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APOLLO USB-SYSTEM

MULTISIGNAL PHASE MODULATION PROBLEMS AND SOLUTIONS

BY
B. KRUGER

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GODDARD SPACE FLIGHT CENTER
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SUMMARY

The spectrum of a phase modulated carrier is analyzed. The analysis is based on the RMS phase deviation only, so that knowledge of the frequency spectrum of the modulating time function is not required. Upper and lower limits of the amplitudes of the spectral components of the phase modulated carrier are derived at, the main emphasis being given to the component of carrier frequency. The usefulness and simplicity of the equations derived at is demonstrated in the analysis of a carrier phase modulated by a pseudo random code and two phase modulated subcarriers. The calculations are carried out without introducing approximations such as substituting voice or other information by a single frequency. author

CONTENTS

LIST OF ILLUSTRATIONS

Figure		Page

APOLLO USB-SYSTEM

Multisignal Phase Modulation Problems and Solutions

INTRODUCTION

For the Apollo missions, both tracking and communication will be handled by the Unified S-band System (USBS). In this system, one main carrier is phase modulated by a base band consisting of a pseudo random range code and a number of phase modulated subcarriers [1]. In this paper the frequency spectrum of the modulated carriers is analyzed. Special attention is given to the calculation of the main carrier component because the success of the USBS hinges on the capability of the receivers to lock on to and track the main carrier component. To analyze the power in the main carrier component under worst case conditions is therefore of vital importance. Although the analysis presented here was made with the USBS in mind, the results are, of course, applicable to any system utilizing multisignal phase modulation.

The conventional way of analysis of multisignal phase modulation is to expand the base band into a Fourier series and then to treat the problem as multi-tone phase modulation. A number of papers (see Reference [3], [4], [5], and [6]) have been written about multi-tone phase modulation. Unfortunately, the equations derived at are long and hard to evaluate as soon as the modulating time function consists of more than two or three frequency components. Another drawback of this method is that it is often difficult or impossible to expand the base band into a Fourier series. This is the case for a pseudo random code or a phase modulated subcarrier, which is modulated with an unknown periodic time function. However, the power of a pseudo random code is constant and the power of a phase modulated subcarrier is constant and independent of the modulating time function.* The power in the base band is thus easily calculated and is independent of the information modulated onto the subcarriers. A method of analysis of phase modulation, based on the power of the modulating time function (= base band) rather on the Fourier spectrum, has therefore been developed and is presented in this paper. With this type of analysis the exact value of the main carrier component cannot be calculated, but upper and lower limits can be calculated. This is not a limitation, but an advantage, because the upper and lower limits, rather than any particular value in between, are essential for circuit design and system analysis.

*This is only true if the Fourier spectrum of the modulating time function does not contain a component of the carrier frequency.

The analysis has been based on complex Fourier analysis because of the simplicity in derivations and lucidity of results obtained with the complex method. A brief recapitulation of complex analysis of phase modulation* is given in the next chapter, which may be omitted by the reader completely familiar with the subject.

Only phase modulation has been considered in this paper because frequency modulation with $f(t)$ is identical to phase modulation with $\int^t f(t)\, dt$. The integral of a function is more well behaved than the function itself and some time functions can therefore only be treated with phase modulation and not with frequency modulation. An example is the frequently used phase modulation with a square wave, also known as phase shift modulation. The concept of phase modulation thus completely covers frequency modulation.

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*The complex analysis of phase modulation is based on lectures given by Professor E. Löfgren in 1952 at the Royal Institute of Technology, Stockholm, Sweden.

2. General Analysis of Multi-Tone Phase Modulation

In this chapter we will analyze the case where a carrier $\cos(\omega_0 t + \psi_0)$ is phase modulated by a time function $f(t)$. We assume that $f(t)$ can be expanded in a Fourier series

$$f(t) = \sum_{n=1}^{n=N} \varphi_n \sin(\omega_n t + \psi_n) \tag{2-1}$$

where N may go to infinity. The D-C or average term of $f(t)$ is included in the phase angle ψ_0 of the carrier and $f(t)$ may thus be considered to be an A-C signal. If we normalize the peak amplitude of the modulated carrier to 1, then the instantaneous amplitude of the modulated carrier is

$$v = \cos(\omega_0 t + \psi_0 + f(t)) \tag{2-2}$$

or

$$v = \cos\left[\omega_0 t + \psi_0 + \sum_{n=1}^{n=N} \varphi_n \sin(\omega_n t + \psi_n)\right]$$

Using complex notation, we obtain

$$V = e^{j\left[\omega_0 t + \psi_0 + \sum_1^N \varphi_n \sin(\omega_n t + \psi_n)\right]} \tag{2-3}$$

where

$$v = \mathrm{Re}\{V\} \tag{2-4}$$

In appendix A it is shown that V can be expanded in an infinite series

$$V = e^{j(\omega_0 t + \psi_0)} \prod_{n=1}^{n=N} \sum_{m=-\infty}^{m=\infty} J_m(\varphi_n)\, e^{jm(\omega_n t + \psi_n)} \tag{2-5}$$

where J_m are the Bessel functions of the first kind and order m. This is a very general expression for the frequency spectrum resulting from a phase modulation. Unfortunately, it is very difficult to obtain any information about the amplitudes of the frequency spectrum from Equ. (2-5) if N is a large number. In the following paragraphs, Equ. (2-5) will be analyzed for some cases of practical importance and simplified and useful expressions will be derived at. Before going into this, let us apply Equ. (2-5) to two special cases:

Case A Let $f(t)$ be a sinusoidal function

$$f(t) = \hat{\Phi} \sin(pt + \psi) \tag{2-6}$$

For this case Equ. (2-5) reduces to

$$V = e^{j(\omega_0 t + \psi_0)} \sum_{m=-\infty}^{m=\infty} J_m(\hat{\Phi})\, e^{jm(pt + \psi)} \tag{2-7}$$

Using the relation [7]

$$J_{-m}(x) = (-1)^m J_m(x)$$

we can rewrite Equ. (2-7)

$$V = e^{j(\omega_0 t + \psi_0)} \left\{ J_0(\hat{\Phi}) + 2 \sum_{n=2}^{\infty}{}_{e} J_m(\hat{\Phi}) \cos m(pt + \psi) \right.$$
$$\left. + j2 \sum_{m=1}^{\infty}{}_{0} J_m(\hat{\Phi}) \sin m(pt + \psi) \right\} \tag{2-8}$$

where Σ_e denotes summarization over even m and Σ_0 denotes summarization over odd m. Equation (2-8) can be interpreted by a phasor diagram utilizing the complex plain as shown in Fig. 1. The parenthesis in Equ. (2-8) is represented by the complex quanity Z, which is composed of infinitely many real and imaginary components. Z is of magnitude 1 due to the normalization introduced in the beginning of this chapter and performs an angular oscillation within the limits $\pm\hat{\Phi}$. The factor $e^{j(\omega_0 t + \psi_0)}$ is represented by the rotating time line t. In Equ. (2-2), we assumed a cosine function for v and v is therefore the projection of Z on the time line t. If we instead had assumed a sine function in Equ. (2-2) then v would be the projection of Z on a line perpendicular to the time line t.

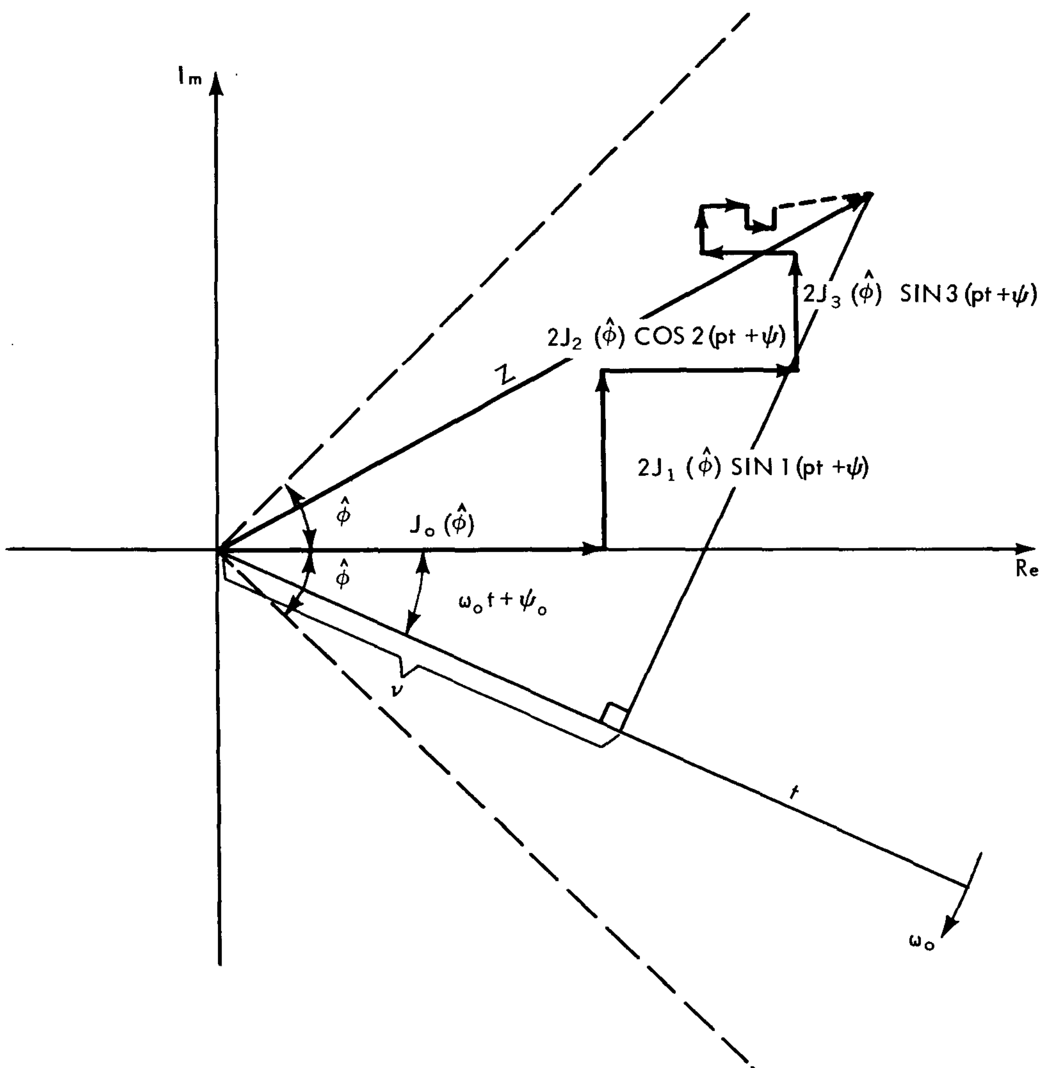

Figure 1—Phasor diagram for phase modulation of a carrier ω_0 with one sinewave of angular frequency p and amplitude $\hat{\Phi}$

