X-547-65-81

NASA TM X-55244

FACILITY FORM 602

N65-29803 (ACCESSION NUMBER) (THRU)

25 (PAGES) 1 (CODE)

(NASA CR OR TMX OR AD NUMBER) 13 (CATEGORY)

ON THE DIFFERENCE BETWEEN PERIGEE AND THE POINT OF MINIMUM HEIGHT ABOVE THE EARTH'S SURFACE

GPO PRICE $

CFSTI PRICE(S) $

Hard copy (HC) 2.00

Microfiche (MF) .50

ff 653 July 65

APRIL 1965

NASA

GODDARD SPACE FLIGHT CENTER

GREENBELT, MARYLAND

X-547-65-81

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by

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April 1965

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ON THE DIFFERENCE BETWEEN PERIGEE AND THE POINT OF MINIMUM HEIGHT ABOVE THE EARTH'S SURFACE

The point of minimum height above the earth's surface does not occur, in general, at perigee. The combination of the dynamical and geometrical effects of the oblateness will cause this to be true. The result holds, however, even by virtue of the geometrical effect alone. In other words, this effect is associated not only with a perturbed orbit, but also with the mean elliptic orbit which approximates to it. This paper will show that, in this sense, the geodetic latitude of minimum height and the geodetic latitude at perigee are different, in general. Also, formulas for finding the minimum distance between two ellipses will be developed.

In our development we will use the model of the earth as an oblate spheriod and thus the intersection of the orbit plane and the earth forms an ellipse. Consider this ellipse and the ellipse representing the satellite orbit as shown in the following diagram. The center of the small ellipse is at one focus of the large ellipse.

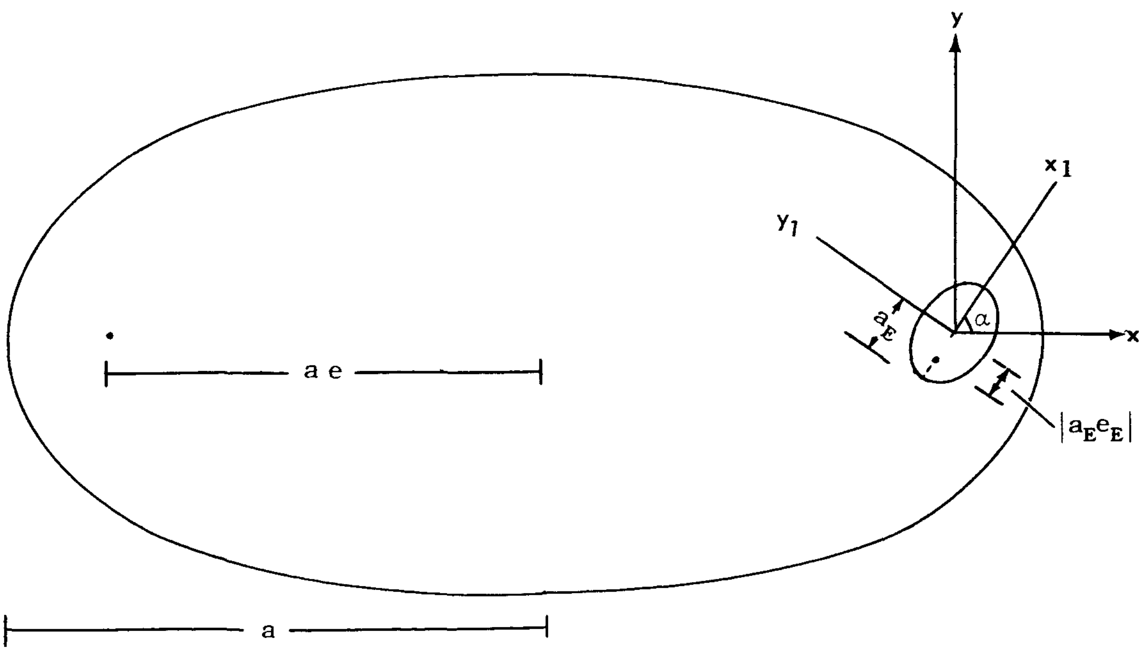

Figure 1

Let α be the angle between the major axis of the large ellipse and the major axis of the small ellipse measured in the counterclockwise direction. Then:

$$x_1 = x \cos \alpha + y \sin \alpha$$

$$y_1 = - x \sin \alpha + y \cos \alpha$$

Now for the small ellipse

$$\frac{x_1^2}{a_E^2} + \frac{y_1^2}{a_E^2 (1 - e_E^2)} = 1$$

$$\therefore \frac{(x \cos \alpha + y \sin \alpha)^2}{a_E^2} + \frac{(-x \sin \alpha + y \cos \alpha)^2}{a_E^2 (1 - e_E^2)} = 1$$

Solving for x, we find that

$$x = \frac{y e_E^2 \sin \alpha \cos \alpha \pm \sqrt{(1-e_E^2)} \sqrt{a_E^2 - a_E^2 e_E^2 \cos^2 \alpha - y^2}}{1 - e_E^2 \cos^2 \alpha}$$

Let (x_E, y_E) be any point on the small ellipse. Then:

$$x_E = \frac{y_E e_E^2 \sin \alpha \cos \alpha \pm \sqrt{1 - e_E^2} \sqrt{a_E^2 - a_E^2 e_E^2 \cos^2 \alpha - y_E^2}}{1 - e_E^2 \cos^2 \alpha} \tag{1}$$

$$\left.\frac{\partial x}{\partial y}\right|_{y = y_E} = \frac{e_E^2 \sin\alpha \cos\alpha \sqrt{a_E^2 - a_E^2 e_E^2 \cos^2 \alpha - y_E^2} \mp y_E \sqrt{1 - e_E^2}}{(1 - e_E^2 \cos^2 \alpha) \sqrt{a_E^2 - a_E^2 e_E^2 \cos^2 \alpha - y_E^2}} \tag{2}$$

$$\left.\frac{\partial^2 x}{\partial y^2}\right|_{y=y_E} = \frac{\mp\sqrt{1-e_E^2}}{1-e_E^2\cos^2\alpha}\left\{\left[a_E^2 - a_E^2 e_E^2 \cos^2\alpha - y_E^2\right]^{-1/2}\right.$$

$$\left. + y_E^2 \left[a_E^2 - a_E^2 e_E^2 \cos^2\alpha - y_E^2\right]^{-3/2}\right\}$$

and thus

$$\left.\frac{\partial^2 x}{\partial y^2}\right|_{y=y_E} = \mp\sqrt{1-e_E^2}\ \frac{a_E^2}{(a_E^2 - a_E^2 e_E^2 \cos^2\alpha - y_E^2)^{3/2}} \tag{3}$$

