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GEORGE C. MARSHALL SPACE FLIGHT CENTER

UNDER CONTRACT NAS8-11255

NORTHROP

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by

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HUNTSVILLE DEPARTMENT
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PART I

INFLUENCE OF DAMPING AND INITIAL CONDITIONS UPON THE DYNAMIC STABILITY OF A UNIFORM FREE-FREE BEAM UNDER A GIMBALED THRUST OF PERIODICALLY VARYING MAGNITUDE

TABLE OF CONTENTS

NOTATION

a	Longitudinal acceleration
A	Beam cross-sectional area
$\left.\begin{matrix} A_o \\ A_n \\ B_n \\ C_n \end{matrix}\right\}$	Boundary value constants depending upon initial conditions
$c_k^{(m)}$	The k^{th} element in the matrix $\left\{c_k\right\}^{(m)}$
C	Damping factor
$D_{i,j}$	Square array of elements appearing in characteristic determinant
EI	Bending stiffness of uniform beam
$f(x)$	Displacement function at time $t = 0$
$g(x)$	Velocity function at time $t = 0$
E^i_{jk}	Coefficient in the j^{th} row and k^{th} column of matrix $\left[E^i_{jk}\right]$ $i = 1,2,\ldots$
F_{jk}	Coefficient in the j^{th} row and k^{th} column of matrix $\left[F_{jk}\right]$
g	Acceleration due to gravity
G_{jk}	Coefficient in the j^{th} row and k^{th} column of matrix $\left[G_{jk}\right]$
H^i_{jk}	= Coefficient in the j^{th} row and k^{th} column of matrix $\left[H^i_{jk}\right]$ $i = 1,2,\ldots$
G.F.	Growth factor
$H(\alpha)$	Function formed from $\Delta(\alpha)$ to eliminate singularities
$\left[I\right]$	Identity matrix
K_j	Constants evaluated so as to eliminate singularities of $H(\alpha)$

K_θ	Directional control factor determining thrust vector gimbal angle
ℓ	Length of uniform beam
m	Mass per unit length of beam
M	Moment distribution in beam; also, range of index m in evaluation of $\Delta_j(\hat{\omega}_j)$
N	Number of bending degrees of freedom assumed in numerical analysis
p	Lateral force on beam arising from component of thrust due to gimbaling
P	Axial force distribution in beam
Pr	Product of diagonal elements of $\Delta(\alpha)$
q_A, q_B	Rigid-body generalized coordinates
q_n	n^{th} bending generalized coordinate
R	Modulus of $z \pm \sqrt{z^2 - 1}$
S	Sum of nondiagonal elements of $\Delta(\alpha)$
t	Real-time variable
T_o	Amplitude of constant thrust
T_1	Amplitude of sinusoidally varying thrust
T_c	Critical end thrust for cantilever beam with direction of thrust parallel to axis of beam
$\bar{T}_o$	Nondimensional thrust parameter $= T_o\ell^2/EI$
u(x,t)	Longitudinal displacement of particles of beam measured in Lagrangian coordinate system
V	Lateral force distribution in beam
x	Lagrangian coordinate defining position of particles in unstrained beam relative to one end of the beam
x_G	x-coordinate corresponding to the location of direction-sensing element in the beam

$y(x,t)$	Lateral displacement of axis of beam from fixed reference line
z	$= \cos 2\pi\alpha$
ζ_i	$= \cos 2\pi \bar{F}_{jj}^{\frac{1}{2}}$
α^2	$= M/AE$
α	Characteristic exponent whose value indicates the stability of a system whose motion is represented by linear differential equations with periodic coefficients
β	Argument of $z \pm \sqrt{z^2 - 1}$
γ	$= T_1/T_o$
$\delta(\xi)$	Dirac delta function
$\Delta(\alpha)$	Value of the infinite determinant of coefficients obtained from series expansion of $\psi_k(\tau)$
$\Delta_j(\alpha)$	$= \Delta(\alpha) \cdot (-\alpha^2 + \bar{F}_{jj})$
ε	A small quantity
θ	Gimbal angle, equal to rotation of thrust vector from a tangent to the beam-deflection curve
λ_n^4	Uniform beam frequency parameter $= \omega_n^2 \frac{m\ell^4}{EI}$
ξ	Nondimensional coordinate $= x/\ell$
τ	Nondimensional time variable $= \omega_1 t$
ϕ_n	Mode shape of n^{th} vibration mode of uniform free-free beam
$\phi(\xi)$	A function defining the longitudinal force distribution in a uniform beam arising from the varying thrust component
$\Psi(x)$	Rotation of the beam element located at x
Ψ_G	Rotation of the beam element located at x_G
$\Psi_k(\tau)$	A matrix whose elements have a periodic variation of 2π in τ

ω	Beat frequency of Ω & ω_L
ω_L	Fundamental longitudinal frequency of free-free beam
ω_n	Lateral bending frequency of n^{th} mode of free-free beam
$\omega_{(n)}$	Lateral bending frequency of n^{th} mode of free-free beam with end thrust
$\bar{\omega}$	Nondimensional frequency parameter used to represent general form of solution
$\bar{\omega}_L$	Nondimensional longitudinal frequency $= \omega_L/\omega_1$
$\bar{\omega}_n$	Nondimensional bending frequency $= \omega_n/\omega_1$
$\bar{\omega}_{(n)}$	Nondimensional bending frequency $= \omega_{(n)}/\omega_1$
$\hat{\omega}_k$	$= \bar{\omega}_{(k)}/\bar{\Omega}$
Ω	Frequency of thrust variation
$\bar{\Omega}$	Nondimensional frequency of thrust variation $= \Omega/\omega_1$
$\cdot$	$= \frac{d}{d\tau}$ or $\frac{\partial}{\partial\tau}$
$'$	$= \frac{d}{d\xi}$ or $\frac{\partial}{\partial\xi}$

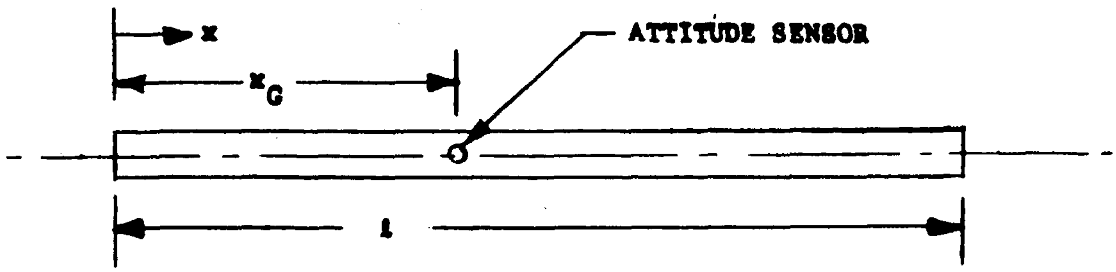
1.0 INTRODUCTION

The dynamic stability of an uniform free-free beam under a gimbaled thrust of periodically varying magnitude was investigated in a thesis by Thomas Reynolds Beal [1]. As an extension of this preliminary problem this study will include the effects of damping and arbitrary initial conditions upon stability and response of a beam on the lateral bending mode of vibration.

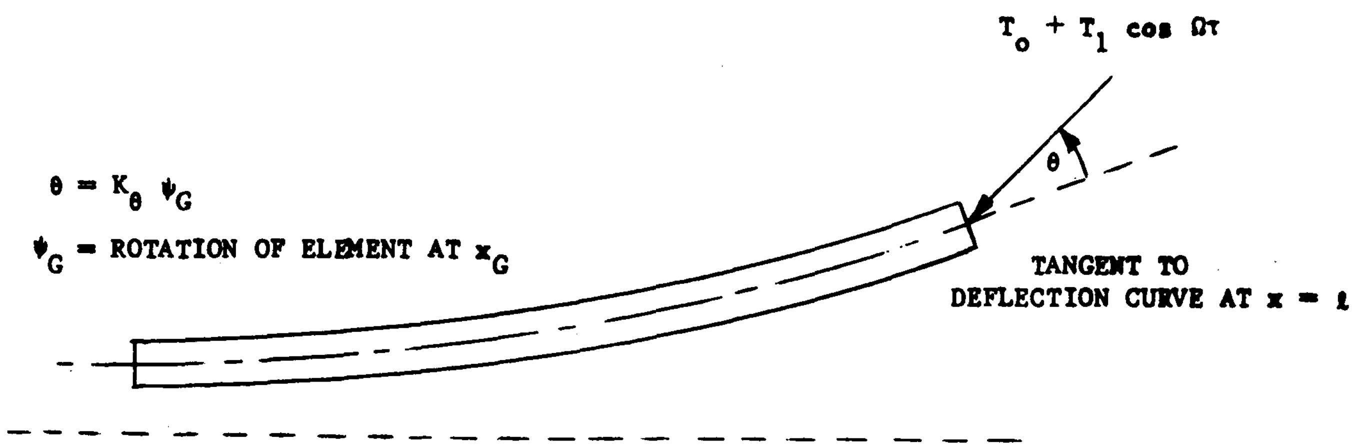
As the most general configuration a slender uniform beam is accelerated through space by a non-conservative force, $T_o + T_1 \cos \Omega t$, starting from rest. T_o and T_1 are constants with T_1 being the amplitude of the periodically varying thrust component. An attitude sensor controls the thrust angle by means of a simple feedback system thereby achieving rigid body stability. It is assumed that a linear relationship exists between the gimbal angle (or thrust angle) and the element rotation where the attitude sensor is located (see Fig. 1). Linearity of the resultant differential equations of motion is to be maintained, but the effects of longitudinal compliance upon the transverse motion of the beam is to be incorporated. The normal assumption of small angles and slopes is maintained to insure linearity.

Analysis is simplified by the following restrictions:

1. Slender uniform beam
2. Simple feedback system is maintaining rigid body directional control
3. Two dimensional motion



(a) UNDEFORMED STATE



(b) DEFORMED STATE - WITH CONTROL SYSTEM

FIG. 1 FREE BEAM WITH END THRUST

4. Shear deformation and rotary inertia effects are neglected
5. Damping in longitudinal motion is neglected.

By including damping and taking initial conditions into account an analytical model which more closely approximates modern space vehicle structure and environment is achieved. One of the most serious over-simplifications is the choice of an uniform beam, but it is felt that the theory and analysis presented herein may be extended to include beams with concentrated masses and discontinuous stiffness distribution.

The equation of motion derived for the beam under consideration is reduced to an infinite set of linear ordinary differential equations by the method of Galerkin [13]. A method of solution for an infinite set of Mathieu type differential equations was the primary contribution of Thomas R. Beal [1] where damping was neglected. By a suitable change of variable the problem with damping may be reduced to a form that may be solved by the methods set forth by Beal.

This technical memo is prepared mainly through the effort of Mr. Jim Kincaid, with other members of the team participating in the work from time to time. This report represents the progress to date of this portion of the contract work. The formulation enclosed has been programmed for the IBM 7094 computer and verified. At the present time, a full parameter study is being initiated and will be completed in the near future.

2.0 ANALYTICAL DEVELOPMENT

2.1 Equations of Motion

The coordinate system of the beam under analysis, shown in Fig. 2a, relates particle or element displacement to a Lagrangian coordinate system. In its initial state the x coordinate locates particle position while the y coordinate, a function of x & t, measures particle lateral motion relative to a fixed reference line. Particle displacement u(x,t) measures particle motion parallel to the fixed reference line.

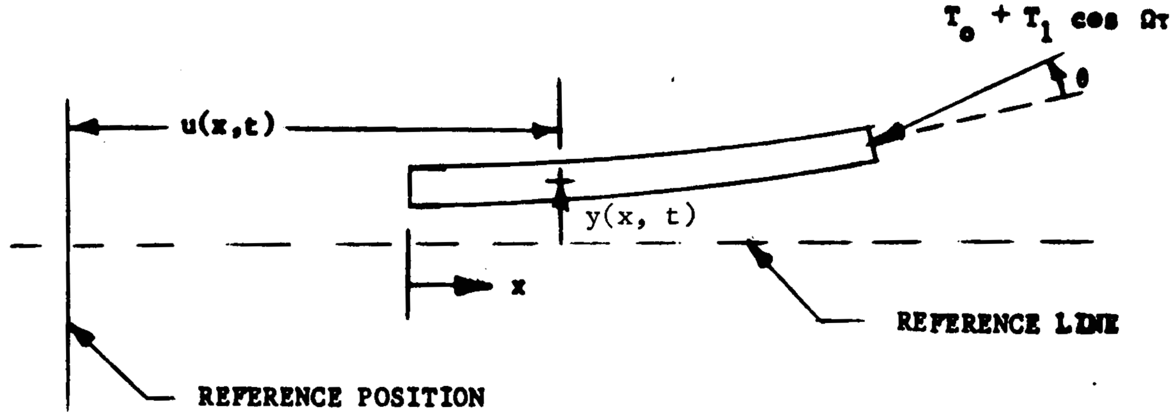
It is assumed that the beam is uniform in cross section and mass distribution, that simple beam theory is applicable, and that shear and rotary inertia effects may be considered negligible.

It was previously stated in the introduction that the beam maintains rotational stability by a simple feedback system or in equation form

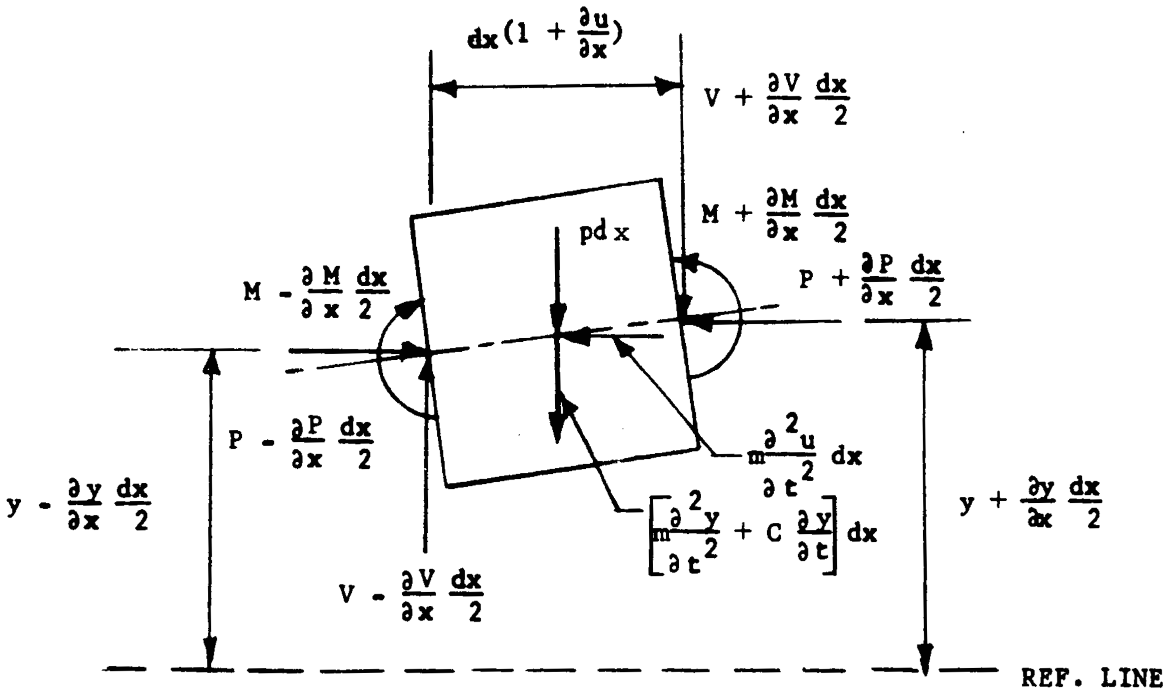
$$\Theta = K_\theta \psi_G \tag{2.1}$$

where K_Θ is the constant of proportionality between the gimbal angle Θ and the attitude sensor rotation angle ψ_G located at station x = G. In Fig. 2b an equilibrium diagram of forces on a beam element is shown. Considering this element at station x = G, the angle ψ_G is seen to be

$$\psi_G = \tan^{-1} \frac{\frac{\partial y}{\partial x}}{(1 + \frac{\partial u}{\partial x})} \tag{2.2}$$



(a) DISPLACEMENTS IN LAGRANGIAN COORDINATE SYSTEM



(b) FORCES ON BEAM ELEMENTS

FIG. 2 CONTROLLED BEAM WITH THRUST OF PERIODICALLY VARYING MAGNITUDE and the gimbal angle may be written as

$$\theta = K_\theta \tan^{-1} \left[\frac{\partial y}{\partial x} \bigg/ (1 + \frac{\partial y}{\partial x}) \right] \tag{2.3}$$

An element of the beam shown in Fig. 2b is in equilibrium at some arbitrary position and time. Lateral forces acting on the element are the inertial force $m\frac{\partial^2 y}{\partial t^2}$ dx, the applied side load pdx, and the damping term $C \frac{\partial y}{\partial t}$.

The equations of equilibrium are readily obtained by the summation of forces and moments as

$$\frac{\partial P}{\partial x} + m \frac{\partial^2 u}{\partial t^2} = 0 \tag{2.4}$$

$$\frac{\partial V}{\partial x} + m \frac{\partial^2 y}{\partial t^2} + C \frac{\partial y}{\partial t} + p = 0 \tag{2.5}$$

$$\frac{\partial M}{\partial x} + P \frac{\partial y}{\partial x} - V (1 + \frac{\partial u}{\partial x}) = 0. \tag{2.6}$$

In simple beam theory the moment at any cross section may be written as

$$M = EI \frac{\partial^2 y}{\partial x^2} \tag{2.7}$$

and from elasticity

$$P = - AE \frac{\partial u}{\partial x} . \tag{2.8}$$

If the angle of rotation $\frac{\partial y}{\partial x}$ and strain $\frac{\partial u}{\partial x}$ are considered small in comparison to unity which are assumptions used to obtain Eqs. (2.7 & 2.8) then Eq. (2.6) may be written as

$$\frac{\partial M}{\partial x} + P\frac{\partial y}{\partial x} - V = 0 \tag{2.6a}$$

Substituting Eq. (2.8) into Eq. (2.4) we obtain the familiar wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{m}{AE}\frac{\partial^2 u}{\partial t^2} \tag{2.9}$$

which is the differential equation of longitudinal vibrations.

The longitudinal displacement function u is seen to be dependent upon the forcing function $T_o + T_1 \cos \Omega t$ and the gimbal angle by equation (2.8). If the gimbal angle is restricted to small angles then at

$$x = \ell: \quad \frac{\partial u}{\partial x} = -\frac{T_o}{AE} - \frac{T_1}{AE}\cos \Omega t \tag{2.10a}$$

and at

$$x = 0: \quad \frac{\partial u}{\partial x} = 0 \tag{2.10b}$$

Let a solution u_1 be a product of X and T which are functions of x and t respectively.

$$u_1 = XT \tag{2.11}$$

Substitution of Eq. (2.11) into Eq. (2.9) yields

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{\alpha^2}{T}\frac{d^2T}{dt^2} = \text{Const.} \tag{2.12}$$

where $\alpha^2 = \frac{m}{AE}$. Let the const. term in Eq. (2.12) equal $-\nu^2$ to achieve a periodic solution. Solutions of Eq. (2.12) may be written as

$$X = C_1 \cos \nu x + C_2 \sin \nu x \tag{2.13}$$

$$T = C_3 \cos \frac{\nu}{\alpha}t + C_4 \sin \frac{\nu}{\alpha}t \tag{2.14}$$

Therefore by Eq. (2.11)

$$u_1 = (C_1 \cos \nu x + C_2 \sin \nu x)(C_3 \cos \frac{\nu}{\alpha} t + C_4 \sin \frac{\nu}{\alpha} t) \tag{2.15}$$

$$\frac{\partial u_1}{\partial x} = (-\nu C_1 \sin \nu x + \nu C_2 \cos \nu x)(C_3 \cos \frac{\nu}{\alpha} t + C_4 \sin \frac{\nu}{\alpha} t) \tag{2.16}$$

Boundary conditions applicable to a periodic solutions are at

$$x = \ell: \quad \frac{\partial u_1}{\partial x} = - \frac{T_1}{AE} \cos \Omega t \tag{2.17a}$$

and at

$$x = 0: \quad \frac{\partial u_1}{\partial x} = 0 \tag{2.17b}$$

Using the B.C. of Eq. (2.17b) and Eq. (2.16)

$$C_2 = 0 \tag{2.18a}$$

and the B.C. of Eq. (2.17a) and Eq. (2.16)

$$C_4 = 0 \tag{2.18b}$$

$$\nu = \alpha\Omega$$

u_1 may then be written as

$$u_1 = \frac{T_1 \alpha}{\Omega m} \frac{\cos \alpha\Omega x}{\sin \alpha\Omega} \cos \Omega t \tag{2.19}$$

Let u_2 be of the form

$$u_2 = X + T \tag{2.20}$$

subject to the B.C. that at

$$x = \ell: \quad \frac{\partial u_2}{\partial x} = - \frac{T_o}{AE} \tag{2.21a}$$

$$x = 0: \quad \frac{\partial u_2}{\partial x} = 0 \tag{2.21b}$$

It is easily shown that

$$u_2 = - \frac{T_o x^2}{2AE\ell} - \frac{T_o t^2}{2AE\ell\alpha^2} \tag{2.22}$$

plus some constants that do not affect $\frac{\partial u}{\partial x}$.

