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SUMMARY

The stability of an incompressible two-fluid wheel flow to infinitesimal helical disturbances is considered for the case where the inner fluid is heavy and the outer fluid is light. This situation may be viewed as a Rayleigh-Taylor problem in the frame of the rotating fluid and is dynamically unstable. As the density ratio of inner to outer fluid increases, the calculated growth rates increase and approach a limiting value as the density ratio becomes infinite. Growth rates also increase with increasing axial as well as azimuthal wave number. Furthermore, the presence of a fixed boundary outside the light fluid tends to have negligible influence on the growth rates as the density ratio becomes very large.

INTRODUCTION

In some of the proposed gaseous nuclear rocket schemes, vortex containment is suggested for the maintenance of a critical mass of fissioning gaseous fuel with minimal losses (refs. 1 to 3). The present study is based upon Evvard's concept of the wheel-flow reactor (ref. 3). In this reactor concept, a core of heavy fissioning gas in solid-body rotation is surrounded by an outer light gas coolant also in solid-body rotation at the same angular velocity. Further description of the hydrodynamic, nucleonic, and heat-transfer aspects of the wheel-flow reactor is given in reference 3.

Unfortunately, the steady-state two-fluid wheel flow of a heavy gas core surrounded by a lighter, outer gas is unstable. The nature of this instability is the subject of this report.

The stability of rotating flows is considered in detail by Chandrasekhar (ref. 4), but the reported treatments of wheel flow are limited to a single fluid in a bounded geometry. On the other hand, Chandrasekhar also considers the Rayleigh-Taylor stability problem for a stratified but nonrotating flow. Of interest here is the combination of these two cases which may in fact be viewed as a Rayleigh-Taylor problem relative to the rotating fluid. This fact was also pointed out by Melcher (ref. 5) in his general formulation for electrohydrodynamic waves in rotating systems.

The wheel flow treated herein was idealized to facilitate obtaining solu-

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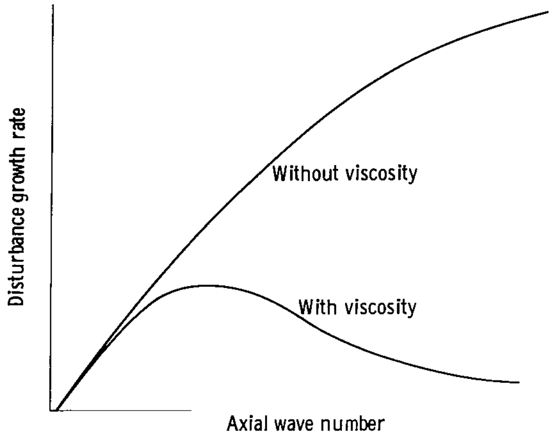

Figure 1. - Effect of viscosity on growth rates of infinitesimal disturbances.

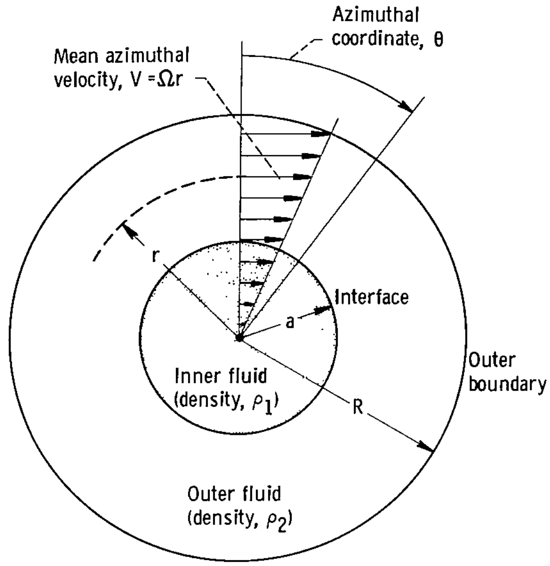

Figure 2. - Geometry of two-fluid wheel flow.

tions; it consisted of two incompressible, immiscible fluids separated by a cylindrical interface. While the time-independent wheel flow is compatible with the complete Navier-Stokes equations including effects of viscosity, the viscous effects were omitted in this treatment of the stability of the configuration. While this is a serious omission, the results yielded by this simplification are likely to be pessimistic. As pointed out by Chandrasekhar (ref. 4) for rotating fluids and also by Bellman and Pennington (ref. 6) in the case of the Rayleigh-Taylor problem, the effects of viscosity on the growth rates of infinitesimal disturbances tend to be as in figure 1. The salient effect of viscosity is a severe diminuation of the growth rate of short wavelength disturbances such that the maximum growth rate occurs at some finite axial wavelength. The results obtained herein should be reasonably correct in the long wavelength limit. The effect of azimuthal wave number on the growth rate will suggest whether large or small core fragments tend to break away more rapidly.

BASIC FLOW

The geometry of the two-fluid wheel flow is shown in figure 2. The problem is most conveniently handled in cylindrical coordinates. The axial or z-coordinate is directed into the paper. The inner fluid is given the subscript 1, while the subscript 2 refers to the outer fluid. The interface between them is at radius a. The outer boundary at radius R bounds the outer fluid.

The idealized basic flow is considered herein to be only azimuthal and solely a function of the radial coordinate. The following equations are the pertinent continuity and momentum equations, respectively:

Continuity:

$$\frac{\partial V^*}{\partial \theta} = 0 \tag{1a}$$

Radial momentum:

$$\frac{V^{*2}}{r^*} = \frac{1}{\rho^*} \frac{\partial p^*}{\partial r^*} \tag{1b}$$

Azimuthal momentum:

$$0 = \nu^*\left(\frac{\partial^2 V^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial V^*}{\partial r^*} - \frac{V^*}{r^{*2}}\right) \tag{1c}$$

(All symbols are defined in appendix A.)

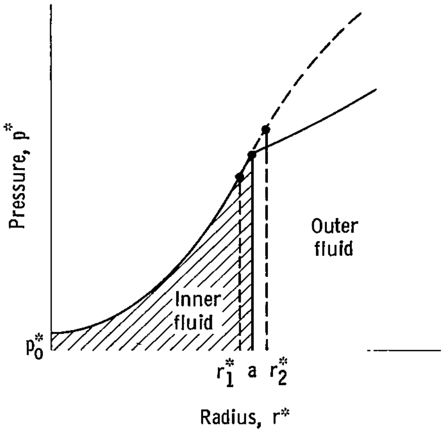

Figure 3. - Pressure distribution for inner fluid density greater than outer fluid density.