Now for the large ellipse $x'/a^2 + y'/b^2 = 1$ where $x' = x + ae$, $y' = y$.

$$\therefore \frac{(x+ae)^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

Solving for x, we find that

$$x = \pm\sqrt{a^2 - \frac{y^2}{1-e^2}} - ae$$

Let (x_0, y_0) be any point on the large ellipse. Thus:

$$x_0 = \pm\sqrt{a^2 - \frac{y_0^2}{1-e^2}} - ae \tag{4}$$

Now

$$\left.\frac{\partial x}{\partial y}\right|_{y=y_0} = \mp\frac{y_0}{\sqrt{a^2 - \frac{y_0^2}{1-e^2}}\,(1-e^2)} \tag{5}$$

$$\left.\frac{\partial^2 x}{\partial y^2}\right|_{y=y_0} = \mp \frac{\left[a^2 - \dfrac{y_0^2}{1 - e^2}\right]^{-1/2} + \dfrac{y_0^2}{1 - e^2}\left[a^2 - \dfrac{y_0^2}{1 - e^2}\right]^{-3/2}}{1 - e^2}$$

$$\therefore \left.\frac{\partial^2 x}{\partial y^2}\right|_{y=y_0} = \mp \frac{a^2}{(1 - e^2)\left[a^2 - \dfrac{y_0^2}{1 - e^2}\right]^{3/2}} \tag{6}$$

we will let

$$\left.\frac{\partial x}{\partial y}\right|_{y=y_0} = \frac{\partial x_0}{\partial y_0}, \quad \left.\frac{\partial^2 x}{\partial y^2}\right|_{y=y_0} = \frac{\partial^2 x_0}{\partial y_0^2}, \quad \left.\frac{\partial x}{\partial y}\right|_{y=y_E} = \frac{\partial x_E}{\partial y_E} \quad \text{and} \quad \left.\frac{\partial^2 x}{\partial y^2}\right|_{y=y_E} = \frac{\partial^2 x_E}{\partial y_E^2}$$

next

$$d = \sqrt{(x_0 - x_E)^2 + (y_0 - y_E)^2}$$

$$\frac{\partial d}{\partial y_E} = -\frac{(x_0 - x_E)\dfrac{\partial x_E}{\partial y_E} + (y_0 - y_E)}{\sqrt{(x_0 - x_E)^2 + (y_0 - y_E)^2}} \tag{7}$$

$$\frac{\partial^2 d}{\partial y_E^2} = -\frac{\left[(x_0 - x_E)\dfrac{\partial^2 x_E}{\partial y_E^2} - \left(\dfrac{\partial x_E}{\partial y_E}\right)^2 - 1\right]\cdot d - \left[(x_0 - x_E)\dfrac{\partial x_E}{\partial y_E} + (y_0 - y_E)\right]\dfrac{\partial d}{\partial y_E}}{(x_0 - x_E)^2 + (y_0 - y_E)^2}$$

$$\frac{\partial^2 d}{\partial y_E^2} = -\frac{\left[(x_0 - x_E)\dfrac{\partial^2 x_E}{\partial y_E^2} - \left(\dfrac{\partial x_E}{\partial y_E}\right)^2 - 1\right]\left[(x_0 - x_E)^2 + (y_0 - y_E)^2\right] + \left[(x_0 - x_E)\dfrac{\partial x_E}{\partial y_E} + (y_0 - y_E)\right]^2}{\left[(x_0 - x_E)^2 + (y_0 - y_E)^2\right]^{3/2}} . \tag{8}$$

also

$$\frac{\partial d}{\partial y_0} = \frac{(x_0 - x_E)\dfrac{\partial x_0}{\partial y_0} + (y_0 - y_E)}{\sqrt{(x_0 - x_E)^2 + (y_0 - y_E)^2}} \tag{9}$$

$$\frac{\partial^2 d}{\partial y_0^2} = \frac{\left[(x_0 - x_E)\dfrac{\partial^2 x_0}{\partial y_0^2} + \left(\dfrac{\partial x_0}{\partial y_0}\right)^2 + 1\right] \cdot d - \left[(x_0 - x_E)\dfrac{\partial x_0}{\partial y_0} + (y_0 - y_E)\right]\dfrac{\partial d}{\partial y_0}}{(x_0 - x_E)^2 + (y_0 - y_E)^2}$$

$$\frac{\partial^2 d}{\partial y_0^2} = \frac{\left[(x_0 - x_E)\dfrac{\partial^2 x_0}{\partial y_0^2} + \left(\dfrac{\partial x_0}{\partial y_0}\right)^2 + 1\right]\left[(x_0 - x_E)^2 + (y_0 - y_E)^2\right] - \left[(x_0 - x_E)\dfrac{\partial x_0}{\partial y_0} + (y_0 - y_E)\right]^2}{\left[(x_0 - x_E)^2 + (y_0 - y_E)^2\right]^{3/2}} \tag{10}$$

Now a point on the small ellipse and a point on the large ellipse are a minimum distance apart only if the line connecting them is normal to both ellipses. If the point on the large ellipse is at the pericenter then the normal to the large ellipse at that point must be the major axis of the large ellipse. For this major axis to be normal to the small ellipse either $\alpha = \pm\pi k$ or $\alpha = \pm\pi(2k+1)/2$, $k = 0, 1, 2, \cdots$. We will show that a minimum will occur in the case $\alpha = \pm\pi k$, $k = 0, 1, 2, \cdots$ and will occur under certain conditions in the case $\alpha = \pm\pi(2k+1)/2$, $k = 0, 1, 2, \cdots$. Let $y_E = y_0 = 0$ and $x_0 = a - ae$.

