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SOLUTION OF DELAUNAY'S EQUATIONS OF TYPE II

BY

THEODORE L. FELSENTREGER

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NASA GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

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SUMMARY

An analytic solution of Delaunay's Equations of Type II is obtained. The results can be applied to artificial satellite problems, including satellites of small eccentricity.

CONTENTS

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INTRODUCTION

Delaunay, in the expansion of his lunar theory, found it necessary to solve a particular pair of differential equations involving eccentricity and the argument of a trigonometric term in the disturbing function (Reference 1). These he called his "Equations of Type II"—they differ from the Equations of Type I in that eccentricity appears also as a divisor. They have the general form

$$\frac{de}{dt} = M\left(1 + \sum_{i=1}^{\infty} M_i e^{2i}\right)\sin\theta$$

$$\frac{d\theta}{dt} = N\left(1 + \sum_{i=1}^{\infty} N_i e^{2i}\right) + \frac{M}{e}\left(1 + \sum_{i=1}^{\infty} P_i e^{2i}\right)\cos\theta \ , \tag{1}$$

where e is the eccentricity, θ is the argument, M is the small parameter of the problem, and N, M_i, N_i, P_i are all of zero order with respect to M. Furthermore, the coefficients are assumed to be independent of e, θ, and time.

These equations were also discussed by Tisserand (Reference 2) and Brown (Reference 3). The results derived in this paper are an extension of those presented by previous authors—the solutions for $e\cos\theta$ and $e\sin\theta$ are complete through order $e_0^4 Q^3$, and the formulas for e and θ are also extended. In addition, the method of solution is outlined in greater detail for the convenience of the reader.

SOLUTION FOR $e\cos\theta$, $e\sin\theta$

To avoid difficulties arising from the presence of e as a divisor, we consider the variables

$$x = e\cos\theta \ , \qquad y = e\sin\theta \ .$$

Then, from Equations (1),

$$\frac{dx}{dt} = NQ\left(\sum_{i=1}^{\infty} M_i\, e^{2i-2} - \sum_{i=1}^{\infty} P_i\, e^{2i-2}\right) xy - N\left(1 + \sum_{i=1}^{\infty} N_i\, e^{2i}\right) y$$

$$\frac{dy}{dt} = NQ + NQ\left(y^2 \sum_{i=1}^{\infty} M_i\, e^{2i-2} + x^2 \sum_{i=1}^{\infty} P_i\, e^{2i-2}\right) + N\left(1 + \sum_{i=1}^{\infty} N_i\, e^{2i}\right) x\,, \tag{2}$$

where $Q = M/N$.

A solution will be derived for x and y which is of the form

$$x = E_0 + \left(e_0 + E_1\right) \cos \overline{\theta} + \sum_{n=2}^{\infty} E_n \cos n\overline{\theta}$$

$$y = e_0 \sin \overline{\theta} + \sum_{n=2}^{\infty} E_n' \sin n\overline{\theta} \qquad \left[\overline{\theta} = \theta_0\,(t + c)\right]\,, \tag{3}$$

where e_0, c are arbitrary constants, θ_0 is of order zero, and E_n, E_n' are of the order $e_0^n\,Q$ for even n and order $e_0^n\,Q^2$ for odd n. Assuming the convergence of Equations (3) is such that term-by-term differentiation is admissible, we have

$$\frac{dx}{dt} = -\left(e_0 + E_1\right) \theta_0 \sin \overline{\theta} - \sum_{n=2}^{\infty} n\, E_n\, \theta_0 \sin n\overline{\theta}$$

$$\frac{dy}{dt} = e_0\, \theta_0 \cos \overline{\theta} + \sum_{n=2}^{\infty} n\, E_n'\, \theta_0 \cos n\overline{\theta}\,. \tag{4}$$

A solution complete through order $e_0^4\, Q^3$ necessitates the inclusion of terms with arguments $\overline{\theta}$, $2\overline{\theta}$, $3\overline{\theta}$, and $4\overline{\theta}$ in Equations (3). Equations (3) and (4) then become

$$x = E_0 + \left(e_0 + E_1\right) \cos \overline{\theta} + E_2 \cos 2\overline{\theta} + E_3 \cos 3\overline{\theta} + E_4 \cos 4\overline{\theta}$$

$$y = e_0 \sin \overline{\theta} + E_2' \sin 2\overline{\theta} + E_3' \sin 3\overline{\theta} + E_4' \sin 4\overline{\theta} \tag{5}$$

and

$$\frac{dx}{dt} = -\left(e_0 + E_1\right)\theta_0 \sin\bar{\theta} - 2E_2\theta_0 \sin 2\bar{\theta} - 3E_3\theta_0 \sin 3\bar{\theta} - 4E_4\theta_0 \sin 4\bar{\theta}$$

$$\frac{dy}{dt} = e_0\theta_0 \cos\bar{\theta} + 2E_2'\theta_0 \cos 2\bar{\theta} + 3E_3'\theta_0 \cos 3\bar{\theta} + 4E_4'\theta_0 \cos 4\bar{\theta} \,. \tag{6}$$

Equations (2) are

$$\frac{dx}{dt} = NQ\Big[\left(M_1 - P_1\right)xy + \left(M_2 - P_2\right)xye^2 + \left(M_3 - P_3\right)xye^4\Big] - N\left(y + N_1 ye^2 + N_2 ye^4 + N_3 ye^6\right)$$

$$\frac{dy}{dt} = NQ + NQ\left(M_1 y^2 + P_1 x^2 + M_2 y^2 e^2 + P_2 x^2 e^2 + M_3 y^2 e^4 + P_3 x^2 e^4\right) + N\left(x + N_1 xe^2 + N_2 xe^4 + N_3 xe^6\right) \,, \tag{7}$$

where

$$\begin{aligned} x^2 = {} & \left(E_0^2 + \frac{1}{2}e_0^2 + e_0E_1 + \frac{1}{2}E_2^2\right) + \left(2e_0E_0 + 2E_0E_1 + e_0E_2 + E_1E_2\right)\cos\bar{\theta} \\ & + \left(\frac{1}{2}e_0^2 + e_0E_1 + 2E_0E_2 + e_0E_3\right)\cos 2\bar{\theta} + \left(e_0E_2 + E_1E_2 + 2E_0E_3\right)\cos 3\bar{\theta} \\ & + \left(2E_0E_4 + e_0E_3 + \frac{1}{2}E_2^2\right)\cos 4\bar{\theta} \end{aligned}$$

$$\begin{aligned} y^2 = {} & \left(\frac{1}{2}e_0^2 + \frac{1}{2}E_2'^2\right) + e_0E_2'\cos\bar{\theta} + \left(-\frac{1}{2}e_0^2 + e_0E_3'\right)\cos 2\bar{\theta} \\ & - e_0E_2'\cos 3\bar{\theta} - \left(e_0E_3' + \frac{1}{2}E_2'^2\right)\cos 4\bar{\theta} \end{aligned}$$

$$\begin{aligned} e^2 = {} & x^2 + y^2 \\ = {} & \left(e_0^2 + E_0^2 + e_0E_1 + \frac{1}{2}E_2^2 + \frac{1}{2}E_2'^2\right) + \left(2e_0E_0 + 2E_0E_1 + e_0E_2 + e_0E_2' + E_1E_2\right)\cos\bar{\theta} \\ & + \left(e_0E_1 + 2E_0E_2 + e_0E_3 + e_0E_3'\right)\cos 2\bar{\theta} + \left(e_0E_2 - e_0E_2' + E_1E_2 + 2E_0E_3\right)\cos 3\bar{\theta} \\ & + \left(2E_0E_4 + e_0E_3 - e_0E_3' + \frac{1}{2}E_2^2 - \frac{1}{2}E_2'^2\right)\cos 4\bar{\theta} \end{aligned}$$