The particular solution of Eq. (2.9) is written

$$u_p = u_1 + u_2$$

$$u_p = - \frac{T_o x^2}{2AE\ell} - \frac{T_o t^2}{2AE\alpha^2\ell} + \frac{T_1 \alpha}{\Omega m} \frac{\cos \alpha\Omega x}{\sin \alpha\Omega} \cos \Omega t \tag{2.23}$$

To obtain a general solution a term u_o must be added to the particular solution u_p such that the sum will satisfy arbitrary initial conditions.

Choose u_o of the form

$$u_o = XT \tag{2.24}$$

subject to the boundary conditions

$$u_o\ (x,0) = f(x) - u_p\ (x,0) \tag{2.25a}$$

$$\frac{\partial u_o}{\partial t}\ (x,0) = g(x) - \frac{\partial u_p}{\partial t}\ (x,0) \tag{2.25b}$$

$$\frac{\partial u_o}{\partial x}\ (0,t) = 0 \tag{2.25c}$$

$$\frac{\partial u_o}{\partial x}\ (\ell,t) = 0 \tag{2.25d}$$

Substitution of Eq. (2.24) into the wave equation yields

$$\frac{X''}{X} = \alpha^2\ \frac{T''}{T} = -\ k^2 \text{ a const.} \tag{2.26}$$

and

$$u_o = (C_1 \cos kx + C_2 \sin kx)(C_3 \cos \frac{k}{\alpha}\ t + C_4 \sin \frac{k}{\alpha}\ t) \tag{2.27}$$

$$\frac{\partial u_o}{\partial x} = (-C_1 k \sin kx + C_2 k \cos kx)(C_3 \cos \frac{k}{\alpha}\ t + C_4 \sin \frac{k}{\alpha}\ t) \tag{2.28}$$

From Eq. (2.25c, 2.25d) and Eq. (2.28)

$$C_2 = 0 \tag{2.29a}$$

$$k = \frac{n\pi}{\ell} \tag{2.29b}$$

and

$$u_o = A_o + \sum_{n=1}^{\infty}\left[A_n \cos\frac{n\pi t}{\alpha\ell} + B_n \sin\frac{n\pi t}{\alpha\ell}\right] \cos\frac{n\pi x}{\ell} \tag{2.30}$$

$$\frac{\partial u_o}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi}{\alpha\ell}\left[- A_n \sin\frac{n\pi t}{\alpha\ell} + B_n \cos\frac{n\pi t}{\alpha\ell}\right] \cos\frac{n\pi x}{\ell} \tag{2.31}$$

Eq. (2.25a) and Eq. (2.30) yields

$$f(x) - u_p(x,0) = A_o + \sum_{n=1}^{\infty} A_m \cos\frac{m\pi x}{\ell} \tag{2.32}$$

Multiplying by $\cos\frac{n\pi x}{\ell}$ and integrating over the length yields

$$A_n = \frac{2}{\ell}\int_o^{\ell} \left[f(x) - u_p(x,0)\right] \cos\frac{n\pi x}{\ell}\, dx \tag{2.33}$$

In a similar manner Eq. (2.25b) and Eq. (2.31) yields

$$B_n = \frac{2\alpha}{n\pi}\int_o^{\ell} \left[g(x) - \frac{\partial u_p}{\partial t}(x,0)\right] \cos\frac{n\pi x}{\ell}\, dx \tag{2.34}$$

The general solution becomes

$$u = u_p + u_o = - \frac{T_o x^2}{2AE\ell} - \frac{T_o t^2}{2AE\alpha^2\ell} + \frac{T_1}{AE}\frac{\cos \alpha\Omega x}{\sin \alpha\Omega\ell} \cos \Omega t$$

$$+ A_o + \sum_{n=1}^{\infty}\left[A_n \cos\frac{n\pi}{\alpha\ell}t + B_n \sin\frac{n\pi}{\alpha\ell}t\right] \cos\frac{n\pi x}{\ell} \tag{2.35}$$

where A_n & B_n are defined in Eqs. (2.33 & 2.34). The normal force P in Eq. (2.8) due to the longitudinal vibration may now be expressed as

$$P = \frac{T_o x}{\ell} + T_1\alpha\Omega \frac{\sin \alpha\Omega x}{\sin \alpha\Omega\ell} \cos \Omega t + AE \sum_{n=1}^{\infty}\frac{n\pi}{\alpha\ell}\left[A_n \cos\frac{n\pi}{\ell\alpha} t + B_n \sin\frac{n\pi}{\alpha\ell} t\right]$$

$$\sin\frac{n\pi x}{\ell} \tag{2.36}$$

The shear term V is eliminated between Eqs. (2.5 & 2.6) resulting in the equation of motion for beam vibration

$$EI \frac{\partial^4 y}{\partial x^4} + \frac{\partial}{\partial x}\left(P \frac{\partial y}{\partial x}\right) + m \frac{\partial^2 y}{\partial t^2} + C \frac{\partial y}{\partial t} + p = 0 \tag{2.37}$$

where P is defined in Eq. 2.36.

This is an opportune point to simplify and dimensionalize our derived equations. We now make the following substitution recalling that α was previously defined as $\alpha^2 = \frac{m}{AE}$.

Let

$$\gamma = \frac{T_1}{T_o} \tag{2.38}$$

$$\xi = \frac{x}{\ell} \tag{2.39}$$

$$\tau = t\omega_1 \tag{2.40}$$

where ω_1 is the fundamental transverse frequency of a free-free beam.

$$\omega_L = \frac{\pi}{\ell\alpha} = \frac{\pi}{\ell}\sqrt{\frac{AE}{m}} \tag{2.41}$$

ω_L is the fundamental frequency of longitudinal vibration of a free-free beam.

$$\bar{\sigma} = \frac{\Omega}{\omega_L} \tag{2.42}$$

$$\bar{\Omega} = \frac{\Omega}{\omega_1} \tag{2.43}$$

$$\bar{T}_o = \frac{T_o \ell^2}{EI} \tag{2.44}$$

$$\lambda^4_n = \omega^2_n \frac{m\ell^4}{EI} \tag{2.46}$$

$$\eta = \frac{C}{m\omega_1} \tag{2.47}$$

$$\Phi(\xi) = \frac{\sin(\pi\bar{\sigma}\xi)}{\sin(\pi\bar{\sigma})} \tag{2.48}$$

Eqs. (2.33, 2.34, 2.36 & 2.37) may be rewritten as

$$A_n = 2\ell \int_o^1 \left[f(\xi) - u_p(\xi, o) \right] \cos(n\pi\xi)\, d\xi \tag{2.49}$$

$$B_n = \frac{2\ell}{n} \frac{\omega_1}{\omega_L} \int_o^1 \left[g(\xi) - \frac{\partial u_p}{\partial \tau}(\xi, 0) \right] \cos(n\pi\xi) d\xi \tag{2.50}$$

$$P(\xi) = T_o \left\{ \xi + \gamma\Phi(\xi) \cos \bar{\Omega}\tau + \frac{AE\pi}{T_o} \sum_{n=1}^{\infty} n \left[A_n \cos n \frac{\omega_L}{\omega_1} \tau + B_n \sin n \frac{\omega_L}{\omega_1}\tau \right] \sin(n\pi\xi) \right\} \tag{2.51}$$

$$\frac{\partial^4 y}{\partial \xi^4} + \frac{\ell^2}{EI} \frac{\partial}{\partial \xi} \left(P \frac{\partial y}{\partial \xi} \right) + \lambda_1^4 \frac{\partial^2 y}{\partial \tau^2} + \eta\lambda_1^4 \frac{\partial y}{\partial \tau} + \frac{\ell^4}{EI} p = 0 \tag{2.52}$$

Let

$$C_n \cos\left(n \frac{\omega_L}{\omega_1}\tau - \psi_n\right) = A_n \cos n \frac{\omega_L}{\omega_1}\tau + B_n \sin n \frac{\omega_L}{\omega_1}\tau \tag{2.53}$$

as shown in the figure below:

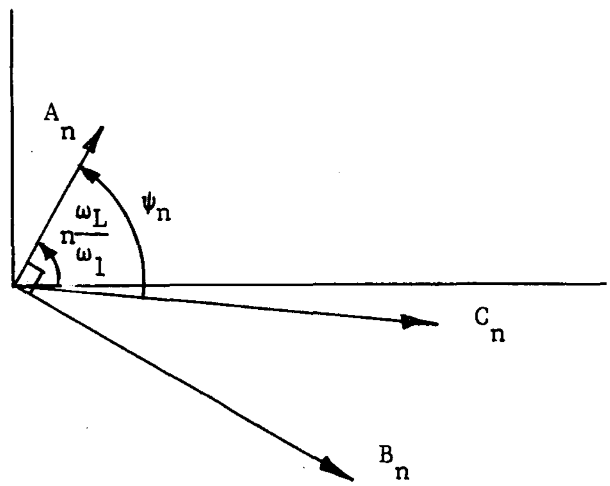


Fig. 2.1

The dimensionalized thrust vector $T_o + T_1 \cos \bar{\Omega}\tau$ applied at $\xi = 1$ has a normal component $(T_o + T_1 \cos \bar{\Omega}\tau) \sin \theta$ acting on the beam at $\xi = 1$. See the figure below

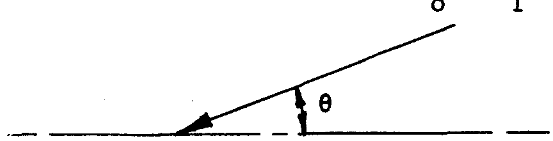


Fig. 2.2

For small gimbal angles $\sin \theta = \theta$ and the normal side load p acting at ξ may be written as

$$p = (T_o + T_1 \cos \bar{\Omega}\tau) \frac{\theta}{\ell} \delta (\xi - 1) \tag{2.54}$$

but for small rotations, and considering $\frac{\partial u}{\partial x} << 1$

$$\theta = \frac{K_\theta}{\ell} \frac{\partial y(\xi,\tau)_G}{\partial \xi}$$

as shown previously in Eq. 2.2.

Therefore

$$p = \frac{T_o}{\ell^2} (1 + \gamma \cos \bar{\Omega}\tau) K_\theta \frac{\partial y(\xi,\tau)_G}{\partial \xi} \delta(\xi-1) \qquad (2.55)$$

Substitution of Eq. (2.48 & 2.51) into Eq. (2.52), making use of the relationship of Eq. (2.49), then simplifying yields

$$\frac{\partial^4 y}{\partial \xi^4} + \bar{T}_o \frac{\partial}{\partial \xi} \left\{ \frac{\partial y}{\partial \xi} (\xi + \gamma\Phi(\xi) \cos \pi\bar{\Omega}\tau) + \frac{AE\pi}{T_o \ell} \sum_{n=1}^{\infty} \right.$$

$$\left. \left[n\, C_n \cos \left(n \frac{\omega_L}{\bar{\omega}_1}\tau - \psi_n\right) \right] \sin (n\pi\xi) \right\} + \lambda_1^4 \frac{\partial^2 y}{\partial \tau^2} +$$

$$\eta\lambda_1^4 \frac{\partial y}{\partial \tau} + \bar{T}_o (1 + \gamma \cos \bar{\Omega}\tau) K_\theta \frac{\partial y(\xi,\tau)_G}{\partial \xi} \delta(\xi-1) = 0 \qquad (2.56)$$

We note from physical considerations that the following boundary conditions on y are to be satisfied.

$$\text{at } \xi = 0: \quad \frac{\partial^2 y}{\partial \xi^2} = 0; \quad \frac{\partial^3 y}{\partial \xi^3} = 0 \qquad (2.57)$$

$$\text{at } \xi = 1: \quad \frac{\partial^2 y}{\partial \xi^2} = 0; \quad \frac{\partial^3 y}{\partial \xi^3} = 0 \qquad (2.58)$$

Notice that no restriction has been placed on the translation term $y(\xi,\tau)$. Therefore arbitrarily large values of $y(\xi,\tau)$ are permitted.

Eq. (2.56) is the governing linear partial differential equation representing lateral vibratory motion of a beam subjected to conditions defined in this study. We now proceed to a method of solution.

2.2 Reduction of Equation (2.56) to a System of Ordinary Differential Equations by Galerkin's Method

A description of the Galerkin Method indicates that the solution of Eq. (2.56) can be adequately approximated by expressing the deflection

$$y_N(\xi,\tau) = q_A + q_B\xi + \sum_{n=1}^{\infty} q_n(\tau)\ \phi_n(\xi) \tag{2.59}$$

as a sum of some translation term $q_A(\tau)$, rotation term $q_B(\tau)$, and a function $\phi_n(\xi)$ as the n^{th} vibration mode shape of a free-free beam that satisfies the boundary conditions of Eqs. (2.57 & 2.58). $q_n(\tau)$ is the generalized coordinate associated with $\phi_n(\xi)$ of the form

$$\phi_n = \cosh R_n\xi + \cos R_n\xi - \alpha_n (\sinh R_n\xi + \sin R_n\xi). \tag{2.59a}$$

Properties of this function $\phi_n(\xi)$ are set forth in several texts but for our purposes Ref. [9] is quite adequate. We observe that $\phi_n(\xi)$ satisfies the differential equation

$$\frac{d^4\phi_n}{d\xi^4} = \lambda_n^4\ \phi_n \tag{2.60}$$

where $\lambda_n^4 = \omega_n^2 \frac{m\ell^4}{EI}$ was previously defined on page 13. Also $\phi_n(\xi)$ is seen to satisfy the following boundary conditions at

$$\xi = 0: \quad \frac{d^2\phi_n}{d\xi^2} = 0 \qquad \frac{d^3\phi_n}{d\xi^3} = 0 \tag{2.61}$$

$$\xi = 1: \quad \frac{d^2\phi_n}{d\xi^2} = 0 \qquad \frac{d^3\phi_n}{d\xi^3} = 0 \tag{2.62}$$

as do all the functions of Eq. (2.59).

Galerkin's method requires that the error inherent to the approximate solution Eq. (2.59) be orthogonal to the weighing function $\frac{\partial y_N}{\partial q_i}$. In equation form

$$\int_0^1 \S(y_N) \frac{\partial y_N}{\partial q_i} d\xi = 0 \qquad i = q_A, q_B, q_n \tag{2.63}$$

or

$$\int_0^1 \S(y_N) d\xi = 0 \tag{2.63a}$$

$$\int_0^1 \S(y_N)\xi d\xi = 0 \tag{2.63b}$$

$$\int_0^1 \S(y_N)\phi_n(\xi)d\xi \qquad n = 1,2,3,\ldots \tag{2.63c}$$

where $\S(y_N)$ is the error resulting when Eq. (2.59) is substituted into Eq. (2.56). The theory behind this process is explained in Ref. [13].

Without going into detail Eqs. (2.63a, 2.63b, & 2.64c) may be integrated to obtain

$$\lambda_1^4 \ddot{q}_A + \eta\lambda_1^4 \mathring{q}_A + \frac{1}{2}\lambda_1^4 \ddot{q}_B + \frac{\eta}{2}\mathring{q}_B +$$

$$\overline{T}_o\,(1 + \gamma\cos\overline{\Omega}\tau)\left[q_B + \sum_{n=1}^{N} q_n\phi_n'\,(\xi)_1\right] +$$

$$\overline{T}_o\,(1 + \gamma\cos\overline{\Omega}\tau)\,K_\theta\left[q_B + \sum_{n=1}^{N} q_n\,\phi_n'\,(\xi)_G\right] = 0 \qquad (2.64a)$$

$$\frac{\lambda_1^4}{2}\ddot{q}_A + \frac{\eta}{2}\lambda_1^4\mathring{q}_A + \frac{\lambda_1^4}{3}\ddot{q}_B + \frac{\eta}{3}\lambda_1^4\mathring{q}_B$$

$$+ \overline{T}_o\,(1 + \gamma\cos\overline{\Omega}\tau)\left[q_B + \sum_{n=1}^{N} q_n\,\phi_n'(\xi)_1\right]$$

$$- \frac{\overline{T}_o}{2} q_B - T_o\,\gamma\,\cos\overline{\Omega}\tau\left[\frac{1 - \cos\pi\overline{\sigma}}{\pi\overline{\sigma}\ \ \sin\pi\overline{\sigma}}\right] q_B$$

$$- \overline{T}_o\,(1 + \gamma\,\cos\overline{\Omega}\tau)\sum_{n=1}^{N} q_n\,\phi_n(\xi)_1$$

$$+ \overline{T}_o\,\gamma\,\cos\overline{\Omega}\tau\sum_{n=1}^{N} q_n\int_o^1 \phi_n\phi'\,d\xi$$

$$+ \overline{T}_o(1 + \gamma\,\cos\overline{\Omega}\tau)K_\theta\left[q_B + \sum_{n=1}^{N} q_n\,\phi_n'(\xi)_G\right]$$

$$+ \bar{T}_o \, q_B \sum_{s=1}^{\infty} \frac{C_s}{s\pi} \left[(-1)^s - 1 \right] \cos\left(s \frac{\omega_L}{\omega_1}\tau + \psi_s\right)$$

$$+ \bar{T}_o \sum_{n=1}^{N} q_n \sum_{s=1}^{\infty} \pi s \, C_s \int_o^1 \phi_n(\xi) \cos(s\pi\xi) d\xi \quad \bullet$$

$$\cos\left(s \frac{\omega_L}{\omega_1} \tau + \psi_s\right) = 0 \qquad (2.64b)$$

$$\lambda_1^4 \ddot{q}_k + \eta \, \lambda_1^4 \mathring{q}_k + \lambda_k^4 q_k$$

$$+ \bar{T}_o (1 + \gamma \cos \bar{\Omega}\tau) \left[q_B + \sum_{n=1}^{N} q_n \, \phi_n'(\xi)_1 \right] \phi_k(\xi)_1$$

$$- \bar{T}_o \, \phi_k(\xi)_1 \, q_B - \bar{T}_o \, \gamma \cos \bar{\Omega}\tau \left[\phi_k(\xi)_1 - \int_o^1 \phi_k \, \Phi' d\xi \right] q_B$$

$$- \bar{T}_o \sum_{n=1}^{N} q_n \int_o^1 \xi\phi_n' \, \phi_k' \, d\xi - \bar{T}_o \, \gamma \cos \bar{\Omega}\tau \sum_{n=1}^{N} q_n \int_o^1 \Phi\phi_n' \, \phi_k' \, d\xi$$

$$+ \bar{T}_o (1 + \gamma \cos \bar{\Omega}\tau) \, K_\theta \left[q_B + \sum_{n=1}^{N} q_N \, \phi_n'(\xi_G) \right] \phi_k(\xi)_1$$

$$- \bar{T}_o \, q_B \sum_{s=1}^{\infty} C_s \cos\left(s \frac{\omega_L}{\omega_1} \tau + \psi_s\right) \int_o^1 \phi_k' \sin(s\pi\xi) d\xi$$

$$- \bar{T}_o \sum_{n=1}^{N} q_n \sum_{s=1}^{\infty} C_s \cos\left(s \frac{\omega_L}{\omega_1} + \psi_s\right) \int_o^1 \phi_n' \, \phi_k' \sin(s\pi\xi) d\xi = 0 \quad (2.64c)$$

$$k = 1, 2, \ldots N$$

where ψ_s is defined in Eq. (2.53).

Note that Eq. (2.56) has now been reduced to a set of ordinary linear differential equations as given above. The coordinate q_A can be eliminated between Eqs. (2.64a & 2.64b) without losing any important modes since translation is uncontrolled.