Case B. Let $f(t)$ be a subcarrier ω_1 phase modulated with $\hat{\Phi}_1 \sin pt$, thus

$$f(t) = \hat{\Phi}_0 \sin(\omega_1 t + \psi_1 + \hat{\Phi}_1 \sin pt) \tag{2-9}$$

or

$$f(t) = \hat{\Phi}_0 \operatorname{Im}\{e^{j(\omega_1 + \psi_1 + \hat{\Phi}_1 \sin pt)}\} \tag{2-10}$$

The series expansion of Equ. (2-10) is the same as Equ. (2-7) and we therefore have

$$f(t) = \hat{\Phi}_0 \sum_{n=-\infty}^{n=\infty} J_n(\hat{\Phi}_1) \sin((\omega_1 + np)\, t + \psi_1) \tag{2-11}$$

Equ. (2-11) is of the same form as Equ. (2-1) with

$$\varphi_n = \hat{\Phi}_0 J_n(\hat{\Phi}_1)$$

$$\omega_n = \omega_1 + np$$

$$\psi_n = \psi_1$$

Substitution into Equ. (2-5) yields

$$V = e^{j(\omega_0 t + \psi_0)} \prod_{n=-\infty}^{n=\infty} \sum_{m=-\infty}^{m=\infty} J_m(\hat{\Phi}_0 J_n(\hat{\Phi}_1))\, e^{jm((\omega_1 + np)\, t + \psi_1)} \tag{2-12}$$

where

$\hat{\Phi}_0$ = peak phase swing of main carrier modulation

$\hat{\Phi}_1$ = peak phase swing of subcarrier modulation

ω_0 = angular main carrier frequency

ω_1 = angular subcarrier frequency

p = angular frequency of subcarrier modulation

ψ_1 = constant phase angle

This equation gives the complete frequency spectrum of a carrier, which is phase modulated with a phase modulated subcarrier.

The complexity of Equ. (2-5) and (2-12) indicate that a better method of analysis is desirable. In the next chapters the analysis will be based on the RMS value or average power ϕ rather than on the Fourier spectrum of the modulating time function.

3. Phase Modulation with Harmonically Unrelated Frequencies

If the multiplications of the sums in Equ. (2-5) are carried out, an expression of the form

$$V = e^{j(\omega_0 t + \psi_0)} \left\{ \pm A_0 + \sum_{i=-\infty}^{\infty} \pm A_i e^{j(\omega_i t + \psi_i)} \right\} \qquad (3\text{-}1)$$

is obtained, where

$$A_0 = \prod_{n=1}^{n=\infty} |J_0(\varphi n)| \qquad (3\text{-}2)$$

$$\omega_i = \pm M_1 \omega_1 \pm M_2 \omega_2 \pm \ldots \pm M_n \omega_n \pm \ldots \pm M_N \omega_N \qquad (3\text{-}3)$$

ω_n = angular frequencies of the modulating function f(t)

M_n = an integer, including 0

If ω_i is always different from zero if any one or several of the integers M_n are different from zero, then the angular frequencies are harmonically unrelated. We see from Equ. (3-1) that in this case the amplitude of the component of carrier frequency is A_0. From Equ. (2-5) we see that ω_i is zero only for $m = 0$ (provided, of course, that $\omega_1 + np \neq 0$, see footnote on page 1). The carrier component amplitude can thus be obtained from Equ. (2-5) by putting $m = 0$, which leads to Equ. (3-2). An example of a time function with harmonically unrelated frequency spectrum is a subcarrier phase-modulated by $\hat{\Phi}_1 \sin pt$. The frequency spectrum for this time function was given by Equ. (2-12). The carrier component is obtained by putting $m = 0$ in Equ. (2-12). The resulting expression can be brought in the same form as Equ. (3-2) after change of variables.

If, on the other hand, the frequencies are harmonically related, the amplitude of the carrier component is no longer given by Equ. (3-2). This case will be analyzed in the next chapter.

If the power of the modulating time function (= base band) is ϕ, then

$$\frac{1}{2}\sum_{n=1}^{N} \varphi_n^2 = \phi^2 \tag{3-4}$$

Under this condition it is shown in Appendix B, Equ. (B-12) and (B-15), that

$$J_0(\sqrt{2}\,\phi) \leq \prod_{n=1}^{n=N} J_0(\varphi_n) \leq \left[J_0\left(\sqrt{\frac{2}{N}}\,\phi\right)\right]^N \tag{3-5}$$

provided all $\varphi_n \leq 5.1$ for $N \geq 2$. If $N = 1$ then equ. (3-5) reduces to an identity, which is true for all values of φ_n. It can further be shown, (see Appendix B) that

$$\lim_{N\to\infty} \left[J_0\left(\sqrt{\frac{2}{N}}\,\phi\right)\right]^N = e^{-\phi^2/2} \tag{3-6}$$

$J_0(\sqrt{2}\,\phi)$ is always positive in the region $0 \leq \sqrt{2}\,\phi \leq 2.4048$, where 2.4048 is the first zero of $J_0(\sqrt{2}\,\phi)$, and we can combine Equ. (3-2), (3-5), and (3-6) into

$$\left.\begin{aligned} &J_0(\sqrt{2}\,\phi) \leq A_0 \leq e^{-\phi^2/2} \\ &0 \leq \sqrt{2}\,\phi \leq 2.4048 \end{aligned}\right\} \tag{3-7}$$

For $\sqrt{2}\,\phi >$ 2.4048 we obtain instead

$$\left.\begin{array}{l} 0 \leq A_0 \leq \begin{cases} |J_0(\sqrt{2}\,\phi)| \\ e^{-\phi^2/2} \end{cases} \quad \text{whichever is greater} \\ \\ \qquad\qquad 2.4048 \leq \sqrt{2}\,\phi \leq 5.1\sqrt{2} \end{array}\right\} \tag{3-8}$$

The upper limit for ϕ is obtained from equ. (3-4) for N = 2 and $\varphi_n = 5.1$.

Equations (3-7) and (3-8) thus show that upper and lower limits of the carrier component can be calculated if only the power of the modulating time function is known. The lower limit is obtained if the modulating power is contained in one modulating frequency and the upper limit is obtained if the power is divided equally among infinitely many modulating frequencies.

The Equ. (3-7) and (3-8) are shown graphically in Fig. 2. It is seen that for $\phi < 1$ the upper and lower limits are very close together. The amplitude of the carrier component varies therefore only very little in this region if the frequency spectrum of the modulating time function is changed.

The reduction of the main carrier component due to modulating is commonly referred to as modulation loss. The modulation loss L_m for the carrier can be defined as the ratio of total output power and the power of the carrier component Expressing L_m in db we obtain

$$L_m = 10 \log \frac{\text{Total output power}}{\text{Carrier component power}} \text{ db}$$

Using the limits for the carrier component given in Equ. (3-7) and (3-8) we obtain

$$10 \log \frac{1}{J_0(\sqrt{2}\,\phi)^2} \geq L_m \geq 10 \log e^{\phi^2} \tag{3-7a}$$

for

$$0 \leq \sqrt{2}\,\phi \leq 2.4048$$

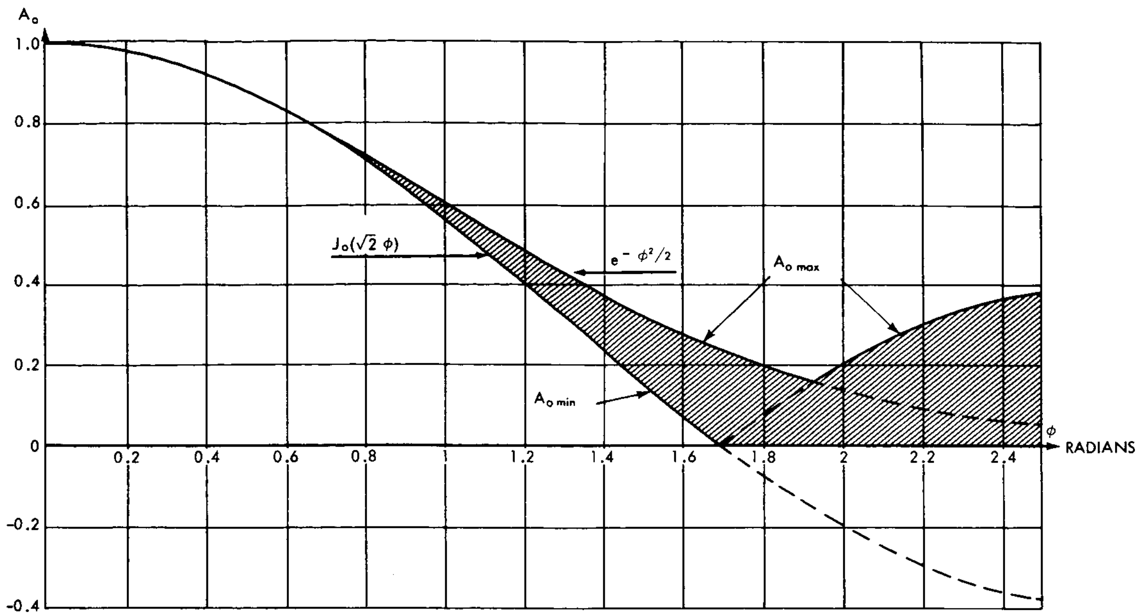

Figure 2—Upper limit $A_{0\ max}$ and lower limit $A_{0\ min}$ of the carrier component amplitude A_0. The carrier is phase modulated with RMS phase swing ϕ by a time function f(t), whose frequency spectrum is not harmonically related.

and

$$L_m \geq \begin{cases} 10 \log e^{\phi^2} \\ \\ 10 \log \dfrac{1}{J_0(\sqrt{2}\,\phi)^2} \end{cases} \quad \text{whichever is smaller} \qquad (3\text{-}8a)$$

for

$$2.4048 \leq \sqrt{2}\ \phi \leq 5.1\ \sqrt{2}$$

Equ. (3-7a) is shown graphically in Fig. 3. The graphs in Fig. 3 show both the upper and lower limit for the modulation loss as a function of RMS phase deviation ϕ. All information needed for analysis of the main carrier component is