$$\begin{aligned} xy = {} & \left(e_0E_0 + \frac{1}{2}e_0E_2' - \frac{1}{2}e_0E_2 + \frac{1}{2}E_1E_2'\right)\sin\bar{\theta} \\ & + \left(\frac{1}{2}e_0^2 + \frac{1}{2}e_0E_1 + E_0E_2' + \frac{1}{2}e_0E_3' - \frac{1}{2}e_0E_3\right)\sin 2\bar{\theta} \\ & + \left(\frac{1}{2}e_0E_2 + \frac{1}{2}e_0E_2' + \frac{1}{2}E_1E_2' + E_0E_3'\right)\sin 3\bar{\theta} + \left(E_0E_4 + \frac{1}{2}e_0E_3' + \frac{1}{2}E_2E_2' + \frac{1}{2}e_0E_3\right)\sin 4\bar{\theta} \end{aligned}$$

$$e^4 = e_0^2\left(e_0^2 + 4E_0^2 + 2e_0 E_1 + 2E_0 E_2 + 2E_0 E_2'\right) + 2e_0 E_0\left(2e_0^2 + 2E_0^2 + 5e_0 E_1 + 3E_0 E_2 + E_0 E_2'\right)\cos\overline{\theta}$$

$$+ 2e_0^2\left(E_0^2 + e_0 E_1 + 4E_0 E_2\right)\cos 2\overline{\theta} + 2e_0 E_0\left(e_0 E_1 + 3E_0 E_2 - E_0 E_2'\right)\cos 3\overline{\theta}$$

$$+ \left(2e_0^2 E_0 E_2 - 2e_0^2 E_0 E_2'\right)\cos 4\overline{\theta}$$

$$xye^2 = \frac{3}{2} e_0^3 E_0 \sin\theta_0 (t + c) + \frac{1}{2} e_0^2\left(e_0^2 + 3E_0^2 + 2e_0 E_1 + 6E_0 E_2'\right)\sin 2\overline{\theta}$$

$$+ \frac{1}{2} e_0^3 E_0 \sin 3\overline{\theta} + \left(\frac{3}{2} e_0^2 E_0 E_2 + \frac{1}{4} e_0^3 E_1\right)\sin 4\overline{\theta}$$

$$ye^2 = e_0\left(e_0^2 + E_0^2 + \frac{1}{2} e_0 E_1 + E_0 E_2' - E_0 E_2\right)\sin\overline{\theta}$$

$$+ \left(e_0^2 E_0 + e_0 E_0 E_1 + 2e_0^2 E_2' + E_0^2 E_2' + e_0 E_1 E_2' + e_0 E_0 E_3' - e_0 E_0 E_3\right)\sin 2\overline{\theta}$$

$$+ \frac{1}{2} e_0\left(e_0 E_1 + 2E_0 E_2 + 2E_0 E_2'\right)\sin 3\overline{\theta}$$

$$+ \left(\frac{1}{2} e_0^2 E_2 - \frac{1}{2} e_0^2 E_2' + \frac{1}{2} e_0 E_1 E_2 + e_0 E_0 E_3 + \frac{1}{2} e_0 E_1 E_2' + E_0 E_2 E_2' + e_0 E_0 E_3' + E_0^2 E_4'\right)\sin 4\overline{\theta}$$

$$xe^2 = \frac{1}{2}\left(4e_0^2 E_0 + 2E_0^3 + 6e_0 E_0 E_1 + e_0^2 E_2 + e_0^2 E_2' + 3e_0 E_1 E_2 + e_0 E_1 E_2' + 3E_0 E_2^2 + E_0 E_2'^2\right)$$

$$+ \frac{1}{2} e_0\left(2e_0^2 + 6E_0^2 + 5e_0 E_1 + 6E_0 E_2 + 2E_0 E_2'\right)\cos\overline{\theta}$$

$$+ \left(e_0^2 E_0 + 3e_0 E_0 E_1 + 2e_0^2 E_2 + 3E_0^2 E_2 + 3e_0 E_1 E_2 + 3e_0 E_0 E_3 + e_0 E_0 E_3'\right)\cos 2\overline{\theta}$$

$$+ \frac{1}{2} e_0\left(e_0 E_1 + 6E_0 E_2 - 2E_0 E_2'\right)\cos 3\overline{\theta} + \left(3E_0^2 E_4 + 3e_0 E_0 E_3 - e_0 E_0 E_3' + \frac{1}{2} e_0^2 E_2\right.$$

$$\left. - \frac{1}{2} e_0^2 E_2' + \frac{3}{2} E_0 E_2^2 - \frac{1}{2} E_0 E_2'^2 + \frac{3}{2} e_0 E_1 E_2 - \frac{1}{2} e_0 E_1 E_2'\right)\cos 4\overline{\theta}$$

$$x^2 e^2 = \frac{1}{4} e_0^2\left(2e_0^2 + 14E_0^2 + 7e_0 E_1 + 10E_0 E_2 + 4E_0 E_2'\right) + \frac{7}{2} e_0^3 E_0 \cos\overline{\theta}$$

$$+ \frac{1}{2} e_0^2\left(e_0^2 + 5E_0^2 + 4e_0 E_1 + 14E_0 E_2\right)\cos 2\overline{\theta} + \frac{1}{2} e_0^3 E_0 \cos 3\overline{\theta}$$

$$+ \left(\frac{1}{4} e_0^3 E_1 + \frac{5}{2} e_0^2 E_0 E_2 - e_0^2 E_0 E_2'\right)\cos 4\overline{\theta}$$

$$y^2 e^2 = \frac{1}{4} e_0^2 \left(2e_0^2 + 2E_0^2 + e_0 E_1 + 4E_0 E_2' - 2E_0 E_2\right) + \frac{1}{2} e_0^3 E_0 \cos\overline{\theta}$$

$$+ \frac{1}{2} e_0^2 \left(- e_0^2 - E_0^2 + 2E_0 E_2\right) \cos 2\overline{\theta} - \frac{1}{2} e_0^3 E_0 \cos 3\overline{\theta}$$

$$- \left(\frac{1}{4} e_0^3 E_1 + \frac{1}{2} e_0^2 E_0 E_2 + e_0^2 E_0 E_2'\right) \cos 4\overline{\theta}$$

$$xe^4 = e_0^2 E_0 \left(3e_0^2 + 6E_0^2 + 9e_0 E_1 + 6E_0 E_2 + 3E_0 E_2'\right) + 9e_0^3 E_0^2 \cos\overline{\theta}$$

$$+ 2e_0^2 E_0 \left(e_0^2 + 2E_0^2 + 5e_0 E_1 + 9E_0 E_2\right) \cos 2\overline{\theta} + e_0^3 E_0^2 \cos 3\overline{\theta}$$

$$+ \left(e_0^3 E_0 E_1 + 6e_0^2 E_0^2 E_2 - 3e_0^2 E_0^2 E_2'\right) \cos 4\overline{\theta}$$

$$ye^4 = 3e_0^3 E_0^2 \sin\overline{\theta} + 2e_0^2 E_0 \left(e_0^2 + E_0^2 + 2e_0 E_1 + 3E_0 E_2'\right) \sin 2\overline{\theta} + e_0^3 E_0^2 \sin 3\overline{\theta}$$