The general solution for azimuthal velocity obtained from equation (1c) is

$$V^* = Ar^* + \frac{B}{r^*} \tag{2}$$

The wheel flow $V = \Omega r$ is the pertinent solution for the inner fluid. If the outer boundary is assumed to rotate at velocity ΩR, then $V = \Omega r$ is the complete solution and is compatible with the Navier-Stokes equations including viscosity. The pressure distribution in the wheel flow from solution of equation (1b) is

$$p^* = p_0^* + \rho_1 \frac{\Omega^2 r^{*2}}{2} \qquad 0 \leq r^* \leq a \tag{3a}$$

$$= p_0^* + \rho_1 \frac{\Omega^2 a^2}{2} + \rho_2 \frac{\Omega^2 (r^{*2} - a^2)}{2} \qquad a \leq r^* \leq R \tag{3b}$$

For a heavy fluid core surrounded by a light fluid, the inertial nature of the instability is apparent from the pressure distribution for $\rho_1 > \rho_2$ (fig. 3). If an element of heavy fluid from just inside the interface $(r^* = r_1^*)$ is displaced to just outside the interface $(r^* = r_2^*)$ while conserving its angular momentum, a pressure gradient $dp^*/dr^* = \rho_1/r_2^*\left[V_1^*(r_1^*/r_2^*)\right]^2$ (represented by the dashed line in fig. 3) is required for radial balance. However, the available pressure gradient at r_2^* is $dp^*/dr^* = \rho_2(V_2^{*2}/r_2^*)$ (solid line), which for wheel flow is $\rho_2/r_2^*\left[V_1^*(r_2^*/r_1^*)\right]^2$. Thus in the limit of infinitesimal displacement $\left[(r_2^* - r_1^*)/r_1^* \ll 1\right]$ as long as $\rho_1/\rho_2 > 1$ the heavy fluid encounters a pressure gradient inadequate for radial balance on penetrating the interface and the element of heavy fluid will continue its motion outward. This behavior is characteristic of Rayleigh-Taylor instabilities.[1]

FORMULATION OF STABILITY PROBLEM

Disturbance Equations

The equations governing infinitesimal disturbances are obtained from a linearization of the complete equations of motion about the assumed basic flow. As previously mentioned, only inviscid disturbance motions will be considered in this treatment. For the basic wheel flow $V^* = \Omega r^*$, the disturbance equations for each of the fluids are:

Continuity:

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^*u^*) + \frac{1}{r^*}\frac{\partial v^*}{\partial \theta} + \frac{\partial w^*}{\partial z^*} = 0 \tag{4a}$$

Momentum:

$$\frac{\partial u^*}{\partial t^*} + \Omega\frac{\partial u^*}{\partial \theta} - 2\Omega v^* = -\frac{1}{\rho}\frac{\partial \pi^*}{\partial r^*} \tag{4b}$$

$$\frac{\partial v^*}{\partial t^*} + \Omega\frac{\partial v^*}{\partial \theta} + 2\Omega u^* = -\frac{1}{\rho r^*}\frac{\partial \pi^*}{\partial \theta} \tag{4c}$$

$$\frac{\partial w^*}{\partial t^*} + \Omega\frac{\partial w^*}{\partial \theta} = -\frac{1}{\rho}\frac{\partial \pi^*}{\partial z^*} \tag{4d}$$

These equations may also be written in dimensionless form for later convenience. When the disturbance quantities and the independent variables are referred to the following reference values,

$$v^*_{ref} = \Omega a \tag{5a}$$

$$L^*_{ref} = a \tag{5b}$$

$$p^*_{ref} = \rho\Omega^2 a^2 \tag{5c}$$

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[1]The argument presented here parallels that given by von Karman (see Lin, ref. 7) in support of Rayleigh's stability criterion for rotating fluids. The equivalent to Rayleigh's criterion for a stratified incompressible ideal fluid (inviscid and not thermally conducting) is that the flow configuration is stable if $d/dr(\rho V^2 r^2) > 0$. The two-fluid wheel flow under consideration $(\rho_1/\rho_2 > 1)$ is clearly unstable by this criterion.

$$t^*_{ref} = \frac{1}{\Omega} \tag{5d}$$

$$\omega^*_{ref} = \Omega \tag{5e}$$

The following equations are the dimensionless disturbance equations:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \tag{6a}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial \theta} - 2v = -\frac{\partial \pi}{\partial r} \tag{6b}$$

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial \theta} + 2u = -\frac{1}{r}\frac{\partial \pi}{\partial \theta} \tag{6c}$$

$$\frac{\partial w}{\partial t} + \frac{\partial w}{\partial \theta} = -\frac{\partial \pi}{\partial z} \tag{6d}$$

Since the coefficients of equations (6) are independent of t, θ, and z, the equations allow disturbances of the form $q(r)e^{i(kz+m\theta-\omega t)}$, where $q(r)$ is a complex disturbance amplitude, k and m are the axial and azimuthal wave numbers, respectively, and ω is a complex frequency. The wave numbers k and m are real numbers; m is an integer. The real part of ω is the rotational frequency of the disturbance flow, while the imaginary part ω_i represents the amplification rate. Disturbances are growing, neutral, or decaying according to whether ω_i is positive, zero, or negative, respectively. For the assumed disturbance form, equations (6) become

$$u' + \frac{u}{r} + \frac{imv}{r} + ikw = 0 \tag{7a}$$

$$i\tilde{\omega}u + 2v = \pi' \tag{7b}$$

$$i\tilde{\omega}v - 2u = \frac{im\pi}{r} \tag{7c}$$

$$i\tilde{\omega}w = ik\pi \tag{7d}$$

where primes denote differentiation with respect to r and $\tilde{\omega} \equiv \omega - m$ is related to the angular velocity of the disturbance pattern relative to the mean rotation.[2] Thus, to an observer at a given axial location z and moving with the mean rotational velocity Ω, the problem may be viewed as a Rayleigh-Taylor

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[2]In terms of dimensional quantities, $\tilde{\omega}^*_r/m = \theta/t^* - \Omega$ where (θ/t^*) is the absolute angular velocity of the disturbance pattern. To an observer at a given axial station z, the quantity $\tilde{\omega}_r = (m/\Omega)(\theta/t^* - \Omega)$ is proportional to the angular velocity of the disturbance pattern relative to the mean rotation. The dimensionless angular velocity of the wave is simply $\theta/\Omega t^* = 1 + \tilde{\omega}_r/m$.

problem with a centrifugal rather than a gravitational driving force.