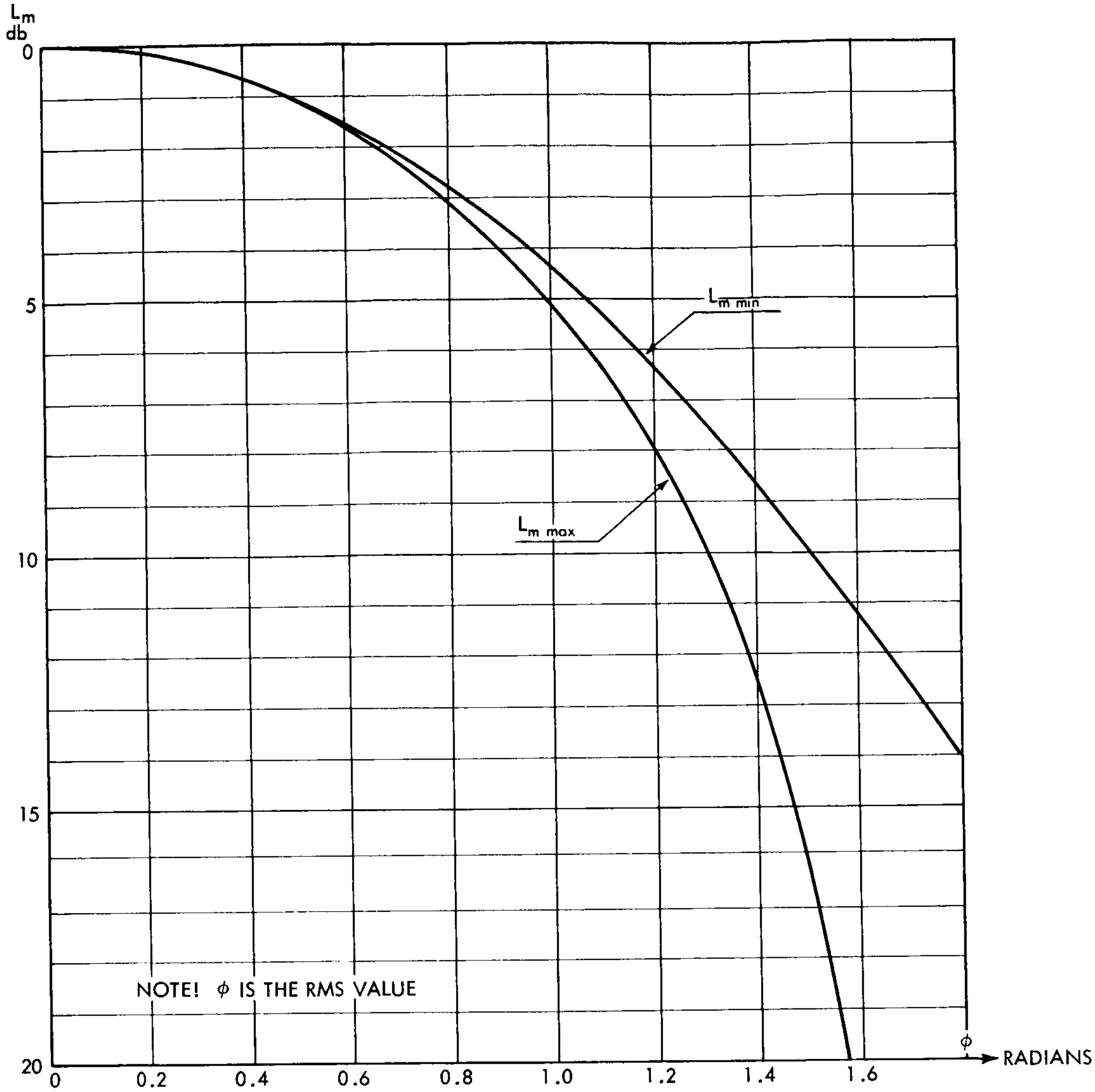

Figure 3—Upper and lower limits of modulation loss L_m for a carrier phase modulated with RMS phase swing ϕ

thus contained in these graphs. The use of the graphs in analysis work will be demonstrated in Chapter 5.

It should once more be pointed out, that the results of this paragraph are valid only if the modulating frequencies are harmonically unrelated (see Equ. (3-3)). Modulation with harmonically related frequencies is treated in the next chapter.

4. Phase Modulation with Harmonically Related Frequencies

If the modulating frequencies are harmonically related, we can find integers M_n which are not all zero, so that

$$\omega_i = \pm M_1 \omega_1 \pm M_2 \omega_2 \pm \cdots \pm M_n \omega_n \pm \cdots \pm M_N \omega_N = 0 \qquad (4\text{-}1)$$

This implies that in calculating the carrier component from equ. (2-5) more terms then

$$A_0 = \prod_1^N |J_0(\varphi_n)|$$

have to be included. For example, assume that the modulating function consists of three frequency components

$$f(t) = \varphi_1 \sin(\omega t + \psi_1) + \varphi_2 \sin(2\omega t + \psi_2) \qquad (4\text{-}2)$$

$$+ \varphi_3 \sin(3\omega t + \psi_3)$$

In this case the amplitude of the carrier component is

$$\begin{aligned} A_0 = \Big| & J_0(\varphi_1) J_0(\varphi_2) J_0(\varphi_3) - 2J_0(\varphi_2) J_1(\varphi_3) J_2(\varphi_1) \cos(2\psi_1 - \psi_2) \\ & + 2J_1(\varphi_1) J_1(\varphi_3) J_2(\varphi_2) \cos(\psi_1 - 2\psi_2 + \psi_3) \\ & - 2J_1(\varphi_2) J_1(\varphi_3) J_5(\varphi_1) \cos(5\psi_1 - \psi_2 - \psi_3) \\ & - j\, 2J_0(\varphi_3) J_1(\varphi_2) J_2(\varphi_1) \sin(2\psi_1 - \psi_2) \\ & + j\, 2J_0(\varphi_3) J_2(\varphi_2) J_4(\varphi_1) \sin(4\psi_1 - 2\psi_2) \\ & - j\, 2J_1(\varphi_1) J_1(\varphi_2) J_1(\varphi_3) \sin(\psi_1 + \psi_2 - \psi_3) \Big| \end{aligned} \qquad (4\text{-}3)$$

In the derivation, Bessel functions of higher order than 5 have been neglected.

Equ. (4-3) shows that with only three harmonically related modulating frequencies the expression for the carrier component already becomes a monstrosity and a better method of analysis is desirable. If the modulating frequencies are harmonically related, they can be combined into a periodic function $f(t)$

$$f(t) = f(t + T)$$

The complex notation for the modulated carrier is

$$V = e^{j(\omega_0 t + \psi_0)} e^{jf(t)} \tag{4-4}$$

If $f(t)$ is periodic, then $e^{jf(t)}$ is also periodic and the carrier component A_0 can be found by complex Fourier analysis

$$A_0 = \left| \frac{1}{T} \int_0^T e^{jf(t)} \, dt \right| \tag{4-5}$$

Because $|e^{jf(t)}| \leq 1$ the max value of A_0 is

$$A_{0\,max} \leq 1 \tag{4-6}$$

This max value is, for instance, obtained if $f(t)$ is a narrow square pulse of width τ as shown in Fig. 4, and we obtain

$$\lim_{\tau \to 0} A_0 = 1 \tag{4-7}$$

independent of the RMS phase deviation ϕ.

It has been proven in Appendix C that

$$A_0 \geq \cos \phi \tag{4-8}$$

where ϕ is the RMS phase deviation, thus

$$\frac{1}{T} \int_0^T f(t)^2 \, dt = \phi^2$$

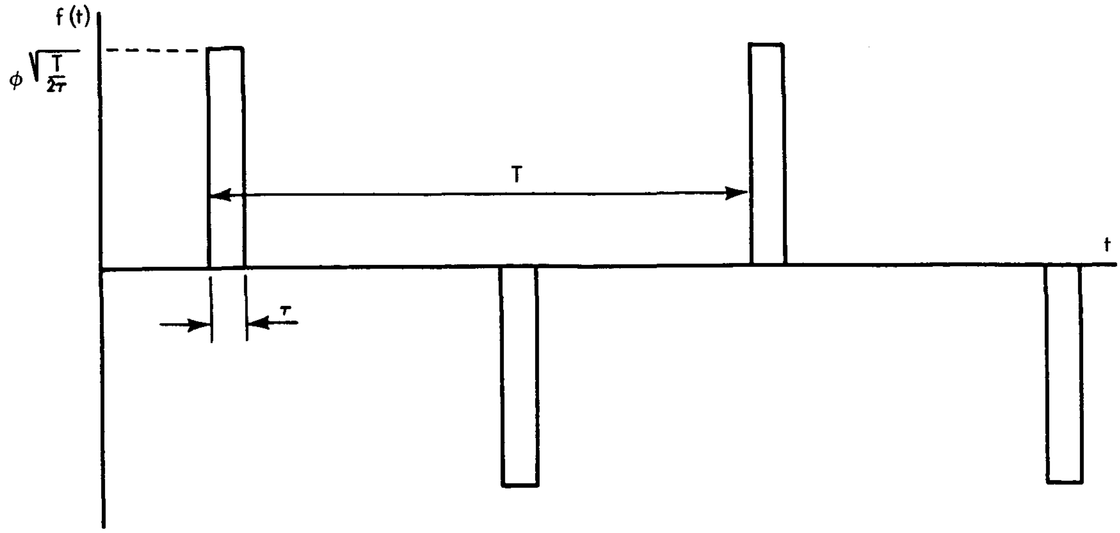

Figure 4—Narrow pulses for which $A_0 = A_{0\ max} = 1$ if $\tau \rightarrow 0$

The proof holds true if

$$|f(t)| \leq 1.432\,\pi \tag{4-9}$$

and

$$\phi \leq 1.432\,\pi \tag{4-10}$$

Observing that cos ϕ becomes negative for $\phi > \pi/2$ we can combine Equ. (4-6) and (4-8) to

$$\left.\begin{array}{lll} \cos\phi \leq A_0 \leq 1 & \text{for} & \phi \leq \frac{\pi}{2} \\ \text{and} & & \\ A_0 \leq 1 & \text{for} & \phi \geq \frac{\pi}{2} \end{array}\right\} \tag{4-12}$$

It is also shown in Appendix C that $A_{0\ min}$ occurs if f(t) is an A-C signal with constant amplitude

$$\left.\begin{array}{l} |f(t)| = \phi \\ \int_0^T f(t)\,dt = 0 \end{array}\right\} \tag{4-11}$$

as illustrated in Fig. 5.