$$+ \left(e_0^3 E_0 E_1 + 3e_0^2 E_0^2 E_2\right) \sin 4\overline{\theta}$$

$$e^6 = 9e_0^4 E_0^2 + 18e_0^3 E_0^3 \cos\overline{\theta} + 6e_0^4 E_0^2 \cos 2\overline{\theta} + 2e_0^3 E_0^3 \cos 3\overline{\theta}$$

$$xye^4 = 4e_0^3 E_0^3 \sin\overline{\theta} + 4e_0^4 E_0^2 \sin 2\overline{\theta} + 2e_0^3 E_0^3 \sin 3\overline{\theta} + \frac{1}{2} e_0^4 E_0^2 \sin 4\overline{\theta}$$

$$y^2 e^4 = \frac{3}{2} e_0^4 E_0^2 + e_0^3 E_0^3 \cos\overline{\theta} - e_0^4 E_0^2 \cos 2\overline{\theta} - e_0^3 E_0^3 \cos 3\overline{\theta} - \frac{1}{2} e_0^4 E_0^2 \cos 4\overline{\theta}$$

$$x^2 e^4 = \frac{15}{2} e_0^4 E_0^2 + 17e_0^3 E_0^3 \cos\overline{\theta} + 7e_0^4 E_0^2 \cos 2\overline{\theta} + 3e_0^3 E_0^3 \cos 3\overline{\theta} + \frac{1}{2} e_0^4 E_0^2 \cos 4\overline{\theta}$$

$$ye^6 = 8e_0^4 E_0^3 \sin 2\overline{\theta} + e_0^4 E_0^3 \sin 4\overline{\theta}$$

$$xe^6 = 18e_0^4 E_0^3 + 16e_0^4 E_0^3 \cos 2\overline{\theta} + e_0^4 E_0^3 \cos 4\overline{\theta} .$$

The procedure shall be to equate the coefficients of similar trigonometric terms in Equations (6) and (7), and then to equate similar orders of $e_0^i Q^j$.

First of all, we note from Equation (6) that the constant term of dy/dt is zero. Therefore, equating the constant term of dy/dt in Equation (7) to zero and grouping terms of similar order, we have

$$O = \left(Q + E_0\right) + e_0^2\left(\frac{1}{2} QM_1 + \frac{1}{2} QP_1 + 2E_0 N_1\right) + e_0^2\left(\frac{1}{2} E_2 N_1 + \frac{1}{2} E_2' N_1 + \frac{1}{2} e_0^2 QM_2 + \frac{1}{2} e_0^2 QP_2 + 3e_0^2 E_0 N_2\right)$$

$$+ \left(QE_0^2 P_1 + E_0^3 N_1\right) + \left(e_0 QE_1 P_1 + 3e_0 E_0 E_1 N_1 + \frac{1}{2} e_0^2 QE_0^2 M_2 + \frac{7}{2} e_0^2 QE_0^2 P_2 + 6e_0^2 E_0^3 N_2\right)$$

$$+ \left(\frac{1}{2} QE_2'^2 M_1 + \frac{1}{2} QE_2^2 P_1 + \frac{3}{2} e_0 E_1 E_2 N_1 + \frac{1}{2} e_0 E_1 E_2' N_1 + \frac{3}{2} E_0 E_2^2 N_1 + \frac{1}{2} E_0 E_2'^2 N_1 + \frac{1}{4} e_0^3 QE_1 M_2\right.$$

$$+ e_0^2 QE_0 E_2' M_2 - \frac{1}{2} e_0^2 QE_0 E_2 M_2 + \frac{7}{4} e_0^3 QE_1 P_2 + \frac{5}{2} e_0^2 QE_0 E_2 P_2 + e_0^2 QE_0 E_2' P_2 + 9e_0^3 E_0 E_1 N_2$$

$$\left. + 6e_0^2 E_0^2 E_2 N_2 + 3e_0^2 E_0^2 E_2' N_2 + \frac{3}{2} e_0^4 QE_0^2 M_3 + \frac{15}{2} e_0^4 QE_0^2 P_3 + 18e_0^4 E_0^3 N_3\right) , \tag{8}$$

where the groupings are in the sequence Q, $e_0^2 Q$, $e_0^4 Q$, Q^3, $e_0^2 Q^3$, $e_0^4 Q^3$. To order $e_0^2 Q$, we have

$$E_0 = Q\left[-1 - \frac{1}{2}\left(M_1 - 4N_1 + P_1\right) e_0^2\right] . \tag{9}$$

Equating the coefficients of $\cos \overline{\theta}$ in the expressions for dy/dt, we obtain

$$e_0 \theta_0 = Ne_0\left(1 + N_1 e_0^2\right) + N\left(3e_0 N_1 E_0^2 + 2e_0 QP_1 E_0 + E_1\right) + N\left(e_0 QM_1 E_2' + e_0 QP_1 E_2\right.$$

$$\left. + \frac{5}{2} e_0^2 N_1 E_1 + 3e_0 N_1 E_0 E_2 + e_0 N_1 E_0 E_2' + \frac{1}{2} e_0^3 QM_2 E_0 + \frac{7}{2} e_0^3 QP_2 E_0 + 9e_0^3 N_2 E_0^2\right) . \tag{10}$$

Equating the coefficients of $\sin \overline{\theta}$ in the formulas for dx/dt yields

$$-\left(e_0 + E_1\right)\theta_0 = -Ne_0\left(1 + N_1 e_0^2\right) + N\left[e_0 Q\left(M_1 - P_1\right) E_0 - e_0 N_1 E_0^2\right]$$

$$+ N\left[\frac{1}{2} e_0 Q\left(M_1 - P_1\right) E_2' - \frac{1}{2} e_0 Q\left(M_1 - P_1\right) E_2 - \frac{1}{2} e_0^2 N_1 E_1 - e_0 N_1 E_0 E_2' + e_0 N_1 E_0 E_2\right.$$

$$\left. + \frac{3}{2} e_0^3 Q\left(M_2 - P_2\right) E_0 - 3e_0^3 N_2 E_0^2\right] . \tag{11}$$

In the last two equations, the groupings are in the order e_0, e_0^3, $e_0 Q^2$, $e_0^3 Q^2$. To obtain approximations to order $e_0 Q^2$ for E_1 and Q^2 for θ_0, set

$$e_0 \theta_0 = Ne_0 \left(1 + N_1 e_0^2\right) + N\left(3e_0 N_1 E_0^2 + 2e_0 QP_1 E_0 + E_1\right)$$

$$-\left(e_0 + E_1\right)\theta_0 = -Ne_0\left(1 + N_1 e_0^2\right) + N\left[e_0 Q\left(M_1 - P_1\right) E_0 - e_0 N_1 E_0^2\right],$$

obtaining

$$E_1 = \frac{1}{2} Q^2 \left(M_1 - 2N_1 + P_1\right) e_0 \tag{12}$$

$$\theta_0 = N\left[\left(1 + N_1 e_0^2\right) + \frac{1}{2} Q^2 \left(M_1 + 4N_1 - 3P_1\right)\right]. \tag{13}$$