From equations (7b), (7c), and (7d), the velocity fluctuation amplitudes may be expressed in terms of π:

$$u = \frac{i\left(\tilde{\omega}\pi' - \frac{2m\pi}{r}\right)}{4 - \tilde{\omega}^2} \tag{8a}$$

$$v = \frac{2\pi' - \tilde{\omega}\frac{m\pi}{r}}{4 - \tilde{\omega}^2} \tag{8b}$$

$$w = \frac{k\pi}{\tilde{\omega}} \tag{8c}$$

Substitution of equations (8) into the disturbance continuity equation (7a) yields the differential equation for the pressure fluctuation amplitude:

$$\pi'' + \frac{\pi'}{r} - \left(k^2\Delta^2 + \frac{m^2}{r^2}\right)\pi = 0 \tag{9}$$

where

$$\Delta^2 \equiv 1 - \frac{4}{\tilde{\omega}^2} \tag{10}$$

The general solution to equation (9) is any linear combination of Bessel functions of the first and second kind of the proper order and argument. The form used herein is

$$\pi = AJ_m(ik\Delta r) + BH_m^{(1)}(ik\Delta r) \tag{11}$$

where the first term is regular at the origin while the second is regular at infinity, provided that the real part of $k\Delta$ is positive for large values of $k\Delta$.

In the inner fluid, the second term is inadmissable so that

$$\pi_1 = A_1J_m(ik\Delta r) \tag{12}$$

In the outer fluid, the general solution is

$$\pi_2 = A_2J_m(ik\Delta r) + B_2H_m^{(1)}(ik\Delta r) \tag{13}$$

If the outer boundary is at infinity, the coefficient A_2 must vanish as a consequence of the proper satisfaction of boundary conditions.

Boundary Conditions

The boundary condition at $r = 0$ has already been applied. The remaining conditions are that the radial velocity must vanish at $r = R$ and that the radial velocity and normal stress be continuous at the interface $r = 1$ ($r^* = a$). These conditions are treated consecutively.

Radial velocity $u = 0$ at $r = R$. - From equations (8a) and (13),

$$0 = A_2\left\{\tilde{\omega}\left[J_m(ik\Delta R)\right]' - \frac{2mJ_m(ik\Delta R)}{R}\right\} + B_2\left\{\tilde{\omega}\left[H_m^{(1)}(ik\Delta R)\right]' - \frac{2mH_m^{(1)}(ik\Delta R)}{R}\right\}$$

or

$$A_2 = -B_2\frac{\mathcal{H}_m(R)}{\mathcal{J}_m(R)} \tag{14}$$

where

$$\mathcal{H}_m(r) \equiv \left[H_m^{(1)}(ik\Delta r)\right]' - \frac{2mH_m^{(1)}(ik\Delta r)}{\tilde{\omega}r} \tag{15}$$

and

$$\mathcal{J}_m(r) \equiv \left[J_m(ik\Delta r)\right]' - \frac{2mJ_m(ik\Delta r)}{\tilde{\omega}r} \tag{16}$$

As R approaches infinity, A_2 approaches zero, as seen from equation (14).

Radial velocity $u_1 = u_2$ at $r = 1$. - From equations (8a), (12), (13), (15), and (16), this condition yields the following relation between A_1 and B_2:

$$\frac{A_1}{B_2} = \frac{\mathcal{H}_m(1)}{\mathcal{J}_m(1)} - \frac{\mathcal{H}_m(R)}{\mathcal{J}_m(R)} \tag{17}$$

Continuity of normal force at interface. - The satisfaction of this boundary condition is accomplished by obtaining the sum of pressure plus centrifugal forces for each fluid at the displaced location of the interface and then equating these sums. Because two separate fluids with different densities are involved, the force balance is carried out in physical variables:

$$p_0 + \rho_1 \frac{\Omega^2 a^2}{2}\left[1 + \xi_1(1)\right]^2 + \rho_1 \Omega^2 a^2 \pi_1(1)$$

$$= p_0 + \rho_1 \frac{\Omega^2 a^2}{2} + \frac{\rho_2 \Omega^2 a^2}{2}\left\{\left[1 + \xi_2(1)\right]^2 - 1\right\} + \rho_2 \Omega^2 a^2 \pi_2(1) \tag{18}$$

When squares of disturbance quantities are dropped, the linearized form of equation (18) becomes

$$\rho_1\left[\xi_1(1) + \pi_1(1)\right] = \rho_2\left[\xi_2(1) + \pi_2(1)\right] \tag{19}$$

The amplitude of the radial interface displacement ξ is obtained by a time integral of the velocity in the rotating frame; that is,

$$\xi = \frac{iu}{\tilde{\omega}} \tag{20}$$

General Dispersion Relation

After the appropriate substitutions are made into equation (19), the following dispersion relation is obtained:

$$\tilde{\omega}^2 - 4 = \frac{\dfrac{\rho_1}{\rho_2} - 1}{\dfrac{\dfrac{H_m(ik\Delta)}{\mathcal{J}_m(1)} - \dfrac{\mathcal{H}_m(R)}{\mathcal{J}_m(R)}\dfrac{J_m(ik\Delta)}{\mathcal{J}_m(1)}}{\dfrac{\mathcal{H}_m(1)}{\mathcal{J}_m(1)} - \dfrac{\mathcal{H}_m(R)}{\mathcal{J}_m(R)}} - \dfrac{\rho_1}{\rho_2}\dfrac{J_m(ik\Delta)}{\mathcal{J}_m(1)}} \tag{21}$$

where the $\mathcal{H}_m$ and $\mathcal{J}_m$ functions are those defined in equations (15) and (16) and Δ is defined in equation (10). The dispersion relation (21) is a complex equation. In fact, since the complex unknown $\tilde{\omega}$ appears in the arguments of the Bessel functions through Δ, solutions to the dispersion relation (21) are probably best obtained by an iterative trial-and-error procedure. Such a procedure has been employed to obtain numerical solutions for arbitrary complex frequencies where the Bessel functions for arbitrary complex arguments are evaluated by the method of reference 8.

The dispersion relation (21) has multiple roots, not all of which have physical significance. This difficulty arises particularly because of the assumption regarding the positiveness of the real part of Δ that was made in choosing the Hankel function of the first kind $H_m^{(1)}$ rather than the Hankel function of the second kind $H_m^{(2)}$ for the general solution (11). The appropriate roots of the dispersion relation will be sorted out by examination of various limiting solutions of the dispersion relation (21).