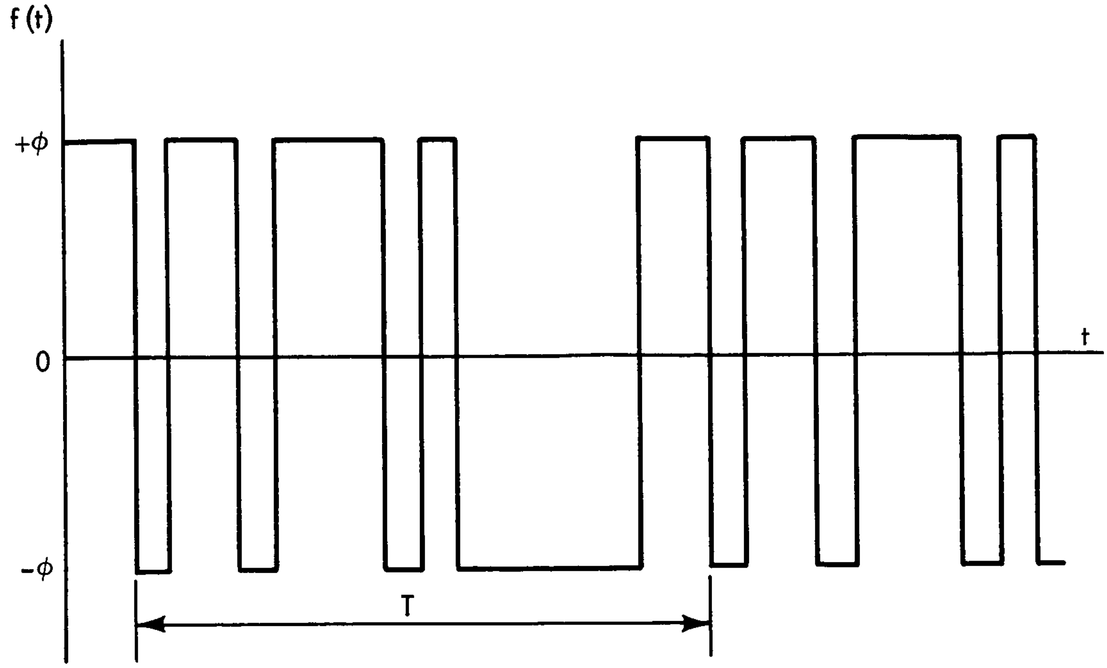

Figure 5– Example of periodic time function f(t) for which $|f(t)| = \phi$ and

$$\int_0^T f(t)\,dt = 0, \text{ so that } A_0 = A_{0\ min} = \cos\phi$$

The modulation loss L_m for the carrier component is thus

$$\left.\begin{aligned} 10 \log \frac{1}{\cos^2\phi} \geq L_m \geq 0 \quad &\text{for } \phi \leq \frac{\pi}{2} \\ \text{and} \qquad L_m \geq 0 \quad &\text{for } \phi \geq \frac{\pi}{2} \end{aligned}\right\} \qquad (4\text{-}13)$$

Equ. (4-13) is shown graphically in Fig. 6. For comparison, also the modulation loss for the harmonically unrelated case is shown with dotted lines. We see that the upper and lower limits for the harmonically unrelated case are much closer together than for the harmonically related case. It is therefore of importance to distinguish between the harmonically related and unrelated case, especially for larger values of ϕ. All A-C square waves, including pseudo random codes, and most amplitude limited signals satisfy Equ. (4-11) and must therefore be treated as harmonically related and the solid curves in Fig. 6 apply. An example where the modulating time function consists of both harmonically related and unrelated frequencies is treated in Chapter 5.

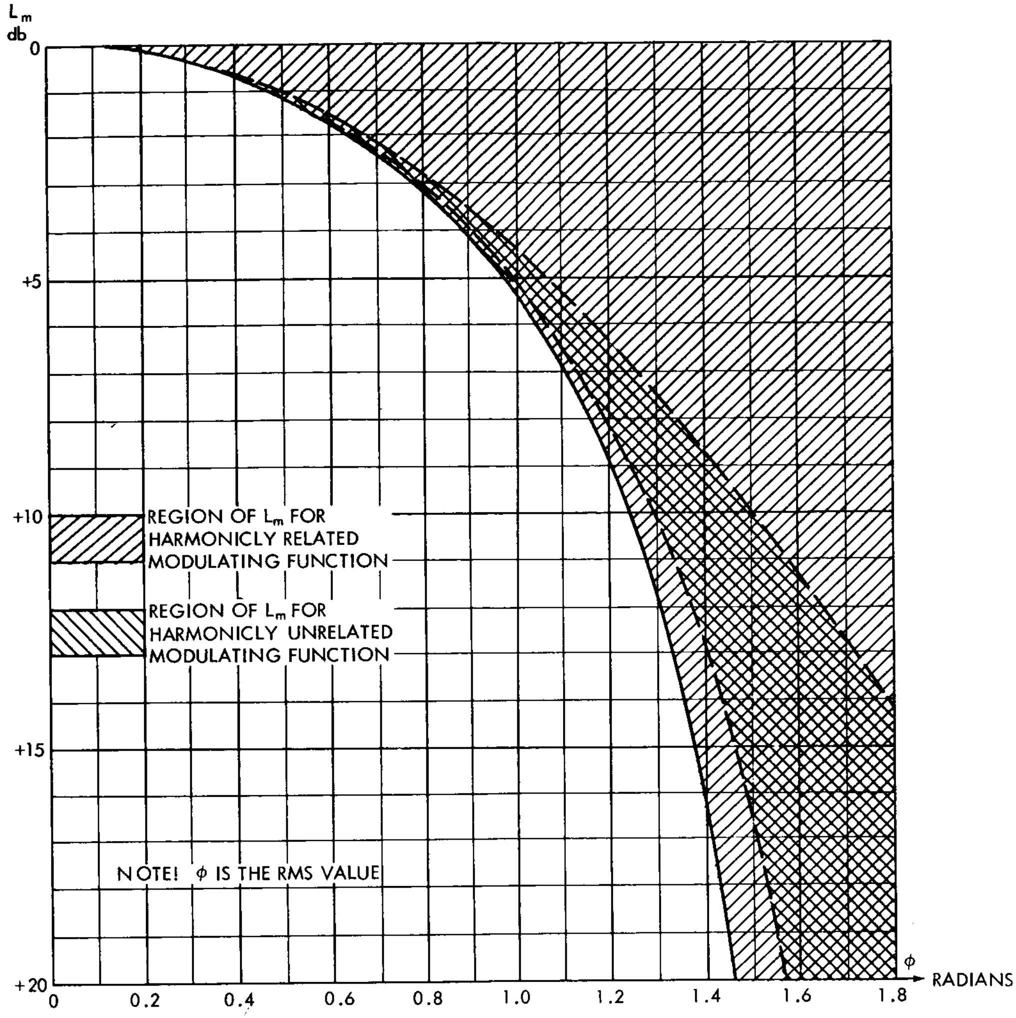

Figure 6–Upper and lower limits of modulation loss L_m for a carrier phase modulated with RMS phase swing ϕ

5. Applications

Example 1

As an example, let us analyze the case where a carrier is phase modulated with a time function consisting of a pseudo random code $f_1(t)$ with amplitude ϕ_1 and two phase modulated subcarriers $f_2(t)$ and $f_3(t)$ with peak phase swings $\hat{\phi}_2$ and $\hat{\phi}_3$ respectively. The total modulating time function $f(t)$ is thus

$$f(t) = f_1(t) + f_2(t) + f_3(t) \tag{5-1}$$

We are interested in calculating the amplitude of the main carrier component ω_0 for tracking purposes. Of interest are also the subcarrier components in the spectrum of the main carrier, i.e., $\omega_0 \pm \omega_2$ and $\omega_0 \pm \omega_3$ where ω_2 and ω_3 are the angular frequencies of the subcarriers. We want to know the minimum of the main carrier components in order to insure receiver lock under worst conditions and the maximum of the subcarrier components in order to compute the risk of locking on to them instead of the main carrier.

The phase modulated main carrier can be written

$$V = e^{j(\omega_0 t + \psi_0)}\, e^{jf(t)}$$

or

$$V = e^{j(\omega_0 t + \omega_0)}\, e^{jf_1(t)}\, e^{jf_2(t)}\, e^{jf_3(t)} \tag{5-2}$$

The Fourier expansion of the factor $e^{jf_1(t)}$ has a D-C component, whose magnitude A_{01} is given by Equ. (4-8)

$$A_{01} = \cos \phi_1 \tag{5-3}$$

because $f_1(t)$ is a square wave. The factors $e^{jf_2(t)}$ and $e^{jf_3(t)}$ have their minimum D-C components $A_{02\ min}$ and $A_{03\ min}$ given by Equ. (3-7)

$$\begin{aligned} A_{02\ min} &= J_0(\sqrt{2}\,\phi_2) \\ A_{03\ min} &= J_0(\sqrt{2}\,\phi_3) \end{aligned} \tag{5-4}$$

It is assumed that $\phi_1 < \pi/2$, $\hat{\phi}_2 < 2.40/\sqrt{2}$, and $\hat{\phi}_3 < 2.40/\sqrt{2}$ so that Equ. (3-7) and (4-8) are applicable. If V given by Equ. (5-2) is expanded in a series, the coefficient for $e^{j(\omega_0 t + \psi_0)}$ will be

$$A_0 = A_{01}\, A_{02}\, A_{03} \tag{5-5}$$

and the minimum carrier component is thus

$$A_{0\ min} = A_{01\ min} A_{02\ min} A_{03\ min} \tag{5-6}$$

or

$$A_{0\ min} = \cos \phi_1 \ J_0(\sqrt{2}\,\phi_2)\ J_0(\sqrt{2}\,\phi_3)$$

The maximum modulation loss $L_{m\ max}$ corresponding to $A_{0\ min}$ is found from Fig. 6. With

$$\phi_1 = 0.6 \text{ rad}$$

$$\hat{\phi}_2 = \hat{\phi}_3 = 1.22 \text{ rad}$$

we obtain

$$L_{m1\ max} = 1.7 \text{ dB}$$

$$L_{m2\ max} = 3.6 \text{ dB}$$

$$\underline{L_{m3\ max} = 3.6 \text{ dB}}$$

and thus $L_{m\ max} = 8.9$ dB

where L_{m1} corresponds to A_{01} etc. The minimum modulation loss is, according to Fig. 6, $L_{1\ min} = L_{1\ max} = 1.7$ dB

$$L_{2\ min} = 3.25 \text{ dB}$$

$$L_{3\ min} = \underline{3.25 \text{ dB}}$$

$$L_{m\ min} = 8.2 \text{ dB}$$

The modulation loss for the main carrier will thus vary between 8.2 and 8.9 dB depending on the subcarrier modulation. Note that $L_{m\ max}$ is obtained for unmodulated subcarriers and $L_{m\ min}$ for modulated subcarriers.