Now, setting the coefficients of $\sin 2\bar{\theta}$ in the equations for dx/dt equal results in the expression

$$\begin{aligned} -2E_2 \theta_0 = {} & N\left[\frac{1}{2} e_0^2 Q\left(M_1 - P_1\right) - e_0^2 N_1 E_0 - E_2'\right] + N\left[\frac{1}{2} e_0^4 Q\left(M_2 - P_2\right) - 2e_0^2 N_1 E_2' - 2e_0^4 N_2 E_0\right] \\ & + N\left[Q\left(M_1 - P_1\right) E_0 E_2' + \frac{1}{2} e_0 Q\left(M_1 - P_1\right) E_1 - e_0 N_1 E_0 E_1 - N_1 E_0^2 E_2' + \frac{3}{2} e_0^2 Q\left(M_2 - P_2\right) E_0^2 - 2e_0^2 N_2 E_0^3\right] \\ & + N\left[\frac{1}{2} e_0 Q\left(M_1 - P_1\right) E_3' - \frac{1}{2} e_0 Q\left(M_1 - P_1\right) E_3 - e_0 N_1 E_1 E_2' - e_0 N_1 E_0 E_3' + e_0^3 Q\left(M_2 - P_2\right) E_1\right. \\ & \left. + 3e_0^2 Q\left(M_2 - P_2\right) E_0 E_2' - 4e_0^3 N_2 E_0 E_1 - 6e_0^2 N_2 E_0^2 E_2' + e_0 N_1 E_0 E_3 + 4e_0^4 Q\left(M_3 - P_3\right) E_0^2 - 8e_0^4 N_3 E_0^3\right]. \end{aligned} \tag{14}$$

Equating the coefficients of $\cos 2\bar{\theta}$ in the equations for dy/dt, we get

$$\begin{aligned} 2E_2' \theta_0 = {} & N\left[-\frac{1}{2} e_0^2 Q\left(M_1 - P_1\right) + e_0^2 N_1 E_0 + E_2\right] + N\left[-\frac{1}{2} e_0^4 Q\left(M_2 - P_2\right) + 2e_0^2 N_1 E_2 + 2e_0^4 N_2 E_0\right] \\ & + N\left[2QP_1 E_0 E_2 + e_0 QP_1 E_1 + 3e_0 N_1 E_0 E_1 + 3N_1 E_0^2 E_2 - \frac{1}{2} e_0^2 Q\left(M_2 - 5P_2\right) E_0^2 + 4e_0^2 N_2 E_0^3\right] \\ & + N\left[e_0 QM_1 E_3' + e_0 QP_1 E_3 + 3e_0 N_1 E_1 E_2 + 3e_0 N_1 E_0 E_3 + e_0 N_1 E_0 E_3' + 2e_0^3 QP_2 E_1 + e_0^2 Q\left(M_2 + 7P_2\right) E_0 E_2\right. \\ & \left. + 10e_0^3 N_2 E_0 E_1 + 18e_0^2 N_2 E_0^2 E_2 - e_0^4 QM_3 E_0^2 + 7e_0^4 QP_3 E_0^2 + 16e_0^4 N_3 E_0^3\right]. \end{aligned} \tag{15}$$

In Equations (14) and (15), the orders are $e_0^2 Q$, $e_0^4 Q$, $e_0^2 Q^3$, and $e_0^4 Q^3$. Making use of Equations (9) and (13), to order $e_0^2 Q$ the Equations (14) and (15) are

$$- 2E_2 + E_2' = \frac{1}{2} e_0^2 Q(M_1 - P_1) + e_0^2 QN_1$$

$$E_2 - 2E_2' = \frac{1}{2} e_0^2 Q(M_1 - P_1) + e_0^2 QN_1 \quad ,$$

the solution being

$$E_2 = E_2' = - \frac{1}{2} Q(M_1 + 2N_1 - P_1) e_0^2 \quad . \tag{16}$$

For the coefficients of $\sin 3\bar{\theta}$ and $\cos 3\bar{\theta}$ (order $e_0^3 Q^2$), we have, respectively,

$$- 3E_3 \theta_0 = N\left[\frac{1}{2} e_0 Q(M_1 - P_1)(E_2 + E_2') - \frac{1}{2} e_0 N_1 (e_0 E_1 + 2E_0 E_2 + 2E_0 E_2') + \frac{1}{2} e_0^3 Q(M_2 - P_2) E_0 - e_0^3 N_2 E_0^2 - E_3'\right] \tag{17}$$

$$3E_3' \theta_0 = N\left[- e_0 QM_1 E_2' + e_0 QP_1 E_2 + \frac{1}{2} e_0 N_1 (e_0 E_1 + 6E_0 E_2 - 2E_0 E_2') - \frac{1}{2} e_0^3 Q(M_2 - P_2) E_0 + e_0^3 N_2 E_0^2 + E_3\right] . \tag{18}$$

Employing Equations (9), (12), (13), and (16), we deduce

$$- 3E_3 + E_3' = E_3 - 3E_3' = - \frac{1}{4} e_0^3 Q^2 (2M_1^2 + 6N_1^2 + 2P_1^2 + 9M_1 N_1 - 4M_1 P_1 - 7N_1 P_1 + 2M_2 + 4N_2 - 2P_2) \quad ,$$

which yields

$$E_3 = E_3' = \frac{1}{8} Q^2 (2M_1^2 + 6N_1^2 + 2P_1^2 + 9M_1 N_1 - 4M_1 P_1 - 7N_1 P_1 + 2M_2 + 4N_2 - 2P_2) e_0^3 \quad . \tag{19}$$

Finally, we obtain the following equations for E_4 and E_4' (orders $e_0^4 Q$ and $e_0^4 Q^3$):

$$\begin{aligned} - 4E_4 \theta_0 = & - N\left(E_4' + \frac{1}{2} e_0^2 N_1 E_2 - \frac{1}{2} e_0^2 N_1 E_2'\right) + N\Big[Q(M_1 - P_1)\left(E_0 E_4 + \frac{1}{2} e_0 E_3' + \frac{1}{2} e_0 E_3 + \frac{1}{2} E_2 E_2'\right) \\ & + Q(M_2 - P_2)\left(\frac{3}{2} e_0^2 E_0 E_2 + \frac{1}{4} e_0^3 E_1\right) - N_1 \Big(\frac{1}{2} e_0 E_1 E_2 + e_0 E_0 E_3 + \frac{1}{2} e_0 E_1 E_2' + E_0 E_2 E_2' \\ & + e_0 E_0 E_3' + E_0^2 E_4'\Big) - N_2 (e_0^3 E_0 E_1 + 3e_0^2 E_0^2 E_2) + \frac{1}{2} Q(M_3 - P_3) e_0^4 E_0^2 - N_3 e_0^4 E_0^3\Big] \end{aligned} \tag{20}$$

$$4E_4'\theta_0 = N\left(E_4 + \frac{1}{2}e_0^2 N_1 E_2 - \frac{1}{2}e_0^2 N_1 E_2'\right) + N\Big[-QM_1\left(e_0 E_3' + \frac{1}{2}E_2'^2\right) + QP_1\left(2E_0 E_4 + e_0 E_3 + \frac{1}{2}E_2^2\right)$$

$$- QM_2\left(\frac{1}{4}e_0^3 E_1 + \frac{1}{2}e_0^2 E_0 E_2 + e_0^2 E_0 E_2'\right) + QP_2\left(\frac{1}{4}e_0^3 E_1 + \frac{5}{2}e_0^2 E_0 E_2 - e_0^2 E_0 E_2'\right) + N_1\Big(3E_0^2 E_4$$

$$+ 3e_0 E_0 E_3 - e_0 E_0 E_3' + \frac{3}{2}E_0 E_2^2 - \frac{1}{2}E_0 E_2'^2 + \frac{3}{2}e_0 E_1 E_2 - \frac{1}{2}e_0 E_1 E_2'\Big) + N_2\Big(e_0^3 E_0 E_1$$

$$+ 6e_0^2 E_0^2 E_2 - 3e_0^2 E_0^2 E_2'\Big) - \frac{1}{2}QM_3 e_0^4 E_0^2 + \frac{1}{2}QP_3 e_0^4 E_0^2 + N_3 e_0^4 E_0^3\Big]. \tag{21}$$

From these equations, it is concluded that the $e_0^4 Q$ parts of E_4 and E_4' are both zero.