Now from (1)

$$x_E\Big|_{y_E=0} = \frac{\pm\sqrt{1 - e_E^2}\,\sqrt{a_E^2 - a_E^2 e_E^2 \cos^2\alpha}}{1 - e_E^2 \cos^2\alpha}$$

$$\therefore x_E\Big|_{y_E=0} = \pm a_E \sqrt{\frac{1 - e_E^2}{1 - e_E^2 \cos^2\alpha}} \tag{11}$$

From (2)

$$\left.\frac{\partial x_E}{\partial y_E}\right|_{y_E=0} = \frac{e_E^2 \sin \alpha \cos \alpha}{(1 - e_E^2 \cos^2 \alpha)} \tag{12}$$

From (3)

$$\left.\frac{\partial^2 x_E}{\partial y_E^2}\right|_{y_E=0} = \frac{\mp \sqrt{1 - e_E^2}}{a_E \, (1 - e_E^2 \cos^2 \alpha)^{3/2}} \tag{13}$$

From (5)

$$\left.\frac{\partial x_0}{\partial y_0}\right|_{y_0=0} = 0$$

From (6)

$$\left.\frac{\partial^2 x_0}{\partial y_0^2}\right|_{y_0=0} = \mp \frac{1}{a\,(1 - e^2)} \tag{14}$$

From (7)

$$\left.\frac{\partial d}{\partial y_E}\right|_{\substack{y_0=0\\ y_E=0}} = - \frac{(x_0 - x_E) \left.\dfrac{\partial x_E}{\partial y_E}\right|_{y_E=0}}{x_0 - x_E}$$

$$\therefore \left.\frac{\partial d}{\partial y_E}\right|_{\substack{y_0=0\\ y_E=0}} = - \frac{e_E^2 \sin \alpha \cos \alpha}{(1 - e_E^2 \cos^2 \alpha)} \tag{15}$$

We will now show that minimum distance may occur at the pericenter when $\alpha = \pm \pi \ (2k + 1)/2, k = 0, 1, 2, \cdots$. For simplicity let $\alpha = \pi/2$.

From (15)

$$\left.\frac{\partial d}{\partial y_E}\right|_{\substack{y_E = 0 \\ y_0 = 0 \\ \alpha = \pi/2}} = 0$$

From (11)

$$\left. x_E \right|_{\substack{y_E = 0 \\ \alpha = \pi/2}} = \pm a_E \sqrt{1 - e_E^2} \tag{16}$$

From (12)

$$\left.\frac{\partial x_E}{\partial y_E}\right|_{\substack{y_E = 0 \\ \alpha = \pi/2}} = 0$$

From (13)

$$\left.\frac{\partial^2 x_E}{\partial y_E^2}\right|_{\substack{y_E = 0 \\ \alpha = \pi/2}} = \frac{\mp \sqrt{1 - e_E^2}}{a_E} \tag{17}$$

From (4)

$$x_0\Big|_{\substack{y_0=0\\ \alpha=\pi/2}} = \pm\, a - a\, e \tag{18}$$

$\therefore$ from (8)

$$\frac{\partial^2 d}{\partial y_E^2}\Bigg|_{\substack{y_E=0\\ y_0=0\\ \alpha=\pi/2}} = -\,\frac{\left[\mp\dfrac{\sqrt{1-e_E^2}}{a_E}\,(x_0 - x_E) - 1\right](x_0 - x_E)^2}{(x_0 - x_E)^3}$$

$$\frac{\partial^2 d}{\partial y_E^2}\Bigg|_{\substack{y_E=0\\ y_0=0\\ \alpha=\pi/2}} = -\left[\frac{\mp\sqrt{1-e_E^2}}{a_E} - \frac{1}{x_0 - x_E}\right] \tag{19}$$

From (10)

$$\frac{\partial^2 d}{\partial y_0^2}\Bigg|_{\substack{y_E=0\\ y_0=0\\ \alpha=\pi/2}} = \frac{\mp 1}{a\,(1 - e^2)} + \frac{1}{x_0 - x_E} \tag{20}$$

From (9)

$$\frac{\partial^2 d}{\partial y_E\,\partial y_0} = \frac{\partial}{\partial y_E}\,\frac{(x_0 - x_E)\dfrac{\partial x_0}{\partial y_0} + (y_0 - y_E)}{d}$$

$$\frac{\partial^2 d}{\partial y_E\, \partial y_0} = \frac{\left[-\dfrac{\partial x_E}{\partial y_E}\dfrac{\partial x_0}{\partial y_0} - 1\right]\cdot d - \dfrac{\partial d}{\partial y_E}\left[(x_0 - x_E)\dfrac{\partial x_0}{\partial y_0} + (y_0 - y_E)\right]}{d^2}$$

$$\therefore \left.\frac{\partial^2 d}{\partial y_E\, \partial y_0}\right|_{\substack{y_0=0\\ y_E=0\\ \alpha=\pi/2}} = \frac{-d}{d^2}$$

Then

$$\left.\frac{\partial^2 d}{\partial y_E\, \partial y_0}\right|_{\substack{y_0=0\\ y_E=0\\ \alpha=\pi/2}} = \frac{-1}{x_0 - x_E} \tag{21}$$