It has been assumed in the above derivation that the frequencies of $f_1(t)$ are harmonically related to each other and that the frequencies of $f_2(t)$ are not harmonically related to each other as well as the frequencies of $f_3(t)$. In addition, Equ. (5-5) is only true if none of the frequencies of $f_1(t)$ are harmonically related to the frequency $f_2(t)$ or $f_3(t)$ and the frequencies of $f_2(t)$ are not harmonically related to the frequencies of $f_3(t)$.

If the frequencies are harmonically related the total RMS phase swing ϕ can be computed by

$$\phi = \sqrt{\phi_1^2 + \frac{1}{2}\hat{\phi}_2^2 + \frac{1}{2}\hat{\phi}_3^2} \tag{5-7}$$

and the carrier amplitude is then given by Equ. (4-8)

$$A_{0\ min} = \cos \phi \tag{4-8}$$

This is the absolute worst case. Using the same numbers as before we obtain

$$\phi = \sqrt{0.6^2 + \frac{1}{2}\,1.22^2 + \frac{1}{2}\,1.22^2} = 1.363 \text{ rad}$$

and from Fig. 6

$$L_{m\ max\ worst\ case} = 13.8 \text{ dB}$$

which is 4.9 dB more than the $L_{m\ max}$ computed under the assumption that the subcarrier and pseudo random code frequencies are harmonically unrelated. This example shows the importance of distinguishing between the harmonically related and unrelated case and of choosing suitable subcarrier frequencies.

Let us also compute the subcarrier components in the spectrum of the modulated main carrier.

The amplitude A_2 of the frequency component $\omega_0 \pm \omega_2$ is

$$A_2 = A_{01} A_{12} A_{03} \tag{5-8}$$

where

$$A_{12} = \frac{J_1(\varphi_1)}{J_0(\varphi_1)} \prod_{1}^{N} J_0(\varphi_n)$$

is obtained from Equ. (A-11) in Appendix A, and where φ_n are the coefficients in the Fourier expansion of $f_2(t)$. Again, Equ. (5-8) holds true only if the above assumption about the harmonic relation of the frequency components hold true.

Because $f_1(t)$ is a square wave, A_{01} is constant. $A_{12\ max}$ is obtained for $\varphi_1 = \sqrt{2}\,\phi_2$ and all other $\varphi_n = 0$, thus

$$A_{12\ max} = J_1(\sqrt{2}\,\phi_2)$$

$A_{03\ max}$ is

$$A_{03\ max} = e^{-\frac{\phi_3^2}{2}}$$

according to Equ. (3-7). Hence

$$A_{2\ max} = A_{01}\, J_1(\sqrt{2}\,\phi_2)\, e^{-\frac{\phi_3^2}{2}} \tag{5-9}$$

The ratio of $A_{2\ max}$ to $A_{0\ min}$ is thus

$$\frac{A_{2\ max}}{A_{0\ min}} = \frac{J_1(\sqrt{2}\,\phi_2)\, e^{-\frac{\phi_3^2}{2}}}{J_0(\sqrt{2}\,\phi_2)\, J_0(\sqrt{2}\,\phi_3)} \tag{5-10}$$

and is obtained for subcarrier ω_2 unmodulated and subcarrier ω_3 modulated. With the same numbers as before for ϕ_1, $\hat{\phi}_2$, and $\hat{\phi}_3$, we obtain

$$\frac{A_{2\ max}}{A_{0\ min}} = 0.794 = -2.0 \text{ db}$$

The subcarrier components $\omega_0 \pm \omega_2$ or $\omega_0 \pm \omega_3$ may thus only be 2 db below the main carrier component ω_0.

Example 2

Assume that the RMS phase swing ϕ is changed. How much does the modulation loss L_m change?

For the harmonically related case we have (worst case)

$$L_m = -20 \log \cos \phi \qquad \left(\phi \leq \frac{\pi}{2}\right)$$

Differentiation yields

$$\frac{dL_m}{d\phi} = 20(\log e) \tan \phi \quad \text{db/rad} \tag{5-11}$$

For the harmonically unrelated cases we have (worst case)

$$L_m = -20 \log J_0(\sqrt{2}\,\phi)$$

and after differentiation

$$\frac{dL_m}{d\phi} = 20(\log e) \frac{\sqrt{2}\, J_1(\sqrt{2}\,\phi)}{J_0(\sqrt{2}\,\phi)} \tag{5-12}$$

Again we consider the case where the modulating time function consists of a pseudo random code with RMS phase deviation ϕ_1 and two subcarriers with RMS phase deviation ϕ_2 and ϕ_3 as in example 1. If the phase deviations have the same relative change δ, we obtain

$$\Delta\phi_1 = \phi_1 \delta$$

$$\Delta\phi_2 = \phi_2 \delta \tag{5-13}$$

$$\Delta\phi_3 = \phi_3 \delta$$

and from Equ. (5-11) and (5-12) to a first degree approximation

$$\Delta L_{m1} = 8.69\, \phi_1 (\tan \phi_1)\, \delta \text{ db}$$

$$\Delta L_{m2} = 8.69 \frac{\sqrt{2}\, \phi_2\, J_1(\sqrt{2}\, \phi_2)}{J_0(\sqrt{2}\, \phi_2)}\, \delta \text{ db} \tag{5-14}$$

$$\Delta L_{m3} = 8.69 \frac{\sqrt{2}\, \phi_3\, J_1(\sqrt{2}\, \phi_3)}{J_0(\sqrt{2}\, \phi_3)}\, \delta \text{ db}$$

Assuming the worst case with all $\Delta L_{m\ 1\text{-}3}$ having the same sign, we thus obtain

$$\Delta L_m = \Delta L_{m1} + \Delta L_{m2} = \Delta L_{m3} = 20\delta \text{ db}$$

for

$$\phi_1 = 0.6 \text{ rad}$$

$$\sqrt{2}\, \phi_2 = \sqrt{2}\, \phi_3 = 1.22 \text{ rad}$$

Example 3

What is the spectrum of a phase modulated carrier, if f(t) is a symmetric square wave G(t) as shown in Fig. 7?

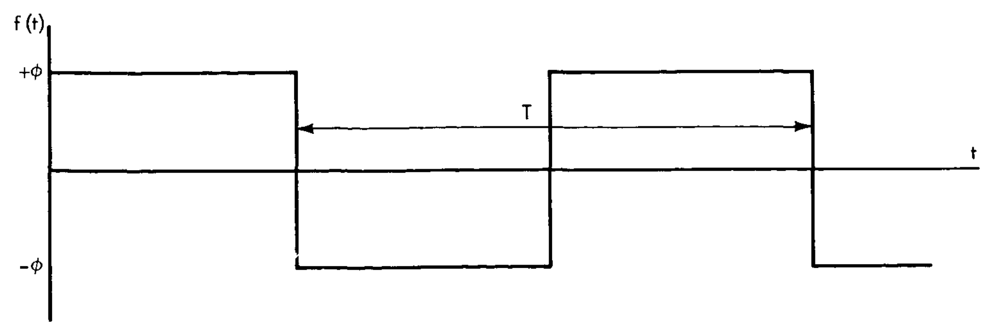

Figure 7—Symmetric square wave

The modulated carrier is

$$v = \cos(\omega_0 t + \psi_0 + G(t)) = \mathrm{Re}\left\{e^{j(\omega_0 t + \psi_0)}\, e^{jG(t)}\right\} \tag{5-11}$$

$e^{jG(t)}$ is a periodic function with period T and we can therefore expand $e^{jG(t)}$ into a complex Fourier series

$$e^{jG(t)} = \sum_{n=-\infty}^{n=\infty} C_n\, e^{j\frac{2n\pi}{T}t} \tag{5-12}$$

where

$$C_n = \frac{1}{T}\int_0^T e^{j\left(G(t) - \frac{2n\pi}{T}t\right)}\, dt \tag{5-13}$$

$$= \frac{1}{T}\int_{T/2}^{T} e^{-j\left(\phi + \frac{2n\pi}{T}\right)t}\, dt + \frac{1}{T}\int_0^{T/2} e^{j\left(\phi - \frac{2n\pi}{T}t\right)}\, dt$$

and hence

$$C_n = \frac{\sin n\pi \cos\phi}{n\pi} + \frac{1 - (-1)^n}{n\pi}\sin\phi$$

Using complex notations we can thus write the modulated carrier

$$V = e^{j(\omega_0 t + \psi_0)}\left(\cos\phi + \sum_{n=-\infty}^{n=\infty}{}_0\, \frac{2}{n\pi}\sin\phi\, e^{j\frac{2n\pi}{T}t}\right) \tag{5-14}$$

where Σ_0 denotes summation over odd n only.

This example demonstrates that it is easier in some cases to perform the complex Fourier analysis, Equ. (5-13), without expanding the modulating function into a Fourier series. Another example for which this is true, is a symmetric triangular modulating time function with peak amplitude $\hat{\phi}$ and period T. The complex Fourier expansion yields

$$V = e^{j(\omega_0 t + \psi_0)} \sum_{n=-\infty}^{n=\infty} \frac{2\hat{\phi}}{n^2\pi^2 - 4\hat{\phi}^2}\left(e^{j\hat{\phi}} - (-1)^n\, e^{-j\hat{\phi}}\right) e^{j\left(\frac{2n\pi}{T}t + \frac{\pi}{2}\right)} \tag{5-15}$$

6. Generalizations

So far the analysis has been concentrated on the component of carrier frequency. In this chapter an outline will be given how to analyze the other components in the spectrum. The harmonically unrelated and related cases will be treated separately as before.