Now, let

$$\Delta E_0 = e_0^4 QA + Q^3 B + e_0^2 Q^3 C .$$

Substituting the expressions in Equations (9), (12), (13), and (16) into Equation (8), we get

$$0 = e_0^4 Q\left[A - \frac{1}{2}N_1\left(3M_1 - 6N_1 + P_1\right) + \frac{1}{2}\left(M_2 + P_2\right) - 3N_2\right] + Q^3\left(B + P_1 - N_1\right)$$

$$+ e_0^2 Q^3\left[C + 2N_1 B + 9N_1^2 + \frac{3}{2}P_1^2 - 3M_1 N_1 + \frac{3}{2}M_1 P_1 - 8N_1 P_1 + \frac{1}{2}M_2 - 6N_2 + \frac{7}{2}P_2\right],$$

from which A, B, and C can be determined. The result is

$$\Delta E_0 = \frac{1}{2}Q\left(3M_1 N_1 - 6N_1^2 + N_1 P_1 - M_2 + 6N_2 - P_2\right)e_0^4 + Q^3\left(N_1 - P_1\right)$$

$$+ \frac{1}{2}Q^3\left(-22N_1^2 - 3P_1^2 + 6M_1 N_1 - 3M_1 P_1 + 20N_1 P_1 - M_2 + 12N_2 - 7P_2\right)e_0^2 . \tag{22}$$

Let $\Delta\theta_0$ be the $e_0^2 Q^2$ part of θ_0 and ΔE_1 the $e_0^3 Q^2$ part of E_1. With the aid of Equations (9), (12), (13), and (16), Equations (10) and (11) become

$$e_0\Delta\theta_0 - N\Delta E_1 = \frac{1}{4}e_0^3 Q^2 N\left(-2M_1^2 - 42N_1^2 - 2P_1^2 + 21M_1 N_1 - 4M_1 P_1 + 21N_1 P_1 - 2M_2 + 36N_2 - 14P_2\right)$$

$$-e_0\Delta\theta_0 - N\Delta E_1 = \frac{1}{4}e_0^3 Q^2 N\left(-2M_1^2 + 14N_1^2 + 2P_1^2 + 5M_1 N_1 - 11N_1 P_1 - 6M_2 - 12N_2 + 6P_2\right),$$

from which it follows that

$$\Delta\theta_0 = \frac{1}{4}Q^2 N\left(-28N_1^2 - 2P_1^2 + 8M_1 N_1 - 2M_1 P_1 + 16N_1 P_1 + 2M_2 + 24N_2 - 10P_2\right)e_0^2 \tag{23}$$

$$\Delta E_1 = \frac{1}{4} Q^2 \left(2M_1^2 + 14N_1^2 - 13M_1 N_1 + 2M_1 P_1 - 5N_1 P_1 + 4M_2 - 12N_2 + 4P_2\right) e_0^3 . \qquad (24)$$

Next, let

$$\Delta E_2 = e_0^4 QD + e_0^2 Q^3 F , \qquad \Delta E_2' = e_0^4 QG + e_0^2 Q^3 H .$$

Substituting the approximations obtained so far for θ_0, E_0, E_1, and E_2 into Equations (14) and (15), there results

$$e_0^4 Q(-2D + G) + e_0^2 Q^3 (-2F + H) = \frac{1}{2} e_0^4 Q\left(-4N_1^2 + M_1 N_1 + N_1 P_1 + M_2 + 4N_2 - P_2\right)$$

$$+ \frac{1}{4} e_0^2 Q^3 \left(M_1^2 - 20N_1^2 - 5P_1^2 - 6M_1 N_1 + 4M_1 P_1 + 22N_1 P_1 + 6M_2 + 8N_2 - 6P_2\right)$$

$$e_0^4 Q(-D + 2G) + e_0^2 Q^3 (-F + 2H) = \frac{1}{2} e_0^4 Q\left(4N_1^2 - M_1 N_1 - N_1 P_1 - M_2 - 4N_2 + P_2\right)$$

$$+ \frac{1}{2} e_0^2 Q^3 \left(M_1^2 + 10N_1^2 + 2P_1^2 - M_1 P_1 - 10N_1 P_1 - M_2 - 8N_2 + 5P_2\right) ,$$

from which D, F, G, and H can be computed. Thus,

$$\Delta E_2 = \frac{1}{2} Q\left(4N_1^2 - M_1 N_1 - N_1 P_1 - M_2 - 4N_2 + P_2\right) e_0^4 + \frac{1}{6} Q^3 \left(30N_1^2 + 7P_1^2 + 6M_1 N_1 - 5M_1 P_1 - 32N_1 P_1 - 7M_2 - 16N_2 + 11P_2\right) e_0^2 \qquad (25)$$

$$\Delta E_2' = \frac{1}{2} Q\left(4N_1^2 - M_1 N_1 - N_1 P_1 - M_2 - 4N_2 + P_2\right) e_0^4 + \frac{1}{12} Q^3 \left(3M_1^2 + 60N_1^2 + 13P_1^2 + 6M_1 N_1 - 8M_1 P_1 - 62N_1 P_1 - 10M_2 - 40N_2 + 26P_2\right) e_0^2 . \qquad (26)$$

Finally, the $e_0^4 Q^3$ parts of E_0, E_2, E_2', E_4, and E_4' are obtained by substituting into Equations (8), (14), (15), (20), and (21) the results gotten so far. The complete expressions for E_0, θ_0, etc. are as follows:

$$E_0 = Q\Big[-1 - \frac{1}{2} \left(M_1 - 4N_1 + P_1\right) e_0^2 + \frac{1}{2} \left(3M_1 N_1 - 6N_1^2 + N_1 P_1 - M_2 + 6N_2 - P_2\right) e_0^4\Big]$$

$$+ Q^3 \Big[\left(N_1 - P_1\right) + \frac{1}{2} \left(-22N_1^2 - 3P_1^2 + 6M_1 N_1 - 3M_1 P_1 + 20N_1 P_1 - M_2 + 12N_2 - 7P_2\right) e_0^2 - \frac{1}{8} \left(M_1^3 - 436N_1^3\right.$$

$$+ 3P_1^3 - 27M_1^2 N_1 + 5M_1^2 P_1 + 240M_1 N_1^2 + 7M_1 P_1^2 + 356N_1^2 P_1 - 65N_1 P_1^2 - 124M_1 N_1 P_1 + 7M_1 M_2$$

$$- 144M_1 N_2 + 49M_1 P_2 - 66N_1 M_2 + 528N_1 N_2 - 174N_1 P_2 + 19P_1 M_2 - 168P_1 N_2 + 37P_1 P_2 + 12M_3 - 144N_3$$

$$\left. + 60P_3\right) e_0^4\Big]$$

$$\theta_0 = N\left(1 + N_1 e_0^2\right) + NQ^2\left[\frac{1}{2}\left(M_1 + 4N_1 - 3P_1\right) + \frac{1}{2}\left(-14N_1^2 - P_1^2 + 4M_1N_1 - M_1P_1 + 8N_1P_1 + M_2 + 12N_2 - 5P_2\right)e_0^2\right]$$