The set of Equations (2.64) may be represented in matrix form as

$$\left\{\ddot{q}_k\right\} + \eta \left\{\dot{q}_k\right\} + \left[F_{jk}\right] \left\{q_k\right\} + \gamma \cos \bar{\Omega}\tau \left[G_{jk}\right] \left\{q_k\right\}$$

$$+ \sum_{i=1}^{\infty} \left[H^i_{jk}\right] \cos \left(i\, \bar{\omega}_L \tau + \psi_i\right) \left\{q_k\right\} = 0 \tag{2.65}$$

where

$$\left\{q_k\right\} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \\ q_B \end{Bmatrix}$$

and $\left[F_{jk}\right]$, $\left[G_{jk}\right]$, & $\left[H^i_{jk}\right]$ are matrices of order N + 1. The elements of these matrices are defined as follows.

$$F_{jk} = \omega_j^2\, \delta_{jk} + \frac{\bar{T}_o}{\lambda_1^4} \Big[\phi_j\, (1)\, \phi_k'\, (1) - \int_o^1 \xi\, \phi_k'\, \phi_k'\, d\xi$$

$$+ K_\theta \phi_j\, (\xi)_1\, \phi_k'\, (\xi)_G \Big] \qquad \begin{aligned} j &= 1,2,\ldots N \\ k &= 1,2,\ldots N \\ \delta_{jk} &= 1 \quad j = k \\ \delta_{jk} &= 0 \quad j \neq k \end{aligned} \tag{2.66}$$

$$F_{j,N+1} = \frac{\overline{T}_o}{\lambda_1^4} K_\theta \; \phi_j(\xi)_1 \qquad j = 1,2,\ldots N \tag{2.67}$$

$$F_{N+1,k} = 12 \frac{\overline{T}_o}{\lambda_1^4} \left[\frac{1}{2} \phi_k' (\xi)_1 - \phi_k (\xi)_1 + \frac{1}{2} K_\theta \phi_k' (\xi)_G \right] \tag{2.68}$$

$$k = 1,2,\ldots N$$

$$F_{N+1,N+1} = \frac{6\overline{T}_o K_\theta}{\lambda_1^4} \tag{2.69}$$

$$G_{jk} = \frac{\overline{T}_o}{\lambda^4} \left[\phi_j (\xi)_1 \phi_k (\xi)_1 - \int_o^1 \Phi\phi_j' \phi_k' d\xi + K_\theta \phi_j (1) \phi_k' (\xi)_G \right] \tag{2.70}$$

$$j = 1,2,\ldots N$$

$$k = 1,2,\ldots N$$

$$G_{j,N+1} = \frac{\overline{T}_o}{\lambda_1^4} \left[\int_o^1 \phi_j \Phi' d\xi + K_\theta \phi_j(1) \right] \tag{2.71}$$

$$j = 1,2,\ldots N$$

$$G_{N+1,k} = 12 \frac{\overline{T}_o}{\lambda_1^4} \left[\frac{1}{2} \phi_k' (\xi)_1 - \phi_k (\xi)_1 + \int_o^1 \phi_k \Phi' d\xi + \frac{1}{2} K_\theta \phi_k' (\xi)_G \right] \qquad k = 1,2,\ldots N \tag{2.72}$$

$$G_{N+1,N+1} = 12 \frac{\overline{T}_o}{\lambda_1^4} \left[\frac{1}{2} - \frac{1 - \cos \pi\overline{\sigma}}{\pi\overline{\sigma} \sin \pi\overline{\sigma}} + \frac{1}{2} K_\theta \right] \tag{2.73}$$

$$H_{jk}^i = \frac{AE}{\overline{T}_o \ell} \qquad E_{jk}^i C_i \qquad j = 1,2,\ldots N+1 \tag{2.74}$$

$$k = 1,2,\ldots N+1$$

$$i = 1,2,\ldots$$

$$E^{i}_{jk} = -\frac{\overline{T}_o}{\lambda^4_1} \int_o^1 \phi'_k \, \phi'_j \sin (i\pi\xi) d\xi \qquad \begin{array}{l} j = 1,2,\ldots N \\ k = 1,2,\ldots N \\ i = 1,2,\ldots \end{array} \tag{2.75}$$

$$E^{i}_{j,N+1} = -\frac{\overline{T}_o}{\lambda^4_1} \int_o^1 \phi'_j \sin (i\pi\xi) d\xi \qquad \begin{array}{l} j = 1,2,\ldots N \\ i = 1,2,\ldots \end{array} \tag{2.76}$$

$$E^{i}_{N+1,k} = 12 \frac{\overline{T}_o}{\lambda^4_1} \pi s \int_o^1 \phi_k (\xi) \cos (i\pi\xi) d\xi \tag{2.77}$$

$$k = 1,2,\ldots N$$

$$i = 1,2,\ldots$$

$$E^{i}_{N+1,N+1} = 12 \frac{\overline{T}_o}{\lambda^4_1} \left[(-1)^i - 1 \right] \frac{1}{\pi i} \qquad i = 1,2,\ldots \tag{2.78}$$

Eq. (2.65) represents a system of ordinary linear differential equations of the damped Mathieu type whose solutions are required.

3.0 METHOD OF SOLUTION

3.1 Form of Equation

The system of equations (2.65) can not be classified as Mathieu's or Hill's equation precisely but their method of solution may be employed. For reference we rewrite Eq. (2.56).

$$\left\{\ddot{q}_k\right\} + \eta \left\{\dot{q}_k\right\} + \left[F_{jk}\right] \left\{q_k\right\} + \gamma \cos \bar{\Omega}\tau \left[G_{jk}\right] \left\{q_k\right\}$$

$$+ \sum_{i=1}^{\infty} \left[H^i_{jk}\right] \cos \left(\pi i \bar{\omega}_L + \psi_i\right) \tau \left\{q_k\right\} = 0$$

It becomes convenient to remove the phase angle ψ_i in the last term of the preceding expression. This is done by recalling that $C_i \cos (n\, \bar{\omega}_L \tau + \psi_i) = A_i \cos n\, \bar{\omega}_L + B_i \cos n\, \bar{\omega}_L \tau$.

The damping term η can best be handled by the change of variable

$$q_k = e^{-\frac{\eta}{2}\tau}\, u_k \tag{3.1}$$

as suggested in Ref. [12] . Proceeding we now write

$$\left\{\ddot{u}_k\right\} - \frac{\eta^2}{4} \left\{u_k\right\} + \left[F_{jk}\right]\left\{u_k\right\} + \gamma \cos \bar{\Omega}\tau \left[G_{jk}\right] \left\{u_k\right\} \tag{3.2}$$

$$+ \sum_{i=1}^{\infty} \left[H^i_{jk}\right] \bar{A}_i \cos i\, \bar{\omega}_L \tau \left\{u_k\right\} + \sum_{i=1}^{\infty} \left[H^i_{jk}\right] \bar{B}_i \cos n\, \bar{\omega}_L \tau \left\{u_k\right\} = 0$$

3.2 Method of Solution

In accordance with [5] we may take a solution of the form

$$\left\{u_k\right\} = e^{i\alpha\bar{\Omega}\tau}\left\{\psi_k(\tau)\right\} \tag{3.3}$$

where

$$\left\{\psi_k\right\} = \sum_{m=-\infty}^{\infty}\left\{c_k\right\}^{(m)} e^{im\bar{\Omega}\tau} \tag{3.4}$$

with a period of $2\pi/\bar{\Omega}$.

Eq. (3.3) becomes

$$\left\{u_k\right\} = \sum_{m=-\infty}^{\infty}\left\{c_k\right\}^{(m)} e^{i(\alpha + m)} \tag{3.5}$$

We have introduced a new constant α which may be real, imaginary or complex, its type determining the stability of the system. See Section 3.3.

Let us now express the forcing frequency and the longitudinal natural frequency as a ratio

$$\frac{\bar{\Omega}}{\bar{\omega}_L} = \frac{p}{q} \tag{3.6}$$

where p & q are integers but restricted to non-interger values of p/q. Therefore $\bar{\omega}_L$ & $\bar{\Omega}$ may be written as

$$\bar{\Omega} = p\omega \tag{3.6a}$$

$$\bar{\omega}_L = q\omega \tag{3.6b}$$

where ω is the beat frequency.

Substitution of Eqs. (3.5 & 3.6) into Eqs. (3.2) yields

$$- \omega^2 \sum_{m=-\infty}^{\infty} \left\{c_k\right\}^{(m)} (\alpha + m)^2 e^{i(\alpha + m)\omega\tau}$$

$$+ \left[\left[F_{jk}\right] - \frac{\eta^2}{4}[I]\right] \sum_{m=-\infty}^{\infty} \left\{c_k\right\}^{(m)} e^{i(\alpha + m)\omega\tau}$$

$$+ \frac{\gamma}{2} \left[G_{jk}\right] \sum_{m=-\infty}^{\infty} \left\{c_k\right\}^{(m)} e^{i(\alpha + m + p)\omega\tau}$$

$$+ \frac{\gamma}{2} \left[G_{jk}\right] \sum_{m=-\infty}^{\infty} \left\{c_k\right\}^{(m)} e^{i(\alpha + m - p)\omega\tau}$$

$$+ \frac{1}{2} \sum_{s=1}^{\infty} \left[H^s_{jk}\right] A_s \sum_{m=-\infty}^{\infty} \left\{c_k\right\}^{(m)} e^{i(\alpha + m + sq)\omega\tau}$$

$$+ \frac{1}{2} \sum_{s=1}^{\infty} \left[H^s_{jk}\right] A_s \sum_{m=-\infty}^{\infty} \left\{c_k\right\}^{(m)} e^{i(\alpha + m - sq)\omega\tau}$$

$$- \frac{i}{2} \sum_{s=1}^{\infty} \left[H^s_{jk}\right] B_s \sum_{m=-\infty}^{\infty} \left\{c_k\right\}^{(m)} e^{i(\alpha + m + sq)\omega\tau}$$

$$+ \frac{i}{2} \sum_{s=1}^{\infty} \left[H^s_{jk}\right] B_s \sum_{m=-\infty}^{\infty} \left\{c_k\right\}^{(m)} e^{i(\alpha + m - sq)\omega\tau} = 0 \qquad (3.7)$$

Equation (3.7) is satisfied for all τ if collected coefficients of like exponentials are equal to zero, or in equation form

$$- (\alpha + m)^2 \left\{c_k\right\}^{(m)} + \frac{1}{\omega^2}\left[\left[F_{jk}\right] - \frac{\eta^2}{4}\left[I\right]\right]\left\{c_k\right\}^{(m)}$$

$$+ \frac{\gamma}{2\omega^2}\left[G_{jk}\right]\left\{c_k\right\}^{(m-p)} + \frac{\gamma}{2\omega^2}\left[G_{jk}\right]\left\{c_k\right\}^{(m+p)}$$

$$+ \frac{1}{\omega^2}\sum_{s=1}^{\infty}\left[H^s_{jk}\right](A_s - iB_s)\left\{c_k\right\}^{(m-sq)}$$

$$+ \frac{1}{\omega^2}\sum_{s=1}^{\infty}\left[H^s_{jk}\right](A_s + iB_s)\left\{c_k\right\}^{(m+sq)} = 0 \qquad (3.8)$$

$$m = \ldots -3, -2, -1, 0, 1, 2, 3 \ldots$$

$$j = 1,2,3,\ldots N+1$$

$$k = 1,2,3,\ldots N+1$$

The system of equations represented by Eq. (3.8) can be expanded into a single matrix equation as shown on the following page. To ensure that the determinant of the matrix of coefficients is absolutely convergent Eq. (3.8) is first divided by the factor

$$\frac{1}{\omega^2}(F_{kk} - \frac{\eta^2}{4}) - (\alpha + m)^2.$$

The importance of absolute convergence becomes evident as the development of the method of solution proceeds.

Eq. (3.9) below is the matrix equivalent of the system of equations given by Eq. (3.8). As shown the index m ranges from -1 through +1. There is no upper limit except computer capacity but we can choose a finite value of m, say M which will give adequate convergence.

$$\begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & D_{-1,-1} & \frac{1}{\omega^2} D_{-1,0} & \frac{1}{\omega^2} D_{-1,1} & \cdots \\ \cdots & \frac{1}{\omega^2} D_{0,-1} & D_{0,0} & \frac{1}{\omega^2} D_{0,1} & \cdots \\ \cdots & \frac{1}{\omega^2} D_{1,-1} & \frac{1}{\omega^2} D_{1,0} & D_{1,1} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix} \begin{Bmatrix} \vdots \\ C_k^{(-1)} \\ C_k^{(0)} \\ C_k^{(1)} \\ \vdots \end{Bmatrix} = 0 \qquad (3.9)$$

$$k = 1,2,3,\ldots N+1$$

We here define $\Delta(\alpha)$ as the determinant of the matrix of coefficients and $D_{j,k}$ as square arrays of elements within $\Delta(\alpha)$. The arrays $D_{j,k}$ are defined by the following equation.

$$D_{m,m} = \begin{vmatrix} 1 & \dfrac{F_{12}}{\bar{F}_{11}-(\alpha + m)^2} & \cdots & \dfrac{F_{1,N+1}}{\bar{F}_{11}-(\alpha + m)^2} \\ \dfrac{F_{21}}{\bar{F}_{22}-(\alpha + m)^2} & 1 & \cdots & \dfrac{F_{2,N+1}}{\bar{F}_{22}-(\alpha + m)^2} \\ \vdots & \vdots & & \vdots \\ \dfrac{F_{N+1,1}}{\bar{F}_{33}-(\alpha + m)^2} & \dfrac{F_{N+1,2}}{\bar{F}_{33}-(\alpha + m)^2} & \cdots & 1 \end{vmatrix}$$

$$D_{m,m\pm p} = \begin{vmatrix} \dfrac{G_{11}(\gamma/2)}{\bar{F}_{11}-(\alpha + m)^2} & \dfrac{G_{12}(\gamma/2)}{\bar{F}_{11}-(\alpha + m)^2} & \cdots & \dfrac{G_{1,N+1}(\gamma/2)}{\bar{F}_{11}-(\alpha + m)^2} \\ \dfrac{G_{21}(\gamma/2)}{\bar{F}_{22}-(\alpha + m)^2} & \dfrac{G_{22}(\gamma/2)}{\bar{F}_{22}-(\alpha + m)^2} & \cdots & \dfrac{G_{2,N+1}(\gamma/2)}{\bar{F}_{22}-(\alpha + m)^2} \\ \vdots & \vdots & & \vdots \\ \dfrac{G_{N+1,1}(\gamma/2)}{\bar{F}_{33}-(\alpha + m)^2} & \dfrac{G_{N+1,2}(\gamma/2)}{\bar{F}_{33}-(\alpha + m)^2} & \cdots & \dfrac{G_{N+1,N+1}(\gamma/2)}{\bar{F}_{33}-(\alpha + m)^2} \end{vmatrix} \qquad (3.10)$$

$$D_{m,m+sq} = \begin{vmatrix} \dfrac{H^s_{11}(A_s + iB_s)}{\bar{F}_{11}-(\alpha + m)^2} & \dfrac{H^s_{12}(A_s + iB_s)}{\bar{F}_{11}-(\alpha + m)^2} & \cdots & \dfrac{H^s_{1,N+1}(A_s + iB_s)}{\bar{F}_{11}-(\alpha + m)^2} \\ \dfrac{H^s_{21}(A_s + iB_s)}{\bar{F}_{22}-(\alpha + m)^2} & \dfrac{H^s_{22}(A_s + iB_s)}{\bar{F}_{s}-(\alpha + m)^2} & \cdots & \dfrac{H^s_{2,N+1}(A_s + iB_s)}{\bar{F}_{22}-(\alpha + m)^2} \\ \vdots & \vdots & & \vdots \\ \dfrac{H^s_{N+1,1}(A_s + iB_s)}{\bar{F}_{33}-(\alpha + m)^2} & \dfrac{H^s_{N+L,2}(A_s + iB_s)}{\bar{F}_{33}-(\alpha + m)^2} & \cdots & \dfrac{H^s_{N+1,N+1}(A_s + iB_s)}{\bar{F}_{33}-(\alpha + m)^2} \end{vmatrix}$$

$s = 1,2,3,\ldots$

$$D_{m,m-sq} = \begin{vmatrix} \dfrac{H^s_{11}(A_s - iB_s)}{\bar{F}_{11}-(\alpha+m)^2} & \dfrac{H^s_{12}(A_s - iB_s)}{\bar{F}_{11}-(\alpha+m)^2} & \cdots & \dfrac{H^s_{1,N+1}(A_s - iB_s)}{\bar{F}_{11}-(\alpha+m)^2} \\ \\ \dfrac{H^s_{21}(A_s - iB_s)}{\bar{F}_{22}-(\alpha+m)^2} & \dfrac{H^s_{22}(A_s - iB_s)}{\bar{F}_{22}-(\alpha+m)^2} & \cdots & \dfrac{H^s_{N+1,2}(A_s - iB_s)}{\bar{F}_{22}-(\alpha+m)^2} \\ \vdots & \vdots & & \vdots \\ \dfrac{H^s_{N+1,1}(A_s - iB_s)}{\bar{F}_{33}-(\alpha+m)^2} & \dfrac{H^s_{N+1,2}(A_s - iB_s)}{\bar{F}_{33}-(\alpha+m)^2} & \cdots & \dfrac{H^s_{N+1,N+1}(A_s - iB_s)}{\bar{F}_{33}-(\alpha+m)^2} \end{vmatrix}$$

$$s = 1,2,3,\ldots$$

where

$$\bar{F}_{jj} = \frac{1}{\omega^2}\left(F_{jj} - \frac{\eta^2}{4}\right) \tag{3.11}$$

j corresponding to the row within each array.

The system of equations represented by the matrix Eq. (3.9) can have a non trivial solution only if the determinant of the matrix of coefficients equals zero, that is

$$\Delta(\alpha) = 0 \tag{3.12}$$

where only the values of α which satisfy Eq. (3.12) are permitted. Our problem then is to solve Eq. (3.12) for α.

To ensure the validity of our solution the infinite determinant represented by $\Delta(\alpha)$ must converge absolutely. It has been proven in Ref. [2] that an infinite determinant converges if the product of the diagonal elements and the sum of the non-diagonal elements are absolutely convergent.

The product of the diagonal elements is identically equal to 1 as seen from Eq. (3.10) since all elements along the diagonal are equal to 1. This condition was deliberately obtained by dividing Eq. (3.8) by the appropriate factor.

The sum of the non-diagonal elements of $\Delta(\alpha)$ is

$$S = \frac{1}{\omega^2} \sum_{\substack{j=1 \\ j\neq k}}^{N+1} \sum_{k=1}^{N+1} \sum_{m=-\infty}^{\infty} \frac{F_{jk} + \gamma/2\, G_{jk} + H^s_{jk}(A_s - iB_s)}{\overline{F}_{jj} - (\alpha + m)^2}$$

$$+ \frac{1}{\omega^2} \sum_{j=1}^{N+1} \sum_{m=-\infty}^{\infty} \frac{\gamma/2\, G_{jj} + H^s_{jj}(A_s - iB_s)}{\overline{F}_{jj} - (\alpha + m)^2} \tag{3.12}$$

which is convergent by comparison with the series $\sum_{n=-\infty}^{\infty} \frac{1}{n^2}$. We may therefore conclude that $\Delta\alpha$ is convergent for all α except where the factor $\overline{F}_{jj} - (\alpha + n)^2$ is equal to zero.

3.2.1 Determination of the Characteristic Values

We have previously defined the characteristic determinant $\Delta(\alpha)$ in terms of arrays $D_{j,k}$. Each array is defined in terms of its corresponding elements in Eq. (3.10). Referring now to the elements of Eq. (3.10) and keeping in mind their position as elements in $\Delta(\alpha)$ we may deduce the following pertinent points.

1. Writing $-\alpha$ for α and $-m$ for m does not alter $(\alpha+m)^2$. Transposition of the elements containing $+m$ and $-m$ leaves $\Delta(\alpha)$ unaltered since m assumes all integral values from $-\infty$ to ∞. Therefore $\Delta(\alpha) = \Delta(-\alpha)$ so that $\Delta(\alpha)$ is an even function of α.

2. $\left[(\alpha+1)+m\right]^2 = \left[\alpha + (m+1)\right]^2$, therefore replacing $m+1$ by m gives $(\alpha+m)^2$. Hence $\Delta(\alpha) = \Delta(\alpha+1)$ proving that $\Delta\alpha$ has period 1.

3. Singularities exist only when α satisfies $(\alpha+m)^2 - \bar{F}_{jj} = 0$. That is when $\alpha = \pm(\bar{F}_{jj})^{\frac{1}{2}} - m$. We exclude the possibility that $(\bar{F}_{jj})^{\frac{1}{2}} = m$ therefore there is no singularity at $\alpha = 0$. We now define a function $\rho_j(\alpha) = \dfrac{1}{\cos 2\alpha\pi - \cos \pi \bar{F}_{jj}^{\frac{1}{2}}}$ that has the singularities and period of $\Delta(\alpha)$.