A. The Harmonically Unrelated Case

Let us analyze the component of frequency $\omega_0 + \omega_i$, where ω_i is given by Equ. (3-3).

$$\omega_i = \pm M_1 \omega_1 \pm M_2 \omega_2 \pm \cdots \pm M_n \omega_n \pm \cdots \pm M_N \omega_N$$

ω_n = angular frequency components of modulating function f(T)

M_n = integers, including 0

The amplitude A_i is obtained from Equ. (2-5)

$$A_i = \prod_{n=1}^{N} J_{M_n}(\varphi_n) \tag{6-1}$$

where φ_n = the amplitude of component ω_n

If all $M_n = 0$ then $A_i = A_0$. This case has been treated earlier. We therefore assume here that not all M_n are zero. The lower limit, $A_{i\ min}$, is zero, because if $M \neq 0$, then

$$J_M(\varphi) \to 0 \quad \text{if} \quad \varphi \to 0$$

Thus

$$A_{i\ min} = 0 \tag{6-2}$$

The maximum of A_i may be calculated using Lagrange's multiplier with the side condition

$$\frac{1}{2} \Sigma \varphi_n^2 = \phi^2 \tag{6-3}$$

We obtain N equations

$$\frac{\partial A_i}{\partial \varphi_n} + \lambda \varphi_n = 0$$

Performing the differentiation yields

$$\frac{J_{M_n-1}(\varphi_n) - J_{M_n-1}(\varphi_n)}{J_{M_n}(\varphi_n)} + \frac{2\lambda}{A_i}\varphi_n = 0 \tag{6-4}$$

Together with (6-2), we have N + 1 equations, which is sufficient to eliminate the Lagrange's multiplier λ and solve for all φ_n. Insertion in Equ. (6-1) will yield $A_{i\ max}$.

B. The Harmonically Related Case

If the modulating time function $f(t)$ is periodic with periodicity T, then a complex Fourier analysis of the modulated carrier is possible, resulting in

$$V = e^{j(\omega_0 t + \psi_0)} \Sigma C_n e^{j\frac{2n\pi}{T}t} \tag{6-4}$$

where

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{j\left(f(t) - \frac{2n\pi}{T}t\right)} dt \tag{6-5}$$

For some time functions such as square waves, triangular waves and sinusoidal waves, C_n is easy to evaluate. This is demonstrated in Chapter 5, Example 3. Our problem is to find $f(t)$ such that $|C_n|$ has a maximum or minimum. For $n \neq 0$ we have

$$C_{n\ min} = 0$$

because $f(t)$ can be chosen so that all the power is in C_0 and therefore none in C_n. The upper limit, $C_{n\ max}$ has not been established so far. The problem is to find $f(t)$, subject to the condition

$$\int_0^T f(t)^2 dt = \phi^2$$

such that $|C_n|$ given by Equ. (6-5) has a maximum and to calculate that maximum.

The work is continuing along the above scheduled outlines (A) and (B) and more complete results will be published as soon as they are available.

Another interesting generalization can be obtained from the fact that $f(t)$ may not only be expanded into a Fourier series, but into any series consisting of orthogonal functions, provided, of course, that the set is complete. Let the orthogonal functions be $g_n(t)$

$$f(t) = \sum_{0}^{\infty} \varphi_n g_n(t) \tag{6-6}$$

The modulated carrier is then

$$V = e^{j(\omega_0 t + \psi_0)} \prod_{n=1}^{n=N} e^{j(\varphi_n g_n(t))} \tag{6-7}$$

each factor $e^{j(\varphi_n g_n(t))}$ can be expanded in a complex Fourier series

$$e^{j(\varphi_n g_n(t))} = \sum_{m=-\infty}^{m=\infty} C_{nm} e^{j\frac{2n\pi}{T}t} \tag{6-8}$$

and

$$V = e^{j(\omega_0 t + \psi_0)} \prod_{n=1}^{n=N} \sum_{m=-\infty}^{m=\infty} C_{nm} e^{j\frac{2nm\pi}{T}t} \tag{6-9}$$

If the $g_n(t)$'s are sinusoidal functions, then the C_{nm}'s are Bessel functions. If the $g_n(t)$'s are square waves, the C_{nm}'s are sine functions as shown in Chapter 5, Example 3. An analysis of phase modulation without involving Bessel functions is thus possible.

7. Conclusions

It has been demonstrated that upper and lower limits of the spectral amplitudes of a phase modulated carrier can be calculated if the frequency spectrum of the modulating time function is not known and only the RMS phase deviation (= RMS of modulating time function) is known. Calculations can thus be carried out without introducing approximations such as substituting voice or other information by a single frequency. In addition, upper and lower limits are generally more important in circuit design and systems analysis than any value in between the limits, which exists only under very special conditions. The method is also suitable for stability and drift analysis, i.e., to analyze the change in the spectrum of modulated carrier if the phase deviation changes.

APPENDIX A

Phase Modulation with with Multiple Frequencies

A carrier with angular frequency ω_0 is phase modulated with N signals of angular frequency ω_n and amplitude φ_n. The instantaneous normalized amplitude v of the carrier is

$$v = \cos\left(\omega_0 t + \psi_0 + \sum_{n=1}^{n=N} \varphi_n \sin(\omega_n t + \psi_n)\right) \tag{A-1}$$

or with complex notations

$$V = e^{j\left(\omega_0 t + \psi_0 + \sum_{n=1}^{n=N} \varphi_n \sin(\omega_n t + \psi_n)\right)} \tag{A-2}$$

which also can be written

$$V = e^{j(\omega_0 t + \psi_0)} \prod_{n=1}^{n=N} e^{j\varphi_n \sin(\omega_n t + \psi_n)} \tag{A-3}$$

where $\prod_{n=1}^{n=N}$ denotes the product of the N factors $e^{j\varphi_n \sin(\omega_n t + \psi_n)}$

Each of the factors is a periodic function in t and can therefore be expressed as a complex Fourier series

$$e^{j\varphi \sin(\omega t + \psi)} = \sum_{m=-\infty}^{m=\infty} C_m e^{jm\omega t} \tag{A-4}$$

Substituting $\omega t + \psi$ by θ we obtain

$$e^{j\varphi \sin \theta} = \sum C_m e^{jm(\theta - \psi)} \tag{A-5}$$

where the complex constants C_m are given by

$$C_m = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{j(\varphi \sin \theta - m\theta + m\psi)} \, d\theta \tag{A-6}$$

From reference [7] we obtain

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{j(\varphi \sin \theta - m\theta)} \, d\theta = J_m(\varphi) \tag{A-7}$$

where $J_m(\varphi)$ is the Bessel function of the first kind and order m. Hence

$$C_m = e^{jm\psi} J_m(\varphi) \tag{A-8}$$

and

$$e^{j\varphi \sin(\omega t + \psi)} = \sum_{m=-\infty}^{m=\infty} J_m(\varphi) e^{jm(\omega t + \psi)} \tag{A-9}$$

Substitution into Equ. (A-3) finally yields

$$V = e^{j(\omega_0 t + \psi_0)} \prod_{n=1}^{n=N} \sum_{m=-\infty}^{m=\infty} J_m(\varphi_n) \, e^{jm(\omega_n t + \psi_n)} \tag{A-10}$$

The carrier amplitude A_0 is obtained for $m = 0$, thus

$$A_0 = \prod_{n=1}^{n=N} J_0(\varphi_n) \tag{A-11}$$

provided that the angular frequencies ω_n are not harmonically related. Under this condition the amplitude A_{nk} for the frequency component $\omega_0 \pm n\omega_k$ is

$$A_{nk} = \frac{J_n(\varphi_k)}{J_0(\varphi_k)} \prod_{n=1}^{n=N} J_0(\varphi_n) \tag{A-12}$$

provided that $n\omega_k$ cannot be obtained by adding or subtracting multiples of the other frequencies:

$$n\omega_k \neq M_1\,\omega_1 \pm M_2\,\omega_2 \pm M_3\,\omega_3 \pm \cdots \tag{A-13}$$

where M_1 etc are integers.

APPENDIX B

The problem is to find maxima and minima of

$$A_0 = \prod_{n=1}^{n=N} J_0(\varphi_n) \tag{3-3}$$

where the independent variables φ_n are subject to the condition

$$\frac{1}{2} \sum_{n=1}^{n=N} \varphi_n^2 = \phi^2 \tag{3-4}$$

Using the method of Lagranges multiplier, we obtain

$$\left.\begin{aligned} \frac{\partial A_0}{\partial \varphi_k} + \lambda \frac{\partial}{\partial \varphi_k}\left(\frac{1}{2} \sum_{n=1}^{n=N} \varphi_n^2 - \phi^2\right) = 0 \\ \frac{\partial A_0}{\partial \varphi_\ell} + \lambda \frac{\partial}{\partial \varphi_\ell}\left(\frac{1}{2} \sum_{n=1}^{n=N} \varphi_n^2 - \phi^2\right) = 0 \end{aligned}\right\} \tag{B-1}$$

where φ_k and φ_ℓ are any two of the variables φ_n. Observing that

$$\frac{d}{d\varphi} J_0(\varphi) = -J_1(\varphi)$$

we obtain

$$\left.\begin{aligned} \frac{-J_1(\varphi_k)}{J_0(\varphi_k)} A_0 + \lambda \varphi_k = 0 \\ \frac{-J_1(\varphi_\ell)}{J_0(\varphi_\ell)} A_0 + \lambda \varphi_\ell = 0 \end{aligned}\right\} \tag{B-2}$$

and after elimination of λ and A_0

$$\frac{J_1(\varphi_k)}{\varphi_k J_0(\varphi_k)} = \frac{J_1(\varphi_\ell)}{\varphi_\ell J_0(\varphi_\ell)} \tag{B-3}$$

The obvious solution is

$$\varphi_k = \varphi_\ell \tag{B-4}$$

Because φ_k and φ_ℓ were arbitrarily chose, Equ. (B-4) holds for all φ_n. A_0 has thus an extreme value for

$$\varphi_n = \varphi_0 = \sqrt{\frac{2}{N}}\,\phi \tag{B-5}$$

valid for all n. We still need to show that Equ. (B-3) has only one solution and we have to determine whether the extreme value of A_0 is a maximum, minimum or sadelpoint. By plotting

$$y = \frac{J_1(\varphi)}{\varphi J_0(\varphi)} \tag{B-6}$$

as is done in Fig. 8, we see that

$$y = \text{constant}$$

has only one solution $\varphi \leq 5.1$. By limiting ϕ to

$$\phi \leq \frac{5.1}{\sqrt{2}} \leq \frac{5.1}{\sqrt{2}}\,N \tag{B-6}$$

Equ. (B-3) is thus limited to only one solution for all positive values of the integer N.