Before proceeding, some remarks are in order on the effect of the outermost boundary at $r = R$. The boundary condition that $u = 0$ at $r = R$ enters the dispersion relation (21) through the $\mathcal{H}_m(R)/\mathcal{J}_m(R)$ terms in the first grouping of the denominator of the right side. For large inner to outer-density ratio $\rho_1/\rho_2 \gg 1$, the first portion of the denominator is small compared with that term which has ρ_1/ρ_2 as its coefficient, and its effect on the result is very weak. Furthermore for large arguments $(k\Delta R)$, $\left[\mathcal{H}_m(R)/\mathcal{J}_m(R)\right] \sim e^{-2k\Delta R}$ becomes quite small. For the values of ρ_1/ρ_2 and R of interest in wheel-flow reactors, the stability charactersitics are essentially those of the unbounded configuration. The solutions presented herein are, therefore, for the unbounded configuration $R \to \infty$.

SOLUTIONS TO DISPERSION RELATION FOR UNBOUNDED CONFIGURATION

For the unbounded configuration, $\left[\mathcal{H}_m(R)/\mathcal{J}_m(R)\right] \to 0$ and the dispersion relation (21) reduces to

$$\tilde{\omega}^2 - 4 = \frac{\dfrac{\rho_1}{\rho_2} - 1}{\dfrac{1}{\dfrac{\left[H_m^{(1)}(ik\Delta r)\right]'_{r=1}}{H_m^{(1)}(ik\Delta)} - \dfrac{2m}{\tilde{\omega}}} - \dfrac{\dfrac{\rho_1}{\rho_2}}{\dfrac{\left[J_m(ik\Delta r)\right]'_{r=1}}{J_m(ik\Delta)} - \dfrac{2m}{\tilde{\omega}}}} \tag{22}$$

Solutions to dispersion relation (22) are obtained and discussed for axially symmetric disturbances ($m = 0$) and then for the helical disturbances with flute number m different from zero.

Axially Symmetric Disturbances

For axially symmetric disturbances ($m = 0$), equation (22) becomes

$$\tilde{\omega}^2 - 4 = \frac{-\left(\dfrac{\rho_1}{\rho_2} - 1\right)(k\Delta)}{\left[\dfrac{iH_0^{(1)}(ik\Delta)}{-H_1^{(1)}(ik\Delta)}\right] + \dfrac{\rho_1}{\rho_2}\left[\dfrac{J_0(ik\Delta)}{-iJ_1(ik\Delta)}\right]} \tag{23}$$

In this particular case $\tilde{\omega} = \omega$ since $m = 0$. Solutions are now sought subject to the condition that the real part of Δ is greater than zero. Assume for the moment that for $m = 0$, Δ is real and positive. Then, the bracketed ratios of Bessel functions in equation (23) are real and positive so that the right side of equation (23) is real but always negative. Since the left side of equation (23) is just $\tilde{\omega}^2\Delta^2$, a solution is possible for negative real values of ω^2 or purely imaginary values of ω, which are obtained for

$$k\Delta \geq \frac{4\left\{\left[\dfrac{iH_0^{(1)}(ik\Delta)}{-H_1^{(1)}(ik\Delta)}\right] + \dfrac{\rho_1}{\rho_2}\left[\dfrac{J_0(ik\Delta)}{-iJ_1(ik\Delta)}\right]\right\}}{\dfrac{\rho_1}{\rho_2} - 1} \tag{24}$$

From the definition of Δ^2, these are the only admissible solutions with $\Delta_r > 0$. Thus the axially symmetric disturbances may be described as standing waves ($\omega_r = 0$) that are amplified ($\omega_i > 0$).[3]

Solutions to dispersion relation (23) were obtained (by hand calculation) for density ratios ρ_1/ρ_2 of 2, 10, 100, and ∞. These results are shown in figure 4. As expected from the qualitative discussion in the INTRODUCTION, the growth rates increase with increasing density ratio. For density ratios ρ_1/ρ_2 greater than about 10, the results are not very different from those for an infinite density ratio, which indicates that at these density ratios the pressure in the outer fluid has negligible effect in decelerating a displaced fluid element. For large axial wave numbers, the results are well represented by the asymptotic solution to equation (23) for large $k\Delta$ obtained in appendix B, namely,

$$\omega_i = \sqrt{\frac{(k\Delta)}{\mathcal{R}} - \left(4 + \frac{1}{2\mathcal{R}^2} + \ldots\right)} \tag{25}$$

and

$$k = \frac{k\Delta}{\sqrt{1 + \dfrac{4}{\omega_i^2}}} \tag{26}$$

where

$$\mathcal{R} = \frac{\dfrac{\rho_1}{\rho_2} + 1}{\dfrac{\rho_1}{\rho_2} - 1} \tag{27}$$

[3]Both amplification ($\omega_i > 0$) and decay ($\omega_i < 0$) of the disturbances are possible. Only the possible amplification, however, is relevant to the assessment of the stability of the configuration.

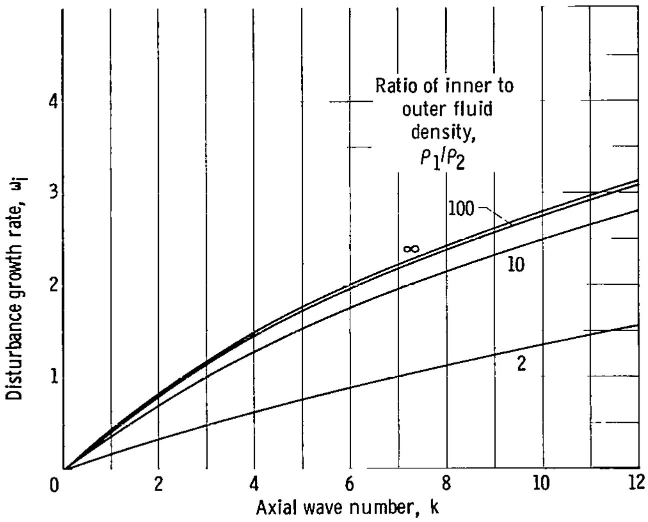


Figure 4. - Growth rates of axially symmetric disturbances in unbounded configuration.

Solutions for very long wavelengths $k \to 0$ are obtained at finite $k\Delta$ at the value corresponding to the equal sign in equation (24). Since $k\Delta$ for axially symmetric disturbances is always greater than 4, the solutions (25) to (27) are good representations of the results of figure 4 for all wave numbers and density ratios.