We have a relative minimum only if $B^2 - AC < 0$ and $A + C > 0$ where A is the right hand side of (19), C is the right hand side of (20), B is the right hand side of (21). Thus,

$$B^2 = \frac{1}{(x_0 - x_E)^2} \tag{22}$$

Now both (14) and (17) must have the same sign since maximum x_0 and x_E occur when $y_0 = 0$, $y_E = 0$ and $x_0 > 0$, $x_E > 0$ while minimum x_0 and x_E occur when $y_0 = 0$, $y_E = 0$ and $x_0 < 0$, $x_E < 0$. By our knowledge of maxima and minima we can say when (14) and (17) are positive, $x_0 < 0$, $x_E < 0$ and $(x_0 - x_E) < 0$; when (14) and (17) are negative, $x_0 > 0$, $x_E > 0$ and $(x_0 - x_E) > 0$. Therefore:

$$\left.\begin{aligned} A &= -\frac{\sqrt{1 - e_E^2}}{a_E} + \frac{1}{x_0 - x_E} \\ C &= \frac{1}{a(1 - e^2)} + \frac{1}{x_0 - x_E} \end{aligned}\right\} x_0 < 0,\ x_E < 0 \tag{23}$$

or

$$\left.\begin{aligned} A &= \frac{\sqrt{1-e_E^2}}{a_E} + \frac{1}{x_0 - x_E} \\ C &= -\frac{1}{a\,(1-e^2)} + \frac{1}{x_0 - x_E} \end{aligned}\right\} x_0 > 0,\ x_E > 0 \qquad (24)$$

From (16) and (18) we have for $x_0 < 0$ and $x_E < 0$

$$x_0 - x_E = -a - a\,e + a_E \sqrt{1-e_E^2}$$

Using (22) and (23) we find that

$$B^2 - AC = -\left\{\frac{-\sqrt{1-e_E^2}}{a_E} \cdot \frac{1}{a\,(1-e^2)} + \left(\frac{1}{a\,(1-e^2)} - \frac{\sqrt{1-e_E^2}}{a_E}\right) \frac{1}{\left(-a - a\,e + a_E\sqrt{1-e_E^2}\right)}\right\}$$

$$= \frac{a\,e\sqrt{1-e_E^2} + a_E\,e_E^2 + a\,e^2\sqrt{1-e_E^2}}{a_E\,a\,(1-e^2)\left(a + a\,e - a_E\sqrt{1-e_E^2}\right)} > 0\,.$$

For Eq. (23) $B^2 - AC > 0$ and thus we do not have a minimum.

Consider (22) and (24), where

$$x_0 - x_E = a - a\,e - a_E\sqrt{1-e_E^2} > 0$$

$$B^2 - AC = -\left\{-\frac{\sqrt{1-e_E^2}}{a_E} \cdot \frac{1}{a\,(1-e^2)} + \left(\frac{\sqrt{1-e_E^2}}{a_E} - \frac{1}{a\,(1-e^2)}\right) \frac{1}{\left(a - a\,e - a_E\sqrt{1-e_E^2}\right)}\right\}$$

$$= \frac{-a\,e\sqrt{1-e_E^2} + a_E\,e_E^2 + a\,e^2\sqrt{1-e_E^2}}{a_E\,a\,(1-e^2)\left(a - a\,e - a_E\sqrt{1-e_E^2}\right)}$$

Now $B^2 - AC > 0$ only if

$$a_E e_E^2 - a e \sqrt{1-e_E^2}\,(1-e) > 0$$

This will occur if

$$\frac{a_E e_E^2}{\sqrt{1-e_E^2}} > a e (1-e)$$

and in this case we would not have a minimum. In the case

$$\frac{a_E e_E^2}{\sqrt{1-e_E^2}} < a e (1-e)$$

we would either have a minimum or maximum depending on whether $A + C > 0$ or $A + C < 0$ respectively.

Now

$$A + C = \frac{\sqrt{1-e_E^2}}{a_E} + \frac{1}{x_0 - x_E} - \frac{1}{a\,(1-e^2)} + \frac{1}{x_0 - x_E}$$

$$= \frac{a^2(1-e)\sqrt{1-e_E^2}\,(1-e^2) + a_E a e_E^2 (1-e^2) + a_E a e (1-e) + a_E^2 \sqrt{1-e_E^2}}{(x_0 - x_E)\; a_E\, a (1-e^2)} > 0.$$

$\therefore$ we have a minimum distance for $\alpha = \pm \pi\,(2k+1)/2$, $k = 0, 1, 2, \cdots$ when $x_0 > 0$, $x_E > 0$ and

$$\frac{a_E e_E^2}{\sqrt{1-e_E^2}} < a e\,(1-e).$$

We will now show that minimum distance does occur at the pericenter if $\alpha = \pm\pi k$, $k = 0, 1, 2, \cdots$. For simplicity let $\alpha = 0$.

$$\left.\frac{\partial^2 d}{\partial y_E^2}\right|_{\substack{y_E=0\\ y_0=0\\ \alpha=0}} = \mp \frac{1}{a_E(1 - e_E^2)} - \frac{1}{x_0 - x_E}$$

Let

$$A_1 = \mp \frac{1}{a_E(1 - e_E^2)} - \frac{1}{x_0 - x_E}$$

From (20)

$$\left.\frac{\partial^2 d}{\partial y_0^2}\right|_{\substack{y_E=0\\ y_0=0\\ \alpha=0}} = \mp \frac{1}{a(1 - e^2)} + \frac{1}{x_0 - x_E}$$

Let

$$C_1 = \mp \frac{1}{a(1 - e^2)} + \frac{1}{x_0 - x_E}$$

$$\left.\frac{\partial^2 d}{\partial y_E \partial y_0}\right|_{\substack{y_E=0\\ y_0=0\\ \alpha=0}} = - \frac{1}{x_0 - x_E}$$