In order to determine the type of extreme value of A_0, we study A_0 in the neighborhood of $\varphi_n = \varphi_0$ by putting

$$\varphi_n = \varphi_0 + \epsilon_n \tag{B-7}$$

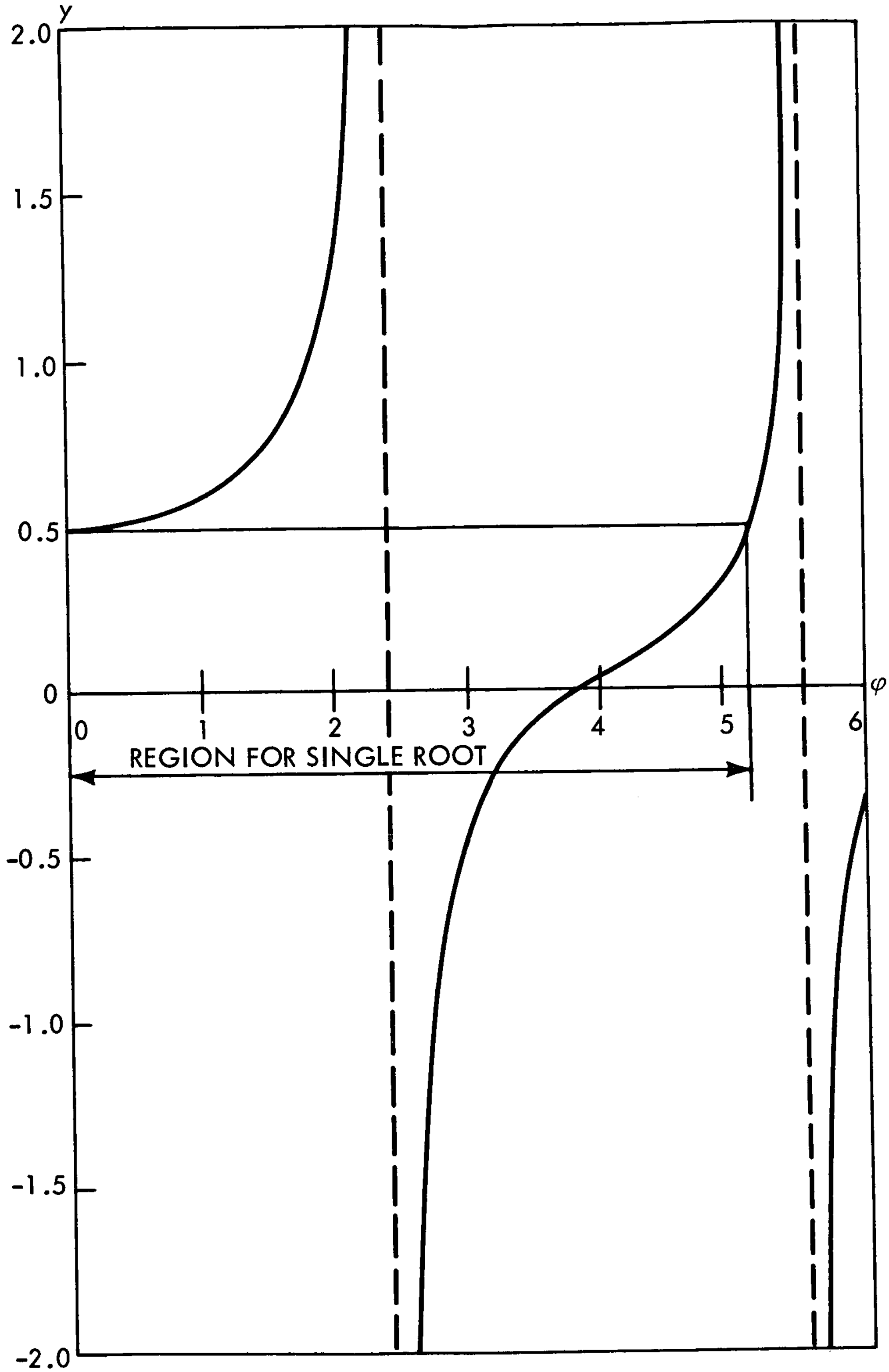

Figure 8–$y = \dfrac{J_1(\varphi)}{\varphi J_0(\varphi)}$ y = constant has only one root $\varphi \leq 5.1$

where ϵ_n is an arbitrarily small quantity. From Equ. (3-4) we obtain

$$\frac{1}{2} \sum_{n=1}^{n=N} (\varphi_0^2 + 2\varphi_0 \epsilon_n + \epsilon_n^2) = \phi^2$$

or (B-8)

$$\varphi_0 \sum_{n=1}^{n=N} \epsilon_n + \frac{1}{2} \sum_{n=1}^{n=N} \epsilon_n^2 = 0$$

The expansion of $J_0(\varphi_0 + \epsilon_n)$ up to second order terms yields

$$J_0(\varphi_n) = J_0(\varphi_0 + \epsilon_n) = J_0(\varphi_0) - J_1(\varphi_0)\epsilon_n - \frac{1}{2}\left(\frac{J_1(\varphi_0)}{\varphi_0} - J_2(\varphi_0)\right)\epsilon_n^2 + \dots \qquad \text{(B-9)}$$

Taking the logarithm of Equ. (3-3) yields

$$\ell n\, A_0 = \sum_{n=1}^{n=N} \ell n\,(J_0(\varphi_n))$$

and substituting $J_0(\varphi_n)$ from Equ. (B-9)

$$\ell n\, A_0 = N\ell n(J_0(\varphi_0)) + \sum \ell n\left[\left(1 - \frac{J_1(\varphi_0)}{J_0(\varphi_0)}\right)\epsilon_n - \frac{1}{2}\left(\frac{J_1(\varphi_0)}{\varphi_0 J_0(\varphi_0)} - \frac{J_2(\varphi_0)}{J_0(\varphi_0)}\right)\epsilon_n^2 + \dots\right] \qquad \text{(B-10)}$$

For any $|x| < 1$ the logarithm $\ell n(1 - x)$ can be expanded in a series

$$\ell n(1 - x) = - x - \frac{1}{2} x^2 \dots\dots$$

and Equ. (B-10) can therefore be written, including up to second order terms

$$\ell n\, A_0 = N\,\ell n(J_0(\varphi_0)) - \frac{J_1(\varphi_0)}{J_0(\varphi_0)} \sum_{n=1}^{n=N} \epsilon_n$$

$$- \frac{1}{2}\left[\frac{J_1(\varphi_0)}{\varphi_0 J_0(\varphi_0)} - \frac{J_2(\varphi_0)}{J_0(\varphi_0)}\right] \sum_{n=1}^{n=N} \epsilon_n^2$$

Substituting $\Sigma\epsilon_n$ from Equ. (B-8) yields

$$\ell n\, A_0 = N\,\ell n(J_0(\varphi_0)) - \frac{1}{2J_0(\varphi_2)^2}\left[J_1(\varphi_0)^2 - J_0(\varphi_0)J_2(\varphi_0)\right]\sum \epsilon_n^2 - \ldots \quad \text{(B-11)}$$

Using the identitics

$$J_0(\varphi_0) = \frac{1}{\varphi_0} J_1(\varphi_0) + \frac{d}{d\varphi_0} J_1(\varphi_0)$$

$$J_2(\varphi_0) = \frac{1}{\varphi_0} J_1(\varphi_0) - \frac{d}{d\varphi_0} J_1(\varphi_0)$$

we obtain

$$J_1(\varphi_0)^2 - J_0(\varphi_0)\,J_2(\varphi_0) = \left(1 - \frac{1}{\varphi_0^2}\right) J_1(\varphi_0)^2 + \left[\frac{d}{d\varphi_0} J_1\ (\varphi_0)\right]^2$$

The brackets [ ] in front of $\Sigma\epsilon_n^2$ is thus positive for $|\varphi_0| \geq 1$. [ ] is also positive for $|\varphi_0| \leq 1$, which can be shown in the following way:

For $\varphi_0 = 0$ we find [ ] = 0.

and

$$\frac{d}{d\varphi_0}[\;] = \frac{2}{\varphi_0}\, J_0\ (\varphi_0)\, J_2\ (\varphi_2) \geq 0$$

for $|\varphi_0| \leq 1$. As [ ] is an even function of φ_0, we thus conclude that [ ] ≥ 0 for all φ_0.

Hence,

$$A_0 = e^{-C\sum_{n=1}^{n=N}\epsilon_n^2}\left[J_0(\varphi_0)\right]^N \quad \text{(B-12)}$$

where C is a positive quantity. The extreme value is therefore a maximum. All φ_n are equal at the maximum and we thus have

$$A_{0\ max} \leq \lim_{N\to\infty}\left(J_0\left(\sqrt{\frac{2}{N}}\,\phi\right)\right)^N \quad \text{(B-13)}$$

Expanding $J_0\left(\sqrt{\frac{2}{N}}\,\phi\right)$ into a series yields

$$J_0\left(\sqrt{\frac{2}{N}}\,\phi\right) = 1 - \frac{\phi^2}{2N} + \frac{\phi^4}{16N^2} - \ldots \tag{B-14}$$

we also have

$$\lim_{p\to\infty}\left(1 - \frac{x}{p}\right)^p = e^{-x}$$

with

$$x = \frac{1}{N}\left(\frac{\phi^2}{2} - \frac{\phi^4}{16N} + \ldots\right)$$

we obtain

$$\lim_{N\to\infty}\left(J_0\left(\sqrt{\frac{2}{N}}\,\phi\right)\right)^N = \lim_{N\to\infty} e^{-\frac{\phi^2}{2} + \frac{\phi^4}{16N} - \ldots} = e^{-\frac{\phi^2}{2}} \tag{B-15}$$