Nonaxially Symmetric

Disturbances

For the nonaxially symmetric disturbances, the dispersion relation (22) is solved numerically by trial and error on an IBM 7094 computer with the pertinent Bessel functions for arbitrary complex arguments calculated according to the method of reference 8. The results for density ratios 2, 10, and 100 are shown in figure 5. The numerical results of these densities agree well with asymptotic solutions to dispersion relation (22) in both the long-wavelength $k \to 0$ and the short-wavelength limits (appendixes C and D, respectively).

Limiting solution for long wavelength. - Long-wavelength solutions $k \to 0$ for nonaxially symmetric disturbances are sought in the limit $k\Delta \to 0$, despite the aforementioned result for $m = 0$. The pertinent solution (from appendix C) is

$$\tilde{\omega}_r = -\frac{1}{\Re} \tag{28a}$$

or

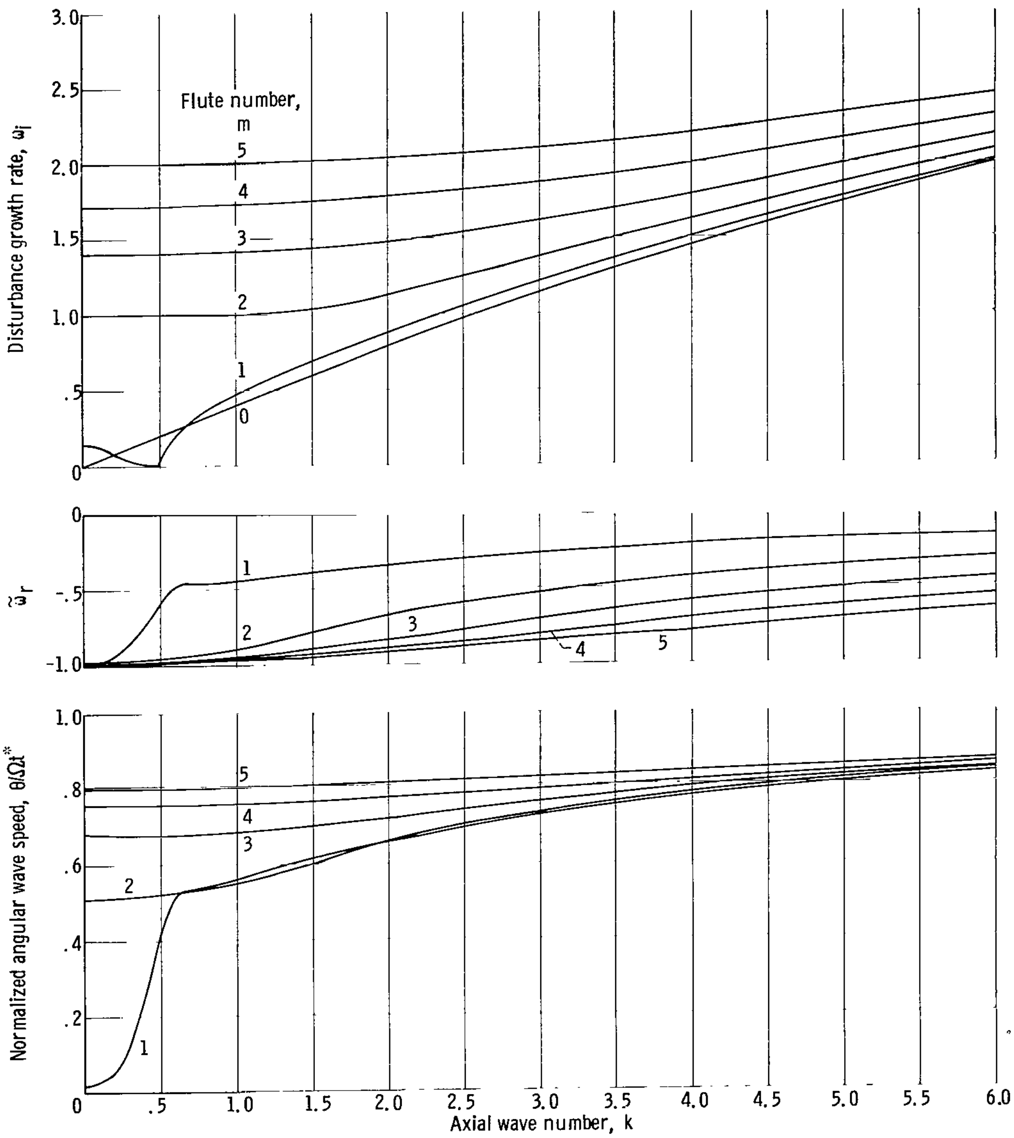
$$\omega_r = m - \frac{1}{\Re} \tag{28b}$$

and

$$\omega_i = \frac{\sqrt{m\Re - 1}}{\Re} \tag{29}$$

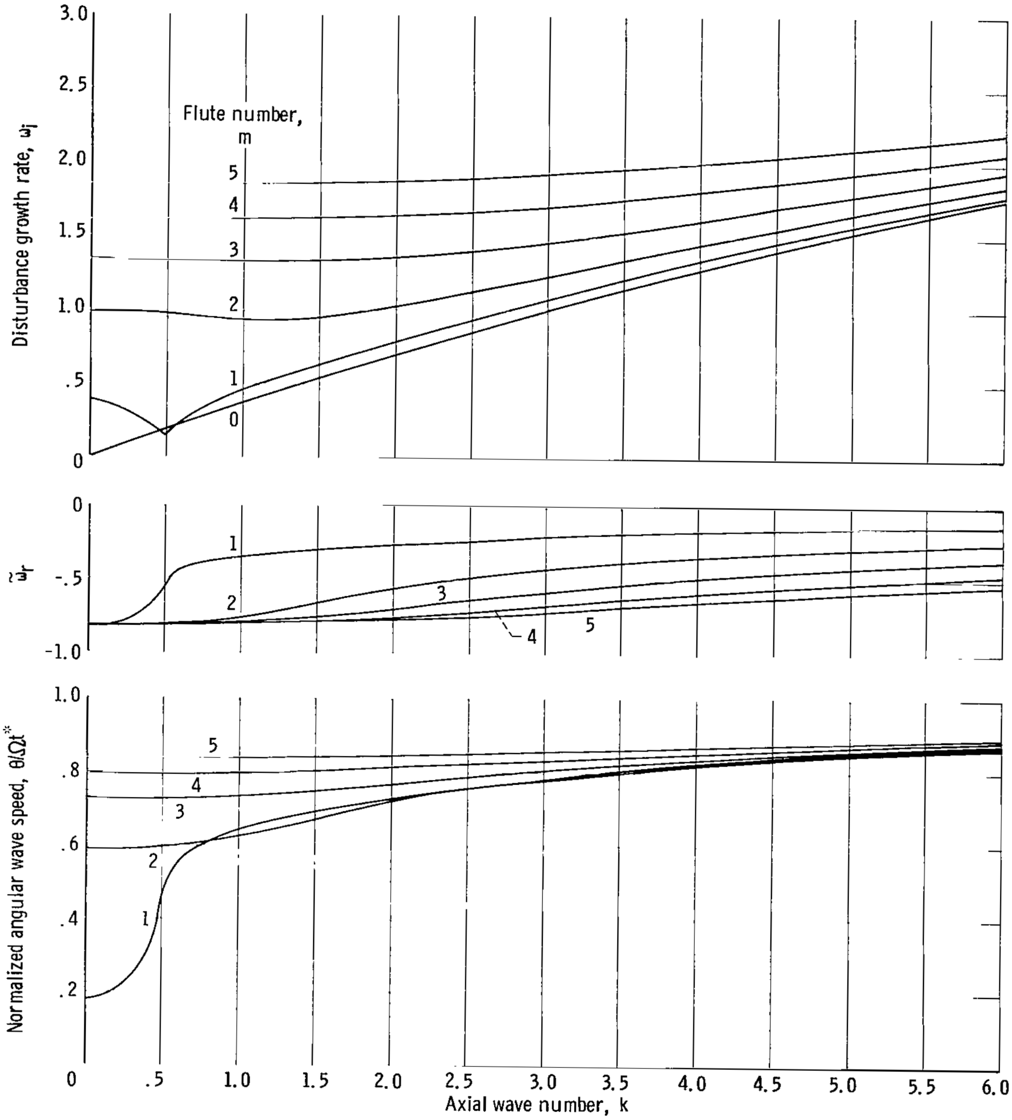
where $\Re$ is defined in equation (27). This solution satisfies the condition $\Delta_r > 0$ (appendix C).