From 3 it follows that

$$H(\alpha) = \Delta(\alpha) - \sum_{j=1}^{N+1} K_j \rho_j(\alpha)$$

will have no singularities if the constants K_j are suitably chosen. We may now employ Liouville's theorem to conclude that $H(\alpha)$ must be a constant. By letting $\alpha \to i\infty$, $H(\alpha)$ may be evaluated as

$$H(\alpha) = \lim_{\alpha\to i\infty} \Delta(\alpha) - \sum_{j=1}^{N+1} K_j \rho_j(\alpha) = 1$$

since all the elements off the diagonal of $\Delta(\alpha)$ tend to zero as α approaches $i\infty$.

The constants K_1, K_2, K_3, ... K_{n+1} are evaluated by allowing α to approach $(\bar{F}_{11})^{\frac{1}{2}}$, $(\bar{F}_{22})^{\frac{1}{2}}$, ... $(\bar{F}_{n+1,\ N+1})^{\frac{1}{2}}$ respectively. Letting $\alpha = (\bar{F}_{jj})^{\frac{1}{2}} + \varepsilon$,

$$K_j = -2\pi \sin 2\pi(\bar{F}_{jj})^{\frac{1}{2}} \lim_{\varepsilon\to 0} \varepsilon\Delta\,(\bar{F}_{jj}{}^{\frac{1}{2}} + \varepsilon) \tag{3.13}$$

To evaluate Eq.(3.13) it becomes necessary to define a new determinant

$$\Delta_j(\alpha) = (-\alpha^2 + \bar{F}_{jj})\Delta(\alpha) \tag{3.14}$$

This new determinant does not have a singularity at $\alpha = (\bar{F}_{jj})^{\frac{1}{2}}$. Substitution of Eq. (3.14) into Eq. (3.13) and taking the limit yields

$$K_j = \frac{\pi \sin 2\pi(\bar{F}_{jj})^{\frac{1}{2}}}{\bar{F}_{jj}} \Delta_j(\bar{F}_{jj}{}^{\frac{1}{2}}) \tag{3.15}$$

We may now write the characteristic equation as

$$1 - \sum_{j=1}^{N+1} \frac{K_j}{\xi_j - z} = 0 \tag{3.16}$$

where

$$\xi_j = \cos 2\pi(\bar{F}_{jj})^{\frac{1}{2}} \tag{3.17a}$$

$$z = \cos 2\pi\alpha \tag{3.17b}$$

3.3 Stability of Solution

The characteristic values α of the solutions

$$u_k = e^{i\alpha\omega\tau} \Psi_k \qquad k = 1,2,\ldots N+1$$

are evaluated from Eq. (3.17b) as

$$\alpha = -\frac{i}{2} \ln\left(z \pm \sqrt{z^2 - 1} \right) \tag{3.18}$$

Solutions to our original differential equation may now be obtained from the relationship

$$\left\{q_k\right\} = e^{-\frac{1}{2}\eta\tau} \left\{u_k\right\}$$

of Eq. (3.1). q_k may now be written in terms of the characteristic values of u_k.

$$\left\{q_k\right\} = e^{-\frac{1}{2}\eta\tau} \cdot e^{i\alpha\omega\tau} \left\{\Psi_k\right\} \tag{3.19a}$$

$$\left\{q_k\right\} = e^{\tau(i\alpha\omega - \frac{1}{2}\eta)} \left\{\Psi_k\right\} \tag{3.19b}$$

We now establish the stability criteria governing the set of solutions of Eq. (3.19b)

1. A solution is defined as unstable if q_k tends to $\pm\infty$ as τ approaches $+\infty$.

2. A solution is defined to be stable if q_k tends to zero or remains bounded as τ approaches $+\infty$.

3. A solution with period 2π is neutral but q_k may be regarded as a special case of a stable solution.

The elements in the column matrix Ψ_k have a period $\frac{2\pi}{\omega}$ which is neutral, therefore stability depends upon $e^{\tau(i\alpha\omega - \frac{1}{2}\eta)}$ or more specifically upon the relative values of η & α. The term $(i\alpha\omega - \frac{1}{2}\eta)$ may in general have any real, imaginary or complex value depending upon α & η.

α is determined as a function of the computed z value from Eq. (3.18). z may be real or complex so for convenience Eq. (3.18) is written as

$$\alpha = - \frac{i}{2\pi} \ln (Re^{i\beta}) \tag{3.18a}$$

$$\alpha = - \frac{i}{2\pi} \ln R + \frac{\beta}{2\pi} \tag{3.18b}$$

where R is the modulus of $z \pm \sqrt{z^2 - 1}$ and β the argument. Then

$$\left\{q_k\right\} = e^{\tau(\frac{\omega}{2\pi} \ln R - \frac{\eta}{2})} \cdot e^{(\frac{i\beta}{2\pi})} \left\{\Psi_k\right\} . \tag{3.19c}$$

From the criteria established a <u>stable solution requires that</u>

$$\eta \geq \frac{\omega}{\pi} \ln R \tag{3.20}$$

or for zero damping that

$$\frac{\omega}{2\pi} \ln R \leq 0 \tag{3.21}$$

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PART II

RESPONSE OF A CIRCULAR CYLINDRICAL SHELL SUBJECTED TO A GIMBALED PERIODICALLY VARYING END THRUST

TABLE OF CONTENTS

LIST OF SYMBOLS

L	- Half length of cylindrical shell
r	- Radius of cylindrical shell
h	- Thickness of cylindrical shell
ρ	- Mass per unit volume of shell material
E	- Young's modulus of shell material
υ	- Poisson's ratio of shell material
g	- Acceleration of gravity
M	- Total mass of the shell
I	- Mass moment of inertia of shell about a line through the center of mass perpendicular to elements of the cylinder
ω	- Any convenient reference frequency
x	- Axial shell coordinate
θ	- Tangential shell coordinate
t	- Time
(X, Y)	- Coordinates of the center of mass
$\alpha\ (t)$	- Rotation of a line element through the center of mass
$\xi = x/L$	- Dimensionless axial shell coordinate
$\tau = \omega t$	- Dimensionless time
$\tau_1 = \frac{\Omega t}{2}$	- Dimensionless time
$X_o = X/L$	- Dimensionless coordinate of center of mass
$Y_o = Y/L$	- Dimensionless coordinate of center of mass
$\sigma = h/L$	- Dimensionless thickness parameter
$\lambda = L/r$	- Dimensionless radius parameter

$\mu = \rho L^2 \omega^2/E$	- Dimensionless parameter
T_o	- Magnitude of steady state thrust per unit of length applied around the bottom of shell
γ	- Ratio of the magnitude of the sinusoidal time varying thrust per unit length to T_o
Ω	- Circular frequency of the sinusoidal component of the applied thrust
K	- Directional control factor determining the direction of the thrust
$a = 8\pi r L T_o K/I\Omega^2$	- Dimensionless parameter of the Mathieu equation that governs $\alpha(t)$
$q = \gamma a/2$	- Dimensionless parameter of the Mathieu equation that governs $\alpha(t)$
β	- Stability parameter of the solution for $\alpha(t)$
N_x, N_θ, $N_{x\theta}$, $N_{\theta x}$	- Stress resultants in shell
M_x, M_θ, $M_{x\theta}$, $M_{\theta x}$	- Moment resultants in shell
Q_x, Q_θ	- Shear resultants in shell
P_x, P_θ, P_z	- Axial, circumferential and normal components of surface force applied to shell
a_x, a_θ, a_z	- Axial, circumferential and normal components of acceleration of shell element
u, v, w	- Axial, circumferential and normal components of shell relative to moving reference frame
$\bar{u} = u/L$, $\bar{v} = v/L$, $\bar{w} = w/L$	- Dimensionless components of displacement of shell element
$U_{mn}(\tau)$, $V_{mn}(\tau)$, $W_{mn}(\tau)$	- Generalized coordinates of $\bar{u}$, $\bar{v}$, $\bar{w}$
u_{mn}, v_{mn}, w_{mn}	- Initial values of $U_{mn}(\tau)$, $V_{mn}(\tau)$, and $W_{mn}(\tau)$
α_o	- Initial value of $\alpha(t)$

δ_{jk}	- Kronecker delta
$(\dot{\ })$	- Differentiation with respect to t, τ or τ_1
$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$	- Laplacian in the shell coordinates
$\bar{\nabla}^2 = \frac{\partial^2}{\partial \xi^2} + \lambda^2 \frac{\partial^2}{\partial \theta^2}$	- Laplacian in dimensionless shell coordinates

I. INTRODUCTION

The dynamic response of a large rocket booster during its powered flight is investigated by studying a simplified model of the booster. In this analysis, a circular cylindrical shell subjected to a gimbaled time-varying end thrust is accepted as a model of the booster in flight. The direction of the gimbaled end thrust is controlled by a simple proportional feedback system as indicated in Figure 1. The time variation of the end thrust is assumed to be sinusoidal about some average thrust. This model will demonstrate, at least qualitatively, the structural performance of a large booster.

The formulation contained in this technical memo was mainly the effort of Dr. James Hill with other team members participating in the work from time to time. This report represents the progress to date of this portion of the contract work. The formulation enclosed is presently being programmed for the IBM 7094 computer. Upon completion of programming and checkout a full parameter study will be initiated.

II. EQUATIONS OF MOTION

The motion of the shell will be described in a moving coordinate system as indicated in Figure 1. The coordinate system is free to move in a plane with the origin always at the center of mass of the shell.

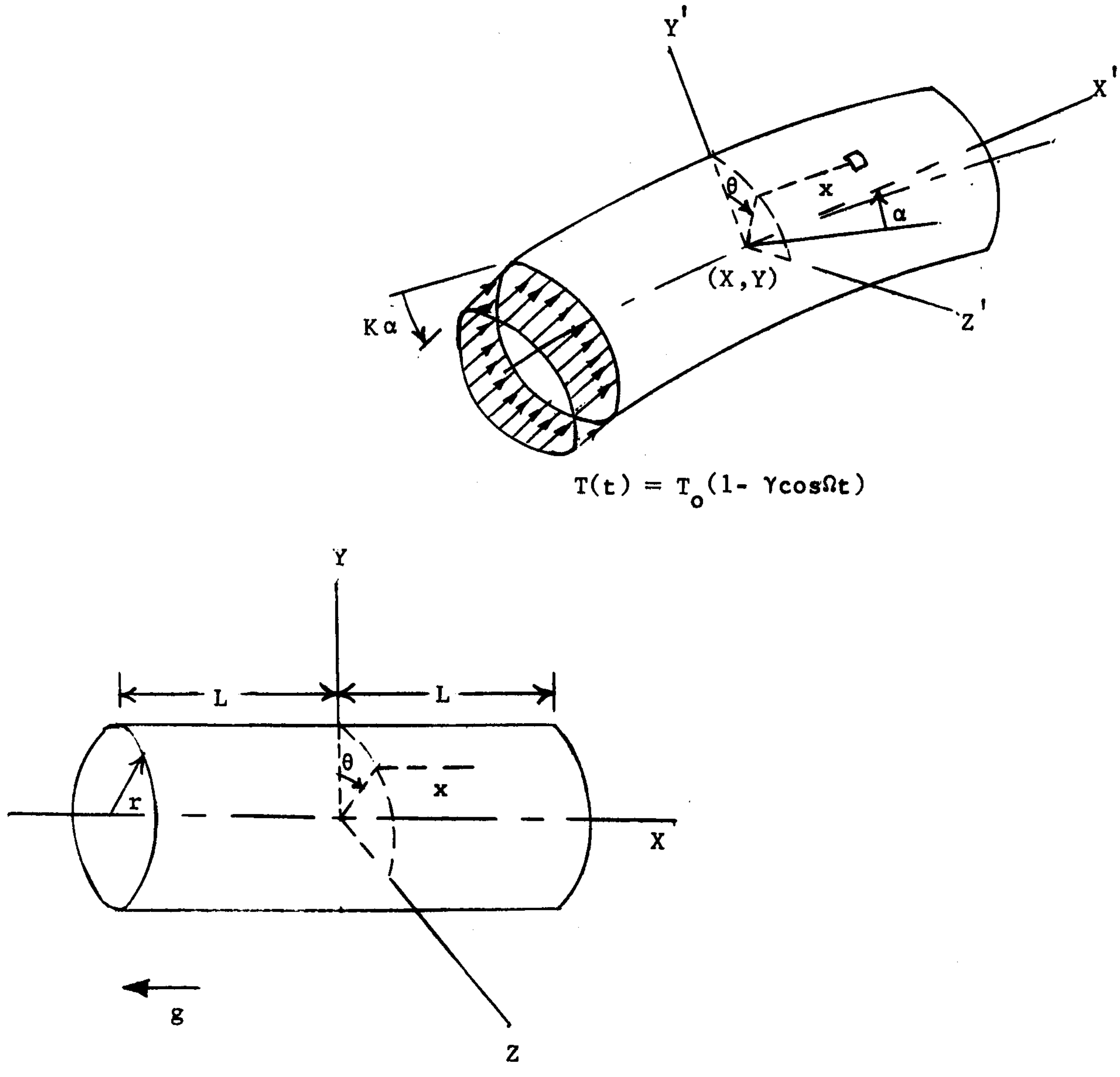

Figure 1

Neglecting the rotary inertia of the shell, the equations of motion of the shell element are (1)*

$$\frac{\partial N_x}{\partial x} + \frac{1}{r}\frac{\partial N_{\theta x}}{\partial \theta} + P_x = \rho h a_x \qquad \text{(a)}$$

$$\frac{\partial N_{x\theta}}{\partial x} + \frac{1}{r}\frac{\partial N_\theta}{\partial \theta} - \frac{1}{r} Q_\theta + P_\theta = \rho h a_\theta \qquad \text{(b)}$$

$$\frac{\partial Q_x}{\partial x} + \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} + \frac{1}{r} N_\theta + P_z = \rho h a_z \qquad \text{(c)}$$

$$\frac{\partial M_x}{\partial x} + \frac{1}{r}\frac{\partial M_{\theta x}}{\partial \theta} - Q_x = 0 \qquad \text{(d)} \qquad (1)$$

$$\frac{\partial M_{x\theta}}{\partial x} - \frac{1}{r}\frac{\partial M_\theta}{\partial \theta} + Q_\theta = 0 \qquad \text{(e)}$$

$$N_{\theta x} - N_{x\theta} = \frac{1}{r} M_{\theta x} \qquad \text{(f)}$$

where ρ is the mass density, h the thickness of the shell, a_x, a_θ, a_z are the components of the acceleration of the element in the axial, circumferential and radial directions, respectively. The radial coordinate z is positive inward. N_x, N_θ, $N_{x\theta}$, $N_{\theta x}$, Q_x, Q_θ, M_x, M_θ, $M_{x\theta}$, and $M_{\theta x}$ are the standard stress resultants. P_x, P_θ and P_z are the components of the

* Numbers in parentheses except where accompanied by the abbreviation eq. refer to references in the Bibliography.

distributed surface force in the axial, circumferential and radial directions, respectively.

Since the moving coordinate system is located at the center of mass of the shell we can determine X, Y, and α by Newton's Laws of plane motion

$$M\ddot{X} = 2\pi r\ T(t)\ \cos(K + 1)\alpha - Mg \qquad \text{(a)}$$

$$M\ddot{Y} = 2\pi r\ T(t)\ \sin(K + 1)\alpha \qquad \text{(b)} \qquad (2)$$

$$I\ddot{\alpha} = -2\pi r\ L\ T(t)\ \sin K\alpha \qquad \text{(c)}$$

Assuming that the deformation does not change the half length nor the mass moment of inertia.

Now we need expressions for (a_x, a_θ, a_z) in terms of (u, v, w) the shell displacements and (X, Y, α) coordinates of the reference frame. Kinematics of a point in a moving reference frame yields

$$a_x = \frac{\partial^2 u}{\partial t^2} + \ddot{X}\cos\alpha + \ddot{Y}\sin\alpha - r\cos\theta\ddot{\alpha} - X\dot{\alpha}^2 + \underline{2\dot{\alpha}\left(\frac{\partial v}{\partial t}\sin\theta + \frac{\partial w}{\partial t}\cos\theta\right)} \qquad \text{(a)}$$

$$a_\theta = \frac{\partial^2 v}{\partial t^2} + (r\cos\theta\dot{\alpha}^2 - X\ddot{\alpha} - \ddot{Y}\cos\alpha + \ddot{X}\sin\alpha)\sin\theta - \underline{2\dot{\alpha}\frac{\partial u}{\partial t}\sin\theta} \qquad \text{(b)} \qquad (3)$$

$$a_z = \frac{\partial^2 w}{\partial t^2} + (r\cos\theta\dot{\alpha}^2 - X\ddot{\alpha} - \ddot{Y}\cos\alpha + \ddot{X}\sin\alpha)\cos\theta - \underline{2\dot{\alpha}\frac{\partial u}{\partial t}\cos\theta} \qquad \text{(c)}$$

The underlined terms are Coriolis terms and will be neglected in the subsequent analysis.

Equations (2) can be solved for X, Y, and α. These are substituted into eqs. (3) to determine expressions for a_x, a_θ, and a_z on the right side of eqs. (1).

III. EQUATIONS OF MOTION IN TERMS OF SHELL DISPLACEMENT

In eq. (1b) the term $\frac{1}{r} Q_\theta$ will be neglected. It is also assumed that $N_{\theta x} = N_{x\theta}$ but not that $M_{\theta x}$ vanishes. Eliminating Q_x and Q_θ from eq. (1c) by eqs. (1d) and (1e) we arrive at

$$\frac{\partial N_x}{\partial x} + \frac{1}{r}\frac{\partial N_{\theta x}}{\partial \theta} + P_x = \rho h \frac{\partial^2 u}{\partial t^2} + f_1(x,\theta,t) \qquad \text{(a)}$$

$$\frac{\partial N_{x\theta}}{\partial x} + \frac{1}{r}\frac{\partial N_\theta}{\partial \theta} + P_\theta = \rho h \frac{\partial^2 v}{\partial t^2} + f_2(x,\theta,t) \qquad (4) \qquad \text{(b)}$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{1}{r}\frac{\partial^2 M_{\theta x}}{\partial x \partial \theta} - \frac{1}{r}\frac{\partial^2 M_{x\theta}}{\partial x \partial \theta} + \frac{1}{r^2}\frac{\partial^2 M_\theta}{\partial \theta^2} + \frac{1}{r} N_\theta + P_z = \rho h \frac{\partial^2 w}{\partial t^2} + f_3(x,\theta,t) \qquad \text{(c)}$$

where

$$f_1(x,\theta,t) = \rho h \,(\ddot{X} \cos\alpha + \ddot{Y} \sin\alpha - r \cos\theta \ddot{\alpha} - x\dot{\alpha}^2) \qquad \text{(a)}$$

$$f_2(x,\theta,t) = \rho h \,(\ddot{X} \sin\alpha - \ddot{Y} \cos\alpha + r \cos\theta \dot{\alpha}^2 - x\ddot{\alpha}) \sin\theta \qquad (5) \qquad \text{(b)}$$

$$f_3(x,\theta,t) = \rho h \,(\ddot{X} \sin\alpha - \ddot{Y} \cos\alpha + r \cos\theta \dot{\alpha}^2 - x\ddot{\alpha}) \cos\theta \qquad \text{(c)}$$

The following approximate relations between the stress resultants and the displacements will be used to write eqs. (4) in terms of the displacements u, v, w. The approximations entailed in these expressions are to neglect higher order terms in "h" and terms containing $\frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{1}{r}\frac{\partial v}{\partial x}$.

$$
\begin{aligned}
N_x &= \frac{Eh}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \frac{\nu}{r}\left(\frac{\partial v}{\partial \theta} - w\right)\right] \\
N_\theta &= \frac{Eh}{1-\nu^2}\left[\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{1}{r} w + \nu \frac{\partial u}{\partial x}\right] \\
N_{x\theta} &= N_{\theta x} = Gh\left[\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x}\right] \\
M_x &= -D\left[\frac{\partial^2 w}{\partial x^2} + \frac{\nu}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right] \\
M_\theta &= -D\left[\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \nu \frac{\partial^2 w}{\partial x^2}\right] \\
M_{x\theta} &= -M_{\theta x} = D\,\frac{(1-\nu)}{r}\,\frac{\partial^2 w}{\partial x \partial \theta}
\end{aligned} \tag{6}
$$