The minimum value of A_0 has to be on the boundary because there was only one extreme value inside the boundary. The minimum value thus occurs for $\varphi_k = \sqrt{2}\,\phi$ and all other $\varphi_n = 0$. Hence

$$J_0(\sqrt{2}\,\phi) \leq A_0 \leq e^{-\frac{\phi^2}{2}} \tag{B-16}$$

for

$$\sqrt{2}\,\phi \leq 2.4048$$

APPENDIX C

The problem is to find a periodic function $f(t)$ such that A_0 has a minimum if A_0 is given by

$$A_0 = \frac{1}{T} \left| \int_0^T e^{j f(t)} \, dt \right| \tag{C-1}$$

and where $f(t)$ is subject to the conditions

$$\int_0^T f(t) \, dt = 0 \tag{C-2}$$

and

$$\frac{1}{T} \int_0^T f(t)^2 \, dt = \phi^2 \tag{C-3}$$

Using

$$e^{jx} = \cos x + j \sin x$$

Equ. (C-1) can be re-written

$$A_0 = \frac{1}{T} \left| \int_0^T \cos f(t) \, dt + j \int_0^T \sin f(t) \, dt \right|$$

and from

$$|x + jy| \geq |x|$$

we obtain

$$A_0 \geq \frac{1}{T} \left| \int_0^T \cos f(t) \, dt \right| \geq \frac{1}{T} \int_0^T \cos f(t) \, dt \tag{C-4}$$

The integral can be considered to be the limit case of a sum

$$\frac{1}{T}\int_0^T \cos f(t)\,dt = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\cos f_n \tag{C-5}$$

where

$$f_n = f(t) \text{ at } t = \frac{n}{N}T$$

as shown in Fig. 9. For the side condition in Equ. (C-3)

$$\frac{1}{T}\int_0^T f(t)^2\,dt = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_n^2 = \phi^2 \tag{C-6}$$

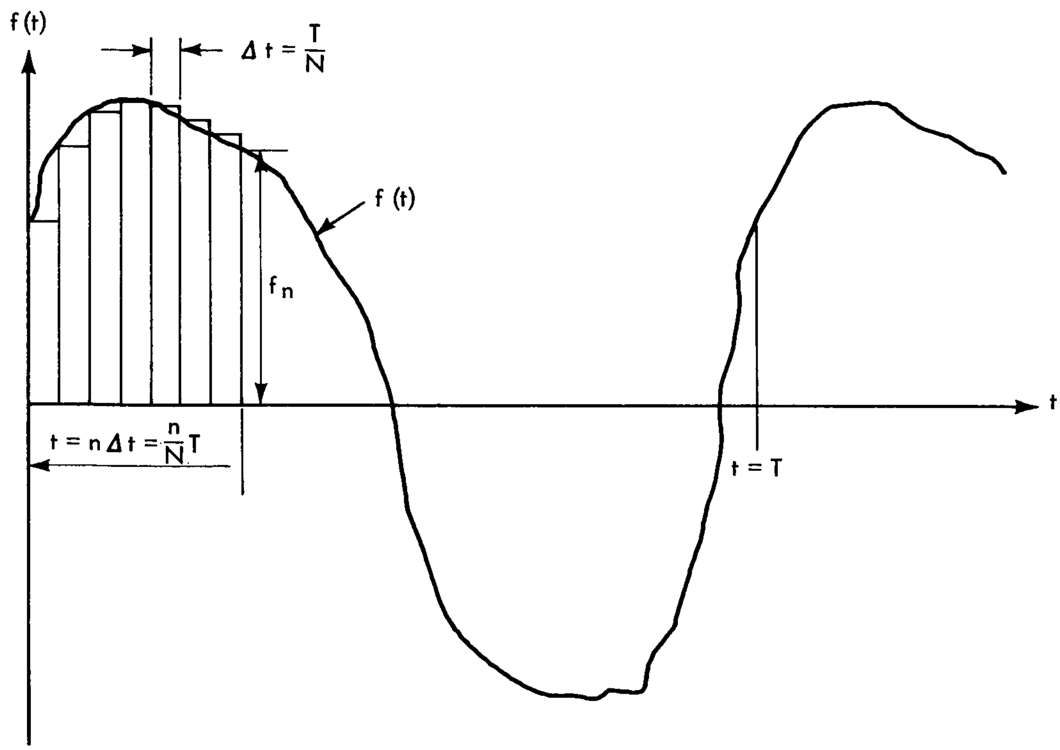

Figure 9–Approximation of f(t) by f_n

is obtained. We consider all the f_n to be independent variables and apply the method of Lagrange Multipliers

$$\frac{\partial}{\partial f_n}\left\{\sum_{1}^{N} \cos f_n + \lambda\left(\sum_{1}^{N} f_n^2 - \phi^2\right)\right\} = 0 \tag{C-7}$$

and thus

$$\left.\begin{aligned} -\sin f_k + 2\lambda\, f_k &= 0 \\ -\sin f_\ell + 2\lambda\, f_\ell &= 0 \end{aligned}\right\} \tag{C-8}$$

After elimination of λ

$$\frac{\sin f_k}{f_k} = \frac{\sin f_\ell}{f_\ell} \tag{C-9}$$

which has only one solution

$$|f_k| = |f_\ell| \tag{C-10}$$

for

$$|f_k| \leq 1.432\pi \quad |f_\ell| \leq 1.432\pi$$

as can be seen from Fig. 10.

From (C-6) follows that

$$f_n = \pm\phi \tag{C-11}$$

Where half of the f_n are positive and the other half are negative in order to satisfy Equ. (C-2). It thus follows that $f(t)$ is an A-C signal with constant amplitude ϕ. For $f(t) = \pm\phi$ we obtain

$$A_0 = \frac{1}{T}\int_0^T \cos\phi\, dt = \cos\phi \tag{C-12}$$

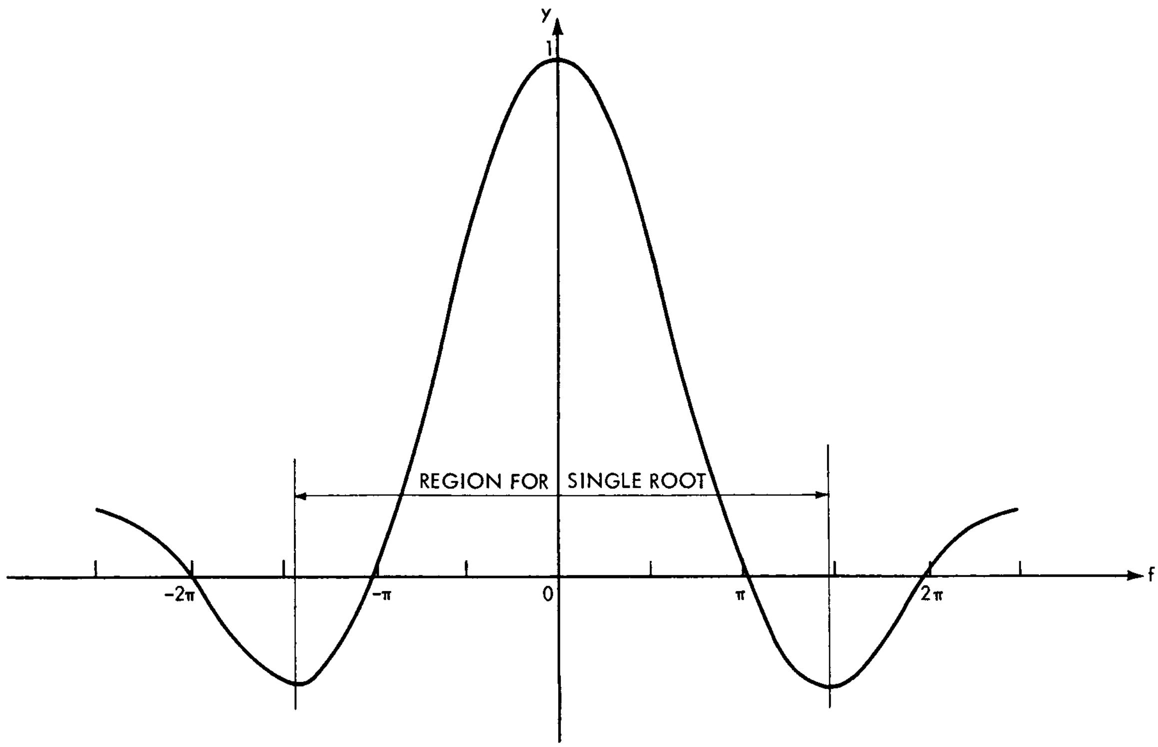

Figure 10– $y = \frac{\sin f}{f}$; y = constant has only one root $|f| \le 1.432\,\pi$

The next step is to show that $\cos\phi$ is a minimum and not a sadelpoint or maximum. Add a small time function ϵ to ϕ^2 so that

$$f(t)^2 = \phi^2 + \epsilon \tag{C-13}$$

From the side condition (C-3) follows

$$\int_0^T \epsilon\, dt = 0 \tag{C-14}$$

Expanding $\cos f(t)$ into a series and inserting in Equ. (C-4) yields

$$A_0 \ge \frac{1}{T} \int_0^T \left[1 - \frac{1}{2!}(\phi^2 + \epsilon) + \frac{1}{4!}(\phi^2 + \epsilon)^2 - \frac{1}{6!}(\phi^2 + \epsilon)^3 + \dots\right] dt$$

or

$$A_0 \geq \frac{1}{T} \int_0^T \left[1 - \frac{1}{2!}\phi^2 + \frac{1}{4!}\phi^4 - \frac{1}{6!}\phi^6 \ldots\right] dt$$

$$+ \frac{1}{T} \int_0^T \left[\frac{1}{4!} - \frac{3}{6!}\phi^2 \ldots\right] \epsilon^2 \, dt \qquad \text{(C-15)}$$

including up to ϵ^2 terms. The ϵ term vanishes due to Equ. (C-14). Thus

$$A_0 \geq \cos\phi + \frac{K}{T} \int_0^T \epsilon^2 \, dt \qquad \text{(C-16)}$$

where

$$K = \left(\frac{1}{4!} - \frac{3}{6!}\phi^2 + \ldots (-1)^n \frac{(n+1)(n+2)}{2(2n+4)!}\phi^{2n} + \ldots\right) \qquad \text{(C-17)}$$

It is easily shown that

$$K = \frac{1}{8\phi^2}\left(\frac{\sin\phi}{\phi} - \cos\phi\right)$$

and

$$K \geq 0$$

for $\phi \leq 1.432\pi$. Thus $A_0 \geq \cos\phi_0$ for $|\phi| \leq 1.432\pi$.

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