Limiting solution for short wavelength. - From appendix D, the leading terms of the solution of the dispersion relation (22) in the limit $k\Delta \to \infty$ are



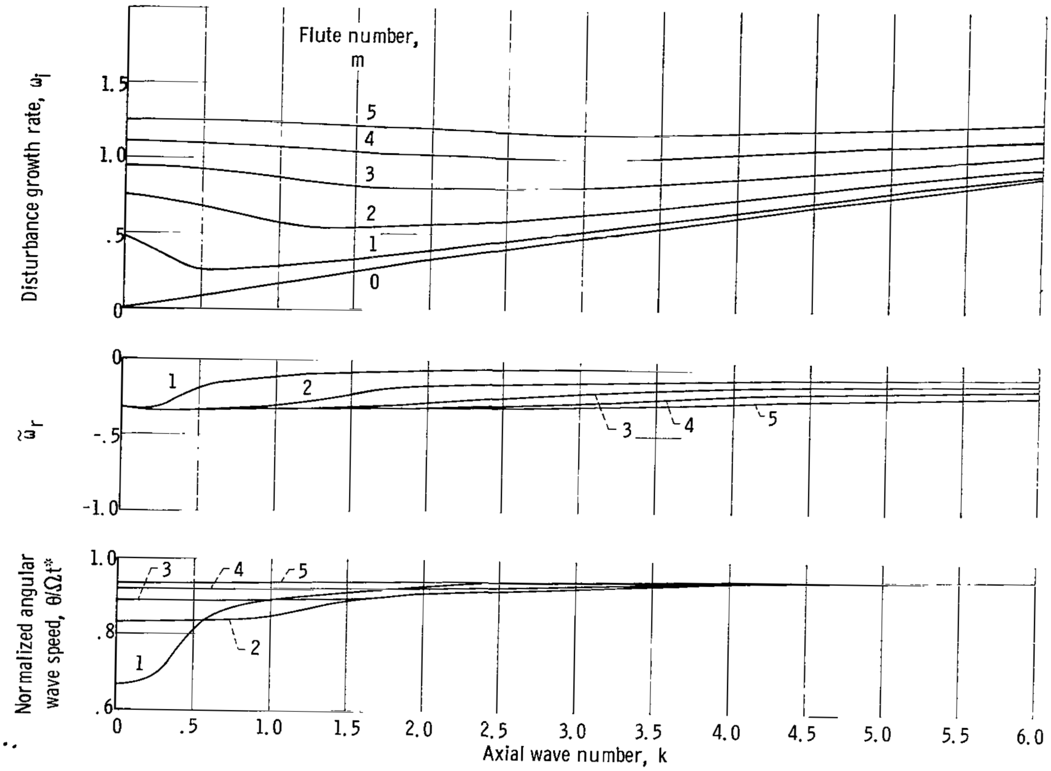
(a) Ratio of inner to outer fluid density, 100.

Figure 5. - Growth rates and wave speeds for unbounded configuration.



(b) Ratio of inner to outer fluid density, 10.

Figure 5. - Continued.



(c) Ratio of inner to outer fluid density, 2.

Figure 5. - Concluded.

$$\tilde{\omega}_r = - \frac{m}{\Re^2 \omega_{i,0}^2} \tag{30a}$$

or

$$\omega_r = m\left(1 - \frac{1}{\Re^2 \omega_{i,0}^2}\right) \tag{30b}$$

and

$$\omega_i = \omega_{i,0} \tag{31}$$

and

$$k = \frac{k\Delta}{\left(1 + \frac{4}{\omega_{i,0}^2}\right)^{1/2}} \tag{32}$$

where

$$\omega_{i,0} = \sqrt{\frac{k\Delta}{\Re} - \left(4 + \frac{1}{2\Re^2} + \ldots\right)} \tag{33}$$

In these expressions, $k\Delta$ is a real number. Note that $\omega_{i,0}$ is independent of the flute number m and is identical to the asymptotic result for axially symmetric disturbances. The dependence of ω_i on flute number m is of higher order, as is the departure of $k\Delta$ from the real axis.

The numerical solutions presented in figure 5 are entirely consistent with these limiting forms. The nonaxially symmetric disturbances are growing waves that for long wavelengths rotate at $1 - 1/m\Re$ of the angular velocity of the basic flow. As the wave number increases, the angular velocity of the wave increases asymptotically and approaches the basic flow angular velocity in the short-wavelength limit. For kink instabilities ($m = 1$), the increase in wave angular velocity occurs rather abruptly at dimensionless axial wave numbers between 0 and 1. The waves at higher flute numbers rotate at higher fractions of basic flow angular velocity and approach that velocity more gradually with an increase in wave number. The growth rates themselves generally increase with an increase in flute number m at all axial wave numbers. An exception occurs for $m = 1$ near $k = 1/2$ for the density ratios 10 and 100. This exception is as yet unexplained. As expected, the level of growth rates increases with increasing density ratio in about the same way that it did for axially symmetric disturbances. For large axial wave numbers, the growth rates approach the $\omega_{i,0}$ given by equation (33) regardless of flute number.