Substitution of eqs. (6) into eqs (4) yields

$$
\frac{\partial^2 u}{\partial x^2} + \frac{(1-\nu)}{2r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{(1+\nu)}{2r}\frac{\partial^2 v}{\partial x \partial \theta} - \frac{\nu}{r}\frac{\partial w}{\partial x} = \frac{(1-\nu^2)}{E}\,\rho\,\frac{\partial^2 u}{\partial t^2} + F_1(x,\theta,t) \qquad \text{(a)}
$$

$$
\frac{(1+\nu)}{2r}\frac{\partial^2 u}{\partial x \partial \theta} + \frac{(1-\nu)}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} - \frac{1}{r^2}\frac{\partial w}{\partial \theta} = \frac{(1-\nu^2)}{E}\,\rho\,\frac{\partial^2 v}{\partial t^2} + F_2(x,\theta,t) \qquad \text{(b)} \qquad (7)
$$

$$
\frac{\nu}{r}\frac{\partial u}{\partial x} + \frac{1}{r^2}\frac{\partial v}{\partial \theta} - \frac{1}{r^2} w - \frac{h^2}{12}\nabla^4 w = \frac{(1-\nu^2)}{E}\,\rho\,\frac{\partial^2 w}{\partial t^2} + F_3(x,\theta,t) \qquad \text{(c)}
$$

where ν is Poisson's ratio, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$, and

$$F_1(x,\theta,t) = \frac{1-\nu^2}{Eh}(f_1 - P_x), \tag{a}$$

$$F_2(x,\theta,t) = \frac{1-\nu^2}{Eh}(f_2 - P_\theta), \qquad (8) \tag{b}$$

$$F_3(x,\theta,t) = \frac{1-\nu^2}{Eh}(f_3 - P_z) \tag{c}$$

Eqs. (7) are rendered dimensionless by the transformations

$$\begin{array}{lll} \xi = x/L, & \tau = \omega t, & \lambda = L/r, \\ \bar{u} = u/L, & \bar{v} = v/L, & \bar{w} = w/L, \\ G_1 = LF_1, & G_2 = LF_2, & G_3 = LF_3, \\ \sigma = h/L, & \mu = \rho\frac{L^2\omega^2}{E}. & \end{array} \tag{9}$$

Eqs. (7) became

$$\frac{\partial^2\bar{u}}{\partial\xi^2} + \frac{(1-\nu)}{2}\lambda^2\frac{\partial^2\bar{u}}{\partial\theta^2} + \frac{(1+\nu)}{2}\lambda\frac{\partial^2\bar{v}}{\partial\xi\partial\theta} - \nu\lambda\frac{\partial\bar{w}}{\partial\xi} = (1-\nu^2)\mu\frac{\partial^2\bar{u}}{\partial\tau^2} + G_1(\xi,\theta,\tau) \tag{10}$$

$$\frac{(1+\nu)}{2}\lambda\frac{\partial^2\bar{u}}{\partial\xi\partial\theta} + \frac{(1-\nu)}{2}\frac{\partial^2\bar{v}}{\partial\xi^2} + \lambda^2\frac{\partial^2\bar{v}}{\partial\theta^2} - \lambda^2\frac{\partial\bar{w}}{\partial\theta} = (1-\nu^2)\mu\frac{\partial^2\bar{v}}{\partial\tau^2} + G_2(\xi,\theta,\tau) \tag{11}$$

$$\nu\lambda\frac{\partial\bar{u}}{\partial\xi} + \lambda^2\frac{\partial\bar{v}}{\partial\theta} - \lambda^2\bar{w} - \frac{\sigma^2}{12}\bar{\nabla}^4\bar{w} = (1-\nu^2)\mu\frac{\partial^2\bar{w}}{\partial\tau^2} + G_3(\xi,\theta,\tau) \tag{12}$$

where

$$\bar{\nabla}^2 = \frac{\partial^2}{\partial\xi^2} + \lambda^2\frac{\partial^2}{\partial\theta^2} \tag{13}$$

Before proceeding with the treatment of eqs. (10, 11, 12) we will determine $\alpha(t)$ from eq. (2c) then eqs. (2a, 2b, 5, and 8) will yield the non homogeneous terms G_1, G_2, and G_3.

IV. SOLUTION OF EQUATION (2c) TO DETERMINE $\alpha(t)$

Knowing $\alpha(t)$, which is the solution of equation (2c), equations (2a) and (2b) render $\ddot{X}$ and $\ddot{Y}$. These functions of time are all that are required to specify f_1, f_2, and f_3 by eqs. (5). Rewriting eq. (2c) we have

$$\frac{d^2\alpha}{dt^2} + \frac{2\pi rLT_o}{I}(1 - \gamma\cos\Omega t)\sin K\alpha = 0 \tag{14}$$

we will restrict α such that $\sin K\alpha \approx K\alpha$ thus

$$\frac{d^2\alpha}{dt^2} + \frac{2\pi rLT_o}{I}K(1 - \gamma\cos\Omega t)\alpha = 0$$

changing to a dimensionless time such that $\tau_1 = \frac{\Omega}{2}t$ and introducing the parameters

$$a = \frac{8\pi rLT_oK}{I\Omega^2}, \qquad q = \frac{\gamma a}{2}, \tag{15}$$

the equation becomes

$$\frac{d^2\alpha}{d\tau_1^{\,2}} + (a - 2q\cos 2\tau_1)\alpha = 0 \tag{16}$$

Eq. (16) is a Mathieu equation and the theory for its solution can be found in several standard references (2), (3). For cylindrical shells that represent large boosters and the magnitude of the thrust they encounter we can restrict the range of a and q to

$$0<a<1 \qquad 0<q\leq 0.05 \tag{17}$$

The nature of the solution of equation (16) greatly depends upon the parameters a and q. For certain values of a and q the solution is periodic with period 2π, for others it is periodic with period π and for other parameters the solution becomes unbounded for large values of τ_1 (unstable) while for even other values the solution remains bounded (stable). A chart of the regions of stability in the (a,q) plane is given in McLachlan's, Theory and Application of Mathieu Functions on page 40. For the range of a and q given by equation (17) the solution is stable and may be taken as

$$\alpha(\tau_1) = e^{i\beta\tau_1} \sum_{r=-\infty}^{\infty} C_{2r} \, e^{2r\tau_1 i} \tag{18}$$

The two independent solutions of eq. (16) are the real and imaginary parts of the solution of eq. (18). Substitution of eq. (18) into eq. (16) yields

$$\sum_{r=-\infty}^{\infty} \Big\{ - (2r + \beta)^2 \, C_{2r} \, e^{i(2r + \beta)\tau_1} + a \, C_{2r} \, e^{i(2r + \beta)\tau_1} -$$

$$- q \, C_{2r} e^{i(2r + 2 + \beta)\tau_1} - q \, C_{2r} e^{i(2r - 2 + \beta)\tau_1} = 0$$

Thus

$$\left[a - (2r + \beta)^2 \right] C_{2r} - q(C_{2r - 2} + C_{2r + 2}) = 0 \tag{19}$$

$$-\infty, \ . \ . \ . \ r \ . \ . \ . \ \infty$$

It can be shown (§4.77 McLachlan) that

$$\frac{C_{2r+2}}{C_{2r}} \simeq q/4(r+1)^2 \qquad \text{for } r \text{ large}$$

$$\longrightarrow 0 \quad \text{as} \quad r \to \infty$$

thus the convergence of the series in eq. (18) is assured.

Equation (19) represents a doubly infinite set of homogeneous linear equations. Consistency requires that the determinant of these equations vanish. This requirement determines the value of β thus β is such that

$$\Delta(\beta) = \begin{vmatrix} \cdots & \xi_{-2} & 1 & \xi_{-2} & \cdots & & & & \\ & \cdots & \xi_{-1} & 1 & \xi_{-1} & \cdots & & & \\ & & \cdots & \xi_{o} & 1 & \xi_{o} & \cdots & & \\ & & & \cdots & \xi_{1} & 1 & \xi_{1} & \cdots & \\ & & & & \cdots & \xi_{2} & 1 & \xi_{2} & \cdots \end{vmatrix} = 0 \qquad (20)$$

where

$$\xi_r = \frac{q}{(2r+\beta)^2 - a}$$

The theory of the infinite determinant in eq. (20) appears in the same references already mentioned. It will suffice here to list some of its properties. The function $\Delta(\beta)$ has the following properties:

1. $\Delta(-\beta) = \Delta(\beta)$, it is an even function of β.
2. $\Delta(\beta+2) = \Delta(\beta)$, it is periodic with period 2.

3. $\Delta(\beta)$ has simple poles at $\beta = a^{\frac{1}{2}} - 2r$ and $\beta = -(a^{\frac{1}{2}} + 2r)$.

4. Since the function

$$\chi(\beta) = \frac{1}{\cos \pi\beta - \cos \pi a^{\frac{1}{2}}}$$

has simple poles at the same points as $\Delta(\beta)$ and has the same periodicity it follows that the function

$$\psi(\beta) = \Delta(\beta) - C\,\chi(\beta)$$

will have no singularities if C is suitable chosen. Observe that the above is an entire function and bounded at infinity, thus by Liouville's Theorem it is a constant.

The value of $\psi(0) = 1$ and $C = \left[\Delta(0)-1\right]\left[1 - \cos\pi a^{\frac{1}{2}}\right]$.

The equation containing $\chi(\beta)$ now results in

$$\sin^2 \tfrac{1}{2}\beta\pi = \Delta(0) \sin^2 \tfrac{1}{2}\pi a^{\frac{1}{2}} \tag{21}$$

For small q, such that $q^2 \ll 1$ by Tisserand, F. [6], we have

$$\Delta(0) \cong 1 + \frac{\pi q^2 \cot\frac{1}{2}\pi a^{\frac{1}{2}}}{4a^{\frac{1}{2}}\,(1-a)} \tag{22}$$

Various numerical examples have indicated that three terms of the series in eq. (18) provide an adequate solution for the range of a and q given in eq. (17). Thus C_4 and all higher order coefficients and C_{-4} and all lower order coefficients are taken to be zero. Eq. (19) yields for r=1

$$C_2 = \frac{q}{a-(2+\beta)^2}\,(C_0 + \cancelto{0}{C_4}) \tag{23}$$

and for r = -1

$$C_{-2} = \frac{q}{\left[a-(\beta-2)^2\right]} \left(\cancelto{0}{C_{-4}}+C_0\right) \tag{24}$$

For r = 0, eq. (19) provides a check, we should have

$$(a-\beta^2) = q^2 \left\{ \frac{1}{\left[a-(2+\beta)^2\right]} + \frac{1}{\left[a-(\beta-2)^2\right]} \right\} \tag{25}$$

how well eq. (25) is satisfied is a measure of the adequacy of the solution.

Since the solutions of eq. (16) are arbitrary to a constant multiplier we can set $C_o = 1$, thus

$$C_{-2} = \frac{q}{\left[a-(\beta-2)^2\right]} \quad , \qquad C_2 = \frac{q}{\left[a-(2+\beta)^2\right]} \tag{26}$$

The general solution of eq. (16) is

$$\begin{aligned} \alpha(\tau_1) = a_1 &\left[C_{-2} \cos (2-\beta)\tau_1 + \cos \beta\tau_1 + C_2 \cos (2+\beta)\tau_1\right] \\ &+ a_2 \left[-C_{-2} \sin (2-\beta)\tau_1 + \sin \beta\tau_1 + C_2 \sin (2+\beta)\tau_1\right] \end{aligned} \tag{27}$$

If the initial conditions are

$$\alpha(0) = \alpha_o \quad , \qquad \left.\frac{d\alpha}{dt}\right|_{t=o} = \omega_o$$

we have

$$\alpha_o = a_1 \left(1+C_2+C_{-2}\right) \tag{28}$$

$$\omega_o = \frac{\Omega}{2} a_2 \left[\beta + (2+\beta)\, C_2 - (2-\beta)\, C_{-2}\right]$$

Eq. (27) is the general solution, a_1 and a_2 are determined by eq. (28), and C_2 and C_{-2} are given by eq. (26) and β is found from eqs. (21) and (22).

V. EXPRESSIONS FOR THE FUNCTIONS $G_1(\xi,\theta,\tau)$, $G_2(\xi,\theta,\tau)$ and $G_3(\xi,\theta,\tau)$

The functions $G_1(x,\theta,t)$, $G_2(x,\theta,t)$ and $G_3(x,\theta,t)$ are given by

$$G_1(x,\theta,t) = \frac{(1-\nu^2)L\ (f_1-Px)}{Eh} \qquad \text{(a)}$$

$$G_2(x,\theta,t) = \frac{(1-\nu^2)L\ (f_2-P_\theta)}{Eh} \qquad \text{(b)} \qquad (29)$$

$$G_3(x,\theta,t) = \frac{(1-\nu^2)L\ (f_3-P_Z)}{Eh} \qquad \text{(c)}$$

Using the dimensionless coordinates of eq. (9) and defining

$$X_o = X/L, \qquad Y_o = Y/L \qquad (30)$$

the inertial forces f_1, f_2 and f_3 are

$$f_1(\xi,\theta,\tau) = \rho hL\omega^2(\ddot{X}_o\cos\alpha + \ddot{Y}_o\sin\alpha - \frac{\ddot{\alpha}}{\lambda}\cos\theta - \dot{\alpha}^2\xi) \qquad \text{(a)}$$

$$f_2(\xi,\theta,\tau) = \rho hL\omega^2(\ddot{X}_o\sin\alpha - \ddot{Y}_o\cos\alpha + \frac{\dot{\alpha}^2}{\lambda}\cos\theta - \ddot{\alpha}\xi)\sin\theta \qquad \text{(b)} \qquad (31)$$

$$f_3(\xi,\theta,\tau) = \rho hL\omega^2(\ddot{X}_o\sin\alpha - \ddot{Y}_o\cos\alpha + \frac{\dot{\alpha}^2}{\lambda}\cos\theta - \ddot{\alpha}\xi)\cos\theta \qquad \text{(c)}$$

The P_x, P_θ, and P_z are components of the force distributed over the surface of the shell. In this problem, we will consider the gimbaled thrust the only distributed force. Thus

$$P_x(x,\theta,t) = T_o\ (1-\gamma\cos\Omega t)\delta(x+L)\cos K\alpha - \rho g h \cos\alpha \qquad (a)$$

$$P_\theta(x,\theta,t) = -T_o\ (1-\gamma\cos\Omega t)\delta(x+L)\sin K\alpha\ \sin\theta - \rho g h \sin\alpha\ \sin\theta \qquad (b) \qquad (32)$$

$$P_z(x,\theta,t) = -T_o\ (1-\gamma\cos\Omega t)\delta(x+L)\sin K\alpha\ \cos\theta - \rho g h \sin\alpha\ \cos\theta \qquad (c)$$

where $\delta(x + L)$ is the Dirac delta function. In terms of dimensionless coordinates

$$P_x(\xi,\theta,\tau) = \frac{T_o}{L}(1-\gamma\cos\bar{\Omega}\tau)\delta(\xi+1)\cos K\alpha - \rho g h\ \cos\alpha \qquad (a)$$

$$P_\theta(\xi,\theta,\tau) = -\frac{T_o}{L}(1-\gamma\cos\bar{\Omega}\tau)\delta(\xi+1)\sin K\alpha\ \sin\theta - \rho g h\ \sin\alpha\ \sin\theta \qquad (b) \qquad (33)$$

$$P_z(\xi,\theta,\tau) = -\frac{T_o}{L}(1-\gamma\cos\bar{\Omega}\tau)\delta(\xi+1)\sin K\alpha\ \cos\theta - \rho g h\ \sin\alpha\ \cos\theta \qquad (c)$$

where

$$\bar{\Omega} = \Omega/\omega \qquad (34)$$

Now we can express $G_1(\xi,\theta,\tau)$, $G_2(\xi,\theta,\tau)$ and $G_3(\xi,\theta,\tau)$ as

$$G_1(\xi,\theta,\tau) = (1-\nu^2)\mu\left[\ddot{X}_o\cos\alpha + \ddot{Y}_o\sin\alpha - \frac{\ddot{\alpha}}{\lambda}\cos\theta - \dot{\alpha}^2\xi\right.$$
$$\left. - \frac{T_o}{\rho h L^2\omega^2}(1-\gamma\cos\bar{\Omega}\tau)\delta(\xi+1)\cos K\alpha\right] + (1-\nu^2)\frac{\rho g L}{E}\cos\alpha \qquad (a)$$

$$G_2(\xi,\theta,\tau) = (1-\nu^2)\mu\left[\ddot{X}_o\sin\alpha - \ddot{Y}_o\cos\alpha + \frac{\dot{\alpha}^2}{\lambda}\cos - \ddot{\alpha}\xi\right.$$
$$\left. + \frac{T_o}{\rho h L^2\omega^2}(1-\gamma\cos\bar{\Omega}\tau)\delta(\xi+1)\sin K\alpha\right]\sin\theta + (1-\nu^2)\frac{\rho g L}{E}\sin\alpha\ \sin\theta \qquad (35)$$
(b)

$$G_3(\xi,\theta,\tau) = (1-\nu^2)\mu\left[\ddot{X}_o \sin\alpha - \ddot{Y}_o \cos\alpha + \frac{\dot{\alpha}^2}{\lambda} \cos\theta - \ddot{\alpha}\xi\right.$$

$$\left. + \frac{T_o}{\rho g L^2\omega^2} (1-\gamma\cos\bar{\Omega}\tau)\delta(\xi+1) \sin K\alpha\right] \cos\theta + (1-\nu^2) \frac{\rho g L}{E} \sin\alpha \cos\theta \quad \text{(c)}$$

From eqs. (2a,b)

$$\ddot{X}_o = \frac{T_o}{2\rho h L^2\omega^2} (1-\gamma\cos\bar{\Omega}\tau) \cos (K+1)\alpha - \frac{g}{L\omega^2} \qquad \text{(a)}$$

$$\ddot{Y}_o = \frac{T_o}{2\rho h L^2\omega^2} (1-\gamma\cos\bar{\Omega}\tau) \sin (K+1)\alpha \qquad \text{(b)} \qquad (36)$$

Substitution of eqs. (36) into eqs. (35) and assuming small values of α, yields

$$G_1(\xi,\theta,\tau) = (1-\nu^2)\mu\left\{\bar{T}_o\left[\tfrac{1}{2} - \delta(\xi+1)\right] (1-\gamma\cos\bar{\Omega}\tau) - \frac{\ddot{\alpha}}{\lambda} \cos\theta - \dot{\alpha}^2\xi\right\} \qquad (37)$$

$$G_2(\xi,\theta,\tau) = (1-\nu^2)\mu\left\{\bar{T}_o\, K\alpha\left[\delta(\xi+1)-\tfrac{1}{2}\right] (1-\gamma\cos\bar{\Omega}\tau) + \frac{\dot{\alpha}^2}{\lambda} \cos\theta - \ddot{\alpha}\xi\right\}\sin\theta \qquad (38)$$

$$G_3(\xi,\theta,\tau) = (1-\nu^2)\mu\left\{\bar{T}_o\, K\alpha\left[\delta(\xi+1)-\tfrac{1}{2}\right] (1-\gamma\cos\bar{\Omega}\tau) + \frac{\dot{\alpha}^2}{\lambda} \cos\theta - \ddot{\alpha}\xi\right\}\cos\theta \qquad (39)$$

where

$$\bar{T}_o = \frac{T_o}{\rho h L^2\omega^2} \qquad (40)$$

VI. GALERKIN'S PROCEDURE FOR THE APPROXIMATE DETERMINATION OF THE DISPLACEMENT COMPONENTS:

In the manner of Donnell (4), eqs. (10), (11), and (12) can be rewritten in a more convenient form, that is, they can be rearranged into one eighth-order, parial differential equation that $\bar{w}$ must satisfy and two fourth-order, partial differential equations that relate $\bar{u}$ to $\bar{w}$ and $\bar{v}$ to $\bar{w}$, respectively. Then if we could determine $\bar{w}$ it would enter into the determination of $\bar{u}$ and $\bar{v}$ as known nonhomogeneous terms. However, to solve the equation for $\bar{w}$ is a formidable task. This equation is of eighth order in the spatial variables and sixth order in time. It contains rather complicated nonhomogeneous terms. An accepted method of handling such complicated equations is the Galerkin procedure (5). If we must resort to this method there is little, if any, advantage to the Donnell separated scheme. We can equally well treat eqs. (10), (11) and (12) by the Galerkin procedure directly. This will be the course of action taken here.

