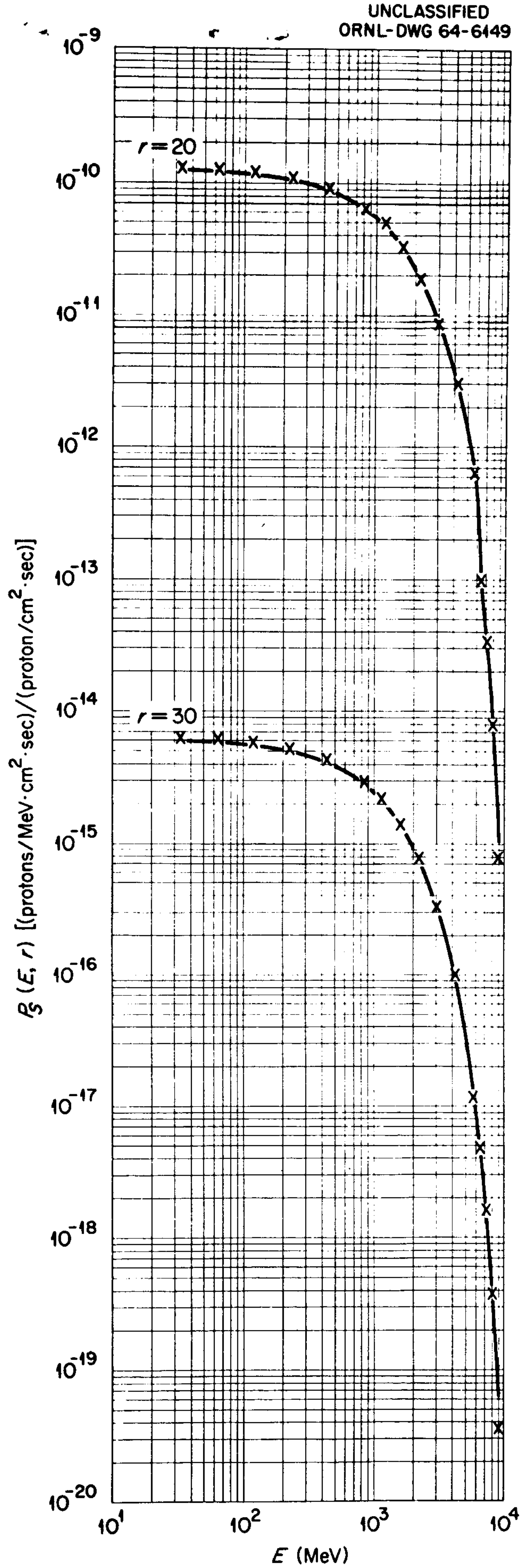


UNCLASSIFIED
ORNL-DWG 64-6243
r = 1
r = 5
r = 10
$N_S(E, r)$ [(neutrons/MeV·cm²·sec)/(proton/cm²·sec)]
E (MeV)



UNCLASSIFIED
ORNL-DWG 64-6244
r = 20
r = 30
$N_S(E, r)$ [(neutrons/MeV·cm²·sec)/(proton/cm²·sec)]
E (MeV)


UNCLASSIFIED
ORNL-DWG 64-6148
r = 1
r = 5
r = 10
$P_S(E, r)$ [(protons/MeV·cm^2·sec)/(proton/cm^2·sec)]
E (MeV)


UNCLASSIFIED
ORNL-DWG 64-6149
r = 20
r = 30
$P_S(E, r)$ [(protons/MeV·cm^2·sec)/(proton/cm^2·sec)]
E (MeV)