RELEVANCE AND SIGNIFICANCE OF RESULTS TO WHEEL-FLOW REACTOR

The wheel-flow reactor (ref. 3) consists of rotating hot compressible gases, where the inner fissionable fluid may be quite a bit hotter than the outer light gas propellant. The interface between these two fluids will not be sharp because of the diffusion of heat and because of species concentration diffusion. The treatment of such a flow in terms of two constant-density immiscible fluids is sufficiently idealized that only qualitative statements may be made to relate the results of the present stability calculations to an actual wheel-flow reactor. Such qualitative statements are expected to have some validity because, as previously stated, the instability is primarily inertial. The statements in this section are based on the premise that the normal mode disturbances with the largest growth rates are expected to be the most troublesome.

For conceivable molecular weights and temperatures, the pertinent density ratios are in the range $10 \lesssim \rho_1/\rho_2 \lesssim 100$. Little qualitative or even quantitative difference exists between the results (figs. 5(a) and (b)) in this range of density ratios. Significant reductions in growth rate are obtained only for $\rho_1/\rho_2 \lesssim 2$.

Although calculated growth rates increase with increasing axial wave number in the present inviscid calculations, the large wave-number disturbances (short wavelength) will be damped by the action of viscosity. This has been verified by the present authors in a preliminary calculation not presented herein.

Accordingly, the disturbances most likely to be amplified are those with axial wavelengths larger than the interface radius, a ($k < 2\pi$). For this range of axial wave numbers, the growth times are of the order of the period of basic flow rotation and increase with azimuthal wave number or flute number. This increase indicates a rather rapid growth of disturbances, with smaller core fragments tending to break away most rapidly. Some of these troublesome long-wavelength disturbances can perhaps by eliminated by choosing the reactor length to be short, of order πa, so that the minimum axial wave number is $k = 1$.

The previous statements should be reexamined with the aid of a less idealized study that might include effects of compressibility and species diffusion. The fissionable core would be substantially ionized and electrically conducting. This suggests the possibility that additional stabilization might be obtained by use of imposed axial magnetic fields.

SUMMARY OF RESULTS

The stability of an incompressible two-fluid wheel flow to infinitesimal inviscid helical disturbances has been considered for the case where the inner fluid is heavy and the outer fluid is light. The following are the important results:

1. For large inner- to outer-fluid density ratio, the presence of a physical boundary outside the light fluid has only a very weak effect on the stability characteristics. For the values of density ratio and outer radius of interest in wheel-flow reactors, the stability characteristics are essentially those of a configuration where the outer fluid is unbounded; this applies both to the axially symmetric disturbances and to the nonaxially symmetric disturbances.

2. The axially symmetric disturbances are identified as standing waves that are amplified. The growth rates increase with increasing density ratio. For density ratios greater than about 10, the results are not very different from those for infinite density ratio.

3. The nonaxially symmetric disturbances are helically propagating waves that are also amplified. For a given density ratio, the growth rates increase with azimuthal wave number. The magnitude of the growth rates varies with density ratio in about the same way as for axially symmetric disturbances.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, May 24, 1965.

APPENDIX A

SYMBOLS

A,B	constants in solution of differential equations
a	radius of two-fluid interface
$\mathcal{H}$	defined by eq. (15)
$\mathcal{J}$	defined by eq. (16)
k	axial wave number
L	reference length
m	azimuthal wave number or flute number
$\mathcal{O}(x)$	order of magnitude of x
p	pressure
q(r)	complex disturbance amplitude
R	radius of outer boundary
$\mathcal{R}$	$\left(\frac{\rho_1}{\rho_2} + 1\right)\Big/\left(\frac{\rho_1}{\rho_2} - 1\right)$
r	radial coordinate
t	time
u	radial disturbance velocity
V	azimuthal velocity of basic flow, Ωr
v	azimuthal disturbance velocity
w	axial disturbance velocity
z	axial coordinate
Δ	defined by eq. (10), $\Delta^2 \equiv 1 - \frac{4}{\tilde{\omega}^2}$
θ	azimuthal coordinate
ν	kinematic viscosity

ξ radial displacement of disturbance flow

π disturbance pressure

ρ density

Ω angular velocity of solid body rotation

ω complex angular disturbance frequency

$\tilde{\omega}$ ω - m

Subscripts:

i imaginary part

r real part

ref reference quantity

1 inner fluid

2 outer fluid

Superscripts:

$*$ dimensional quantity

' differentiation with respect to r

APPENDIX B

SOLUTION FOR LARGE WAVE NUMBER OF DISPERSION RELATION FOR AXIALLY SYMMETRIC DISTURBANCES IN UNBOUNDED CONFIGURATION

The dispersion relation pertinent to axially symmetric disturbances in the unbounded configuration is that of equation (23), namely,

$$\tilde{\omega}^2 - 4 = \frac{-\left(\frac{\rho_1}{\rho_2} - 1\right)(k\Delta)}{\left[\frac{iH_0^{(1)}\ (ik\Delta)}{-H_1^{(1)}\ (ik\Delta)}\right] + \frac{\rho_1}{\rho_2}\left[\frac{J_0(ik\Delta)}{-iJ_1(ik\Delta)}\right]} \tag{23}$$

For large values of $k\Delta$, the functions in brackets are approximately

$$\frac{iH_0^{(1)}\ (ik\Delta)}{-H_1^{(1)}\ (ik\Delta)} = 1 - \frac{1}{2k\Delta} + \ldots \tag{B1}$$

$$\frac{J_0(ik\Delta)}{-iJ_1(ik\Delta)} = 1 + \frac{1}{2k\Delta} + \ldots \tag{B2}$$

The dispersion relation (23) may be written

$$\tilde{\omega}^2 - 4 = -\frac{k\Delta}{\mathfrak{R}}\left(1 - \frac{1}{2\mathfrak{R}k\Delta} + \ldots\right) \tag{B3}$$

where

$$\mathfrak{R} = \frac{\frac{\rho_1}{\rho_2} + 1}{\frac{\rho_1}{\rho_2} - 1} \tag{27}$$