For convenience we will rewrite eqs. (10), (11) and (12) here

$$\frac{\partial^2 \bar{u}}{\partial \xi^2} + \frac{(1-\nu)}{2} \lambda^2 \frac{\partial^2 \bar{u}}{\partial \theta^2} + \frac{(1+\nu)}{2} \lambda \frac{\partial^2 \bar{v}}{\partial \xi \partial \theta} - \nu\lambda \frac{\partial \bar{w}}{\partial \xi} = (1-\nu^2)\mu \frac{\partial^2 \bar{u}}{\partial \tau^2} + G_1(\xi,\theta,\tau) \tag{10}$$

$$\frac{(1+\nu)}{2} \lambda \frac{\partial^2 \bar{u}}{\partial \xi \partial \theta} + \frac{(1-\nu)}{2} \frac{\partial^2 \bar{v}}{\partial \xi^2} + \lambda^2 \frac{\partial^2 \bar{v}}{\partial \theta^2} - \frac{\lambda^2 \partial \bar{w}}{\partial \theta} = (1-\nu^2)\mu \frac{\partial^2 \bar{v}}{\partial \tau^2} + G_2(\xi,\theta,\tau) \tag{11}$$

$$\nu\lambda \frac{\partial \bar{u}}{\partial \xi} + \lambda^2 \frac{\partial \bar{v}}{\partial \theta} - \lambda^2 \bar{w} - \frac{\sigma^2}{12} \nabla^4 \bar{w} = (1-\nu^2)\mu \frac{\partial^2 \bar{w}}{\partial \tau^2} + G_3(\xi,\theta,\tau) \tag{12}$$

In accordance with the Galerkin procedure we chose approximating forms for $\bar{u}$, $\bar{v}$, and $\bar{w}$ as

$$\tilde{u}(\xi,\theta,\tau) = \sum_{m=o}^{M} \sum_{n=o}^{N} U_{mn}(\tau)\, f_{mn}(\xi,\theta) \tag{41}$$

$$\tilde{v}(\xi,\theta,\tau) = \sum_{m=o}^{M} \sum_{n=o}^{N} V_{mn}(\tau)\, g_{mn}(\xi,\theta) \tag{42}$$

$$\tilde{w}(\xi,\theta,\tau) = \sum_{m=o}^{M} \sum_{n=o}^{N} W_{mn}(\tau)\, h_{mn}(\xi,\theta) \tag{43}$$

where the functions $f_{mn}(\xi,\theta)$, $g_{mn}(\xi,\theta)$ and $h_{mn}(\xi,\theta)$ are known functions. It is the role of these functions to ensure satisfaction of the edge conditions at $\xi = \pm 1$.

The Galerkin procedure requires

$$\sum_{m=o}^{M} \sum_{n=o}^{N} \left\langle \left\{(1-\nu^2)\mu \int_{-1}^{1} \int_{0}^{2\pi} f_{mn} f_{jk}\, d\xi d\theta \right\} \ddot{U}_{mn} - \left\{ \int_{-1}^{1} \int_{0}^{2\pi} \left[\frac{\partial^2 f_{mn}}{\partial \xi^2} + \frac{(1-\nu)\lambda^2}{2} \frac{\partial^2 f_{mn}}{\partial \theta^2} \right] f_{jk}\, d\xi d\theta \right\} U_{mn} - \left\{ \frac{(1+\nu)\lambda}{2} \int_{-1}^{1} \int_{0}^{2\pi} \frac{\partial^2 g_{mn}}{\partial \xi \partial \theta} f_{jk}\, d\xi d\theta \right\} V_{mn} + \left\{ \nu\lambda \int_{-1}^{1} \int_{0}^{2\pi} \frac{\partial h_{mn}}{\partial \xi} f_{jk}\, d\xi d\theta \right\} W_{mn} \right\rangle = - \int_{-1}^{1} \int_{0}^{2\pi} G_1(\xi,\theta,\tau)\, f_{jk}(\xi,\theta)\, d\xi d\theta \quad , \tag{44}$$

$$j = 0, 1, \ldots M$$

$$k = 0, 1, \ldots, N$$

$$\sum_{m=o}^{M} \sum_{n=o}^{N} \left\langle -\left\{ \frac{(1+\nu)\lambda}{2} \int_{-1}^{1} \int_{0}^{2\pi} \frac{\partial^2 f_{mn}}{\partial\xi\partial\theta} g_{jk} \, d\xi d\theta \right\} U_{mn} \right.$$

$$+ \left\{ (1-\nu^2)\mu \int_{-1}^{1} \int_{0}^{2\pi} g_{mn} \, g_{jk} \, d\xi d\theta \right\} \ddot{V}_{mn}$$

$$- \left\{ \int_{-1}^{1} \int_{0}^{2\pi} \left[\frac{(1-\nu)}{2} \frac{\partial^2 g_{mn}}{\partial\xi^2} + \lambda^2 \frac{\partial^2 g_{mn}}{\partial\theta^2} \right] g_{jk} \, d\xi d\theta \right\} V_{mn}$$

$$\left. + \left\{ \lambda^2 \int_{-1}^{1} \int_{0}^{2\pi} \frac{\partial h_{mn}}{\partial\theta} g_{jk} \, d\xi d\theta \right\} W_{mn} \right\rangle$$

$$= - \int_{-1}^{1} \int_{0}^{2\pi} G_2(\xi,\theta,\tau) \, g_{jk}(\xi,\theta) \, d\xi d\theta \quad , \tag{45}$$

$j = 0, 1, \ldots M$

$k = 0, 1, \ldots N$

and

$$\sum_{m=o}^{M} \sum_{n=o}^{N} \left\langle -\left\{ \nu\lambda \int_{-1}^{1} \int_{0}^{2\pi} \frac{\partial f_{mn}}{\partial\xi} h_{jk} \, d\xi d\theta \right\} U_{mn} \right.$$

$$- \left\{ \lambda^2 \int_{-1}^{1} \int_{0}^{2\pi} \frac{\partial g_{mn}}{\partial\theta} h_{jk} \, d\xi d\theta \right\} V_{mn} + \left\{ (1-\nu^2)\mu \int_{-1}^{1} \int_{0}^{2\pi} h_{mn} \, h_{jk} \, d\xi d\theta \right\} \ddot{W}_{mn}$$

$$\left. + \left\{ \int_{-1}^{1} \int_{0}^{2\pi} \left[\lambda^2 h_{mn} + \frac{\sigma^2}{12} \bar{\nabla}^4 h_{mn} \right] h_{jk} \, d\xi d\theta \right\} W_{mn} \right\rangle$$

$$= - \int_{-1}^{1} \int_{0}^{2\pi} G_3(\xi,\theta,\tau) \, h_{jk}(\xi,\theta) \, d\xi d\theta \tag{46}$$

$j = 0, 1, \ldots M$

$k = 0, 1, \ldots N$

The system of ordinary differential equations represented by eqs. (44), (45), and (46) govern the time behavior of the generalized coordinates U_{mn}, V_{mn}, and W_{mn}. In its most general form, that is, for arbitrarily selected coordinate functions f_{mn}, g_{mn}, and h_{mn}, this system has coupling in both the inertia and elastic terms. Not only are the coordinates U_{mn} coupled with the coordinates U_{jk} but also to the coordinates V_{mn} and W_{mn}. It behooves us at this point to place some restrictions on the coordinate functions that will simplify the form of eqs. (44), (45), and (46). Requiring f_{mn}, g_{mn} and h_{mn} to be orthogonal over the region of the shell will eliminate the inertia coupling. Not only do we want the functions f_{mn}, g_{mn}, and h_{mn} to be such as to simplify eqs. (44), (45), and (46) but they must also assure the satisfaction of the edge conditions at $\xi = \pm 1$.

One interesting set of edge conditions is

$$\bar{w}(\pm 1,\theta,\tau) = 0, \qquad M\xi(\pm 1,\theta,\tau) = 0,$$

$$\bar{v}(\pm 1,\theta,\tau) = 0, \qquad N\xi(\pm 1,\theta,\tau) = 0 \tag{47}$$

This is the simply supported edge conditions with no axial load on the ends of the shell. A choice of $f_{mn}(\xi,\theta)$, $g_{mn}(\xi,\theta)$ and $h_{mn}(\xi,\theta)$ that assures the satisfaction of eqs. (47) is

$$f_{mn}(\xi,\theta) = \cos \frac{m\pi}{2} (\xi+1) \cos n\theta \qquad (a)$$

$$g_{mn}(\xi,\theta) = \sin \frac{m\pi}{2} (\xi+1) \sin n\theta \qquad (48) \quad (b)$$

$$h_{mn}(\xi,\theta) = \sin \frac{m\pi}{2} (\xi+1) \cos n\theta \qquad (c)$$

$$m = 0, 1, \ldots M$$

$$n = 0, 1, \ldots N$$

Using the end conditions of eqs. (47) for this problem implies that the center of mass always remains at the mid point of a line that connects the centers of the end sections and that a line element through the center of mass rotates as the line that connects the centers of the end sections rotates.

Substitution of eqs. (48) into eqs. (44), (45), and (46) yields

$$\ddot{U}_{oo}(\tau) = \frac{-1}{4\pi(1-\nu^2)\mu} \int_{-1}^{1} \int_{0}^{2\pi} G_1(\xi,\theta,\tau)\, d\xi d\theta = 0 \tag{49}$$

$$\ddot{U}_{ok}(\tau) + \frac{k^2\lambda^2}{2(1+\nu)\mu}\, U_{ok}(\tau) = \frac{-1}{2\,(1-\nu^2)\mu} \int_{-1}^{1} \int_{0}^{2\pi} G_1(\xi,\theta,\tau) \cos k\theta\, d\xi d\theta$$

$$= \frac{\delta_{1k}}{\lambda}\,\ddot{\alpha} \tag{50}$$

$$k = 1, 2, \ldots N$$

$$\ddot{U}_{jo}(\tau) + \frac{j^2\pi^2}{4(1-\nu^2)\mu}\, U_{jo}(\tau) + \frac{\nu\lambda j\pi}{2(1-\nu^2)\mu}\, W_{jo}(\tau) =$$

$$\frac{-1}{2\pi(1-\nu^2)\mu} \int_{-1}^{1} \int_{0}^{2\pi} G_1(\xi,\theta,\tau) \cos \frac{j\pi}{2}(\xi+1)\, d\xi d\theta =$$

$$\overline{T}_o\,(1-\gamma\cos\overline{\Omega}\tau) - \frac{4}{j^2\pi^2}\left[1-(-1)^j\right]\mathring{\alpha}^2 = P_{jo}(\tau) \tag{51}$$

$$j = 1, 2, \ldots M$$

$$\frac{\nu\lambda j\pi}{2(1-\nu^2)\mu} U_{jo}(\tau) + \ddot{W}_{jo}(\tau) + \frac{192\lambda^2 + \sigma^2 j^4\pi^4}{192(1-\nu^2)\mu} W_{jo}(\tau) =$$

$$\frac{-1}{2\pi(1-\nu^2)\mu} \int_{-1}^{1} \int_{0}^{2\pi} G_3(\xi,\theta,\tau) \sin \frac{j\pi}{2} (\xi+1)\, d\xi d\theta =$$

$$\frac{1}{j\pi\lambda} \left[(-1)^j - 1 \right] \dot{\alpha}^2 = R_{jo}(\tau) \qquad (52)$$

$j = 1, 2, \ldots M$

$$\ddot{U}_{jk}(\tau) + \frac{j^2\pi^2 + 2(1-\nu)\lambda^2 k^2}{4(1-\nu^2)\mu} U_{jk}(\tau) - \frac{jk\pi\lambda}{4(1-\nu)\mu} V_{jk}(\tau) +$$

$$\frac{\nu\lambda j\pi}{2(1-\nu^2)\mu} W_{jk}(\tau) = \frac{-1}{\pi(1-\nu^2)\mu} \int_{-1} \int_{0}^{2\pi} G_1(\xi,\theta,\tau) \cos \frac{j\pi}{2} (\xi+1) \cos k\theta d\xi d\theta = 0 \qquad (53)$$

$j = 1, 2, \ldots M$

$k = 1, 2, \ldots N$

$$\frac{-jk\pi\lambda}{4(1-\nu)} U_{jk}(\tau) + \ddot{V}_{jk}(\tau) + \frac{(1-\nu)\ j^2\pi^2 + 8\ k^2\lambda^2}{8(1-\nu^2)\mu} V_{jk}(\tau) - \frac{k\lambda^2}{(1-\nu^2)\mu} W_{jk}(\tau) =$$

$$\frac{-1}{\pi(1-\nu^2)\mu} \int_{-1}^{1} \int_{0}^{2\pi} G_2(\xi,\theta,\tau) \sin \frac{j\pi}{2} (\xi+1) \sin k\theta d\xi d\theta = \delta_{1k} \left\{ \frac{\overline{T}_o K}{j\pi} \left[1 - (-1)^j \right] \right.$$

$$\left. (1 - \gamma\cos\bar{\Omega}\tau)\alpha(\tau) - \frac{2}{j\pi} \left[1 + (-1)^j \right] \ddot{\alpha}(\tau) \right\} + \frac{\delta_{2k}}{j\pi} \left[(-1)^j - 1 \right] \dot{\alpha}^2(\tau)$$

$$= Q_{jk}(\tau) \qquad (54)$$

$j = 1,2,\ldots M$

$k = 1,2,\ldots N$

$$\frac{j\pi\nu\lambda}{2(1-\nu^2)\mu} U_{jk}(\tau) - \frac{k\lambda^2}{(1-\nu^2)\mu} V_{jk}(\tau) + \ddot{W}_{jk}(\tau) +$$

$$\frac{12\lambda^2 + \sigma^2 (j^2\pi^2/4 + k^2\lambda^2)^2}{12(1-\nu^2)\mu} W_{jk}(\tau) = \frac{-1}{\pi(1-\nu^2)\mu} \int_1^1 \int_0^{2\pi}$$

$$G_3(\xi,\theta,\tau) \sin \frac{j\pi}{2} (\xi+1) \cos k\theta d\xi d\theta = Q_{jk}(\tau) \qquad (55)$$

$$j = 1, 2, \ldots M$$

$$k = 1, 2, \ldots N$$

The initial condition for the sets of differential equations of eqs. (49 thru 55) are given by:

$$\bar{u} (\xi,\theta,0) = \sum_{m=o}^{M} \sum_{n=o}^{N} u_{mn} \cos \frac{m\pi}{2} (\xi+1) \cos n\theta, \quad \frac{\partial \bar{u}}{\partial \tau} (\xi,\theta,0) = 0$$

$$\bar{v} (\xi,\theta,0) = \sum_{m=o}^{M} \sum_{n=o}^{N} v_{mn} \sin \frac{m\pi}{2} (\xi+1) \sin n\theta, \quad \frac{\partial \bar{v}}{\partial \tau} (\xi,\theta,0) = 0 \qquad (56)$$

$$\bar{w} (\xi,\theta,0) = \sum_{m=o}^{M} \sum_{n=o}^{N} w_{mn} \sin \frac{m\pi}{2} (\xi+1) \cos n\theta, \quad \frac{\partial \bar{w}}{\partial \tau} (\xi,\theta,0) = 0$$

Before integrating eqs. (49) through (55) we need an expression for $\alpha(\tau)$. We will restrict our attention to the initial conditions

$$\alpha(0) = \alpha_o \text{ and } \dot{\alpha}(0) = 0 \qquad (57)$$

then eqs. (27) and (28) yield

$$\alpha(\tau) = \sum_{i=1}^{3} A_i \cos \Omega_i \tau \tag{58}$$

where

$$A_1 = A_2\, C_{-2}, \qquad \text{(a)}$$

$$A_2 = \frac{\alpha_o}{1+C_2+C_{-2}} \qquad (59) \quad \text{(b)}$$

$$A_3 = A_2\, C_2 \qquad \text{(c)}$$

and

$$\Omega_2 = \frac{\beta\Omega}{2\omega} \qquad \text{(a)}$$

$$\Omega_1 = (2-\beta)\,\frac{\Omega}{2\omega} \qquad (60) \quad \text{(b)}$$

$$\Omega_3 = (2+\beta)\,\frac{\Omega}{2\omega} \qquad \text{(c)}$$

Now we can integrate these equations. Equation (49) yields

$$U_{oo}(\tau) = u_{oo} = o \tag{61}$$

Eq. (50) yields

$$U_{ok}(\tau) = u_{ok} \cos \omega_{ok}\tau + \frac{\delta_{1k}}{\lambda} \sum_{m=1}^{3} \frac{A_i\, \Omega_i^{\,2}}{\Omega_m^{\,2} - \omega_{ok}^{\,2}} (\cos \Omega_m \tau - \cos \omega_{ok}\tau) \tag{62}$$

$$k = 1,\ 2,\ \ldots\ N$$

$$\omega_{ok} = \frac{k\lambda}{\sqrt{2(1+\nu)\mu}} \tag{63}$$

Eqs. (51) and (52) are a set of differential equations that govern $U_{jo}(\tau)$ and $W_{jo}(\tau)$. We will use the Laplace transformation to integrate this set of equations. The transform of a function will be indicated by a bar over the function, i.e., $\mathcal{L}\left[f(\tau)\right] = \bar{f}$. Transforming the equations we obtain

$$(S^2 + \alpha^j_{11})\, \bar{U}_{jo} + \alpha^j_{12}\, \bar{W}_{jo} = \bar{P}_{jo} + s\, u_{jo}$$

$$\alpha^j_{21}\, \bar{U}_{jo} + (S^2 + \alpha^j_{22})\, \bar{W}_{jo} = \bar{R}_{jo} + s\, w_{jo}$$

$$\begin{vmatrix} s^2 + \alpha^j_{11} & \alpha^j_{12} \\ \alpha^j_{21} & s^2 + \alpha^j_{22} \end{vmatrix} = \left[s^2 + (\omega^1_{jo})^2\right]\left[s^2 + (\omega^2_{jo})^2\right]$$

$$\bar{U}_{jo} = \frac{\bar{P}_{jo}(s^2 + \alpha^j_{22}) - \alpha^j_{12}\, \bar{R}_{jo} + u_{jo} s(s^2 + \alpha^j_{22}) - s\, w_{jo}\, \alpha^j_{12}}{\left[(s^2 + \omega^1_{jo})^2\right]\left[s^2 + (\omega^2_{jo})^2\right]} \tag{64}$$

$$\bar{W}_{jo} = \frac{\bar{R}_{jo}\,(s^2 + \alpha^j_{11}) - \alpha^j_{21}\, \bar{P}_{jo} + w_{jo} s(s^2 + \alpha^j_{11}) - s\, u_{jo}\, \alpha^j_{21}}{\left[s^2 + (\omega^1_{jo})^2\right]\left[s^2 + (\omega^2_{jo})^2\right]} \tag{65}$$

where

$$\alpha^j_{11} = \frac{j^2\pi^2}{4(1-\nu^2)\mu} \qquad \alpha^j_{22} = \frac{192\,\lambda^2 + \sigma^2\, j^4\pi^4}{192(1-\nu^2)\mu}$$

$$\alpha^j_{12} = \alpha^j_{21} = \frac{\nu\lambda j\pi}{2(1-\nu^2)\mu} \tag{66}$$

The inverse of equation (64) is

$$U_{jo}(\tau) = \sum_{i=1}^{2} \frac{(-1)^i}{(\omega_{jo}^1)^2 - (\omega_{jo}^2)^2} \left(\left\{ u_{jo}\left[\alpha_{22}^j - (\omega_{jo}^i)^2\right] - \alpha_{12} w_{jo} \right\} \cos \omega_{jo}^i \tau \right.$$

$$- \left[\alpha_{22}^j - (\omega_{jo}^i)^2\right] \left\{ \frac{T_o}{\omega_{jo}^i} \left[\cos \omega_{jo}^i \tau - 1\right] + \bar{T}_o \gamma\, \omega_{jo}^i \left[\frac{\cos \bar{\Omega}\tau - \cos \omega_{jo}^i \tau}{\bar{\Omega}^2 - (\omega_{jo}^i)^2}\right] \right\}$$

$$+ \left\{ \alpha_{12}^j - \left[\alpha_{22}^j - (\omega_{jo}^i)^2\right] \frac{4\lambda}{j\pi} \left\{ \frac{1}{2j\lambda\pi} \left[(-1)^j - 1\right] \left\{ \sum_{\substack{m=1 \\ m \neq n}}^{3} \sum_{n=1}^{3} A_m A_n \Omega_m \Omega_n \omega_{jo}^i \right. \right. \right.$$

$$\cdot \left[\frac{\cos (\Omega_m + \Omega_n)\tau - \cos \omega_{jo}^i \tau}{(\Omega_m + \Omega_n)^2 - (\omega_{jo}^i)^2} - \frac{\cos (\Omega_m - \Omega_n)\tau - \cos \omega_{jo}^i \tau}{(\Omega_m - \Omega_n)^2 - (\omega_{jo}^i)^2} \right]$$

$$\left. \left. \left. + \sum_{m=1}^{3} A_m^2 \Omega_m^2 \left[\frac{1 - \cos \omega_{jo}^i \tau}{\omega_{jo}^i} + \frac{\omega_{jo}^i (\cos 2\Omega_m \tau - \cos \omega_{jo}^i \tau)}{4\Omega_m^2 - (\omega_{jo}^i)^2} \right] \right\} \right] \right\} \right) \quad (67)$$