Let

$$B_1 = - \frac{1}{x_0 - x_E}$$

Now we have two cases. Either

$$\left.\begin{aligned} A_1 &= -\frac{1}{a_E(1 - e_E^2)} + \frac{1}{x_0 - x_E} \\ C_1 &= \frac{1}{a(1 - e^2)} + \frac{1}{x_0 - x_E} \end{aligned}\right\} \quad x_0 < 0,\ x_E < 0$$

or

$$\left.\begin{aligned} A_1 &= \frac{1}{a_E(1 - e_E^2)} + \frac{1}{x_0 - x_E} \\ C_1 &= -\frac{1}{a(1 - e^2)} + \frac{1}{x_0 - x_E} \end{aligned}\right\} \quad x_0 > 0,\ x_E > 0$$

But in Figure 1, the pericenter is to the right of the y axis and thus we need only consider the second case. For $\alpha = 0,\ x_0 - x_E = a - a e - a_E > 0$.

$$\therefore B_1^2 - A_1 C_1 = \frac{-a e(1 - e) - a_E e_E^2}{(x_0 - x_E) a_E a(1 - e_E^2)(1 - e)^2} \lessdot 0$$

and

$$A_1 + C_1 = \frac{1}{a_E(1 - e_E^2)} + \frac{1}{x_0 - x_E} + \frac{a_E + a e(1 - e)}{(x_0 - x_E) a(1 - e^2)} > 0$$

Therefore, in this case we have a minimum.

Thus we conclude that minimum distance occurs at the pericenter when $\alpha = \pm\pi k$, $k = 0, 1, 2, \cdots$. We also conclude that minimum distance does not occur at the pericenter when $\alpha \neq \pm\pi k$, $k = 0, 1, 2, \cdots$ except in the case $\alpha = \pm\pi\ (2k+1)/2$, $k = 0, 1, 2, \cdots$ and occurs in this case if $x_0 > 0$, $x_E > 0$ and

$$\frac{a_E\, e_E^2}{\sqrt{1-e_E^2}} < a\, e\, (1-e)\,.$$

We wish to find the points (x_0^*, y_0^*), (x_E^*, y_E^*) which are a minimum distance apart in the orbit plane.

$\therefore$ set $\partial d/\partial y_0 = 0$ and $\partial d/\partial y_E = 0$ and solve for y_0 and y_E.

From (7) we get

$$\frac{\partial x_E}{\partial y_E} = -\frac{y_0 - y_E}{x_0 - x_E} \tag{25}$$

From (9) we get

$$\frac{\partial x_0}{\partial y_0} = -\frac{y_0 - y_E}{x_0 - x_E} \tag{26}$$

$$\therefore\ \frac{\partial x_E}{\partial y_E} = \frac{\partial x_0}{\partial y_0} \tag{27}$$

Solve (25) and (26) simultaneously to find y_0 and y_E. From (4), (5) and (26) we have

$$\mp \frac{y_0}{\sqrt{a^2 - \dfrac{y_0^2}{1-e^2}}\ (1-e^2)} = -\frac{y_0 - y_E}{\pm\sqrt{a^2 - \dfrac{y_0^2}{1-e^2}} - a\,e - x_E} \tag{28}$$

From (5) and (27) we have

$$\mp \frac{y_0}{\sqrt{a^2 - \frac{y_0^2}{1-e^2}(1-e^2)}} = \frac{\partial x_E}{\partial y_E}$$

$$\therefore y_0 = \frac{a(1-e^2)\frac{\partial x_E}{\partial y_E}}{\pm\sqrt{1 + (1-e^2)\left(\frac{\partial x_E}{\partial y_E}\right)^2}} \tag{29}$$

Use (28) and let

$$f(y_E) = \mp \frac{y_0}{\sqrt{a^2 - \frac{y_0^2}{1-e^2}(1-e^2)}} \tag{30}$$

and

$$g(y_E) = -\frac{y_0 - y_E}{\pm\sqrt{a^2 - \frac{y_0^2}{1-e^2}} - ae - x_E} \tag{31}$$

Now

$$f(y_E + h) \approx f(y_E) + h f'(y_E)$$

and

$$g(y_E + h) \approx g(y_E) + h g'(y_E)$$

we want

$$f(y_E + h) = g(y_E + h)$$

$$\therefore f(y_E) + h f'(y_E) = g(y_E) + h g'(y_E)$$

$$\therefore h = \frac{g(y_E) - f(y_E)}{f'(y_E) - g'(y_E)} \tag{32}$$

From (30)

$$f'(y_E) = \mp \frac{1}{1-e^2} \frac{\sqrt{a^2 - \frac{y_0^2}{1-e^2}} + \frac{y_0^2}{1-e^2}\left[a^2 - \frac{y_0^2}{1-e^2}\right]^{-1/2}}{a^2 - \frac{y_0^2}{1-e^2}} \frac{\partial y_0}{\partial y_E}$$

Then

$$f'(y_E) = \mp \frac{1}{1-e^2} \frac{a^2}{\left(a^2 - \frac{y_0^2}{1-e^2}\right)^{3/2}} \frac{\partial y_0}{\partial y_E} \tag{33}$$

From (29) we obtain

$$\frac{\partial y_0}{\partial y_E} = \pm a(1-e^2) \frac{\frac{\partial^2 x_E}{\partial y_E^2}\sqrt{1+(1-e^2)\left(\frac{\partial x_E}{\partial y_E}\right)^2} - \frac{\partial x_E}{\partial y_E}\left[1+(1-e^2)\left(\frac{\partial x_E}{\partial y_E}\right)^2\right]^{-1/2} \frac{\partial x_E}{\partial y_E}\frac{\partial^2 x_E}{\partial y_E^2}(1-e^2)}{1+(1-e^2)(\partial x_E/\partial y_E)^2}$$

$$= \pm a(1-e^2) \frac{\frac{\partial^2 x_E}{\partial y_E^2}\left[1+(1-e^2)\left(\frac{\partial x_E}{\partial y_E}\right)^2\right] - \left(\frac{\partial x_E}{\partial y_E}\right)^2 \frac{\partial^2 x_E}{\partial y_E^2}(1-e^2)}{\left[1+(1-e^2)\left(\frac{\partial x_E}{\partial y_E}\right)^2\right]^{3/2}}$$

$$\therefore \frac{\partial y_0}{\partial y_E} = \pm a(1-e^2) \frac{\frac{\partial^2 x_E}{\partial y_E^2}}{\left[1+(1-e^2)\left(\frac{\partial x_E}{\partial y_E}\right)^2\right]^{3/2}} \tag{34}$$