$$E_1 = Q^2\left[\frac{1}{2}\left(M_1 - 2N_1 + P_1\right)e_0 + \frac{1}{4}\left(2M_1^2 + 14N_1^2 - 13M_1N_1 + 2M_1P_1 - 5N_1P_1 + 4M_2 - 12N_2 + 4P_2\right)e_0^3\right]$$

$$\begin{aligned} E_2 = {} & Q\left[-\frac{1}{2}\left(M_1 + 2N_1 - P_1\right)e_0^2 + \frac{1}{2}\left(4N_1^2 - M_1N_1 - N_1P_1 - M_2 - 4N_2 + P_2\right)e_0^4\right] \\ & + Q^3\left[\frac{1}{6}\left(30N_1^2 + 7P_1^2 + 6M_1N_1 - 5M_1P_1 - 32N_1P_1 - 7M_2 - 16N_2 + 11P_2\right)e_0^2 - \frac{1}{72}\left(18M_1^3 + 2268N_1^3 - 18P_1^3\right.\right. \\ & \quad - 93M_1^2N_1 + 6M_1^2P_1 - 624M_1N_1^2 - 30M_1P_1^2 - 2060N_1^2P_1 + 433N_1P_1^2 + 388M_1N_1P_1 + 138M_1M_2 \\ & \quad + 300M_1N_2 - 114M_1P_2 - 256N_1M_2 - 2920N_1N_2 + 1088N_1P_2 + 90P_1M_2 + 1092P_1N_2 - 306P_1P_2 \\ & \quad \left.\left. + 216M_3 + 768N_3 - 360P_3\right)e_0^4\right] \end{aligned}$$

$$\begin{aligned} E_2' = {} & Q\left[-\frac{1}{2}\left(M_1 + 2N_1 - P_1\right)e_0^2 + \frac{1}{2}\left(4N_1^2 - M_1N_1 - N_1P_1 - M_2 - 4N_2 + P_2\right)e_0^4\right] \\ & + Q^3\left[\frac{1}{12}\left(3M_1^2 + 60N_1^2 + 13P_1^2 + 6M_1N_1 - 8M_1P_1 - 62N_1P_1 - 10M_2 - 40N_2 + 26P_2\right)e_0^2 - \frac{1}{72}\left(2268N_1^3 - 18P_1^3\right.\right. \\ & \quad + 39M_1^2N_1 - 6M_1^2P_1 - 780M_1N_1^2 - 24M_1P_1^2 - 2008N_1^2P_1 + 413N_1P_1^2 + 404M_1N_1P_1 + 24M_1M_2 \\ & \quad + 564M_1N_2 - 192M_1P_2 - 104N_1M_2 - 3128N_1N_2 + 1144N_1P_2 + 72P_1M_2 + 1068P_1N_2 - 288P_1P_2 \\ & \quad \left.\left. + 144M_3 + 960N_3 - 432P_3\right)e_0^4\right] \end{aligned}$$

$$E_3 = E_3' = \frac{1}{8}Q^2\left(2M_1^2 + 6N_1^2 + 2P_1^2 + 9M_1N_1 - 4M_1P_1 - 7N_1P_1 + 2M_2 + 4N_2 - 2P_2\right)e_0^3$$

$$\begin{aligned} E_4 = E_4' = {} & -\frac{1}{72}Q^3\left(9M_1^3 + 36N_1^3 - 9P_1^3 + 66M_1^2N_1 - 27M_1^2P_1 + 102M_1N_1^2 + 27M_1P_1^2 - 70N_1^2P_1 + 44N_1P_1^2\right. \\ & - 106M_1N_1P_1 + 27M_1M_2 + 60M_1N_2 - 27M_1P_2 + 46N_1M_2 + 64N_1N_2 - 38N_1P_2 - 21P_1M_2 - 36P_1N_2 + 21P_1P_2 + 12M_3 \\ & \left. + 24N_3 - 12P_3\right)e_0^4 . \end{aligned} \tag{27}$$

SOLUTION FOR e, θ

Now, let

$$\theta = \overline{\theta} + Q\alpha_1 \sin\overline{\theta} + Q^2\alpha_2 \sin 2\overline{\theta} + Q\alpha_3 \sin 3\overline{\theta} = \overline{\theta} + \delta\theta \tag{28}$$

$$e = e_1 + Q\beta_1 \cos\bar{\theta} + Q^2 \beta_2 \cos 2\bar{\theta} + Q\beta_3 \cos 3\bar{\theta} \,, \tag{29}$$

where

$$\alpha_i = O\left(\frac{1}{e_0^{\,i}}\right), \qquad \beta_1 = O(1)\,, \qquad \beta_2 = O\left(\frac{1}{e_0}\right), \qquad \beta_3 = O\left(\frac{1}{e_0^{\,2}}\right), \qquad e_1 = O\left(e_0\right).$$

We will determine α_i and β_i ($i = 1, 2, 3$) by forming expressions for $e \cos\theta$ and $e \sin\theta$ from Equations (28) and (29), and then use Equations (27).

First of all, keeping terms through orders Q^3 and $1/e_0^4$, form

$$\begin{aligned}
\sin\theta &= \sin\bar{\theta}\cos\delta\theta + \cos\bar{\theta}\sin\delta\theta \\
&\sim \sin\bar{\theta}\left[1 - \frac{1}{2}(\delta\theta)^2\right] + \cos\bar{\theta}\left[\delta\theta - \frac{1}{6}(\delta\theta)^3\right] \\
&= \left(1 - \frac{3}{8} Q^2 \alpha_1^2 + \frac{1}{2} Q^2 \alpha_2 + \frac{1}{4} Q^2 \alpha_1 \alpha_3\right)\sin\bar{\theta} + \left(\frac{1}{2} Q\alpha_1 - \frac{1}{24} Q^3 \alpha_1^3 - \frac{1}{2} Q^3 \alpha_1 \alpha_2 + \frac{1}{2} Q\alpha_3\right)\sin 2\bar{\theta} \\
&\qquad + \left(\frac{1}{8} Q^2 \alpha_1^2 + \frac{1}{2} Q^2 \alpha_2 - \frac{1}{4} Q^2 \alpha_1 \alpha_3\right)\sin 3\bar{\theta}
\end{aligned}$$

$$\begin{aligned}
\cos\theta &= \cos\bar{\theta}\cos\delta\theta - \sin\bar{\theta}\sin\delta\theta \\
&\sim \cos\bar{\theta}\left[1 - \frac{1}{2}(\delta\theta)^2\right] - \sin\bar{\theta}\left[\delta\theta - \frac{1}{6}(\delta\theta)^3\right] \\
&= \left(-\frac{1}{2} Q\alpha_1 + \frac{1}{16} Q^3 \alpha_1^3 - \frac{1}{4} Q^3 \alpha_1 \alpha_2\right) + \left(1 - \frac{1}{8} Q^2 \alpha_1^2 - \frac{1}{2} Q^2 \alpha_2 - \frac{1}{4} Q^2 \alpha_1 \alpha_3\right)\cos\bar{\theta} \\
&\qquad + \left(\frac{1}{2} Q\alpha_1 - \frac{1}{12} Q^3 \alpha_1^3 - \frac{1}{2} Q\alpha_3\right)\cos 2\bar{\theta} + \left(\frac{1}{8} Q^2 \alpha_1^2 + \frac{1}{2} Q^2 \alpha_2 - \frac{1}{4} Q^2 \alpha_1 \alpha_3\right)\cos 3\bar{\theta}\,.
\end{aligned}$$