$j = 1, 2, \ldots M$

$i = 1, 2$

$m = 1, 2, 3$

$n = 1, 2, 3$

The inverse of equation (65) is

$$W_{jo}(\tau) = \sum_{i=1}^{2} \frac{(-1)^i}{(\omega_{jo}^i)^2 - (\omega_{jo}^2)^2} \left(\left\{ w_{jo} \left[\alpha_{11}^j - (\omega_{jo}^i)^2 \right] - \alpha_{12}^j u_{jo} \right\} \cos \omega_{jo}^i \tau \right.$$

$$+ \left\{ \left[\alpha_{11}^j - (\omega_{jo}^i)^2 \right] - \alpha_{12}^j \frac{4\lambda}{j\pi} \right\} \left\{ \left[\frac{1}{2j\lambda\pi} \left[(-1)^j - 1 \right] \left\{ \sum_{m=1}^{3} \sum_{\substack{n=1 \\ m \neq n}}^{3} A_m A_n \Omega_m \Omega_n \omega_{jo}^i \right. \right. \right.$$

$$\cdot \left[\frac{\cos(\Omega_m + \Omega_n)\tau - \cos \omega_{jo}^i \tau}{(\Omega_m + \Omega_n)^2 - (\omega_{jo}^i)^2} - \frac{\cos(\Omega_m - \Omega_n)\tau - \cos \omega_{jo}^i \tau}{(\Omega_m - \Omega_n)^2 - (\omega_{jo}^i)^2} \right]$$

$$\left. \left. \left. + \sum_{m=1}^{3} A_m^{\,2}\, \Omega_m^{\,2} \left[\frac{1 - \cos \omega_{jo}^i \tau}{\omega_{jo}^i} + \frac{\omega_{jo}^m (\cos 2\Omega_m \tau - \cos \omega_{jo}^i \tau)}{4\,\Omega_m^2 - (\omega_{jo}^m)^2} \right] \right\} \right] \right\}$$

$$\left. - \alpha_{12}^j \left\{ \frac{\bar{T}_o}{\omega_{jo}^i} \left[\cos \omega_{jo}^i \tau - 1 \right] + T_o \gamma\, \omega_{jo}^i \left[\frac{\cos \bar{\Omega}\tau - \cos \omega_{jo}^i \tau}{\bar{\Omega}^2 - (\omega_{jo}^i)^2} \right] \right\} \right) \qquad (68)$$

$j = 1, 2, \ldots M$

$i = 1, 2$

$m = 1, 2, 3$

$n = 1, 2, 3$

Transforming eqs. (53), (54), and (55) we find

$$(s^2 + a_{11}^{jk})\,\bar{U}_{jk} + a_{12}^{jk}\,\bar{V}_{jk} + a_{13}^{jk}\,\bar{W}_{jk} = su_{jk}$$

$$a_{21}^{jk}\,\bar{U}_{jk} + (s^2 + a_{22}^{jk})\,\bar{V}_{jk} + a_{23}^{jk}\,\bar{W}_{jk} = \bar{Q}_{jk} + sv_{jk} \tag{69}$$

$$a_{31}^{jk}\,\bar{U}_{jk} + a_{32}^{jk}\,\bar{V}_{jk} + (s^2 + a_{33}^{jk})\,\bar{W}_{jk} = \bar{Q}_{jk} + sw_{jk}$$

where

$$a_{11}^{jk} = \frac{j^2\pi^2 + 2(1-\nu)\lambda^2k^2}{4(1-\nu^2)\mu}\ , \quad a_{12}^{jk} = \frac{-\ jk\pi\lambda}{4(1-\nu)\mu}$$

$$a_{13}^{jk} = \frac{\nu\lambda j\pi}{2(1-\nu^2)\mu}\ , \quad a_{21}^{jk} = a_{12}^{jk}\ , \quad a_{22}^{jk} = \frac{(1-\nu)\ j^2\pi^2 + 8\lambda^2k^2}{8(1-\nu^2)\mu}\ ,$$

$$a_{23}^{jk} = \frac{-\ \lambda^2k}{(1-\nu^2)\mu}\ , \quad a_{31}^{jk} = a_{13}^{jk}\ , \quad a_{32}^{jk} = a_{23}^{jk}\ ,$$

$$a_{33}^{jk} = \frac{12\lambda^2 + \sigma^2(j^2\pi^2/4 + \lambda^2k^2)^2}{12(1-\nu^2)\mu}\ , \tag{70}$$

$j = 1, 2, \ldots M$

$k = 1, 2, \ldots N$

We define

$$\Delta(S) = \left[s^2 + (\omega_{jk}^1)^2\right]\left[S^2 + (\omega_{jk}^2)^2\right]\left[S^2 + (\omega_{jk}^3)^2\right] =$$

$$\begin{vmatrix} s^2 + a_{11}^{jk} & a_{12}^{jk} & a_{13}^{jk} \\ a_{21}^{jk} & s^2 + a_{22}^{jk} & a_{23}^{jk} \\ a_{31}^{jk} & a_{32}^{jk} & s^2 + a_{33}^{jk} \end{vmatrix} \tag{71}$$

Using Cramer's rule to solve eqs. (69) we get

$$\bar{U}_{jk} = \frac{\begin{vmatrix} su_{jk} & a_{12}^{jk} & a_{13}^{jk} \\ \bar{Q}_{jk} + sv_{jk} & s^2 + a_{22}^{jk} & a_{23}^{jk} \\ \bar{Q}_{jk} + sw_{jk} & a_{32}^{jk} & s^2 + a_{33}^{jk} \end{vmatrix}}{\Delta(s)}$$

$$\begin{aligned} \bar{U}_{jk} = \Big\{ & su_{jk} \left[(s^2 + a_{22}^{jk})(s^2 + a_{33}^{jk}) - a_{32}^{jk}\, a_{23}^{jk}\right] - \\ & (\bar{Q}_{jk} + sv_{jk}) \left[a_{12}^{jk}(s^2 + a_{33}^{jk}) - a_{13}^{jk}\, a_{23}^{jk}\right] + \\ & (\bar{Q}_{jk} + sw_{jk}) \left[a_{12}^{jk}\, a_{23}^{jk} - a_{13}^{jk}(s^2 + a_{22}^{jk})\right]\Big\} \Big/ \Delta(s) \end{aligned} \tag{72}$$

$$\bar{V}_{jk} = \frac{\begin{vmatrix} s^2 + a_{11}^{jk} & su_{jk} & a_{13}^{jk} \\ a_{21}^{jk} & \bar{Q}_{jk} + sv_{jk} & a_{23}^{jk} \\ a_{31}^{jk} & \bar{Q}_{jk} + sw_{jk} & s^2 + a_{33}^{jk} \end{vmatrix}}{\Delta(s)}$$

$$\begin{aligned} \bar{V}_{jk} = \Big\{ - su_{jk} \left[a_{21}^{jk} (s^2 + a_{33}^{jk}) - a_{13}^{jk} a_{23}^{jk} \right] + \\ (\bar{Q}_{jk} + sv_{jk}) \left[(s^2 + a_{11}^{jk}) (s^2 + a_{33}^{jk}) - (a_{13}^{jk})^2 \right] - \\ (\bar{Q}_{jk} + sw_{jk}) \left[a_{23}^{jk} (s^2 + a_{11}^{jk}) - a_{12}^{jk} a_{13}^{jk} \right] \Big\} \Big/ \Delta(s) \end{aligned} \qquad (73)$$

$$\bar{W}_{jk} = \frac{\begin{vmatrix} s^2 + a_{11}^{jk} & a_{12}^{jk} & su_{jk} \\ a_{21}^{jk} & s + a_{22}^{jk} & Q_{jk} + sv_{jk} \\ a_{31}^{jk} & a_{32}^{jk} & \bar{Q}_{jk} + sw_{jk} \end{vmatrix}}{\Delta(s)}$$

$$\bar{W}_{jk} = \Bigg\{ su_{jk} \left[a_{12}^{jk} a_{23}^{jk} - a_{13}^{jk} (s^2 + a_{22}^{jk})\right] - (\bar{Q}_{jk} + sv_{jk})$$

$$\left[a_{23}^{jk} (s^2 + a_{11}^{jk}) - a_{12}^{jk} a_{13}^{jk}\right] + (\bar{Q}_{jk} + sw_{jk})$$

$$\left[(s^2 + a_{11}^{jk}) (s^2 + a_{22}^{jk}) - (a_{12}^{jk})^2\right]\Bigg\} \Bigg/ \Delta(s) \tag{74}$$

Define

$$D_{jk}^{i} = \prod_{\substack{m=1 \\ m \neq i}}^{3} \left[(\omega_{jk}^{m})^2 - (\omega_{jk}^{i})^2\right] \tag{75}$$

Using standard techniques, the inverse transform of equation (72) is

$$U_{jk}(\tau) = \sum_{i=1}^{3} \frac{1}{D_{jk}^{i}} \Bigg[\Bigg\langle u_{jk} \left\{\left[a_{22}^{jk} - (\omega_{jk}^{i})^2\right]\left[a_{33}^{jk} - (\omega_{jk}^{i})^2\right] - (a_{23}^{jk})^2\right\}$$

$$- v_{jk}\left\{ a_{12}^{jk}\left[a_{33}^{jk} - (\omega_{jk}^{i})^2\right] - a_{13}^{jk} a_{23}^{jk}\right\}$$

$$- w_{jk}\left\{ a_{13}^{jk}\left[a_{22}^{jk} - (\omega_{jk}^{i})^2\right] - a_{12}^{jk} a_{23}^{jk}\right\}\Bigg\rangle \quad \cos \omega_{jk}^{i}\tau$$

$$- \frac{1}{\omega_{jk}^{i}}\left\{ a_{12}^{jk}\left[a_{33}^{jk} - (\omega_{jk}^{i})^2\right] + a_{13}^{jk}\left[a_{22}^{jk} - (\omega_{jk}^{i})^2\right] - a_{23}^{jk} (a_{12}^{jk} + a_{13}^{jk})\right\}$$

$$\cdot\Bigg(\delta_{1k} \sum_{m=1}^{3} \frac{A_m \omega_{jk}^{i}}{j\pi} \left\{\bar{T}_o K\left[(-1)^j - 1\right] - 2\Omega_m^2\left[(-1)^j + 1\right]\right\} \frac{\cos \Omega_m\tau - \cos \omega_{jk}^{i}\tau}{\Omega_m^2 - (\omega_{jk}^{i})^2}$$

$$
+ \delta_{1k} \frac{\bar{T}_o KY}{j} \left[1 - (-1)^j \right] \sum_{m=1}^{3} \frac{A_m \omega_{jk}^i}{2} \left[\frac{\cos (\Omega_m + \bar{\Omega})\tau - \cos \omega_{jk}^i \tau}{(\Omega_m + \bar{\Omega})^2 - (\omega_{jk}^i)^2} \right.
$$

$$
\left. + \frac{\cos(\Omega_m - \bar{\Omega})\tau - \cos \omega_{jk}^i \tau}{(\Omega_m - \bar{\Omega})^2 - (\omega_{jk}^i)^2} \right] + \delta_{2k} \left[(-1)^j - 1 \right] \left\{ \sum_{m=1}^{3} \sum_{\substack{n=1 \\ m \neq n}}^{3} \frac{A_m A_n \Omega_m \Omega_n \omega_{jk}^i}{2} \right.
$$

$$
\left[\frac{\cos (\Omega_m + \Omega_n)\tau - \cos \omega_{jk}^i \tau}{(\Omega_m + \Omega_n)^2 - (\omega_{jk}^i)^2} - \frac{\cos (\Omega_m - \Omega_n)\tau - \cos \omega_{jk}^i \tau}{(\Omega_m - \Omega_n)^2 - (\omega_{jk}^i)^2} \right]
$$

$$
\left. + \sum_{m=1}^{3} \frac{A_m \Omega_m^2}{2} \left[\frac{\omega_{jk}^i (\cos 2\,\Omega_m \tau - \cos \omega_{jk}^i \tau)}{4\,\Omega_m^2 - (\omega_{jk}^i)^2} + \frac{1 - \cos \omega_{jk}^i \tau}{\omega_{jk}^i} \right] \right\} \Bigg) \Bigg\} \qquad (76)
$$

$j = 1, 2, \ldots M$

$k = 1, 2, \ldots N$

$i = 1, 2, 3$

$m = 1, 2, 3$

$n = 1, 2, 3$

The inverse transform of equation (73) is

$$
\begin{aligned}
V_{jk}(\tau) = \sum_{i=1}^{3} \frac{1}{D^{i}_{jk}} \Bigg[\Bigg\langle & v_{jk} \left\{ \left[a^{jk}_{11} - (\omega^{i}_{jk})^2 \right] \left[a^{jk}_{33} - (\omega^{i}_{jk})^2 \right] - (a^{jk}_{13})^2 \right\} \\
& - w_{jk} \left\{ a^{jk}_{23} \left[a^{jk}_{11} - (\omega^{i}_{jk})^2 \right] - a^{jk}_{12}\, a^{jk}_{13} \right\} \\
& - u_{jk} \left\{ a^{jk}_{12} \left[a^{jk}_{33} - (\omega^{i}_{jk})^2 \right] - a^{jk}_{13}\, a^{jk}_{23} \right\} \Bigg\rangle \cos \omega^{i}_{jk}\tau \\
& + \frac{1}{\omega^{i}_{jk}} \left\{ \left[a^{jk}_{11} - (\omega^{i}_{jk})^2 \right] \left[a^{jk}_{33} - (\omega^{i}_{jk})^2 \right] - a^{jk}_{23} \left[a^{jk}_{11} - (\omega^{i}_{jk})^2 \right] \right. \\
& \left. - a^{jk}_{13} \left(a^{jk}_{12} - a^{jk}_{13} \right) \right\} \cdot \Bigg(\delta_{1k} \sum_{m=1}^{3} \frac{A_m\, \omega^{i}_{jk}}{j\pi} \left\{ \bar{T}_o K \left[(-1)^j - 1 \right] - 2\Omega_m^{\ 2} \left[(-1)^j + 1 \right] \right\} \\
& \frac{\cos \Omega_m \tau - \cos \omega^{i}_{jk}\tau}{\Omega_m^{\ 2} - (\omega^{i}_{jk})^2} + \delta_{1k} \frac{\bar{T}_o K\gamma}{j\pi} \left[1 - (-1)^j \right] \sum_{m=1}^{3} \frac{A_m\, \omega^{i}_{jk}}{2} \\
& \left[\frac{\cos (\Omega_m - \bar{\Omega})\tau - \cos \omega^{i}_{jk}\tau}{(\Omega_m + \bar{\Omega})^2 - (\omega^{i}_{jk})^2} + \frac{\cos (\Omega_m - \bar{\Omega}) - \cos \omega^{i}_{jk}\tau}{(\Omega_m - \bar{\Omega})^2 - (\omega^{i}_{jk})^2} \right] \\
& + \delta_{2k} \left[(-1)^j - 1 \right] \Bigg\{ \sum_{m=1}^{3} \sum_{\substack{n=1 \\ m \neq n}}^{3} \frac{A_m A_n \Omega_m \Omega_n\, \omega^{i}_{jk}}{2} \left[\frac{\cos(\Omega_m + \Omega_n)\tau - \cos \omega^{i}_{jk}\tau}{(\Omega_m + \Omega_n)^2 - (\omega^{i}_{jk})^2} \right. \\
& \left. - \frac{\cos (\Omega_m - \Omega_n)\tau - \cos \omega^{i}_{jk}\tau}{(\Omega_m - \Omega_n)^2 - (\omega^{i}_{jk})^2} \right] + \sum_{m=1}^{3} \frac{A_m\, \Omega_m^{\ 2}}{2} \\
& \left[\frac{\omega^{i}_{jk} \left(\cos 2\Omega_m \tau - \cos \omega^{i}_{jk}\tau \right)}{4\, \Omega_m^{\ 2} - (\omega^{i}_{jk})^2} + \frac{1 - \cos \omega^{i}_{jk}\tau}{\omega^{i}_{jk}} \right] \Bigg\} \Bigg) \Bigg]
\end{aligned}
\tag{77}
$$

$j,k = 1,2,\ldots M,N$

$i,m,n = 1,2,3$

The inverse transform of equation (74) is

$$
\begin{aligned}
W_{jk}(\tau) = \sum_{i=1}^{3} \frac{1}{D^{i}_{jk}} \Bigg\{ \Bigg\langle & w_{jk} \left\{ \left[a^{jk}_{11} - (\omega^{i}_{jk})^2 \right] \left[a^{jk}_{22} - (\omega^{i}_{jk})^2 \right] - (a^{jk}_{12})^2 \right\} \\
& + u_{jk} \left\{ a^{jk}_{13} \left[a^{jk}_{22} - (\omega^{i}_{jk})^2 \right] - a^{jk}_{12}\, a^{jk}_{23} \right\} \\
& - v_{jk} \left\{ a^{jk}_{23} \left[a^{jk}_{11} - (\omega^{i}_{jk})^2 \right] - a^{jk}_{12}\, a^{jk}_{13} \right\} \Bigg\rangle \cos \omega^{i}_{jk}\tau \\
& + \frac{1}{\omega^{i}_{jk}} \left\{ \left[a^{jk}_{11} - (\omega^{i}_{jk})^2 \right] \left[a^{jk}_{22} - (\omega^{i}_{jk})^2 \right] - a^{jk}_{23} \left[a^{jk}_{11} - (\omega^{i}_{jk})^2 \right] - a^{jk}_{12}(a^{jk}_{12} - a^{jk}_{13}) \right\} \\
& \cdot \Bigg(\delta_{1k} \sum_{m=1}^{3} \frac{A_m \omega^{i}_{jk}}{j\pi} \left\{ \bar{T}_o K \left[(-1)^j - 1 \right] - 2\Omega_m^{\,2} \left[(-1)^j + 1 \right] \right\} \frac{\cos\Omega_m\tau - \cos \omega^{i}_{jk}\tau}{\Omega_m^{\,2} - (\omega^{i}_{jk})^2} \\
& + \delta_{1k} \frac{T_o K Y}{j\pi} \left[1 - (-1)^j \right] \sum_{m=1}^{3} \frac{A_m \omega^{i}_{jk}}{2} \left[\frac{\cos(\Omega_m + \bar{\Omega})\tau - \cos \omega^{i}_{jk}\tau}{(\Omega_m + \bar{\Omega})^2 - (\omega^{i}_{jk})^2} \right. \\
& \left. + \frac{\cos(\Omega_m - \bar{\Omega})\tau - \cos \omega^{i}_{jk}\tau}{(\Omega_m - \bar{\Omega})^2 - (\omega^{i}_{jk})^2} \right] + \delta_{2k} \left[(-1)^j - 1 \right] \Bigg[\Bigg\{ \sum_{m=1}^{3} \sum_{\substack{n=1 \\ m \neq n}}^{3} \frac{A_m A_n \Omega_m \Omega_n\, \omega^{i}_{jk}}{2} \\
& \left[\frac{\cos(\Omega_m + \Omega_n)\tau - \cos \omega^{i}_{jk}\tau}{(\Omega_m + \Omega_n)^2 - (\omega^{i}_{jk})^2} - \frac{\cos(\Omega_m - \Omega_n)\tau - \cos \omega^{i}_{jk}\tau}{(\Omega_m - \Omega_n)^2 - (\omega^{i}_{jk})^2} \right] \\
& + \sum_{m=1}^{3} \frac{A_m \Omega_m^{\,2}}{2} \left[\frac{\omega^{i}_{jk}(\cos 2\Omega_m - \cos \omega^{i}_{jk}\tau)}{4\,\Omega_m^{\,2} - (\omega^{i}_{jk})^2} + \frac{1 - \cos \omega^{i}_{jk}\tau}{\omega^{i}_{jk}} \right] \Bigg\} \Bigg) \Bigg] \Bigg\}
\end{aligned}
\tag{78}
$$

$$j,k = 1,2,\ldots M,N$$

$$i,m,n = 1,2,3$$

VII. Stability Considerations of the Dynamic Solution

Inspection of the expressions for U_{jk} (τ), V_{jk} (τ) and W_{jk} (τ) indicated that the solution becomes unbounded for a large number of possible values of the frequency of the applied thrust, Ω . When Ω is such that any term in the denominator of the expressions for U_{jk} (τ), V_{jk} (τ) and W_{jk} (τ) vanishes the solution becomes unbounded and thus is unstable. For example, from eq. (62) we see if

$$\Omega_m = \omega_{ok} \tag{79}$$

that U_{ok} (τ) is unstable. We can express Ω_m in terms of Ω as

$$\Omega_m = \left[(m-2)^2 + \varepsilon_m \beta \right] \frac{\Omega}{\omega} \; , \qquad m = 1,2,3 \tag{80}$$

where

$$\varepsilon_m = \begin{cases} -\frac{1}{2} & \text{for } m = 1 \\ \frac{1}{2} & \text{for } m = 2 \\ \frac{1}{2} & \text{for } m = 3 \end{cases} \tag{81}$$

Equation (79) can now be written

$$\left[(m-2)^2 + \varepsilon_m \beta \right] \frac{\Omega}{\omega} = \bar{\omega}_{ok} \tag{82}$$

$$k = 1,2, \cdots N$$

$$m = 1,2,3$$

recall that β is a function of Ω as shown by eqs. (15), (21), and (22). Eq. (82) is a transcendental relation that Ω must satisfy. When Ω is a root of eq. (82) U_{ok} (τ) becomes unbounded.