Next

$$g'(y_E) = - \frac{\left(\frac{\partial y_0}{\partial y_E} - 1\right)\left[\pm\sqrt{a^2 - \frac{y_0^2}{1-e^2}} - ae - x_E\right] - (y_0 - y_E)\left[\mp\left(a^2 - \frac{y_0^2}{1-e^2}\right)^{-1/2} \frac{y_0}{1-e^2}\frac{\partial y_0}{\partial y_E} - \frac{\partial x_E}{\partial y_E}\right]}{\left[\pm\sqrt{a^2 - \frac{y_0}{1-e^2}} - ae - x_E\right]^2}$$

$$= - \frac{\left(a^2 - \frac{y_0^2}{1-e^2}\right)^{1/2}\left(\frac{\partial y_0}{\partial y_E} - 1\right)\left[\pm\left(a^2 - \frac{y_0^2}{1-e^2}\right)^{1/2} - ae - x_E\right] - (y_0 - y_E)\left[\mp\frac{y_0}{1-e^2}\frac{\partial y_0}{\partial y_E} - \frac{\partial x_E}{\partial y_E}\left(a^2 - \frac{y_0^2}{1-e^2}\right)^{1/2}\right]}{\left[\pm\sqrt{a^2 - \frac{y_0^2}{1-e^2}} - ae - x_E\right]^2\left(a^2 - \frac{y_0^2}{1-e^2}\right)^{1/2}}$$

$$\therefore g'(y_E) = \frac{1 - \frac{\partial y_0}{\partial y_E}}{\pm\left(a^2 - \frac{y_0^2}{1-e^2}\right)^{1/2} - ae - x_E} + \frac{(y_0 - y_E)\left[\mp\frac{y_0}{1-e^2}\frac{\partial y_0}{\partial y_E} - \frac{\partial x_E}{\partial y_E}\left(a^2 - \frac{y_0^2}{1-e^2}\right)^{1/2}\right]}{\left[\pm\left(a^2 - \frac{y_0^2}{1-e^2}\right)^{1/2} - ae - x_E\right]\left(a^2 - \frac{y_0^2}{1-e^2}\right)^{1/2}} \tag{35}$$

We can now use the equations $f_{n+1}(y_E) = f_n(y_E + h)$ and $g_{n+1}(y_E) = g_n(y_E + h)$ along with equations (1), (2), (3), (29), (30), (31), (32), (33), (34), and (35) to iterate and thus find y_E^*. Then we can use (29) to find y_0^*, (1) to find x_E^* and (4) to find x_0^*.

We now have the points (x_0^*, y_0^*) and (x_E^*, y_E^*) which are a minimum distance apart in the orbit plane. We next wish to find the geocentric latitude and the longitude of these points.

From Figure 2 and knowledge of spherical trigonometry we have $\sin\phi = \sin i \sin c$ where ϕ is geocentric latitude. Also $\tan(\alpha + c) = y_0^*/x_0^* \therefore \alpha + c = \tan^{-1}(y_0^*/x_0^*) \therefore c = \tan^{-1}(y_0^*/x_0^*) - \alpha$ and $\phi = \sin^{-1}(\sin i \sin c)$.

Next let λ' be the longitude of ascending node. Note λ' is negative in Figure 2. $\sin\lambda_1 = \tan\phi \cot i \therefore \lambda_1 = \sin^{-1}(\tan\phi \cot i)$. In Figure 2 λ_1 is negative. Now $\lambda = \lambda' - \lambda_1 = \lambda' - \sin^{-1}(\tan\phi \cot i)$.

Finally to find geocentric latitude and longitude of (x_E^*, y_E^*), replace x_0^* with x_E^* and y_0^* with y_E^* in the above equations.