Then, retaining terms through orders Q^3 and $1/e_0^3$,

$$\begin{aligned}
e\cos\theta = {} & \left[e_1\left(-\frac{1}{2} Q\alpha_1 + \frac{1}{16} Q^3 \alpha_1^3 - \frac{1}{4} Q^3 \alpha_1 \alpha_2\right) + \frac{1}{2} Q\beta_1\left(1 - \frac{1}{8} Q^2 \alpha_1^2 - \frac{1}{2} Q^2 \alpha_2\right) + \frac{1}{4} Q^3 \alpha_1 \beta_2\right] \\
& + \left[e_1\left(1 - \frac{1}{8} Q^2 \alpha_1^2 - \frac{1}{2} Q^2 \alpha_2 - \frac{1}{4} Q^2 \alpha_1 \alpha_3\right) - \frac{1}{4} Q^2 \alpha_1 \beta_1 + \frac{1}{2} Q^2 \beta_2 + \frac{1}{4} Q^2 \alpha_1 \beta_3 - \frac{1}{4} Q^2 \alpha_3 \beta_1\right]\cos\bar{\theta} \\
& + \left[e_1\left(\frac{1}{2} Q\alpha_1 - \frac{1}{2} Q\alpha_3 - \frac{1}{12} Q^3 \alpha_1^3\right) + \frac{1}{2} Q\beta_1 - \frac{1}{2} Q^3 \alpha_1 \beta_2 + \frac{1}{2} Q\beta_3\right]\cos 2\bar{\theta} \\
& + \left[e_1\left(\frac{1}{8} Q^2 \alpha_1^2 + \frac{1}{2} Q^2 \alpha_2 - \frac{1}{4} Q^2 \alpha_1 \alpha_3\right) + \frac{1}{2} Q\beta_1\left(\frac{1}{2} Q\alpha_1 - \frac{1}{2} Q\alpha_3\right) + \frac{1}{2} Q^2 \beta_2 - \frac{1}{2} Q^2 \alpha_1 \beta_3\right]\cos 3\bar{\theta}
\end{aligned} \tag{30}$$

$$e \sin\theta = \left[e_1\left(1 - \frac{3}{8} Q^2 \alpha_1^2 + \frac{1}{2} Q^2 \alpha_2 + \frac{1}{4} Q^2 \alpha_1 \alpha_3\right) + \frac{1}{2} Q\beta_1\left(\frac{1}{2} Q\alpha_1 + \frac{1}{2} Q\alpha_3\right) - \frac{1}{2} Q^2 \beta_2 - \frac{1}{4} Q^2 \alpha_1 \beta_3\right] \sin\bar{\theta}$$

$$+ \left[e_1\left(\frac{1}{2} Q\alpha_1 + \frac{1}{2} Q\alpha_3 - \frac{1}{24} Q^3 \alpha_1^3 - \frac{1}{2} Q^3 \alpha_1 \alpha_2\right) + \frac{1}{2} Q\beta_1\left(1 - \frac{1}{4} Q^2 \alpha_1^2 + Q^2 \alpha_2\right) - \frac{1}{2} Q\beta_3\right] \sin 2\bar{\theta}$$

$$+ \left[e_1\left(\frac{1}{8} Q^2 \alpha_1^2 + \frac{1}{2} Q^2 \alpha_2 - \frac{1}{4} Q^2 \alpha_1 \alpha_3\right) + \frac{1}{2} Q\beta_1\left(\frac{1}{2} Q\alpha_1 + \frac{1}{2} Q\alpha_3\right) + \frac{1}{2} Q^2 \beta_2\right] \sin 3\bar{\theta} \ . \qquad (31)$$

Comparing Equations (30) and (31) with Equations (5) and grouping terms of the same order, we obtain

$$E_0 = \frac{1}{2} Q(\beta_1 - e_1 \alpha_1) + Q^3\left(\frac{1}{16} e_1 \alpha_1^3 - \frac{1}{4} e_1 \alpha_1 \alpha_2 - \frac{1}{16} \alpha_1^2 \beta_1 - \frac{1}{4} \alpha_2 \beta_1 + \frac{1}{4} \alpha_1 \beta_2\right) \qquad (32)$$

$$e_0 + E_1 = e_1 + Q^2\left(-\frac{1}{8} e_1 \alpha_1^2 - \frac{1}{2} e_1 \alpha_2 - \frac{1}{4} \alpha_1 \beta_1 + \frac{1}{2} \beta_2\right) + Q^2\left(-\frac{1}{4} e_1 \alpha_1 \alpha_3 + \frac{1}{4} \alpha_1 \beta_3 - \frac{1}{4} \alpha_3 \beta_1\right) \qquad (33)$$

$$e_0 = e_1 + Q^2\left(-\frac{3}{8} e_1 \alpha_1^2 + \frac{1}{2} e_1 \alpha_2 + \frac{1}{4} \alpha_1 \beta_1 - \frac{1}{2} \beta_2\right) + Q^2\left(\frac{1}{4} e_1 \alpha_1 \alpha_3 + \frac{1}{4} \alpha_3 \beta_1 - \frac{1}{4} \alpha_1 \beta_3\right) \qquad (34)$$

$$E_2 = \frac{1}{2} Q(\beta_1 + e_1 \alpha_1) + Q^3\left(-\frac{1}{12} e_1 \alpha_1^3 - \frac{1}{2} e_1 \alpha_3 - \frac{1}{2} \alpha_1 \beta_2 + \frac{1}{2} \beta_3\right) \qquad (35)$$

$$E_2' = \frac{1}{2} Q(\beta_1 + e_1 \alpha_1) + Q^3\left(-\frac{1}{24} e_1 \alpha_1^3 - \frac{1}{2} e_1 \alpha_1 \alpha_2 + \frac{1}{2} e_1 \alpha_3 - \frac{1}{8} \alpha_1^2 \beta_1 + \frac{1}{2} \alpha_2 \beta_1 - \frac{1}{2} \beta_3\right) \qquad (36)$$

$$E_3 = Q^2\left(\frac{1}{8} e_1 \alpha_1^2 + \frac{1}{2} e_1 \alpha_2 + \frac{1}{4} \alpha_1 \beta_1 + \frac{1}{2} \beta_2\right) + Q^2\left(-\frac{1}{4} e_1 \alpha_1 \alpha_3 - \frac{1}{4} \alpha_3 \beta_1 - \frac{1}{2} \alpha_1 \beta_3\right) \qquad (37)$$

$$E_3' = Q^2\left(\frac{1}{8} e_1 \alpha_1^2 + \frac{1}{2} e_1 \alpha_2 + \frac{1}{4} \alpha_1 \beta_1 + \frac{1}{2} \beta_2\right) + Q^2\left(-\frac{1}{4} e_1 \alpha_1 \alpha_3 + \frac{1}{4} \alpha_3 \beta_1\right) . \qquad (38)$$

Equation (34) yields a first approximation to e_1:

$$e_1 = e_0 \ . \qquad (39)$$

Equations (32) and (35) give first approximations to α_1 and β_1:

$$\alpha_1 = \frac{1}{e_0 Q}\left(E_2 - E_0\right) = \frac{1}{e_0}\left[1 - \left(3N_1 - P_1\right) e_0^2 + \left(5N_1^2 - 2M_1 N_1 - N_1 P_1 - 5N_2 + P_2\right) e_0^4\right] \qquad (40)$$

$$\beta_1 = \frac{1}{Q}\left(E_2 + E_0\right) = -1 - \left(M_1 - N_1\right) e_0^2 - \left(N_1^2 - M_1 N_1 + M_2 - N_2\right) e_0^4 \ . \qquad (41)$$