Since, in this case, $k\Delta$ is real and ω is purely imaginary,

$$\omega_i = \pm\sqrt{\frac{(k\Delta)}{\mathfrak{R}} - \left(4 + \frac{1}{2\mathfrak{R}^2} + \ldots\right)} \tag{25}$$

Now from the definition of Δ (eq. 10),

$$\tilde{\omega}^2\Delta^2 = \tilde{\omega}^2 - 4 \tag{B4}$$

Thus,

$$\Delta^2 = \frac{-\dfrac{k\Delta}{\mathcal{R}} + \dfrac{1}{2\mathcal{R}^2} + \ldots}{-\dfrac{k\Delta}{\mathcal{R}} + 4 + \dfrac{1}{2\mathcal{R}^2} + \ldots} = 1 + \frac{4}{\dfrac{k\Delta}{\mathcal{R}} - \left(4 + \dfrac{1}{2\mathcal{R}^2}\right)} + \ldots \tag{B5}$$

The wave number k is then given by the expression

$$k = \frac{k\Delta}{\sqrt{1 + \dfrac{4}{\omega_i^2}}} \tag{26}$$

APPENDIX C

SOLUTION OF DISPERSION RELATION FOR UNBOUNDED CONFIGURATION FOR SMALL WAVE NUMBER

The dispersion relation for the unbounded configuration $R \to \infty$ (eq. (22)) is to be solved in the limit of small wave number k. The first step in so doing is the evaluation of

$$\frac{\left[J_m(ik\Delta r)\right]'_{r=1}}{J_m(ik\Delta)}$$

and

$$\frac{\left[H_m^{(1)}\ (ik\Delta r)\right]'_{r=1}}{H_m^{(1)}\ (ik\Delta)}$$

for small values of the argument $k\Delta$. When the appropriate relations are used for $k\Delta \to 0$,

$$\frac{\left[J_m(ik\Delta r)\right]'_{r=1}}{J_m(ik\Delta)} = m + \ldots \tag{C1}$$

and

$$\frac{\left[H_m^{(1)}\ (ik\Delta r)\right]'_{r=1}}{H_m^{(1)}\ (ik\Delta)} = -m + \ldots \tag{C2}$$

The dispersion relation (22) for small $k\Delta$ may be written

$$\left(\tilde{\omega}^2 - 4\right)\left(\Re\tilde{\omega}^2 + 2\tilde{\omega} + m\right) = 0 \tag{C3}$$

where

$$\Re \equiv \frac{\dfrac{\rho_1}{\rho_2} + 1}{\dfrac{\rho_1}{\rho_2} - 1} \tag{27}$$

The solutions of this dispersion relation are

$$\tilde{\omega} = \pm 2 \tag{C4}$$

$$\tilde{\omega} = \frac{1}{\Re}\left(-1 \pm i\sqrt{m\Re - 1}\right) \tag{C5}$$

For solutions (C4), $\Delta^2 = 0$. These solutions are rejected since the conditions $\Delta_r > 0$ is not satisfied, namely, that disturbance amplitudes must decay exponentially as the radius becomes infinite.

Solutions (C5) are both acceptable; that with a positive imaginary part indicates growth of the disturbance, while the other indicates decay. Both are possible, but only the growth solution is of interest here since it persists.

APPENDIX D

SOLUTION OF DISPERSION RELATION FOR UNBOUNDED CONFIGURATION IN LARGE WAVE NUMBER LIMIT

For large values of the argument $k\Delta$,

$$\frac{\left[J_m(ik\Delta r)\right]'_{r=1}}{J_m(ik\Delta)} \approx k\Delta\left(1 - \frac{1}{2k\Delta} + \ . \ . \ .\right) \tag{D1}$$

and

$$\frac{\left[H_m^{(1)}\ (ik\Delta r)\right]'_{r=1}}{H_m^{(1)}\ (ik\Delta)} = -k\Delta\left(1 + \frac{1}{2k\Delta} + \ . \ . \ .\right) \tag{D2}$$

Equations (D1) and (D2) are independent of the flute number m, to the order considered, so that the dispersion relation (22) may be approximated as

$$\tilde{\omega}^2 - 4 \approx -\frac{k\Delta}{\Re} + \frac{1}{\Re^2}\left(\frac{2m}{\tilde{\omega}} + \frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{k\Delta}\right) \tag{D3}$$

The last term in (D3) is of order $1/(k\Delta)^2$ compared with the leading term on the right side of (D3), and it will be neglected.

Equation (D3) may now be alternatively written

$$\tilde{\omega}^3 + \tilde{\omega}\,\omega_{i,0}^2 = \frac{2m}{\Re^2} \tag{D4}$$

where $\omega_{i,0}^2$ represents the grouping of terms

$$\omega_{i,0}^2 = \frac{k\Delta}{\Re} - \left(4 + \frac{1}{2\Re^2}\right) \tag{D5}$$

The right side of equation (D4) is always of unit order. For large values of $k\Delta$ (therefore, large values of $\omega_{i,0}$), the left side becomes very large and suggests a solution of the form

$$\tilde{\omega} = i\omega_{i,0} + \tilde{\omega}_1 \tag{D6a}$$

where

$$\left|\frac{\tilde{\omega}_1}{\omega_{i,0}}\right| \ll 1 \tag{D6b}$$

Substituting (D6a) into (D4) and keeping only the leading terms of the expansion yields

$$\tilde{\omega}_1 = -\frac{m}{\Re^2\omega_{i,0}^2} \tag{D7}$$

which is seen to satisfy (D6b) for $k\Delta \gg 1$ and is the leading contribution to the real part of $\tilde{\omega}$.

To obtain the asymptotic form of the wave number k, the quantity Δ must first be evaluated. From its definition,

$$\Delta^2 = 1 - \frac{4}{\tilde{\omega}^2} \tag{10}$$

With the results (D6a) and (D7),

$$\Delta^2 = 1 + \frac{4}{\omega_{i,0}^2} + \mathcal{O}\left(\frac{1}{\omega_{i,0}^5}\right) \tag{D8}$$

When terms are kept up to order $1/\omega_{i,0}^3$ compared with the leading terms

$$\tilde{\omega} = i\omega_{i,0} - \frac{m}{\Re^2\omega_{i,0}^2} \tag{D9}$$

$$\Delta = \sqrt{1 + \frac{4}{\omega_{i,0}^2}} \tag{D10}$$

and

$$k = \frac{k\Delta}{\sqrt{1 + \frac{4}{\omega_{i,0}^2}}} \tag{D11}$$

To this order of approximation, Δ is a real quantity, and, therefore, $k\Delta$ is real as well.

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