Similarily, eqs. (67), (68), (76), (77), and (78) reveal that instabilities exist when Ω is a root of any of the following equations:

$$\frac{\Omega}{\omega} = \omega^{i}_{jo} , \qquad j = 1, 2, \cdots M \quad i = 1,2 \tag{83}$$

$$\left[(m-2)^2 - (n-2)^2 + (\varepsilon_m - \varepsilon_n)\beta\right] \frac{\Omega}{\omega} = \omega^{i}_{jo} \tag{84}$$

$$\left[(m-2)^2 + (n-2)^2 + (\varepsilon_m + \varepsilon_n)\beta\right] \frac{\Omega}{\omega} = \omega^{i}_{jo} \tag{85}$$

$$j = 1, 2, \ldots, M$$
$$i = 1, 2$$
$$m = 1, 2, 3 \qquad m \neq n$$
$$n = 1, 2, 3$$

$$2\left[(m-2)^2 + \varepsilon_m\beta\right] \frac{\Omega}{\omega} = \omega^{i}_{jo} \tag{86}$$

$$j = 1, 2, \ldots M$$
$$i = 1, 2$$
$$m = 1, 2, 3$$

$$\left[(m-2)^2 + \varepsilon_m\beta\right] \frac{\Omega}{\omega} = \omega^{i}_{jk} \tag{87}$$

$$\left|(m-2)^2 - 1 + \varepsilon_m\beta\right| \frac{\Omega}{\omega} = \omega^{i}_{jk} \tag{88}$$

$$\left|(m-2)^2 + 1 + \varepsilon_m\beta\right| \frac{\Omega}{\omega} = \omega_{jk}^{i} \tag{89}$$

$j = 1, 2, \ldots M$

$k = 1, 2, \ldots N$

$i = 1, 2, 3$

$m = 1, 2, 3$

$$\left|(m-2)^2 - (n-2)^2 + (\varepsilon_m - \varepsilon_n)\beta\right| \frac{\Omega}{\omega} = \omega_{jk}^{i} \tag{90}$$

$$\left|(m-2)^2 + (n-2)^2 + (\varepsilon_m + \varepsilon_n)\beta\right| \frac{\Omega}{\omega} = \omega_{jk}^{i} \tag{91}$$

$j = 1, 2, \ldots M$

$k = 1, 2, \ldots N$

$i = 1, 2, 3$

$m = 1, 2, 3$

$m \neq n$

$n = 1, 2, 3$

$$2\left[(m-2)^2 + \varepsilon_m\beta\right] \frac{\Omega}{\omega} = \omega_{jk}^{i} \tag{92}$$

$j = 1, 2, \ldots M$

$k = 1, 2, \ldots N$

$i = 1, 2, 3$

$m = 1, 2, 3$

All of the above equations, with the exception of eq. (83), contain β. Due to the transcendental dependence of β on Ω, the equations that contain β will have to be solved by numerical iteration to determine the unstable values of the thrust frequency, Ω.

VIII. INITIAL CONDITIONS OF THE DYNAMIC SOLUTION

The initial conditions defined by eqs. (56) are arbitrary; however, one particular case of interest might be the static deflection of the shell before the application or the accelerating thrust. The shell is assumed to be inclined to the vertical at an angle α_o and supported by a distributed surface force applied at the lower end as shown in Figure 2.

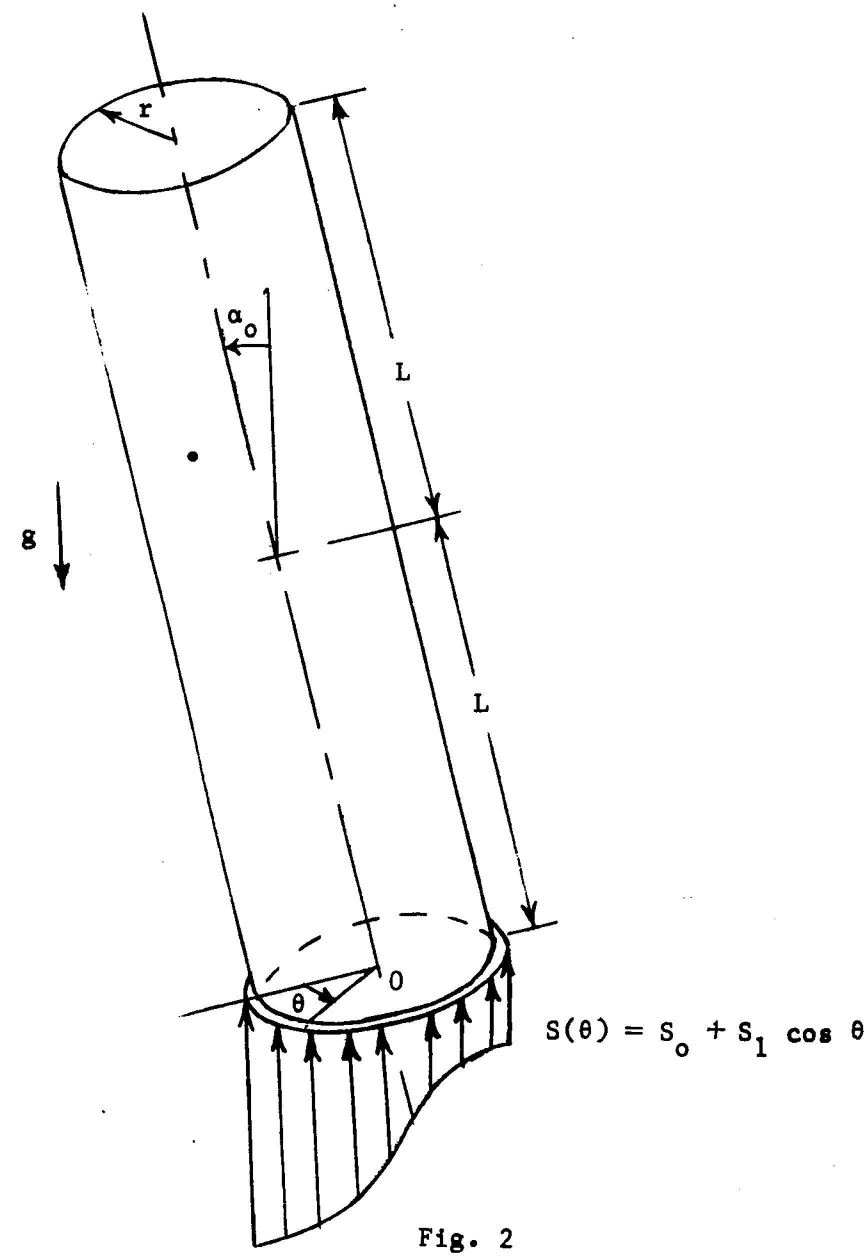

Fig. 2

By static equilibrium we find S_0 and S_1 as

$$S_0 = 2Lh\rho g, \qquad S_1 = 4Lh\rho g\,\lambda \sin\alpha_o \tag{93}$$

The equation of static equilibrium of a shell element is obtained from eqs. (7) by disregarding the inertial terms as

$$\frac{\partial^2 u}{\partial x^2} + \frac{(1-\nu)}{2r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{(1+\nu)}{2r}\frac{\partial^2 v}{\partial x\partial\theta} - \frac{\nu}{r}\frac{\partial w}{\partial x} = -\frac{(1-\nu^2)}{Eh}P_x(x,\theta) \qquad \text{(a)}$$

$$\frac{(1+\nu)}{2r}\frac{\partial^2 u}{\partial x\partial\theta} + \frac{(1-\nu)}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} - \frac{1}{r^2}\frac{\partial w}{\partial\theta} = -\frac{(1-\nu^2)}{Eh}P_\theta(x,\theta) \qquad \text{(b)}$$

$$\frac{\nu}{r}\frac{\partial u}{\partial x} + \frac{1}{r^2}\frac{\partial v}{\partial\theta} - \frac{1}{r^2}w - \frac{h^2}{12}\nabla^4 w = -\frac{(1-\nu^2)}{Eh}P_z(x,\theta) \qquad (94) \quad \text{(c)}$$

For this problem the distributed surface forces are

$$P_x(x,\theta) = \left[-\rho g h + S(\theta)\delta(x+L)\right]\cos\alpha_o\,, \qquad \text{(a)}$$

$$P_\theta(x,\theta) = \left[-\rho g h + S(\theta)\delta(x+L)\right]\sin\alpha_o \sin\theta\,, \qquad \text{(b)}$$

$$P_z(x,\theta) = \left[-\rho g h + S(\theta)\delta(x+L)\right]\sin\alpha_o \cos\theta \qquad (95) \quad \text{(c)}$$

Introducing into eqs. (94) and (95)

$$\bar{u} = u/L, \qquad \bar{v} = v/L, \qquad \bar{w} = w/L, \qquad \xi = x/L$$

$$\bar{g} = \frac{\rho g L}{E}, \qquad \bar{S}_o = \frac{S_o}{Eh}, \qquad \bar{S}_1 = \frac{S_1}{Eh} \tag{96}$$

and substituting eqs. (95) into (94) and assuming α_o small, we obtain

$$\frac{\partial^2 \bar{u}}{\partial \xi^2} + \frac{(1-\nu)}{2} \lambda^2 \frac{\partial^2 \bar{u}}{\partial \theta^2} + \frac{(1+\nu)}{2} \frac{\partial^2 \bar{v}}{\partial \xi \partial \theta} - \nu\lambda \frac{\partial \bar{w}}{\partial \xi} = (1-\nu^2)\left[\bar{g} - (\bar{S}_o + \bar{S}_1 \cos\theta)\delta(\xi+1)\right]$$

$$\frac{(1+\nu)}{2} \lambda \frac{\partial^2 \bar{u}}{\partial \xi \partial \theta} + \frac{(1-\nu)}{2} \frac{\partial^2 \bar{v}}{\partial \xi^2} + \lambda^2 \frac{\partial^2 \bar{v}}{\partial \theta^2} - \lambda^2 \frac{\partial \bar{w}}{\partial \theta} =$$

$$\alpha_o(1-\nu^2)\left[\bar{g} - (\bar{S}_o + \bar{S}_1 \cos\theta)\delta(\xi+1)\right] \sin\theta \tag{97}$$

$$\nu\lambda \frac{\partial \bar{u}}{\partial \xi} + \lambda^2 \frac{\partial \bar{v}}{\partial \theta} - \lambda^2 \bar{w} - \frac{\sigma^2}{12} \bar{\nabla}^4 \bar{w} = \alpha_o \left[\bar{g} - (\bar{S}_o + \bar{S}_1 \cos\theta)\delta(\xi+1)\right] \cos\theta$$

The edge conditions of the static deflection problem are the same as given in eqs. (47) for the dynamic problem

$$\bar{w}(\pm 1,\theta) = 0 \qquad M_\xi(\pm 1,\theta) = 0,$$

$$\bar{v}(\pm 1,\theta) = 0, \qquad N_\xi(\pm 1,\theta) = 0.$$

To satisfy these edge conditions we assume

$$\bar{u}(\xi,\theta) \doteq \sum_{m=o}^{M} \sum_{n=o}^{N} u_{mn} \cos\frac{m\pi}{2}(\xi+1) \cos n\theta \tag{a}$$

$$\bar{v}(\xi,\theta) \doteq \sum_{m=1}^{M} \sum_{n=1}^{N} v_{mn} \sin\frac{m\pi}{2}(\xi+1) \sin n\theta \tag{b}$$

$$\bar{w}(\xi,\theta) \doteq \sum_{m=1}^{M} \sum_{n=o}^{N} w_{mn} \sin\frac{m\pi}{2}(\xi+1) \cos n\theta \tag{98(c)}$$

Substituting the approximations of eqs. (98) into the equilibrium eqs. (97) and applying Galerkin's procedure we obtain

$$(0) \quad u_{oo} = 0 \tag{99}$$

thus u_{oo} is completely arbitrary.

$$u_{ok} = \frac{(1+\nu)}{\lambda^2} \bar{\bar{S}}_1 \ \delta_{1k} \qquad k = 1, 2, \ldots N \tag{100}$$

$$u_{jo} + \frac{2\nu\lambda}{j\pi} w_{jo} = \frac{4}{j\pi} (1-\nu^2) \bar{\bar{S}}_o, \qquad j = 1, 2, \ldots M \tag{101}$$

$$u_{jo} + \frac{(192\lambda^2 + \sigma^2 j^4 \pi^4)}{96 \ \nu\lambda j\pi} w_{jo} = 0, \qquad j = 1, 2, \ldots M \tag{102}$$

$$\left[j^2\pi^2 + 2(1-\nu)\lambda^2 k^2\right] u_{jk} - (1+\nu)\lambda jk\pi \ v_{jk} + 2\nu\lambda j\pi \ w_{jk} =$$
$$4(1-\nu^2) \bar{\bar{S}}_1 \ \delta_{1k} \tag{103}$$

$$-(1+\nu)\lambda jk\pi \ u_{jk} + \left[(1-\nu)\frac{j^2\pi^2}{2} + 4k^2\lambda^2\right] v_{jk} - 4\lambda^2 k \ w_{jk} =$$
$$-\frac{8}{j\pi} \bar{\bar{g}} \ \alpha_o (1-\nu^2)\left[1 - (-1)^j\right] \delta_{1k} \tag{104}$$

$$2\nu\lambda j\pi \ u_{jk} - 4\lambda^2 k \ v_{jk} + \left[4\lambda^2 + \frac{\sigma^2}{12}\left(\frac{j^2\pi^2}{4} + \lambda^2 k^2\right)^2\right] w_{jk} =$$
$$-\frac{16}{j\pi} \bar{\bar{g}} \ \alpha_o (1-\nu^2)\left[1 - (-1)^j\right] \delta_{1k} \tag{105}$$

$$j = 1, 2, \ldots M$$
$$k = 1, 2, \ldots N$$

Solving eqs. (101) and (102) for u_{jo} and w_{jo} we get

$$u_{jo} = - \frac{4\nu(1-\nu^2)\ (192\lambda^2 + \sigma^2 j^4\pi^4)}{j\pi\ \left[\sigma^2 j^4\pi^4 - 192(1-\nu^2)\lambda^2\right]}\ \bar{\bar{S}}_o \tag{106}$$

$$w_{jo} = \frac{384\ (1-\nu^2)\nu\lambda}{\sigma^2\ j^4\pi^4 - 192\lambda^2(1-\nu^2)}\ \bar{\bar{S}}_o \tag{107}$$

$$j = 1, 2, \ldots M$$

Defining

$$\Delta_{jk} = \begin{vmatrix} j^2\pi^2 + 2\lambda^2(1-\nu)k^2 & -jk\pi\lambda(1+\nu) & 2j\pi\nu\lambda \\ -jk\pi\lambda(1+\nu) & \frac{j^2\pi^2}{2}(1-\nu) + 4\lambda^2k^2 & -4\lambda^2 k \\ 2j\pi\nu\lambda & -4\lambda^2 k & \left[4\lambda^2 + \frac{\sigma^2}{12}\left(\frac{j^2\pi^2}{4} + \lambda^2k^2\right)^2\right] \end{vmatrix}$$

$$j = 1, 2, \ldots M \qquad k = 1, 2, \ldots N \tag{108}$$

and using Cramer's rule to solve eqs. (103), (104) and (105) we obtain

$$u_{jk} = \frac{4(1-\nu^2)}{\Delta jk}\,\delta_{1k}\left(\bar{S}_1\left\{\left[\frac{j^2\pi^2}{2}(1-\nu)+4\lambda^2k^2\right]\circ\left[4\lambda^2 + \frac{\sigma^2}{48}(j^2\pi^2+4\lambda^2k^2)^2\right]\right.\right.$$

$$\left. - 16\lambda^4k^2\right\} - 2\,\bar{g}\alpha_o\left[1-(-1)^j\right]\left\{\lambda k(1+\nu)\left[4\lambda^2 + \frac{\sigma^2}{48}(j^2\pi^2+4\lambda^2k^2)^2\right]\right.$$

$$\left.\left. - 8\lambda^3k\nu + 8\lambda^3(1-\nu)k^2 - 2j^2\pi^2\nu\lambda(1-\nu)\right\}\right) \tag{109}$$

$$v_{jk} = \frac{4(1-\nu^2)}{\Delta jk}\,\delta_{1k}\left(\bar{S}_1\left\{jk\pi\lambda(1+\nu)\left[4\lambda^2 + \frac{\sigma^2}{48}(j^2\pi^2+4k^2\lambda^2)^2\right]\right.\right.$$

$$- 8jk\pi\nu\lambda^3 - \frac{2\bar{g}}{j\pi}\alpha_o\left[1-(-1)^j\right]\left\{\left[j^2\pi^2 + 2k^2\lambda^2(1-\nu)\right]\left[4\lambda^2\right.\right.$$

$$\left. + \frac{\sigma^2}{48}(j^2\pi^2+4k^2\lambda^2)^2\right] - 4j^2\pi^2\nu^2\lambda^2 + 8\lambda^2k\left[j^2\pi^2 + 2k^2\lambda^2(1-\nu)\right]$$

$$\left.\left. - 4j^2k\pi^2\lambda^2(1+\nu)\right\}\right) \tag{110}$$

$$w_{jk} = \frac{4(1-\nu^2)}{\Delta jk}\,\delta_{1k}\left(\bar{S}_1\left\{4\lambda^3 jk^2\pi(1+\nu) - 2j\pi\nu\lambda\left[\frac{j^2\pi^2}{2}(1-\nu) + 4\lambda^2k^2\right]\right.\right.$$

$$- \frac{2\bar{g}\alpha_o}{j\pi}\left[1-(-1)^j\right]\cdot\left\{4\lambda^2k\left[j^2\pi^2 + 2\lambda^2k^2(1-\nu)\right] - 2j^2k\pi^2\lambda^2\nu(1+\nu)\right.$$

$$\left.\left. + \left[j^2\pi^2(1-\nu) + 2\lambda^2k^2\right]\left[j^2\pi^2 + 2\lambda^2(1-\nu)k^2\right] - 2j^2k^2\pi^2\lambda^2(1+\nu)^2\right\}\right) \tag{111}$$

$$j = 1, 2, \ldots M$$

$$k = 1, 2, \ldots N$$

This completes the analysis of the problem.

IX. CONCLUSIONS

This work represents an initial step in analyzing the dynamic structural behavior of a large rocket booster during powered flight. As such, the model accepted retained the features that the thrust was time varying and gimbaled for directional control.

The use of Galerkin's procedure rendered the shell equations of motion tractable. The advantages of this analysis are:

1. A complete solution was obtained for possible response studies.
2. Criteria for unstable frequencies of the time varying thrust were readily evident because of the form of the solution.
3. All the natural frequencies of the free vibration of a simply supported circular cylindrical shell were obtained.

The restrictions of the analysis are:

1. The use of Galerkin's procedure to satisfy the equations of motion is some weighted average manner.
2. Requiring that the center of mass always remains at the mid-point of a line that connects the centers of the end sections so that the edge conditions of eqs. (47) are applicable.

Future work on this problem will be to:

1. Generalize the model by using two circular cylinders joined by a rigid connector to simulate a large booster.
2. Allow different end conditions so as to remove the second restriction listed above.

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