Letting Δe_1 equal the Q^2 part of e_1, from Equations (33), (34), (37), (39), (40), and (41) we have

$$-\frac{1}{2} e_0 \alpha_2 + \frac{1}{2} \beta_2 = \frac{1}{8e_0} \left[-3 + 2\left(5N_1 - P_1\right) e_0^2 + \left(2M_1^2 - 23N_1^2 - P_1^2 + 3M_1 N_1 + 7N_1 P_1 + 2M_2 + 10N_2\right) e_0^4\right] \tag{42}$$

$$\frac{1}{2} e_0 \alpha_2 + \frac{1}{2} \beta_2 = \frac{1}{8e_0} \left[1 + 2\left(M_1 - N_1\right) e_0^2 + \left(2M_1^2 + 5N_1^2 + P_1^2 + M_1 N_1 - 2M_1 P_1 - 3N_1 P_1 + 4M_2 + 2N_2 - 2P_2\right) e_0^4\right] \tag{43}$$

$$\Delta e_1 = Q^2 \left(\frac{3}{8} e_0 \alpha_1^2 - \frac{1}{4} \alpha_1 \beta_1 - \frac{1}{2} e_0 \alpha_2 + \frac{1}{2} \beta_2\right). \tag{44}$$

The solution of Equations (42) and (43) is

$$\alpha_2 = \frac{1}{4e_0^2} \left[2 + \left(M_1 - 6N_1 + P_1\right) e_0^2 + \left(14N_1^2 + P_1^2 - M_1 N_1 - M_1 P_1 - 5N_1 P_1 + M_2 - 4N_2 - P_2\right) e_0^4\right] \tag{45}$$

$$\beta_2 = \frac{1}{4e_0} \left[-1 + \left(M_1 + 4N_1 - P_1\right) e_0^2 + \left(2M_1^2 - 9N_1^2 + 2M_1 N_1 - M_1 P_1 + 2N_1 P_1 + 3M_2 + 6N_2 - P_2\right) e_0^4\right]. \tag{46}$$

Substituting these results into Equation (44) gives

$$\Delta e_1 = \frac{Q^2}{8e_0}\left[2 + 2\left(M_1 - 8N_1 + 3P_1\right) e_0^2 + \left(2M_1^2 + 52N_1^2 + 2P_1^2 - 21M_1 N_1 + 2M_1 P_1 - 21N_1 P_1 + 4M_2 - 32N_2 + 8P_2\right)e_0^4\right]. \tag{47}$$

Now, from the Q^2/e_0^3 parts of Equations (37) and (38), we see that

$$\beta_3 = -e_0 \alpha_3 .$$

Therefore, letting $\Delta\alpha_1$, $\Delta\beta_1$ designate the Q^2 parts of α_1 and β_1, Equations (32), (35), and (36) yield

$$\frac{1}{2} Q\left(\Delta\beta_1 - e_0 \Delta\alpha_1\right) = \frac{Q^3}{16e_0^2} \Big[1 - \left(2M_1 + N_1 + 9P_1\right) e_0^2 - \big(M_1^2 + 88N_1^2 + 18P_1^2 - 35M_1 N_1 + 22M_1 P_1$$
$$- 114N_1 P_1 + 10M_2 - 61N_2 + 47P_2\big) e_0^4\Big] \tag{48}$$

$$\frac{1}{2} Q\left(\Delta\beta_1 + e_0 \Delta\alpha_1\right) + Q^3 \beta_3 = \frac{Q^3}{48e_0^2} \Big[-8 + 24\left(3N_1 - P_1\right) e_0^2 + \big(6M_1^2 - 78N_1^2 + 38P_1^2 + 123M_1 N_1$$
$$- 52M_1 P_1 - 109N_1 P_1 - 50M_2 + 4N_2 + 58P_2\big) e_0^4\Big] \tag{49}$$

$$\frac{1}{2} Q\left(\Delta\beta_1 + e_0 \Delta\alpha_1\right) - Q^3 \beta_3 = \frac{Q^3}{48e_0^2} \Big[14 + 6\left(2M_1 - 5N_1 - P_1\right) e_0^2 + \big(12M_1^2 + 222N_1^2 + 46P_1^2 + 63M_1 N_1$$
$$- 56M_1 P_1 - 191N_1 P_1 - 34M_2 - 118N_2 + 68P_2\big) e_0^4\Big]. \tag{50}$$

The solution of these three equations is

$$\Delta\alpha_1 = \frac{Q^2}{4e_0}\left[\left(M_1 + 2N_1 + P_1\right) + \left(M_1^2 + 28N_1^2 + 8P_1^2 - M_1N_1 + M_1P_1 - 41N_1P_1 - M_2 - 20N_2 + 17P_2\right)e_0^2\right] \tag{51}$$

$$\Delta\beta_1 = \frac{Q^2}{8e_0^2}\left[1 + \left(3N_1 - 7P_1\right)e_0^2 + \left(M_1^2 - 32N_1^2 - 2P_1^2 + 33M_1N_1 - 20M_1P_1 + 32N_1P_1 - 12M_2 + 21N_2 - 13P_2\right)e_0^4\right] \tag{52}$$

$$\beta_3 = \frac{1}{48e_0^2}\left[-11 - 3\left(2M_1 - 17N_1 + 3P_1\right)e_0^2 - \left(3M_1^2 + 150N_1^2 + 4P_1^2 - 30M_1N_1 - 2M_1P_1 - 41N_1P_1 + 8M_2 - 61N_2 + 5P_2\right)e_0^4\right] \tag{53}$$

$$\alpha_3 = -\frac{1}{e_0}\beta_3 . \tag{54}$$

The complete expressions for e_1 and the α_i, β_i are as follows:

$$e_1 = e_0 + \frac{Q^2}{8e_0}\left[2 + 2\left(M_1 - 8N_1 + 3P_1\right)e_0^2 + \left(2M_1^2 + 52N_1^2 + 2P_1^2 - 21M_1N_1 + 2M_1P_1 - 21N_1P_1 + 4M_2 - 32N_2 + 8P_2\right)e_0^4\right]$$

$$\alpha_1 = \frac{1}{e_0}\left[1 - \left(3N_1 - P_1\right)e_0^2 + \left(5N_1^2 - 2M_1N_1 - N_1P_1 - 5N_2 + P_2\right)e_0^4\right] + \frac{Q^2}{4e_0}\left[\left(M_1 + 2N_1 + P_1\right) + \left(M_1^2 + 28N_1^2 + 8P_1^2 - M_1N_1 + M_1P_1 - 41N_1P_1 - M_2 - 20N_2 + 17P_2\right)e_0^2\right]$$

$$\beta_1 = -1 - \left(M_1 - N_1\right)e_0^2 - \left(N_1^2 - M_1N_1 + M_2 - N_2\right)e_0^4 + \frac{Q^2}{8e_0^2}\left[1 + \left(3N_1 - 7P_1\right)e_0^2 + \left(M_1^2 - 32N_1^2 - 2P_1^2 + 33M_1N_1 - 20M_1P_1 + 32N_1P_1 - 12M_2 + 21N_2 - 13P_2\right)e_0^4\right]$$

$$\alpha_3 = \frac{1}{48e_0^3}\left[-11 - 3\left(2M_1 - 17N_1 + 3P_1\right)e_0^2 - \left(3M_1^2 + 150N_1^2 + 4P_1^2 - 30M_1N_1 - 2M_1P_1 - 41N_1P_1 + 8M_2 - 61N_2 + 5P_2\right)e_0^4\right]$$

$$\beta_3 = -e_0\alpha_3 . \tag{55}$$

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