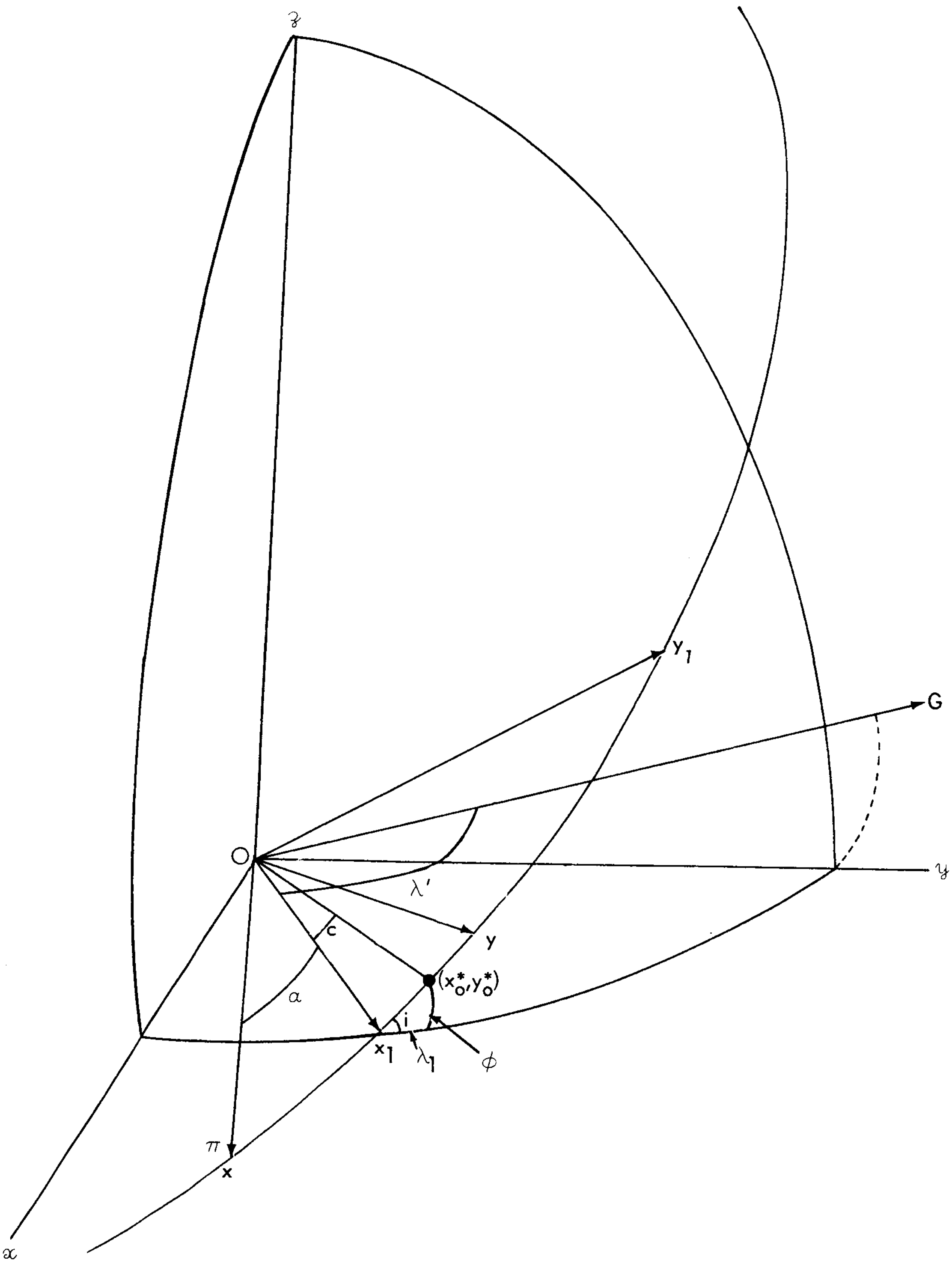

Figure 2

It is interesting to point out that if we desire to find the minimum distance between two orbit ellipses which have the same primary, are in the same plane and do not intersect each other, we can change the equation for the small ellipse from

$$\frac{x_1^2}{a_E^2} + \frac{y_1^2}{a_E^2(1 - e^2)} = 1$$

to

$$\frac{(x_1 + a_E\, e_E)^2}{a_E^2} + \frac{y_1^2}{a_E^2(1 - e_E^2)} = 1$$

and use the derivation given in this paper.

It will now be shown that the point of minimum height above the earth's surface does not occur, in general, at perigee.

Let $d_1 = \sqrt{(x_0 - x_E)^2 + (y_0 - y_E)^2 + (z_0 - z_E)^2}$ where (x_0, y_0, z_0) and (x_E, y_E, z_E) are given in the xyz coordinate system. (x_E, y_E, z_E) is any point on the earth (as an oblate spheriod) and (x_0, y_0, z_0) is any point on the orbit ellipse. Therefore $z_0 = 0$.

Thus

$$d_1 = \sqrt{(x_0 - x_E)^2 + (y_0 - y_E)^2 + z_E^2}$$

$$\therefore 2d_1 \frac{\partial d_1}{\partial y_0} = 2(x_0 - x_E)\frac{\partial x_0}{\partial y_0} + 2(y_0 - y_E) = 0$$

$$(x_0 - x_E)\frac{\partial x_0}{\partial y_0} + (y_0 - y_E) = 0$$

Now at perigee $y_0 = 0$. If minimum height occurs at perigee we would have

$$(x_0 - x_E)\left.\frac{\partial x_0}{\partial y_0}\right|_{y_0=0} - y_E = 0$$

From equation (5) we have

$$\frac{\partial x_0}{\partial y_0} = \mp \frac{y_0}{\sqrt{a^2 - \frac{y_0^2}{1 - e^2}(1 - e^2)}}$$

Thus

$$\left.\frac{\partial x_0}{\partial y_0}\right|_{y_0=0} = 0$$

We immediately conclude that if minimum height occurs at perigee, then $y_E = 0$. This implies that the line of minimum height lies in the xz plane (see Figure 3). We also know that the line of minimum height must be contained in a meridian plane since only then can the line be normal to the earth's surface. Thus the line of minimum height must be contained in the meridian plane containing perigee.

We are primarily interested in the general case where $i \neq 0$ and $\alpha \neq 0$ but we will first consider the special cases where $i = 0$ and where $i \neq 0$ but $\alpha = 0$. For $i = 0$ the meridian plane containing perigee is also the xz plane. The normal from perigee will thus be in the xz plane, the meridian plane and the orbit plane. In this case we do have the minimum height at perigee. If $i \neq 0$ but $\alpha = 0$ then the line of minimum height can be in the meridian plane and the xz plane but it must also be in the orbit plane. We can see from the conclusions of the first part of this paper that this is possible and thus minimum height is at perigee when $i \neq 0$, and $\alpha = 0$.

If $i \neq 0$ and $\alpha \neq 0$ then once again the line of minimum height can be in the meridian plane and the xz plane but only if it is also in the orbit plane. From the conclusions of the first part of this paper we know that it is impossible for the minimum height to be at perigee when $\alpha \neq 0$ unless $\alpha = \pi/2$. Now if $i \neq 0$, and the line of minimum height is not in the orbit plane then it cannot be in both the meridian plane and the xz plane at the same time and we have a contradiction. Thus minimum height can not occur at perigee when $i \neq 0$, $\alpha \neq 0$, $\alpha \neq \pi/2$.

Therefore minimum height cannot occur at perigee unless $i = 0$ or unless either $\alpha = 0$ or $\alpha = \pi/2$. For $\alpha = \pi/2$ the restrictions given earlier in this paper still hold. We thus conclude that the point of minimum height above the earth's surface does not occur, in general, at perigee.

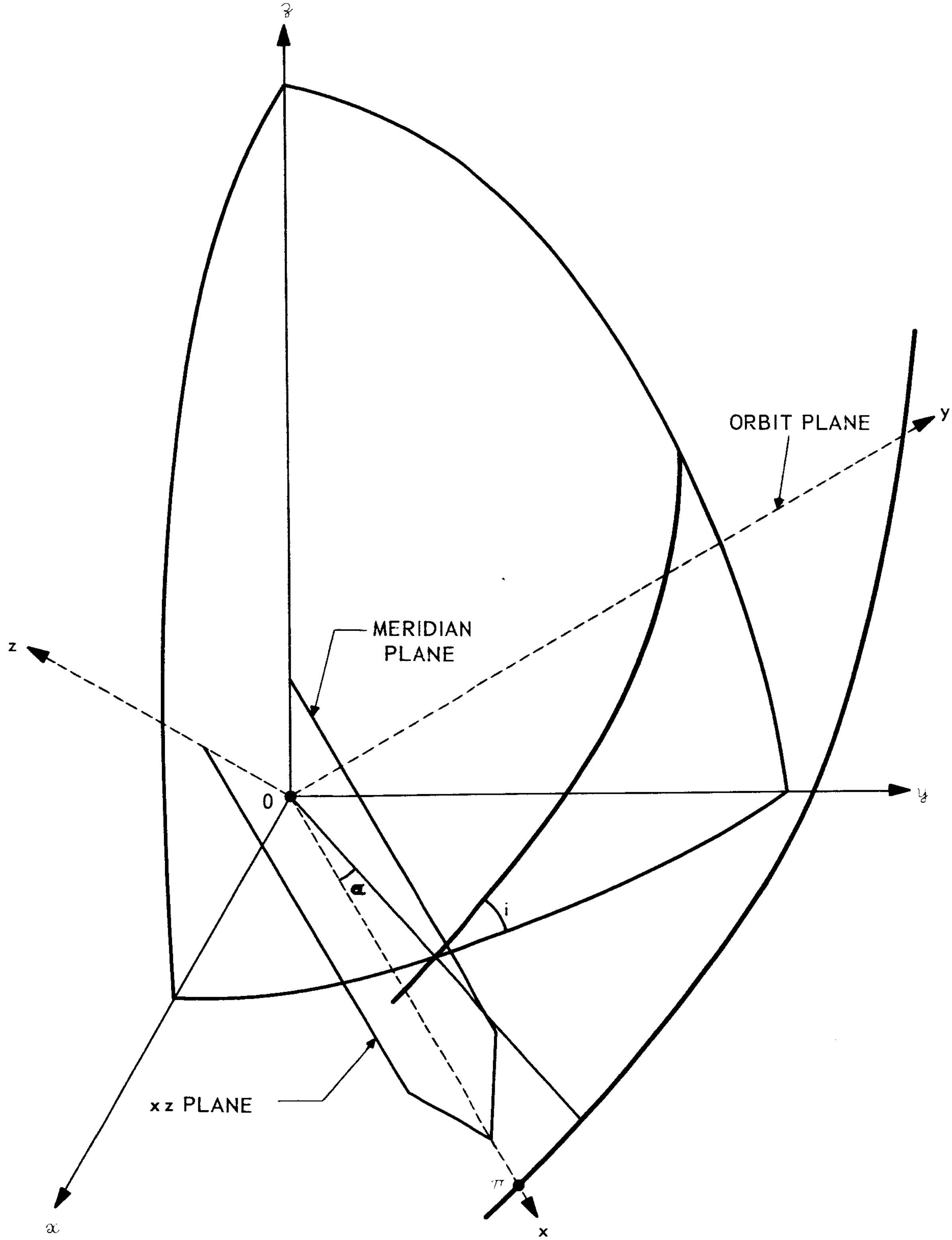

Figure 3

NOMENCLATURE

a = semi-major axis of large ellipse; semi-major axis of satellite orbit

a_E = semi-major axis of small ellipse; equatorial radius of earth

α = angle between major axis of large ellipse and major axis of small ellipse measured in the counterclockwise direction; α or $\pi - \alpha$ is the argument of perigee

c = central angle, measured in the orbit plane, from the ascending node to (x_0^*, y_0^*)

$d = \sqrt{(x_0 - x_E)^2 + (y_0 - y_E)^2}$

$d_1 = \sqrt{(x_0 - x_E)^2 + (y_0 - y_E)^2 + z_E^2}$

e = eccentricity of large ellipse; eccentricity of satellite orbit

e_E = eccentricity of small ellipse

G = intersection of prime meridian and equatorial plane

i = inclination of the orbit plane

λ' = longitude of ascending node

λ_1 = longitude of ascending node minus longitude of (x_0^*, y_0^*)

λ = longitude of (x_0^*, y_0^*)

O = center of earth

ϕ = geocentric latitude of (x_0^*, y_0^*)

π = perigee

(x_E, y_E) = any point on the small ellipse

(x_0, y_0) = any point on the large ellipse

(x_E^*, y_E^*) = point on the small ellipse which is minimum distance from the large ellipse

(x_0^*, y_0^*) = point on the large ellipse which is minimum distance from the small ellipse

(x_E, y_E, z_E) = point on the earth (as an oblate spheriod) given in the xyz coordinate system

$(x_0, y_0, 0)$ = point on the ellipse given in the xyz coordinate system

x, x_1, y, y_1 are the same axes in both Figure 1 and Figure 2.

The author wishes to thank Dr. Joseph W. Siry and Mr. W. S. Soar for the many helpful suggestions which they contributed toward this work.